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Spherically Symmetric Solutions to Compressible Hydrodynamic Flow of Liquid Crystals in N Dimensions*

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Abstract The paper is concerned with the system modeling the compressible hydrodynamic flow of liquid crystals with radially symmetric initial data and non-negative initial density in dimension N $(N \geq 2)$. The authors obtain the existence of global radially symmetric strong solutions in a bounded or unbounded annular domain for any $\gamma > 1$.

Keywords Liquid crystals, Compressible hydrodynamic flow, Global solutions
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1 Introduction

In this paper, we consider the N $(N \geq 2)$ dimensional initial boundary value problem for the hydrodynamic flow equations of liquid crystals:

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, & (1.1a) \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla(P(\rho)) \\ \quad = \mu \Delta \mathbf{u} + (\lambda + \mu)\nabla \operatorname{div}\mathbf{u} - \nu \operatorname{div}\Big(\nabla \mathbf{n} \odot \nabla \mathbf{n} - \dfrac{|\nabla \mathbf{n}|^2}{2} I_N\Big), & (1.1b) \\ \mathbf{n}_t + (\mathbf{u} \cdot \nabla)\mathbf{n} = \theta(\Delta \mathbf{n} + |\nabla \mathbf{n}|^2 \mathbf{n}) & (1.1c) \end{cases}$$

for $(x, t) \in \Omega \times (0, +\infty)$, where Ω is a bounded or unbounded annulus in $\mathbb{R}^N$ and the given data are radially symmetric. More precisely, the domain Ω is given by $\Omega = \{\mathbf{x} \in \mathbb{R}^N : a < |\mathbf{x}| < b\}$ for some constants a and b with $0 < a < b \leq \infty$. The initial conditions are given by

$$(\rho, \mathbf{u}, \mathbf{n})\big|_{t=0} = (\rho_0, \mathbf{u}_0, \mathbf{n}_0), \quad \text{in } \Omega, \tag{1.2}$$

where $\mathbf{n}_0 : \Omega \to S^2$ and

$$\rho_0(\mathbf{x}) = \rho_0(|\mathbf{x}|), \quad \mathbf{u}_0(\mathbf{x}) = u_0(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}, \quad \mathbf{n}_0(\mathbf{x}) = \mathbf{n}_0(|\mathbf{x}|) \quad \text{for } \mathbf{x} \in \Omega.$$

And the boundary conditions are imposed as follows:

$$\begin{aligned} &\mathbf{u} = 0, \quad \frac{\partial \mathbf{n}}{\partial \mathbf{v}} = 0 \ \text{ for } |\mathbf{x}| = a \text{ or } b, \quad t > 0, \ \text{ if } b < \infty, \\ &\mathbf{u} \to 0, \quad \mathbf{n} \to \mathbf{d}_0 \ \text{ as } |\mathbf{x}| \to a \text{ or } b, \quad t > 0, \ \text{ if } b = \infty \end{aligned} \tag{1.3}$$

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for some constant $\mathbf{d}_0 \in S^2$. Here $\rho \geq 0$ denotes the density function, $\mathbf{u}$ denotes the velocity field, and $\mathbf{n}$ denotes the optical axis vector of the liquid crystal which is a unit vector (i.e., $|\mathbf{n}| = 1$). $\mathbf{v}$ is the outward unitary normal vector on $\partial\Omega$. μ and λ are the shear viscosity and bulk viscosity coefficients of the fluid respectively, which satisfy the physical conditions $\mu > 0$ and $2\mu + N\lambda \geq 0$. The constants $\nu > 0$ and $\theta > 0$ are competitive between kinetic and potential energy, and microscopic elastic relaxation time respectively. $P = R\rho^\gamma$, for some constants $\gamma > 1$ and $R > 0$, is the pressure function. The symbols $\otimes$ and $\odot$ denote the tensor product, such that

$$\mathbf{u} \otimes \mathbf{u} = (u^i u^j)_{N\times N}, \quad \nabla\mathbf{n} \odot \nabla\mathbf{n} = (\mathbf{n}_{x_i} \cdot \mathbf{n}_{x_j})_{N\times N}.$$

The main goal of this paper is to prove the global existence of radially symmetric solutions to the initial boundary value problem (1.1)–(1.3), where the initial data satisfy the natural compatibility condition

$$-\mu\Delta\mathbf{u}_0 + \nabla(P(\rho_0)) + \nu \mathrm{div}\Big(\nabla\mathbf{n}_0 \odot \nabla\mathbf{n}_0 - \frac{|\nabla\mathbf{n}_0|^2}{2} I_N\Big) = \rho_0^{\frac{1}{2}}\mathbf{g} \tag{1.4}$$

for some radially symmetric $\mathbf{g} \in L^2(\Omega)$.

The hydrodynamic flow of compressible (or incompressible) liquid crystals was first derived by Ericksen [6] and Leslie [13] in 1960s. However, its rigorous mathematical analysis had not taken place until 1990s, when Lin [14], Lin and Liu [16–18] made some very important progress towards the existence of global weak solutions and partial regularity of the incompressible hydrodynamic flow equation of liquid crystals.

When the Ossen-Frank energy configuration functional reduces to the Dirichlet energy functional, the hydrodynamic flow equation of liquid crystals in $\Omega \subset \mathbb{R}^N$ can be written as (1.1) (see [4, 14]).

The spherically symmetric Cauchy and initial boundary value problems for Navier-Stokes equations have been studied by a number of mathematicians in the last decades. For isothermal flows, Hoff [9] proved the global existence of spherically symmetric weak solutions with strictly positive initial densities in annular domains. Then Jiang and Zhang obtained in [11] global spherically symmetric solutions to the compressible isentropic Navier-Stokes equations for the Cauchy problem for any $\gamma > 1$ with non-negative initial densities. Weigant [22] constructed a radially symmetric strong solution $(\rho, \mathbf{u})$ in $(0,1) \times B_R$ in the case $a = 0$ and $1 < \gamma < 1 + \frac{1}{N-1}$, such that

$$\|\rho\|_{L^\infty(B_R)} \to \infty, \quad \text{as } t \to 1,$$

where $B_R := \{\mathbf{x} \in \mathbb{R}^{\mathrm{N}}, |\mathbf{x}| < R\}$. Choe and Kim [3] showed the global existence of strong solutions to the Dirichlet boundary problem with non-negative bounded densities and the initial data satisfying the natural compatibility condition, where the restriction $\gamma \geq 2$ is requested in order to get the higher regularity about densities. In 2009, Fan, Jiang and Ni [7] proved the global existence of radially symmetric strong solutions for any $\gamma > 1$, improving therefore the corresponding result in [3].

For the beginning, we construct the corresponding system for radial solutions. If we let $r = |\mathbf{x}|$ and take

$$\rho = \rho(r,t), \quad \mathbf{u} = u(r,t)\frac{\mathbf{x}}{r}, \quad \mathbf{n} = \mathbf{n}(r,t),$$

then the system (1.1)–(1.3) becomes

$$
\begin{cases}
\rho_t + (\rho u)_r + \dfrac{m}{r}\rho u = 0, & (1.5\text{a})\\
(\rho u)_t + (\rho u^2)_r + \dfrac{m}{r}\rho u^2 + P_r = \kappa\Big(u_r + m\dfrac{u}{r}\Big)_r - \dfrac{\lambda}{2}(|\mathbf{n}_r|^2)_r - \lambda\dfrac{m}{r}|\mathbf{n}_r|^2, & (1.5\text{b})\\
\mathbf{n}_t + u\mathbf{n}_r = \theta\mathbf{n}_{rr} + \theta|\mathbf{n}_r|^2\mathbf{n} + \theta\dfrac{m}{r}\mathbf{n}_r & (1.5\text{c})
\end{cases}
$$

for $(r,t)\in(a,b)\times(0,+\infty)$, where $m = N-1$ and $\kappa = \lambda + 2\mu > 0$. This problem is subjected to the following initial boundary value conditions:

$$
(\rho,\ u,\ \mathbf{n})\big|_{t=0} = (\rho_0,\ u_0,\ \mathbf{n}_0), \quad \text{in} \ \ [a,b], \tag{1.6}
$$

$$
\begin{aligned}
&u = 0, \quad \ \mathbf{n}_r = 0 \ \ \text{for } r = a \text{ or } b, \quad t>0, \quad \text{if } b<\infty;\\
&u \to 0, \quad \ \mathbf{n}\to\mathbf{d}_0 \ \ \text{as } r\to a \text{ or } b, \quad t>0, \quad \text{if } b=\infty
\end{aligned} \tag{1.7}
$$

for some constant $\mathbf{d}_0\in S^2$

Before stating the main results, we explain the notations and conventions used throughout this paper.

Notation 1.1 (1) $Q_T = \Omega\times(0,T]$, $Q_T^r = (a,b)\times(0,T]$ for $T>0$.

(2) For $p\ge 1$, denote by $L^p = L^p(\Omega)$ the L^p space with the norm $\|\cdot\|_{L^p}$. For $k\ge 1$ and $p\ge 1$, denote by $W^{k,p} = W^{k,p}(\Omega)$ the Sobolev space, whose norm is denoted by $\|\cdot\|_{W^{k,p}}$. Furthermore, let $H^k = W^{k,2}(\Omega)$.

(3) $D^{k,r} = \{v\in L^1_{\text{loc}}(\Omega) : \|\nabla^k v\|_{L^r}<\infty\}$, $D^k = D^{k,2}(\Omega)$ and $D_0^1 = D_0^{1,2}(\Omega)$.

(4) For simplicity, denote

$$
\int_a^b f = \int_a^b f\,\mathrm{d}r \quad \text{and} \quad \int_0^t\int_a^b f = \int_0^t\int_a^b f\,\mathrm{d}r\mathrm{d}t.
$$

Theorem 1.1 *Let $0\le\rho_0\in L^1\cap H^1$, $\mathbf{u}_0\in D_0^1\cap D^2$, $\nabla\mathbf{n}_0\in H^2$ and $|\mathbf{n}_0| = 1$ in $\overline{\Omega}$. If, in addition, we assume that $(\rho_0,\mathbf{u}_0,\mathbf{n}_0)$ satisfies the compatibility condition* (1.4)*, then for any $T>0$, there exists a unique radially symmetric strong solution $(\rho,\mathbf{u},\mathbf{n})$ to the initial boundary value problem* (1.1)–(1.3) *satisfying*

$$
\begin{aligned}
&\rho\in L^\infty(0,T;L^1\cap H^1), \quad \rho_t\in L^\infty(0,T;L^2), \quad \rho\ge 0,\\
&\mathbf{u}\in L^\infty(0,T;D_0^1\cap D^2), \quad \mathbf{u}_t\in L^\infty(0,T;D_0^1), \quad \sqrt{\rho}\,\mathbf{u}_t\in L^\infty(0,T;L^2),\\
&\nabla\mathbf{n}\in L^\infty(0,T;H^2)\cap L^2(0,T;H^3), \quad \mathbf{n}_t\in L^\infty(0,T;H^1)\cap L^2(0,T;H^2).
\end{aligned}
$$

Theorem 1.2 *Let $0\le\rho_0\in L^1\cap H^2$, $\nabla\rho_0^{\frac{\gamma}{2}}\in L^4$, $\mathbf{u}_0\in D_0^1\cap D^2$ and $\nabla\mathbf{n}_0\in H^2$, $|\mathbf{n}_0| = 1$ in $\overline{\Omega}$. If, in addition, we assume that $(\rho_0,\mathbf{u}_0,\mathbf{n}_0)$ satisfies the compatibility condition* (1.4)*, then for any $T>0$, there exists a unique radially symmetric strong solution $(\rho,\mathbf{u},\mathbf{n}) : \Omega\times[0,\infty)\to[0,\infty)\times\mathbb{R}\times S^2$ to the initial boundary value problem* (1.1)–(1.3) *satisfying*

$$
\begin{aligned}
&\rho\in L^\infty(0,T;L^1\cap H^2), \quad \rho_t\in L^\infty(0,T;H^1), \quad \rho\ge 0,\\
&\mathbf{u}\in L^\infty(0,T;D_0^1\cap D^2)\cap L^2(0,T;D^3), \quad \mathbf{u}_t\in L^2(0,T;D_0^1), \quad \sqrt{\rho}\,\mathbf{u}_t\in L^\infty(0,T;L^2),\\
&\nabla\mathbf{n}\in L^\infty(0,T;H^2)\cap L^2(0,T;H^3), \quad \mathbf{n}_t\in L^\infty(0,T;H^1)\cap L^2(0,T;H^2).
\end{aligned}
$$

Remark 1.1 In addition, if $\rho_0 \geq \varepsilon$ in Ω for some constant $\varepsilon > 0$, then the radially symmetric solution to (1.1)–(1.3) under the hypothesis mentioned in Theorem 1.1 additionally satisfies the following regularity:

$$\rho \geq c(\varepsilon) > 0, \quad \mathbf{u} \in L^\infty(0,T; H_0^1 \cap H^2), \quad \mathbf{u}_t \in L^\infty(0,T; L^2) \cap L^2(0,T; H^1).$$

Since the constants R, κ, ν and θ in (1.1) do not play any role in the analysis, we assume henceforth that

$$R = \kappa = \nu = \theta = 1.$$

2 Existence of Local Strong Solutions

In this section, we employ the Schauder fixed point theorem to prove that there exists a unique short time strong solution to the problem (1.5)–(1.7) when ρ_0 has a positive lower bound. We only consider the case $0 < a < b < \infty$ in this section.

For simplicity, we let $L_r^p = L^p(a,b)$, $W_r^{k,p} = W^{k,p}(a,b)$, $W_{0,r}^{k,p} = W_0^{k,p}(a,b)$, $H_{0,r}^k = H_0^k(a,b)$ and $W_{2,r}^{2k,k}(Q_T^r) = W_2^{2k,k}((a,b) \times (0,T])$ below.

Now we state the main result of this section.

Theorem 2.1 *If $\rho_0 \in H_r^2$, $0 < c_0^{-1} \leq \rho_0 \leq c_0$ for some c_0, $\mathbf{u}_0 \in H_{0,r}^1 \cap H_r^2$, $\nabla \mathbf{n}_0 \in H_r^2$ and $|\mathbf{n}_0| = 1$ in $\overline{\Omega}$, then there exists a small time $T^* > 0$, a constant c depending on T^* and a unique strong solution $(\rho, u, \mathbf{n})$ to the initial boundary value problem* (1.5)–(1.7)*, such that*

$$\begin{aligned}
&\rho \in L^\infty(0,T^*; H_r^2), \quad \rho_t \in L^\infty(0,T^*; H_r^1), \quad 0 < c^{-1} \leq \rho \leq c,\\
&u \in L^\infty(0,T^*; H_{0,r}^1 \cap H_r^2) \cap L^2(0,T^*; H_r^3), \quad u_t \in L^\infty(0,T^*; L_r^2) \cap L^2(0,T^*; H_{0,r}^1),\\
&\mathbf{n} \in L^\infty(0,T^*; H_r^3) \cap W_{2,r}^{4,2}(Q_{T^*}^r), \quad \mathbf{n}_t \in L^\infty(0,T^*; H_r^1) \cap L^2(0,T^*; H_r^2).
\end{aligned}$$

To prove Theorem 2.1, we firstly introduce a new variable $\sigma = \rho r^m$, and we can rewrite the problem (1.5)–(1.7) as an equivalent one

$$\begin{cases}
\sigma_t + (\sigma u)_r = 0, & (2.1\text{a})\\
(\sigma u)_t + (\sigma u^2)_r + r^m P_r = r^m \Big(u_r + m\dfrac{u}{r}\Big)_r - \dfrac{1}{2} r^m (|\mathbf{n}_r^2|)_r - m r^{m-1} |\mathbf{n}_r|^2, & (2.1\text{b})\\
\mathbf{n}_t + u\mathbf{n}_r = \mathbf{n}_{rr} + |\mathbf{n}_r|^2 \mathbf{n} + \dfrac{m}{r}\mathbf{n}_r, & (2.1\text{c})
\end{cases}$$

supplemented by the following initial and boundary conditions:

$$(\sigma, u, \mathbf{n})\big|_{t=0} = (r^m \rho_0, u_0, \mathbf{n}_0), \quad \text{in } [a,b], \tag{2.2}$$

$$u(a,t) = u(b,t) = 0, \quad \mathbf{n}_r(a,t) = \mathbf{n}_r(b,t) = 0, \quad \forall t > 0. \tag{2.3}$$

We prove the local existence of a unique strong solution to problem (2.1)–(2.3) by using a standard fixed point argument. Now consider the following linearized problem:

$$\begin{cases}
\sigma_t + (\sigma v)_r = 0, & (2.4\text{a})\\
(\sigma u)_t + (\sigma v u)_r + r^m P_r = r^m \Big(u_r + m\dfrac{u}{r}\Big)_r - \dfrac{1}{2} r^m (|\mathbf{n}_r|^2)_r - m r^{m-1} |\mathbf{n}_r|^2, & (2.4\text{b})\\
\mathbf{n}_t + v\mathbf{n}_r = \mathbf{n}_{rr} + |\mathbf{m}_r|^2 \mathbf{n} + \dfrac{m}{r}\mathbf{n}_r, & (2.4\text{c})
\end{cases}$$

subjected to the initial and boundary conditions (2.2)–(2.3), where $(v, \mathbf{m})$ are known smooth functions which satisfy the boundary value conditions $v(a,t) = v(b,t) = 0$ for $t > 0$, and $\mathbf{m}_r(a,t) = \mathbf{m}_r(b,t) = 0$, while $\sigma(0) \geq c_0^{-1} a^m > 0$ on $[a,b]$. Now we begin to prove the local existence of a strong solution to (2.1)–(2.3).

Define

$$R_{T^*} \equiv \left\{ (v,\mathbf{m}) \left| \begin{array}{l} \|v\|^2_{H^1_{0,r}\cap H^2_r} + \|v_t\|^2_{L^2_r} + \displaystyle\int_0^{T^*} (\|v\|^2_{H^3_r} + \|v_t\|^2_{H^1_r}) \leq K_1, \\ \|\mathbf{m}\|^2_{H^3_r} + \|\mathbf{m}_t\|^2_{H^1_r} + \displaystyle\int_0^{T^*} \|\mathbf{m}_t\|^2_{H^2_r} \leq K_2, \\ v(r,0) = u_0(r), \quad \mathbf{m}(r,0) = \mathbf{n}_0(r), \quad r \in [a,b] \end{array} \right. \right\},$$

where K_1, K_2 and T^* will be decided later. Without loss of generality, assume that $K_1, K_2 > 1$. In this section, we denote by C the constant depending only on a, b and the initial data, but independent of K_1 and K_2.

The existence of the unique strong solution to the hyperbolic equation (2.4a) is well-known. Moreover, the solution σ satisfies the following estimate (see [21]):

$$\sup_{0\leq t\leq T} (\|\sigma(t)\|_{H^2_r} + K_1^{-\frac{1}{2}} \|\sigma_t(t)\|_{H^1_r} + \|\sigma(t)^{-1}\|_{L^\infty}) \leq C \exp(CK_1 T^{\frac{1}{2}}). \tag{2.5}$$

If we choose $T^* > 0$ sufficiently small, such that $T^* \leq T_1 = \frac{1}{K_1^2}$, then we get

$$\exp(CK_1 (T^*)^{\frac{1}{2}}) \leq C. \tag{2.6}$$

Furthermore, we can get the similar estimates about ρ. Note that (2.4c) is a linear parabolic equation for $\mathbf{n}$, and (2.4b) can also be written as a parabolic equation for u: $u_t + vu_r - \sigma^{-1} r^m \big(u_r + \frac{m}{r} u\big)_r = -\sigma^{-1} r^m P_r - \frac{1}{2} \sigma^{-1} r^m (|\mathbf{n}_r|^2)_r - m\sigma^{-1} r^{m-1} |\mathbf{n}_r|^2$. Then by the theory of parabolic equations, we get the existence of the unique strong solution $(u, \mathbf{n})$. Define the map Φ

$$\Phi : R_{T^*} \to R_{T^*}, \quad (v, \mathbf{m}) \mapsto (u, \mathbf{n}).$$

We need some a priori estimates for u and $\mathbf{n}$.

First of all, multiplying (2.4c) by $\mathbf{n}$, we can construct a parabolic equation for $|\mathbf{n}|^2$. Then, using the facts that $|\mathbf{m}_r|^2 \leq K_2$, $|\mathbf{n}_0| = 1$, and the maximum principle, we have

$$|\mathbf{n}|^2 \leq \exp(K_2 T^*). \tag{2.7}$$

Taking T^* sufficiently small such that $T^* \leq T_2 = \min\big\{T_1, \frac{1}{K_2}\big\}$, we have

$$|\mathbf{n}| \leq C. \tag{2.8}$$

Then, multiplying (2.4c) by $\mathbf{n}_{rr}$ and using integration by parts, we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_a^b |\mathbf{n}_r|^2 + \int_a^b |\mathbf{n}_{rr}|^2 \\ &= \int_a^b v(\mathbf{n}_r \cdot \mathbf{n}_{rr}) - \int_a^b |\mathbf{m}_r|^2 (\mathbf{n} \cdot \mathbf{n}_{rr}) - \int_a^b \frac{m}{r} \mathbf{n}_r \cdot \mathbf{n}_{rr} \\ &\leq \frac{1}{2} \int_a^b |\mathbf{n}_{rr}|^2 + C\Big(\|v\|^2_{L^\infty} + \frac{m^2}{a^2}\Big) \int_a^b |\mathbf{n}_r|^2 + C\|\mathbf{n}\|^2_{L^\infty} \int_a^b |\mathbf{m}_r|^4 \\ &\leq \frac{1}{2} \int_a^b |\mathbf{n}_{rr}|^2 + CK_1 \int_a^b |\mathbf{n}_r|^2 + CK_2^2, \end{aligned} \tag{2.9}$$

where we have used the Sobolev embedding $H^1 \hookrightarrow L^\infty$. Integrating the above inequality over $(0,t)$, we obtain

$$\int_a^b |\mathbf{n}_r|^2 + \int_0^t \int_a^b |\mathbf{n}_{rr}|^2 \le \int_a^b |\mathbf{n}_{0r}|^2 + CK_1 \int_0^{T^*} \int_a^b |\mathbf{n}_r|^2 + CK_2^2 t. \tag{2.10}$$

By using the Gronwall inequality, we have

$$\int_a^b |\mathbf{n}_r|^2 \le \Big(\int_a^b |\mathbf{n}_{0r}|^2 + CK_2^2 T^* \Big) \exp(CK_1 T^*) \le C \tag{2.11}$$

for $0 < t < T^*$, where we have chosen T^* sufficiently small, such that $T^* \le T_3 = \min\big\{T_2, \frac{1}{K_2^2}\big\}$. Then (2.10) implies that $\int_0^{T^*}\!\int_a^b |\mathbf{n}_{rr}|^2 \le C$.

Secondly, differentiating (2.4c) with respect to r, multiplying the resulting equation by $\mathbf{n}_{rrr}$, and then integrating it over (a,b), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_a^b |\mathbf{n}_{rr}|^2 + \int_a^b |\mathbf{n}_{rrr}|^2 \\
&= \int_a^b v_r(\mathbf{n}_r \cdot \mathbf{n}_{rrr}) + \int_a^b v(\mathbf{n}_{rr} \cdot \mathbf{n}_{rrr}) - 2\int_a^b (\mathbf{m}_r \cdot \mathbf{m}_{rr})(\mathbf{n} \cdot \mathbf{n}_{rrr}) \\
&\quad - \int_a^b |\mathbf{m}_r|^2 (\mathbf{n}_r \cdot \mathbf{n}_{rrr}) + \int_a^b \frac{m}{r^2}(\mathbf{n}_r \cdot \mathbf{n}_{rrr}) - \int_a^b \frac{m}{r}(\mathbf{n}_{rr} \cdot \mathbf{n}_{rrr}) \\
&\le \frac{1}{2}\int_a^b |\mathbf{n}_{rrr}|^2 + C\|v_r\|_{L^\infty}^2 \int_a^b \mathbf{n}_r^2 + C\|v\|_{L^\infty}^2 \int_a^b |\mathbf{n}_{rr}|^2 \\
&\quad + C\|\mathbf{n}\|_{L^\infty}^2 \|\mathbf{m}_r\|_{L^\infty}^2 \int_a^b |\mathbf{m}_{rr}|^2 + C\|\mathbf{m}_r\|_{L^\infty}^4 \int_a^b |\mathbf{n}_r|^2 + C\frac{m^2}{a^4}\int_a^b |\mathbf{n}_r|^2 + C\frac{m^2}{a^2}\int_a^b |\mathbf{n}_{rr}|^2 \\
&\le \frac{1}{2}\int_a^b |\mathbf{n}_{rrr}|^2 + CK_1 \int_a^b |\mathbf{n}_{rr}|^2 + C(K_1 + K_2^2).
\end{aligned} \tag{2.12}$$

Then, integrating over $(0,t)$, we get

$$\int_a^b |\mathbf{n}_{rr}|^2 + \int_0^t \int_a^b |\mathbf{n}_{rrr}|^2 \le \int_a^b |\mathbf{n}_{0rr}|^2 + CK_1 \int_0^t \int_a^b |\mathbf{n}_{rr}|^2 + C(K_1 + K_2^2)t. \tag{2.13}$$

Then, by the Gronwall inequality, we have

$$\int_a^b |\mathbf{n}_{rr}|^2 \le \Big[\int_a^b |\mathbf{n}_{0rr}|^2 + C\Big(K_1 + K_2^2\Big)T^* \Big] \exp(CK_1 T^*) \le C \tag{2.14}$$

for $0 < t < T^*$, where we have chosen T^* small enough, such that $T^* \le T_4 = \min\big\{T_3, \frac{1}{K_1 + K_2^2}\big\}$. Then (2.13) implies that $\int_0^{T^*}\!\int_a^b |\mathbf{n}_{rrr}|^2 \le C$.

On the other hand, differentiating (2.4c) with respect to t, multiplying the resulting equation by $\mathbf{n}_t$, and then integrating over (a,b), we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_a^b |\mathbf{n}_t|^2 + \int_a^b |\mathbf{n}_{rt}|^2 &= -\int_a^b v(\mathbf{n}_{rt} \cdot \mathbf{n}_t) - \int_a^b v_t(\mathbf{n}_r \cdot \mathbf{n}_t) + \int_a^b |\mathbf{m}_r|^2 |\mathbf{n}_t|^2 \\
&\quad + 2\int_a^b (\mathbf{m}_r \cdot \mathbf{m}_{rt})(\mathbf{n} \cdot \mathbf{n}_t) + \int_a^b \frac{m}{r}(\mathbf{n}_{rt} \cdot \mathbf{n}_t)
\end{aligned}$$

$$\begin{aligned}
&\leq C(\|v\|_{L^\infty}^2 + \|\mathbf{n}_r\|_{L^\infty}^2 + \|\mathbf{m}_r\|_{L^\infty}^2 + \|\mathbf{n}\|_{L^\infty}^2 + 1)\int_a^b |\mathbf{n}_t|^2 \\
&\quad + \frac{1}{2}\int_a^b |\mathbf{n}_{rt}|^2 + C\int_a^b |v_t|^2 + C\|\mathbf{m}_r\|_{L^\infty}^2 \int_a^b |\mathbf{m}_{rt}|^2 \\
&\leq \frac{1}{2}\int_a^b |\mathbf{n}_{rt}|^2 + C(K_1 + K_2)\int_a^b |\mathbf{n}_t|^2 + C(K_1 + K_2^2). \qquad (2.15)
\end{aligned}$$

Then, integrating over $(0, t)$ and using (2.4c), we have

$$\begin{aligned}
\int_a^b |\mathbf{n}_t|^2 + \int_0^{T^*}\int_a^b |\mathbf{n}_{rt}|^2 &\leq \int_a^b |\mathbf{n}_t(x,0)|^2 + C(K_1 + K_2)\int_0^t\int_a^b |\mathbf{n}_t|^2 + C(K_1 + K_2^2)t \\
&\leq C + C(K_1 + K_2)\int_0^t\int_a^b |\mathbf{n}_t|^2 + C(K_1 + K_2^2)t. \qquad (2.16)
\end{aligned}$$

By the Gronwall inequality, one obtains

$$\int_a^b |\mathbf{n}_t|^2 + \int_0^t\int_a^b |\mathbf{n}_{rt}|^2 \leq C \quad \text{for } 0 < t < T^*. \qquad (2.17)$$

Thirdly, differentiating (2.4c) with respect to r, and then multiplying the resulting equation by $\mathbf{n}_{trrr}$, using integration by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b |\mathbf{n}_{rrr}|^2 + \int_a^b |\mathbf{n}_{rrt}|^2 \\
&= \int_a^b v_r(\mathbf{n}_r \cdot \mathbf{n}_{trrr}) + \int_a^b v(\mathbf{n}_{rr} \cdot \mathbf{n}_{trrr}) - 2\int_a^b (\mathbf{m}_r \cdot \mathbf{m}_{rr})(\mathbf{n} \cdot \mathbf{n}_{trrr}) \\
&\quad - \int_a^b |\mathbf{m}_r|^2(\mathbf{n}_r \cdot \mathbf{n}_{trrr}) + \int_a^b \frac{m}{r^2}(\mathbf{n}_r \cdot \mathbf{n}_{trrr}) - \int_a^b \frac{m}{r}(\mathbf{n}_{rr} \cdot \mathbf{n}_{trrr}) \\
&= \sum_{k=1}^{6} \mathrm{I}_k. \qquad (2.18)
\end{aligned}$$

We utilize integration by parts and the Cauchy inequality to get

$$\begin{aligned}
\mathrm{I}_1 &= -\int_1^b v_{rr}(\mathbf{n}_r \cdot \mathbf{n}_{trr}) - \int_a^b v_r(\mathbf{n}_{rr} \cdot \mathbf{n}_{trr}) \\
&\leq C\|v_{rr}\|_{L^2}\|\mathbf{n}_r\|_{L^\infty}\|\mathbf{n}_{trr}\|_{L^2} + C\|v_r\|_{L^\infty}\|\mathbf{n}_{rr}\|_{L^2}\|\mathbf{n}_{trr}\|_{L^2} \\
&\leq \tau\|\mathbf{n}_{trr}\|_{L^2}^2 + C(\tau)K_1, \\
\mathrm{I}_2 &= -\int_a^b v_r(\mathbf{n}_{rr} \cdot \mathbf{n}_{trr}) - \int_a^b v(\mathbf{n}_{rrr} \cdot \mathbf{n}_{trr}) \\
&\leq C\|v_r\|_{L^\infty}\|\mathbf{n}_{rr}\|_{L^2}\|\mathbf{n}_{trr}\|_{L^2} + C\|v\|_{L^\infty}\|\mathbf{n}_{rrr}\|_{L^2}\|\mathbf{n}_{trr}\|_{L^2} \\
&\leq \tau\|\mathbf{n}_{trr}\|_{L^2}^2 + C(\tau)K_1 + C(\tau)K_1\|\mathbf{n}_{rrr}\|_{L^2}^2, \\
\mathrm{I}_3 &= 2\int_a^b |\mathbf{m}_{rr}|^2(\mathbf{n} \cdot \mathbf{n}_{trr}) + 2\int_a^b (\mathbf{m}_r \cdot \mathbf{m}_{rrr})(\mathbf{n} \cdot \mathbf{n}_{trr}) + 2\int_a^b (\mathbf{m}_r \cdot \mathbf{m}_{rr})(\mathbf{n}_r \cdot \mathbf{n}_{trr}) \\
&\leq C\|\mathbf{m}_{rr}\|_{L^4}^2\|\mathbf{n}\|_{L^\infty}\|\mathbf{n}_{trr}\|_{L^2} + C\|\mathbf{m}_r\|_{L^\infty}\|\mathbf{m}_{rrr}\|_{L^2}\|\mathbf{n}\|_{L^\infty}\|\mathbf{n}_{trr}\|_{L^2} \\
&\quad + C\|\mathbf{m}_r\|_{L^\infty}\|\mathbf{m}_{rr}\|_{L^2}\|\mathbf{n}_r\|_{L^\infty}\|\mathbf{n}_{trr}\|_{L^2} \\
&\leq \tau\|\mathbf{n}_{trr}\|_{L^2}^2 + C(\tau)K_2^2,
\end{aligned}$$

$$\begin{aligned}
\mathrm{I}_4 &= 2\int_a^b (\mathbf{m}_r\cdot\mathbf{m}_{rr})(\mathbf{n}_r\cdot\mathbf{n}_{trr}) + \int_a^b |\mathbf{m}_r|^2(\mathbf{n}_{rr}\cdot\mathbf{n}_{trr})\\
&\le C\|\mathbf{m}_r\|_{L^\infty}\|\mathbf{m}_{rr}\|_{L^2}\|\mathbf{n}_r\|_{L^\infty}\|\mathbf{n}_{trr}\|_{L^2} + C\|\mathbf{m}_r\|_{L^\infty}^2\|\mathbf{n}_{rr}\|_{L^2}\|\mathbf{n}_{trr}\|_{L^2}\\
&\le \tau\|\mathbf{n}_{trr}\|_{L^2}^2 + C(\tau)K_2^2,\\
\mathrm{I}_5 &= -\int_a^b \frac{2m}{r^3}(\mathbf{n}_r\cdot\mathbf{n}_{trr}) - \int_a^b \frac{m}{r^2}(\mathbf{n}_{rr}\cdot\mathbf{n}_{trr})\\
&\le C\|\mathbf{n}_r\|_{L^2}\|\mathbf{n}_{trr}\|_{L^2} + C\|\mathbf{n}_{rr}\|_{L^2}\|\mathbf{n}_{trr}\|_{L^2}\\
&\le \tau\|\mathbf{n}_{trr}\|_{L^2}^2 + C(\tau),\\
\mathrm{I}_6 &= -\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \frac{m}{r}(\mathbf{n}_{rr}\cdot\mathbf{n}_{rrr}) + \int_a^b \frac{m}{r}(\mathbf{n}_{trr}\cdot\mathbf{n}_{rrr})\\
&\le -\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \frac{m}{r}(\mathbf{n}_{rr}\cdot\mathbf{n}_{rrr}) + \tau\|\mathbf{n}_{trr}\|_{L^2}^2 + C(\tau)\|\mathbf{n}_{rrr}\|_{L^2}^2.
\end{aligned}$$

Then, choosing $\tau > 0$ small enough, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b |\mathbf{n}_{rrr}|^2 + \int_a^b |\mathbf{n}_{trr}|^2 \le CK_1\|\mathbf{n}_{rrr}\|_{L^2}^2 + C(K_1+K_2^2) - \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \frac{m}{r}(\mathbf{n}_{rr}\cdot\mathbf{n}_{rrr}). \tag{2.19}$$

Integrating over $(0,t)$, we have

$$\begin{aligned}
\int_a^b |\mathbf{n}_{rrr}|^2 + \int_0^t\int_a^b |\mathbf{n}_{trr}|^2 &\le \int_a^b |\mathbf{n}_{0rrr}|^2 + CK_1\int_0^t \|\mathbf{n}_{rrr}\|_{L^2}^2 + C(K_1+K_2^2)t\\
&\quad - \int_a^b \frac{m}{r}(\mathbf{n}_{rr}\cdot\mathbf{n}_{rrr}) + \int_a^b \frac{m}{r}[(\mathbf{n}_0)_{rr}\cdot(\mathbf{n}_0)_{rrr}]\\
&\le C + C(K_1+K_2^2)t + CK_1\int_0^t \|\mathbf{n}_{trr}\|_{L^2}^2 + \frac{1}{2}\|\mathbf{n}_{rrr}\|_{L^2}^2.
\end{aligned} \tag{2.20}$$

It yields from the Gronwall inequality that

$$\int_a^b |\mathbf{n}_{rrr}|^2 + \int_0^t\int_a^b |\mathbf{n}_{trr}|^2 \le C \quad \text{for } 0<t<T^*. \tag{2.21}$$

Finally, differentiating (2.4c) with respect to t, and then multiplying the resulting equation by $\mathbf{n}_{rrt}$ and using integration by parts, the Hölder inequality and the Cauchy inequality, we have $\int_a^b |\mathbf{n}_{rt}|^2 \le C$ for $0<t<T^*$.

Hence, it remains to prove the estimates for u. We first rewrite (2.4b) as

$$\sigma u_t + \sigma v u_r + r^m P_r = r^m\Big(u_r + m\frac{u}{r}\Big)_r - \frac{1}{2}r^m(|\mathbf{n}_r|^2)_r - mr^{m-1}|\mathbf{n}_r|^2. \tag{2.22}$$

Multiplying the above equation by u_t and then integrating over (a,b), we deduce that

$$\begin{aligned}
&\int_a^b \sigma u_t^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \Big(u_r^2 + m\frac{u^2}{r^2}\Big)r^m\\
&= -\int_a^b \sigma v u_r u_t - \int_a^b r^m P_r u_t - \int_a^b r^m(\mathbf{n}_r\cdot\mathbf{n}_{rr})u_t - m\int_a^b r^{m-1}|\mathbf{n}_r|^2 u_t\\
&\le \frac{1}{2}\int_a^b \sigma u_t^2 + C\|\rho\|_{L^\infty}\|v\|_{L^\infty}^2\int_a^b r^m u_r^2 + C\|\sigma\|_{L^\infty}\|\rho\|_{L^\infty}^{2\gamma-4}\int_a^b \rho_r^2\\
&\quad + C\Big\|\frac{1}{\rho}\Big\|_{L^\infty}\|\mathbf{n}_r\|_{L^\infty}^2\int_a^b |\mathbf{n}_{rr}|^2 + Cb^{m-2}\Big\|\frac{1}{\rho}\Big\|_{L^\infty}\int_a^b |\mathbf{n}_r|^4.
\end{aligned} \tag{2.23}$$

We conclude that

$$\int_a^b \sigma u_t^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b r^m\Big(u_r^2 + m\frac{u^2}{r^2}\Big) \le CK_1\int_a^b r^m u_r^2 + C. \tag{2.24}$$

By using the Gronwall inequality, we deduce

$$\|u\|_{H^1_{0,r}} + \int_0^{T^*}\|u_t\|^2_{L^2_r} \le C \quad \text{for } 0 < t < T^*. \tag{2.25}$$

Differentiating (2.22) with respect to t, multiplying the resulting equation by u_t, and then integrating over (a, b), we obtain from the Hölder inequality, the Poincaré inequality and the Cauchy inequality that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \sigma u_t^2 + \int_a^b \Big(u_{tr}^2 + m\frac{u_t^2}{r^2}\Big)r^m \\
&= -2\int_a^b \sigma v u_t u_{rt} - \int_a^b \sigma_t v u_r u_t - \int_a^b \sigma v_t u_r u_t + m\int_a^b r^{m-1}\gamma\rho^{\gamma-1}\rho_t u_t \\
&\quad + \int_a^b r^m\gamma\rho^{\gamma-1}\rho_t u_{rt} + m\int_a^b r^m(\mathbf{n}_r\cdot\mathbf{n}_{rt})u_{rt} - m\int_a^b r^{m-1}(\mathbf{n}_r\cdot\mathbf{n}_{rt})u_t \\
&\le C\|\rho\|_{L^\infty}\|v\|_{L^\infty}\|u_t\|_{L^2_r}\|r^{\frac{m}{2}}u_{rt}\|_{L^2_r} + C\|\sigma_t\|_{L^\infty}\|v\|_{L^\infty}\|u_r\|_{L^2_r}\|u_t\|_{L^2_r} \\
&\quad + C\|\sigma\|_{L^\infty}\|v_t\|_{L^2_r}\|u_r\|_{L^2_r}\|u_t\|_{L^\infty} + C\|\rho\|^{\gamma-1}_{L^\infty}\|\sigma_t\|_{L^2_r}\|u_t\|_{L^2_r} + C\|\rho\|^{\gamma-1}_{L^\infty}\|\sigma_t\|_{L^2_r}\|r^{\frac{m}{2}}u_{rt}\|_{L^2_r} \\
&\quad + C\|\mathbf{n}_r\|_{L^\infty}\|\mathbf{n}_{rt}\|_{L^2_r}\|r^{\frac{m}{2}}u_{rt}\|_{L^2_r} + C\|\mathbf{n}_r\|_{L^\infty}\|\mathbf{n}_{rt}\|_{L^2_r}\|u_t\|_{L^2_r} \\
&\le \frac{1}{2}\|r^{\frac{m}{2}}u_{rt}\|^2_{L^2_r} + CK_1\|u_t\|^2_{L^2_r} + CK_1.
\end{aligned} \tag{2.26}$$

Then, integrating the resulting inequality over $(0, t)$, we have

$$\|u_t\|^2_{L^2_r} + \int_0^t \|u_t\|^2_{H^1_{0,r}} \le CK_1\int_0^t \|u_t\|^2_{L^2_r} + CK_1 t + C, \tag{2.27}$$

where we have used the fact that $\int_a^b \sigma u_t^2(0) \le C(\|\rho_0\|_{H^1_r}, \|u_0\|_{H^2_r}, \|\mathbf{n}_0\|_{H^2_r})$ which we derived from (2.22). Then by the Gronwall inequality, we get

$$\|u_t\|^2_{L^2_r} + \int_0^t \|u_t\|^2_{H^1_{0,r}} \le C \quad \text{for } 0 < t < T^*. \tag{2.28}$$

From (2.22) and the previous estimates, we have

$$\begin{aligned}
\|u_{rr}\|^2_{L^2} &\le C + C\|vu_r\|^2_{L^2} \\
&\le C + C\|v - u_0\|^2_{L^2}\|u_r\|^2_{L^\infty} \\
&\le C + C\Big\|\int_0^t v_t(\|\cdot\|, s)\mathrm{d}s\Big\|^2_{L^2}(1 + \|u_{rr}\|^2_{L^2}) \\
&\le C + C_0K_1t(1 + \|u_{rr}\|^2_{L^2}).
\end{aligned} \tag{2.29}$$

Then we have $\|u_{rr}\|_{L^2} \le C$ provided that $T^* < T_5 = \min\big\{T_4, \frac{1}{2C_0K_1}\big\}$. The remaining regularity estimates for u can be derived in a similar way.

Thus we conclude that there exist some large positive constants K_1 and K_2, such that $\Phi(R_{T^*}) \subset R_{T^*}$ for T^* small enough.

Now we utilize Schauder's fixed point theorem to prove Theorem 2.1. Clearly, R_{T^*} is convex, and it is easy to see that it is closed in $X \equiv C([0,T^*];H_r^1) \times C([0,T^*];H_r^2)$. Moreover, in view of a standard embedding result, R_{T^*} is relatively compact in X. Hence, we need only to prove that Φ is continuous in X. Suppose that $(v_i,\mathbf{m}_i) \in R_{T^*}$, $(v_i,\mathbf{m}_i) \to (v,\mathbf{m})$ in X and set $(u_i,\mathbf{n}_i) \equiv \Phi(v_i,\mathbf{m}_i)$, $(u,\mathbf{n}) \equiv \Phi(v,\mathbf{m})$. Take the difference between the equations for $(\sigma_i,u_i,\mathbf{n}_i)$ and $(\sigma,u,\mathbf{n})$. By an energy argument and the Gronwall lemma, it is easy to show that $(u_i,\mathbf{n}_i)$ converges to $(u,\mathbf{n})$ in $L^\infty([0,T^*];L_r^2) \times L^\infty([0,T^*];H_r^1)$. From the compactness of $\overline{R_{T^*}}$, $(u_i,\mathbf{n}_i)$ converges indeed to $(u,\mathbf{n})$ in X. Hence, Φ is continuous, and it has a fixed point, which is the solution to the problem (1.5)–(1.7) in Q_{T^*}.

Finally, from the parabolic theory, we have $\mathbf{n} \in W_{2,r}^{4,2}(Q_{T^*}^r)$. Furthermore, multiplying (1.5c) by $\mathbf{n}$, we can construct an equation for $(|\mathbf{n}|^2-1)$, and then, multiplying the equation by $|\mathbf{n}|^2-1$ and using integration by parts, we can prove $|\mathbf{n}|=1$ for $(r,t) \in \overline{Q_{T^*}^r}$, by the Gronwall inequality. The uniqueness of the solution can be proved by the standard method similar to [5]. This completes the proof of Theorem 2.1.

3 A Priori Estimates Uniform in b and $\inf \rho_0$

In this section, we derive a priori estimates for radially symmetric solutions to (1.1)–(1.3), which are independent of b. As a corollary, we also prove a global existence result for problem (1.5)–(1.7). Throughout this section, we denote by C_i $(i=1,2,\cdots)$ some generic positive constants depending only on the norms of the initial data N, a and T, but independent of b and $\varepsilon = \inf \rho_0$.

We need the following lemma for the proof.

Lemma 3.1 (see [8]) *There exists a positive constant C, such that the following inequality holds for all $\mathbf{f} \in H_0^1(R_0,\infty)$ and $r^\alpha \mathbf{f}$, $r^\alpha \mathbf{f}_r \in L^2(R_0,\infty)$:*

$$\|r^\beta \mathbf{f}\|_{L^p(R_0,\infty)} \le C\|r^\alpha \mathbf{f}_r\|_{L^2(R_0,\infty)}^b \|r^\alpha \mathbf{f}\|_{L^2(R_0,\infty)}^{1-b} \tag{3.1}$$

if and only if the following relations hold:

$$\begin{aligned} &\frac{1}{p}+\beta = \frac{1}{2}+\alpha-b, \\ &\alpha-\sigma \ge 0, \quad \text{if } b>0, \\ &\alpha-\sigma \le 1, \quad \text{if } b>0, \quad \text{and} \quad \alpha-\frac{1}{2} = \frac{1}{p}+\beta, \end{aligned}$$

where $p>0$, $0 \le b \le 1$, $\alpha > -\frac{1}{2}$, $\beta > -\frac{1}{p}$ and $\beta = b\sigma + (1-b)\alpha$.

For $0<T<+\infty$, let $(\rho,u,\mathbf{n}) : \Omega \times [0,T) \to [0,\infty) \times \mathbb{R} \times S^2$ be the strong solutions obtained by Theorem 2.1. The first estimate we have is the energy law.

Lemma 3.2 (Basic Energy Equality) *For any $0 \le t < T$, it holds*

$$\begin{aligned} &\int_a^b r^m \Big(\frac{\rho u^2}{2} + \frac{\rho^\gamma}{\gamma-1} + \rho + \frac{|\mathbf{n}_r|^2}{2}\Big)(t) \\ &+ \int_0^t \int_a^b r^m \Big(u_r^2 + m\frac{u^2}{r^2} + |\mathbf{n}_{rr} + |\mathbf{n}_r|^2 \mathbf{n}|^2 + m\frac{|\mathbf{n}_r|^2}{r^2}\Big) \end{aligned}$$

$$= \int_a^b r^m \Big(\frac{\rho_0 u_0^2}{2} + \rho_0 + \frac{\rho_0^\gamma}{\gamma - 1} + \frac{|(\mathbf{n}_0)_r|^2}{2} \Big). \tag{3.2}$$

Proof Multiplying (2.1b) by u and integrating the resulting equation over (a, b), we get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_a^b r^m \Big(\frac{\rho u^2}{2} + \frac{\rho^\gamma}{\gamma - 1} \Big) + \int_a^b r^m \Big(u_r^2 + m \frac{u^2}{r^2} \Big) \\ &= - \int_a^b r^m u(\mathbf{n}_r \cdot \mathbf{n}_{rr}) - m \int_a^b r^{m-1} u |\mathbf{n}_r|^2. \end{aligned} \tag{3.3}$$

Then, multiplying (1.5c) by $r^m(\mathbf{n}_{rr} + |\mathbf{n}_r|^2 \mathbf{n})$ and integrating over (a, b), we obtain

$$\begin{aligned} &- m \int_a^b r^{m-1}(\mathbf{n}_t \cdot \mathbf{n}_r) - \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^b r^m |\mathbf{n}_r|^2 + \int_a^b r^m u(\mathbf{n}_r \cdot \mathbf{n}_{rr}) \\ &= \int_a^b r^m |\mathbf{n}_{rr} + |\mathbf{n}_r|^2 \mathbf{n}|^2 - \frac{m(m-1)}{2} \int_a^b r^{m-2} |\mathbf{n}_r|^2, \end{aligned} \tag{3.4}$$

where we have used the fact that $|\mathbf{n}| = 1$ to get

$$(\mathbf{n}_t + u\mathbf{n}_r) \cdot |\mathbf{n}_r|^2 \mathbf{n} = 0, \quad \frac{m}{r} \mathbf{n}_r \cdot |\mathbf{n}_r|^2 \mathbf{n} = 0. \tag{3.5}$$

Multiplying (1.5c) by $mr^{m-1}\mathbf{n}_r$, we get

$$\begin{aligned} &- m \int_a^b r^{m-1}(\mathbf{n}_t \cdot \mathbf{n}_r) \\ &= m \int_a^b r^{m-1} u |\mathbf{n}_r|^2 + \frac{m(m-1)}{2} \int_a^b r^{m-2} |\mathbf{n}_r|^2 - m^2 \int_a^b r^{m-2} |\mathbf{n}_r|^2. \end{aligned} \tag{3.6}$$

Combining (3.4) and (3.6), we have

$$\begin{aligned} &m \int_a^b r^{m-1} u |\mathbf{n}_r|^2 - m \int_a^b r^{m-2} |\mathbf{n}_r|^2 - \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^b r^m |\mathbf{n}_r|^2 + \int_a^b r^m u(\mathbf{n}_r \cdot \mathbf{n}_{rr}) \\ =&\int_a^b r^m |\mathbf{n}_{rr} + |\mathbf{n}_r|^2 \mathbf{n}|^2. \end{aligned} \tag{3.7}$$

In conclusion, it follows from (3.3) and (3.7) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_a^b r^m \Big(\frac{\rho u^2}{2} + \frac{\rho^\gamma}{\gamma - 1} + \frac{|\mathbf{n}_r|^2}{2} \Big) + \int_a^b r^m \Big(u_r^2 + m \frac{u^2}{r^2} + |\mathbf{n}_{rr} + |\mathbf{n}_r|^2 \mathbf{n}|^2 + m \frac{|\mathbf{n}_r|^2}{r^2} \Big) = 0. \tag{3.8}$$

Finally, we get (3.2) from conservation of mass.

Lemma 3.3 *For any $0 \le t < T$, it holds that*

$$\int_0^t \int_a^b r^m |\mathbf{n}_{rr}|^2 \le C_1 \tag{3.9}$$

for some C_1 depending only on a, T and E_0, where

$$E_0 := \int_a^b r^m \Big(\frac{\rho_0 u_0^2}{2} + \frac{\rho_0^\gamma}{\gamma - 1} + \rho_0 + |(\mathbf{n}_0)_r|^2 \Big)$$

denotes the total energy of the initial data.

Proof By using Lemma 3.1, we have

$$\|r^{\frac{m}{2}}\mathbf{n}_r\|_{L_r^4}^4 \le C_1\|r^{\frac{m}{2}}\mathbf{n}_r\|_{L_r^2}^3\|r^{\frac{m}{2}}\mathbf{n}_{rr}\|_{L_r^2}. \tag{3.10}$$

Since $|\mathbf{n}|=1$, one obtains

$$\begin{aligned}\int_a^b r^m|\mathbf{n}_{rr}|^2 &= \int_a^b r^m|\mathbf{n}_{rr}+|\mathbf{n}_r|^2\mathbf{n}|^2 + \int_a^b r^m|\mathbf{n}_r|^4\\ &\le \int_a^b r^m|\mathbf{n}_{rr}+|\mathbf{n}_r|^2\mathbf{n}|^2 + \frac{1}{a^m}\int_a^b r^{2m}|\mathbf{n}_r|^4\\ &\le \frac{1}{2}\int_a^b r^m|\mathbf{n}_{rr}|^2 + \int_a^b r^m|\mathbf{n}_{rr}+|\mathbf{n}_r|^2\mathbf{n}|^2 + C_1\Big(\int_a^b r^m|\mathbf{n}_r|^2\Big)^3.\end{aligned} \tag{3.11}$$

This, combined with (3.2), yields (3.9). The proof of this lemma is completed.

In order to derive further estimates, we outline the following Sobolev inequalities for radially symmetric functions (see [3]):

$$\|\rho\|_{L^\infty}^2 \le C_1\|\rho\|_{H_r^1}^2 \le C_1\int_a^b r^m(\rho^2+\rho_r^2), \tag{3.12}$$

$$\|u\|_{L^\infty}^2 \le C_1(N)\int_a^b r^m\Big(u_r^2+m\frac{u^2}{r^2}\Big), \tag{3.13}$$

$$\|\mathbf{n}_r\|_{L^\infty}^2 \le C_1\|\mathbf{n}_r\|_{H_r^1}^2 \le C_1\int_a^b r^m(|\mathbf{n}_r|^2+|\mathbf{n}_{rr}|^2). \tag{3.14}$$

In order to prove the second one, we make use of the boundary condition $u(a,t)=0$ and get

$$\begin{aligned}|u|r^m = \Big|\int_a^r (us^m)_s\mathrm{d}s\Big| &= \Big|\int_a^r\Big(u_s+m\frac{u}{s}\Big)s^m\mathrm{d}s\Big|\\ &\le \Big[\int_a^r\Big(u_s+m\frac{u}{s}\Big)^2 s^m\mathrm{d}s\Big]^{\frac{1}{2}}\Big[\Big(\frac{r^{m+1}}{m+1}\Big)^{\frac{1}{2}}-\Big(\frac{a^{m+1}}{m+1}\Big)^{\frac{1}{2}}\Big]\\ &\le \Big[\int_a^r\Big(u_s^2+m^2\frac{u^2}{s^2}\Big)s^m\mathrm{d}s\Big]^{\frac{1}{2}}\frac{r^{\frac{m+1}{2}}}{(m+1)^{\frac{1}{2}}}.\end{aligned} \tag{3.15}$$

Then we obtain the following corollary by directly using (1.5c) and (3.13).

Corollary 3.1 *For any $0\le t<T$, it holds that*

$$\int_0^t\int_a^b r^m|\mathbf{n}_t|^2 \le C_1(N). \tag{3.16}$$

Now we turn to prove the higher order energy estimates for $\mathbf{n}$.

Lemma 3.4 *For any $0\le t<T$, it holds that*

$$\int_a^b r^m|\mathbf{n}_{rr}|^2(t) + \int_0^t\int_a^b r^m(|\mathbf{n}_{rt}|^2+|\mathbf{n}_{rrr}|^2) \le C_2 \tag{3.17}$$

for some C_2 depending only on a, T, N, E_0 and $\|(\mathbf{n}_0)_{rr}\|_{L^2}$.

Proof Differentiating (1.5c) with respect to r, multiplying the resulting equation by $r^m\mathbf{n}_{rt}$, and then integrating it over (a,b), we have

$$\begin{aligned}
&\int_a^b r^m|\mathbf{n}_{rt}|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b r^m|\mathbf{n}_{rr}|^2\\
=&\int_a^b r^m[|\mathbf{n}_r|^2(\mathbf{n}_r\cdot\mathbf{n}_{rt})+2(\mathbf{n}_r\cdot\mathbf{n}_{rr})(\mathbf{n}\cdot\mathbf{n}_{rt})]-\int_a^b r^m u_r(\mathbf{n}_r\cdot\mathbf{n}_{rt})\\
&-\int_a^b r^m u(\mathbf{n}_{rr}\cdot\mathbf{n}_{rt})-m\int_a^b r^{m-2}(\mathbf{n}_r\cdot\mathbf{n}_{rt})\\
\le&\frac{1}{2}\int_a^b r^m|\mathbf{n}_{rt}|^2+C_1\int_a^b r^m\Big(|\mathbf{n}_r|^6+|\mathbf{n}_r|^2|\mathbf{n}_{rr}|^2+u_r^2|\mathbf{n}_r|^2+u^2|\mathbf{n}_{rr}|^2+\frac{|\mathbf{n}_r|^2}{r^4}\Big).\\
\le&\frac{1}{2}\int_a^b r^m|\mathbf{n}_{rt}|^2+C_1\int_a^b r^m|\mathbf{n}_r|^6+C_1\|\mathbf{n}_r\|_{L^\infty}^2\int_a^b r^m u_r^2\\
&+C_1(\|\mathbf{n}_r\|_{L^\infty}^2+\|u\|_{L^\infty}^2)\int_a^b r^m|\mathbf{n}_{rr}|^2+C_1\\
\le&\frac{1}{2}\int_a^b r^m|\mathbf{n}_{rt}|^2+C_1\int_1^b r^m u_r^2+C_1\\
&+C_1(N)\Big[1+\int_a^b r^m|\mathbf{n}_{rr}|^2+\int_a^b r^m\Big(u_r^2+m\frac{u^2}{r^2}\Big)\Big]\int_a^b r^m|\mathbf{n}_{rr}|^2,
\end{aligned}\tag{3.18}$$

where we have used Lemma 3.1 to get

$$\|r^{\frac{m}{2}}\mathbf{n}_r\|_{L_r^6}\le C_1\|r^{\frac{m}{2}}\mathbf{n}_r\|_{L_r^2}^{\frac{2}{3}}\|r^{\frac{m}{2}}\mathbf{n}_{rr}\|_{L_r^2}^{\frac{1}{3}}.\tag{3.19}$$

This, combined with Lemmas 3.2–3.3 and the Gronwall inequality, implies that for any $t\in[0,T)$,

$$\int_a^b r^m|\mathbf{n}_{rr}|^2(t)+\int_0^t\int_a^b r^m|\mathbf{n}_{rt}|^2\le C_2.\tag{3.20}$$

By observing that

$$\mathbf{n}_{rrr}=\mathbf{n}_{rt}+u_r\mathbf{n}_r+u\mathbf{n}_{rr}-2(\mathbf{n}_r\cdot\mathbf{n}_{rr})\mathbf{n}-|\mathbf{n}_r|^2\mathbf{n}_r+\frac{m}{r^2}\mathbf{n}_r-\frac{m}{r}\mathbf{n}_{rr},\tag{3.21}$$

we can complete the proof.

Now we want to improve the estimation of the bound of ρ.

Lemma 3.5 *For any $T>0$, it holds that*

$$\|\rho\|_{L^\infty(\Omega\times(0,T))}\le C_3\tag{3.22}$$

for some C_3 depending on $\|\rho_0\|_{L^\infty}$, γ and the parameters of C_2.

Proof This proof is quite similar to the discussion shown by Choe and Kim [3]. We introduce the Lagrangian mass co-ordinates (y,τ) defined by

$$y=\int_a^r\rho(r,t)r^m\mathrm{d}r,\quad \tau=t.\tag{3.23}$$

Then (1.5) can be rewritten in Lagrangian coordinate as follows:

$$\begin{cases}\rho_\tau+\rho^2(r^m u)_y=0,\\ r^{-m}u_\tau+P_y=(\rho(r^m u)_y)_y-\dfrac{1}{2}(r^{2m}\rho^2|\mathbf{n}_y|^2)_y-mr^{m-1}\rho|\mathbf{n}_y|^2,\\ \mathbf{n}_\tau=r^m\rho(r^m\rho\mathbf{n}_y)_y+mr^{m-1}\rho\mathbf{n}_y+r^{2m}\rho^2|\mathbf{n}_y|^2\mathbf{n}.\end{cases}\tag{3.24}$$

Now we only have to show that $\rho\le C$ for $0\le t\le T$ and $0\le y\le Y=\int_a^b r^m\rho_0(r)$.

To begin with, we observe from (3.24) that

$$\begin{aligned}(\log\rho)_{\tau y}&=\Big(\frac{\rho_\tau}{\rho}\Big)_y=-(\rho(r^m u)_y)_y\\ &=-r^{-m}u_\tau-P_y-\frac{1}{2}(r^{2m}\rho^2|\mathbf{n}_y|^2)_y-mr^{m-1}\rho|\mathbf{n}_y|^2\\ &=-(r^{-m}u)_\tau-\frac{m}{r^{m+1}}u^2-P_y-\frac{1}{2}(r^{2m}\rho^2|\mathbf{n}_y|^2)_y-mr^{m-1}\rho|\mathbf{n}_y|^2.\end{aligned}\tag{3.25}$$

Then, integrating over $(0,t)\times(0,y)$, we deduce that

$$\begin{aligned}\log\frac{\rho(y,\tau)}{\rho(0,\tau)}=&\log\frac{\rho_0(y)}{\rho_0(0)}+\int_0^y((r^{-m}u(z,0))-(r^{-m}u)(z,\tau))\mathrm{d}z+\int_0^\tau(P(0,s)-P(y,s))\mathrm{d}s\\ &-\int_0^\tau\int_0^y\Big(\frac{m}{r^{m+1}}u^2+\frac{1}{2}(r^{2m}\rho^2|\mathbf{n}_y|^2)_y+mr^{m-1}\rho|\mathbf{n}_y|^2\Big)\mathrm{d}z\mathrm{d}s.\end{aligned}\tag{3.26}$$

From this identity, we derive a representation formula for ρ,

$$\rho(y,\tau)=K(\tau)Q(y,\tau)\exp\Big(-\int_0^\tau P(y,s)\mathrm{d}s\Big),\tag{3.27}$$

where

$$\begin{aligned}K(\tau)&=\frac{\rho(0,\tau)}{\rho_0(0)}\exp\Big(\int_0^\tau P(0,s)\mathrm{d}s\Big),\\ Q(y,\tau)&=\rho_0(y)\exp\Big(\int_0^y((r^{-m}u(z,0))-(r^{-m}u)(z,\tau))\mathrm{d}z\Big)\\ &\quad\times\exp\Big(-\int_0^\tau\int_0^y\Big(\frac{m}{r^{m+1}}u^2+\frac{1}{2}(r^{2m}\rho^2|\mathbf{n}_y|^2)_y+mr^{m-1}\rho|\mathbf{n}_y|^2\Big)\mathrm{d}y\mathrm{d}\tau\Big).\end{aligned}$$

Moreover, ρ can be represented only in terms of $K(\tau)$ and $Q(y,\tau)$. It follows from (3.27) and $P=\rho^\gamma$ that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\exp\Big(\gamma\int_0^\tau P(y,s)\mathrm{d}s\Big)=\gamma\rho^\gamma\exp\Big(\gamma\int_0^\tau P(y,s)\mathrm{d}s\Big)=\gamma(K(\tau)Q(y,\tau))^\gamma,\tag{3.28}$$

and thus

$$\rho(y,\tau)=\frac{K(\tau)Q(y,\tau)}{(1+\gamma\int_0^\tau(K(\tau)Q(y,\tau))^\gamma\mathrm{d}s)^{\frac{1}{\gamma}}}.\tag{3.29}$$

In order to estimate $K(\tau)$ and $Q(y,\tau)$, we convert back into the Eulerian coordinates and use the previous lemma. Then we have

$$\begin{aligned}\int_0^Y r^{-m}|u|\mathrm{d}y&=\int_a^b\rho|u|\mathrm{d}r=\frac{1}{a^m}\int_a^b r^m\rho^{\frac12}\rho^{\frac12}|u|\mathrm{d}r\\ &\le\frac{1}{a^m}\int_a^b r^m\rho\mathrm{d}r+\frac{1}{a^m}\int_a^b r^m\rho u^2\mathrm{d}r\le C(a,E_0)\quad\text{for }0\le\tau\le T\end{aligned}\tag{3.30}$$

and

$$\begin{aligned}
&\Big| -\int_0^T \int_0^y \Big(\frac{m}{r^{m+1}}u^2 + \frac{1}{2}|(r^{2m}\rho^2|\mathbf{n}_y|^2)_y| + mr^{m-1}\rho|\mathbf{n}_y|^2\Big)\mathrm{d}y\mathrm{d}\tau\Big| \\
&= \int_0^T \int_a^b \Big(\frac{m}{r}\rho u^2 + \frac{1}{2}r^m\rho|(r^{2m}\rho^2|\mathbf{n}_y|^2)_y| + mr^{2m-1}\rho^2|\mathbf{n}_y|^2\Big)\mathrm{d}r\mathrm{d}t \\
&= \int_0^T \int_a^b \Big(\frac{m}{r}\rho u^2 + \frac{1}{2}|(|\mathbf{n}_r|^2)_r| + \frac{m}{r}|\mathbf{n}_r|^2\Big)\mathrm{d}r\mathrm{d}t \\
&= \int_0^T \int_a^b \Big(\frac{m}{r}\rho u^2 + |\mathbf{n}_r\cdot\mathbf{n}_{rr}| + \frac{m}{r}|\mathbf{n}_r|^2\Big)\mathrm{d}r\mathrm{d}t \\
&\le C_2.
\end{aligned} \tag{3.31}$$

Hence, it follows from the definition of $Q(y,\tau)$ that

$$\Big|\log\frac{Q(y,\tau)}{\rho_0(y)}\Big| \le C_2, \tag{3.32}$$

or equivalently,

$$\frac{1}{C_2}\rho_0(y) \le Q(y,\tau) \le C_2\rho_0(y). \tag{3.33}$$

Furthermore, to estimate $K(\tau)$, we observe

$$\int_0^Y \frac{1}{\rho(y,\tau)}\mathrm{d}y = \int_a^b r^m\mathrm{d}r = \frac{b^N - a^N}{N}. \tag{3.34}$$

Then we deduce from (3.29) and (3.33) that

$$\begin{aligned}
\frac{b^N-a^N}{N}K(\tau) &= \int_0^Y \frac{K(\tau)}{\rho(y,\tau)}\mathrm{d}y = \int_0^Y \frac{\big(1+\gamma\int_0^\tau (K(s)Q(y,s))^\gamma\big)^{\frac{1}{\gamma}}}{Q(y,\tau)}\mathrm{d}y \\
&\le \int_0^Y \frac{1}{Q(y,\tau)}\mathrm{d}y + \gamma^{\frac{1}{\gamma}}\int_0^Y \Big(\int_0^\tau K(s)\frac{Q(y,s)}{Q(y,\tau)}\mathrm{d}s\Big)^{\frac{1}{\gamma}}\mathrm{d}y \\
&\le C_3\frac{b^N-a^N}{N} + C_3\Big(\int_0^\tau K(s)^\gamma\mathrm{d}s\Big)^{\frac{1}{\gamma}}.
\end{aligned} \tag{3.35}$$

Therefore, using the Gronwall inequality, we get

$$K(\tau) \le C_3\exp\Big(\frac{C_3}{(b^N-a^N)^\gamma}\Big) \quad \text{for } 0\le\tau\le T. \tag{3.36}$$

We assume that $b \ge a+1$. Hence, the estimate is independent of b. We complete the proof of Lemma 3.4 by combining (3.27), (3.33) and (3.36).

Lemma 3.6 *For any $T>0$, it holds that*

$$\sup_{0\le t\le T}\int_a^b r^m\Big(u_r^2 + m\frac{u^2}{r^2} + |\mathbf{n}_t|^2 + G^2\Big) + \int_0^T\int_a^b r^m(\rho u_t^2 + G_r^2) \le C_4 \tag{3.37}$$

for some C_4 depending only on $\int_a^b r^m(u_{0r}^2 + m\frac{u_0^2}{r^2})$ and the parameters of C_3, where $G(r,t) = u_r + m\frac{u}{r} - P$ denotes the effective viscous flux.

Proof Multiplying (1.5b) by $r^m u_t$, integrating the resulting equation over (a,b), and employing integration by parts, we have

$$\begin{aligned}
&\int_a^b r^m \rho u_t^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b r^m\Big(u_r^2 + m\frac{u^2}{r^2}\Big) \\
&= -\int_a^b r^m \rho u u_r u_t + \int_a^b r^m \rho^\gamma u_{rt} + m\int_a^b r^{m-1}\rho^\gamma u_t \\
&\quad + \frac{1}{2}\int_a^b r^m |\mathbf{n}_r|^2 u_{rt} - \frac{m}{2}\int_a^b r^{m-1}|\mathbf{n}_r|^2 u_t \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III},
\end{aligned} \tag{3.38}$$

where

$$\begin{aligned}
\mathrm{I} &= -\int_a^b r^m \rho u u_r u_t + m\int_a^b r^{m-1}\rho^\gamma u_t \\
&\le \frac{1}{4}\int_a^b r^m \rho u_t^2 + C\|\rho\|_{L^\infty}\|u\|_{L^\infty}^2 \int_a^b r^m u_r^2 + C(a)\|\rho\|_{L^\infty}^{2\gamma-2}\int_a^b r^m \rho \\
&\le \frac{1}{4}\int_a^b r^m \rho u_t^2 + C_3\|u\|_{L^\infty}^2\int_a^b r^m u_r^2 + C_3,
\end{aligned} \tag{3.39}$$

$$\begin{aligned}
\mathrm{II} &= \frac{1}{2}\int_a^b r^m |\mathbf{n}_r|^2 u_{rt} - \frac{m}{2}\int_a^b r^{m-1}|\mathbf{n}_r|^2 u_t \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b r^m |\mathbf{n}_r|^2 u_r - \int_a^b r^m (\mathbf{n}_r\cdot\mathbf{n}_{rt}) u_r - \frac{m}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b r^{m-1}|\mathbf{n}_r|^2 u \\
&\quad + m\int_a^b r^{m-1}(\mathbf{n}_r\cdot\mathbf{n}_{rt}) u \\
&\le \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b r^m |\mathbf{n}_r|^2 u_r - \frac{m}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b r^{m-1}|\mathbf{n}_r|^2 u + C_2\int_a^b r^m |\mathbf{n}_{rt}|^2 \\
&\quad + C_2\int_a^b r^m\Big(u_r^2 + m\frac{u^2}{r^2}\Big),
\end{aligned} \tag{3.40}$$

$$\begin{aligned}
\mathrm{III} &= \int_a^b r^m \rho^\gamma u_{rt} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b r^m \rho^\gamma u_r - \gamma\int_a^b r^m \rho^{\gamma-1}\rho_t u_r \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b r^m \rho^\gamma u_r + m\gamma\int_a^b r^{m-1}\rho^\gamma u u_r + \int_a^b r^m (\rho^\gamma)_r u u_r + \gamma\int_a^b r^m \rho^\gamma u_r^2 \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b r^m \rho^\gamma u_r + m(\gamma-1)\int_a^b r^{m-1}\rho^\gamma u u_r + (\gamma-1)\int_a^b r^m \rho^\gamma u_r^2 - \int_a^b r^m \rho^\gamma u u_{rr} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b r^m \rho^\gamma u_r + m(\gamma-1)\int_a^b r^{m-1}\rho^\gamma u u_r + (\gamma-1)\int_a^b r^m \rho^\gamma u_r^2 \\
&\quad - \int_a^b \rho^\gamma u\Big[r^m \rho u_t + r^m \rho u u_r + r^m(\rho^\gamma)_r - m r^m\Big(\frac{u}{r}\Big)_r + \frac{1}{2}r^m(|\mathbf{n}_r|^2)_r + m r^{m-1}|\mathbf{n}_r|^2\Big] \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b r^m \rho^\gamma u_r + m(\gamma-1)\int_a^b r^{m-1}\rho^\gamma u u_r + (\gamma-1)\int_a^b r^m \rho^\gamma u_r^2 - \int_a^b r^m \rho^{\gamma+1} u u_t \\
&\quad - \int_a^b r^m \rho^{\gamma+1} u^2 u_r + \frac{m}{2}\int_a^b r^{m-1}\rho^{2\gamma} u + \frac{1}{2}\int_a^b r^m \rho^{2\gamma} u_r + m\int_a^b r^{m-1}\rho^\gamma u u_r
\end{aligned}$$

$$- m\int_a^b r^{m-2}\rho^\gamma u^2 - \int_a^b r^m \rho^\gamma u(\mathbf{n}_r\cdot\mathbf{n}_{rr}) - m\int_a^b r^{m-1}\rho^\gamma u|\mathbf{n}_r|^2$$
$$\le \frac{1}{2}\int_a^b r^m\rho u_t^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b r^m\rho^\gamma u_r + C_3\int_a^b r^m u_r^2 + C_3\|u\|_{L^\infty}^2 + C_3. \tag{3.41}$$

Combining the estimates above, we obtain

$$\begin{aligned}&\int_a^b r^m\rho u_t^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b r^m\Big(u_r^2 + m\frac{u^2}{r^2}\Big)\\ \le{}& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b r^m|\mathbf{n}_r|^2u_r + \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b r^m\rho^\gamma u_r - \frac{m}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b r^{m-1}|\mathbf{n}_r|^2 u\\ &+ C_3(\|u\|_{L^\infty}^2+1)\int_a^b r^m\Big(u_r^2+m\frac{u^2}{r^2}\Big) + \int_a^b r^m|\mathbf{n}_{rt}|^2 + C_3\|u\|_{L^\infty}^2 + C_3.\end{aligned} \tag{3.42}$$

Then, integrating (3.42) over $(0,t)$, it yields from the previous lemmas that

$$\begin{aligned}&\int_0^t\int_a^b r^m\rho u_t^2 + \int_a^b r^m\Big(u_r^2+m\frac{u^2}{r^2}\Big)\\ \le{}& \int_a^b r^m\Big(u_{0r}^2+m\frac{u_0^2}{r^2}\Big) + \frac{1}{2}\int_a^b r^m|\mathbf{n}_r|^2u_r - \frac{1}{2}\int_a^b r^m|\mathbf{n}_{0r}|^2u_{0r} + \int_a^b r^m\rho^\gamma u_r\\ &- \int_a^b r^m\rho_0^\gamma u_{0r} - \frac{m}{2}\int_a^b r^{m-1}|\mathbf{n}_r|^2u + \frac{m}{2}\int_a^b r^{m-1}|\mathbf{n}_{0r}|^2u_0\\ &+ C_3\int_0^T(\|u\|_{L^\infty}^2+1)\int_a^b r^m u_r^2 + C_3 + C_3t\\ \le{}& C_4 + \frac{1}{2}\int_a^b r^m\Big(u_r^2+m\frac{u^2}{r^2}\Big) + C_3\int_0^t(\|u\|_{L^\infty}^2+1)\int_a^b r^m\Big(u_r^2+m\frac{u^2}{r^2}\Big) + C_3t.\end{aligned} \tag{3.43}$$

This, combined with the fact that $\int_0^T\|u\|_{L^\infty}^2 \le C_1(N)$ and the Gronwall inequality, implies (3.37) except for the estimates about $\mathbf{n}_t$ and G. Furthermore, (1.5c) infers $\sup\limits_{0\le t\le T}\int_a^b r^m|\mathbf{n}_t|^2 \le C_4$. Finally, the definition of G and (1.5b) imply

$$G_r = \rho u_t + \rho u u_r + \mathbf{n}_r\cdot\mathbf{n}_{rr} + \frac{m}{r}|\mathbf{n}_r|^2. \tag{3.44}$$

We can finish the proof by the previous lemmas.

Lemma 3.7 *For any $T>0$, it holds that*

$$\sup_{0\le t\le T}\int_a^b r^m\rho_r^2 + \int_0^T\|u_r\|_{L^\infty}^2 \le C_5 \tag{3.45}$$

for some C_5 depending only on $\|(\rho_0)_r\|_{L^2}$ and the parameters of C_4. If $\rho_0\ge\varepsilon>0$, then it holds that

$$\inf\rho \ge \varepsilon C_5. \tag{3.46}$$

Proof By the definition of G, we have

$$\begin{aligned}\int_0^T\|u_r\|_{L^\infty}^2 &\le 3\int_0^T\Big(\|G\|_{L^\infty}^2 + \frac{m^2}{a^2}\|u\|_{L^\infty}^2 + \|\rho^\gamma\|_{L^\infty}^2\Big)\\ &\le C_1\int_0^T\Big(\|G\|_{H_r^1}^2 + \frac{m^2}{a^2}\|u\|_{L^\infty}^2 + \|\rho\|_{L^\infty}^{2\gamma}\Big) \le C_4.\end{aligned} \tag{3.47}$$

To estimate $\int_a^b r^m\rho_r^2$, taking the derivative of (1.5a) with respect to r, multiplying the resulting equation by $r^m\rho_r$, integrating over (a,b) and then employing integration by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b r^m\rho_r^2\\
&=-\int_a^b r^m\rho_r(\rho u)_{rr}-\int_a^b r^m\rho_r\Big(\frac{m}{r}\rho u\Big)_r\\
&=-r^m\rho_r\rho u_r|_a^b+m\int_a^b r^{m-1}\rho_r^2u+m\int_a^b r^{m-1}\rho\rho_ru_r+\int_a^b r^m\rho_r\rho_{rr}u\\
&\quad+\int_a^b r^m\rho\rho_{rr}u_r-\int_a^b r^m\rho_r\Big(\frac{m}{r}\rho u\Big)_r\\
&=-r^m\rho_r\rho u_r|_a^b+m\int_a^b r^{m-1}\rho_r^2u+m\int_a^b r^{m-1}\rho\rho_ru_r-\frac{m}{2}\int_a^b r^{m-1}\rho_r^2u-\frac{1}{2}\int_a^b r^m\rho_r^2u_r\\
&\quad+r^m\rho_r\rho u_r|_a^b-m\int_a^b r^{m-1}\rho\rho_ru_r-\int_a^b r^m\rho_r^2u_r-\int_a^b r^m\rho\rho_ru_{rr}+m\int_a^b r^{m-2}\rho\rho_ru\\
&\quad-m\int_a^b r^{m-1}\rho_r^2u-m\int_a^b r^{m-1}\rho\rho_ru_r\\
&=-\frac{m}{2}\int_a^b r^{m-1}\rho_r^2u-\frac{3}{2}\int_a^b r^m\rho_r^2u_r-\int_a^b r^m\rho\rho_r(G_r+P_r)\\
&\le\Big(\frac{m}{2a}\|u\|_{L^\infty}+\frac{3}{2}\|u_r\|_{L^\infty}+\|\rho\|_{L^\infty}^2\Big)\int_a^b r^m\rho_r^2+\int_a^b r^mG_r^2,
\end{aligned}\tag{3.48}$$

where we have used the fact

$$-\int_a^b r^m\rho\rho_rP_r=-\gamma\int_a^b r^m\rho^\gamma\rho_r^2\le 0.\tag{3.49}$$

Then (3.45) follows from (3.48), the Gronwall inequality and the previous lemmas.

In order to prove the lower bound of ρ, we recall the continuity equation (1.1a) that

$$\rho_t+\mathbf{u}\cdot\nabla\rho+\rho\mathrm{div}\mathbf{u}=0,\tag{3.50}$$

which yields

$$\begin{aligned}
\inf\rho(t)&\ge(\inf\rho_0)\exp\Big(-\int_0^t\|\mathrm{div}\mathbf{u}\|_{L^\infty}\Big)=(\inf\rho_0)\exp\Big(-\int_0^t\Big\|u_r+m\frac{u}{r}\Big\|_{L^\infty}\Big)\\
&\ge(\inf\rho_0)\exp\Big(-\int_0^T\Big(\|u_r\|_{L^\infty}+\frac{m}{a}\|u\|_{L^\infty}\Big)\Big)\\
&\ge\varepsilon C_5.
\end{aligned}\tag{3.51}$$

This completes the proof.

Lemma 3.8 *For any $0\le t<T$, it holds that*

$$\int_a^b r^m(\rho u_t^2+u_{rr}^2+G_r^2)+\int_0^t\int_a^b r^m\Big(u_{rt}^2+m\frac{u_t^2}{r^2}+G_{rr}^2\Big)\le C_6\tag{3.52}$$

for some C_6 depending only on $\varphi(\rho_0, \mathbf{u}_0, \mathbf{n}_0)$ and the parameters of C_5, where φ is defined by

$$\varphi(\rho_0, \mathbf{u}_0, \mathbf{n}_0) = \int_a^b r^m \rho_0^{-1} \Big| -\Big(u_{0r} + m\frac{u_0}{r}\Big)_r + (\rho_0^\gamma)_r + \frac{1}{2}(|\mathbf{n}_{0r}|^2)_r + \frac{m}{r}|\mathbf{n}_{0r}|^2\Big|^2.$$

Proof Differentiating (2.1b) with respect to t, multiplying the resulting equation by u_t, integrating over (a, b), and then using integration by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b r^m \rho u_t^2 + \int_a^b r^m \Big(u_{rt}^2 + m\frac{u_t^2}{r^2}\Big) \\
&= \int_a^b r^m (\mathbf{n}_r \cdot \mathbf{n}_{rt}) u_{rt} - m\int_a^b r^{m-1}(\mathbf{n}_r \cdot \mathbf{n}_{rt}) u_t - \frac{1}{2}\int_a^b \sigma_t u_t^2 - \int_a^b \sigma_t u u_r u_t \\
&\quad - \int_a^b \sigma u_t^2 u_r - \int_a^b \sigma u u_t u_{rt} - \int_a^b r^m (\rho^\gamma)_{rt} u_t \\
&= \int_a^b r^m (\mathbf{n}_r \cdot \mathbf{n}_{rt}) u_{rt} - m\int_a^b r^{m-1}(\mathbf{n}_r \cdot \mathbf{n}_{rt}) u_t + \int_a^b (\sigma u)_r u_t^2 + \int_a^b (\sigma u)_r u u_r u_t \\
&\quad - \int_a^b \sigma u_t^2 u_r + \gamma\int_a^b r^m \rho^{\gamma-1}\rho_t u_{rt} + m\gamma\int_a^b r^{m-1}\rho^{\gamma-1}\rho_t u_t \\
&= \int_a^b r^m (\mathbf{n}_r \cdot \mathbf{n}_{rt}) u_{rt} - m\int_a^b r^{m-1}(\mathbf{n}_r \cdot \mathbf{n}_{rt}) u_t - 2\int_a^b \sigma u u_t u_{rt} \\
&\quad - \int_a^b (\sigma u u_r^2 u_t + \sigma u^2 u_{rr} u_t + \sigma u^2 u_r u_{rt}) - \int_a^b \sigma u_t^2 u_r \\
&\quad - \gamma\int_a^b r^m \rho^{\gamma-1}\Big(\rho_r u + \rho u_r + \frac{m}{r}\rho u\Big) u_{rt} - m\gamma\int_a^b r^{m-1}\rho^{\gamma-1}\Big(\rho_r u + \rho u_r + \frac{m}{r}\rho u\Big) u_t \\
&\le \frac{1}{2}\int_a^b r^m \Big(u_{rt} + m\frac{u_t^2}{r^2}\Big) + C\|\mathbf{n}_r\|_{L^\infty}^2 \int_a^b r^m |\mathbf{n}_{rt}|^2 + C\|\rho\|_{L^\infty}\|u\|_{L^\infty}^2 \int_a^b r^m \rho u_t^2 \\
&\quad + C\|u_r\|_{L^\infty}^2 \int_a^b r^m u_r^2 + C\|u\|_{L^\infty}^2 \int_a^b r^m u_{rr}^2 + C\|\rho\|_{L^\infty}\|u\|_{L^\infty}^4 \int_a^b r^m u_r^2 \\
&\quad + C\|u_r\|_{L^\infty} \int_a^b r^m \rho u_t^2 + C(\gamma)\|\rho\|_{L^\infty}^{2\gamma-2}\|u\|_{L^\infty}^2 \int_a^b \rho_r^2 + C(\gamma)\|\rho\|_{L^\infty}^{2\gamma}\int_a^b r^m u_r^2 \\
&\quad + C(\gamma)\|\rho\|_{L^\infty}^{2\gamma}\int_a^b r^{m-2}u^2 + \frac{C(m,\gamma)}{a^2}\|\rho\|_{L^\infty}^{2\gamma-3}\|u\|_{L^\infty}^2 \int_a^b r^m \rho_r^2 + C\int_a^b r^m \rho u_t^2 \\
&\quad + \frac{C(m,\gamma)}{a^2}\|\rho\|_{L^\infty}^{2\gamma-1}\int_a^b r^m u_r^2 + \frac{C(m,\gamma)}{a^2}\|\rho\|_{L^\infty}^{2\gamma-1}\int_a^b r^{m-2}u^2.
\end{aligned} \tag{3.53}$$

Therefore, by observing

$$u_{rr}^2 = \Big(G_r - \frac{m}{r}u_r + \frac{m}{r^2}u + P_r\Big)^2 \le C\Big(G_r^2 + \frac{m^2}{r^2}u_r^2 + \frac{m^2}{r^4}u^2 + \rho^{2\gamma-2}\rho_r^2\Big) \tag{3.54}$$

and by the previous lemmas and the Gronwall inequality, we have

$$\int_a^b r^m \rho u_t^2 + \int_0^t \int_a^b r^m \Big(u_{rt}^2 + m\frac{u_t^2}{r^2}\Big) \le C_6, \tag{3.55}$$

where we have used the fact following from (2.1b) that

$$\int_a^b r^m \rho u_t^2(\tau) \le C\int_a^b r^m \rho u^2 u_r^2 + C\int_a^b r^m \rho^{-1}\Big|\Big(u_r + m\frac{u}{r}\Big)_r - (\rho^\gamma)_r - \frac{1}{2}(|\mathbf{n}_r|^2)_r - \frac{m}{r}|\mathbf{n}_r|^2\Big|^2$$
$$\to C\int_a^b r^m \rho u_0^2 u_{0r}^2 + C\varphi(\rho_0, \mathbf{u}_0, \mathbf{n}_0) \le C_6, \quad \text{as } \tau \to 0. \tag{3.56}$$

Then (3.52) follows by (1.5b) and the definition of G.

Lemma 3.9 *For any* $0 \le t < T$, *it holds that*

$$\int_a^b r^m \mathbf{n}_{rrr}^2 + \int_0^t \int_a^b r^m (\mathbf{n}_{rrt}^2 + \mathbf{n}_{rrrr}^2) \le C_7 \tag{3.57}$$

for some C_7 *depending only on* $\|\mathbf{n}_{0rrr}\|_{L^2}$ *and the parameters of* C_6.

Proof Differentiating (1.5c) with respect to r, multiplying the resulting equation by $r^m \mathbf{n}_{trrr}$, and then using integration by parts, one obtains

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b r^m |\mathbf{n}_{rrr}|^2 + \int_a^b r^m |\mathbf{n}_{rrt}|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b r^m \Big[(u\mathbf{n}_r)_r - (|\mathbf{n}_r|^2\mathbf{n})_r - \Big(\frac{m}{r}\mathbf{n_r}\Big)_r\Big] \cdot \mathbf{n}_{rrr}$$
$$- \int_a^b r^m \Big[(u\mathbf{n}_r)_{rt} - (|\mathbf{n}_r|^2\mathbf{n})_{rt} - \Big(\frac{m}{r}\mathbf{n_r}\Big)_{rt}\Big] \cdot \mathbf{n}_{rrr}, \tag{3.58}$$

where direct calculation yields that

$$\begin{aligned}
&\int_a^b r^m \Big[(u\mathbf{n}_r)_{rt} - (|\mathbf{n}_r|^2\mathbf{n})_{rt} - \Big(\frac{m}{r}\mathbf{n_r}\Big)_{rt}\Big] \cdot \mathbf{n}_{rrr} \\
&= \int_a^b r^m \Big[u_{rt}\mathbf{n}_r + u_r\mathbf{n}_{rt} + u_t \cdot \mathbf{n}_{rr} + u \cdot \mathbf{n}_{rrt} - 2(\mathbf{n}_{rt}\cdot\mathbf{n}_{rr})\mathbf{n} - 2(\mathbf{n}_r \cdot \mathbf{n}_{rrt})\mathbf{n} \\
&\quad - 2(\mathbf{n}_r\cdot\mathbf{n}_{rr})\mathbf{n}_t - 2(\mathbf{n}_r\cdot\mathbf{n}_{rt})\mathbf{n}_r - |\mathbf{n}_r|^2\mathbf{n}_{rt} + \frac{m}{r^2}\mathbf{n}_{rt} - \frac{m}{r}\mathbf{n}_{rrt}\Big] \cdot \mathbf{n}_{rrr} \\
&= \sum_{k=1}^{11} \mathrm{J}_k.
\end{aligned} \tag{3.59}$$

Then we use integration by parts, the Hölder inequality and the Cauchy inequality to get

$$\begin{aligned}
|\mathrm{J}_1| &\le C\|r^{\frac{m}{2}}u_{rt}\|_{L^2}\|\mathbf{n}_r\|_{L^\infty}\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2} \le C_2\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2}^2 + C_2\|r^{\frac{m}{2}}u_{rt}\|_{L^2}^2, \\
|\mathrm{J}_2| &\le \|r^{\frac{m}{2}}u_r\|_{L^2}\|\mathbf{n}_{rt}\|_{L^\infty}\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2} \le C_4(\|r^{\frac{m}{2}}\mathbf{n}_{rt}\|_{L^2} + \|r^{\frac{m}{2}}\mathbf{n}_{rrt}\|_{L^2})\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2} \\
&\le \tau\|r^{\frac{m}{2}}\mathbf{n}_{rrt}\|_{L^2}^2 + C_4(\tau)\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2}^2 + C_4\|r^{\frac{m}{2}}\mathbf{n}_{rt}\|_{L^2}^2, \\
|\mathrm{J}_3| &\le C\|u_t\|_{L^\infty}\|r^{\frac{m}{2}}\mathbf{n}_{rr}\|_{L^2}\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2} \le C_2\Big(\int_a^b r^m\Big(u_{rt}^2 + m\frac{u_t^2}{r^2}\Big)\Big)^{\frac{1}{2}}\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2} \\
&\le C_2\int_a^b r^m\Big(u_{rt}^2 + m\frac{u_t^2}{r^2}\Big) + C_2\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2}^2, \\
|\mathrm{J}_4| &\le C\|u\|_{L^\infty}\|r^{\frac{m}{2}}\mathbf{n}_{rrt}\|_{L^2}\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2} \le \tau\|r^{\frac{m}{2}}\mathbf{n}_{rrt}\|_{L^2}^2 + C_4(\tau)\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2}^2, \\
|\mathrm{J}_5| &\le C\|r^{\frac{m}{2}}\mathbf{n}_{rt}\|_{L^2}\|\mathbf{n}_{rr}\|_{L^\infty}\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2} \le C_2\|r^{\frac{m}{2}}\mathbf{n}_{rt}\|_{L^2}(1 + \|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2})\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2} \\
&\le C_2(\|r^{\frac{m}{2}}\mathbf{n}_{rt}\|_{L^2}^2 + 1)\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2}^2 + C_2\|r^{\frac{m}{2}}\mathbf{n}_{rt}\|_{L^2}^2, \\
|\mathrm{J}_6| &\le C\|\mathbf{n}_r\|_{L^\infty}\|r^{\frac{m}{2}}\mathbf{n}_{rrt}\|_{L^2}\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2} \le \tau\|r^{\frac{m}{2}}\mathbf{n}_{rrt}\|_{L^2}^2 + C_2(\tau)\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2}^2,
\end{aligned}$$

$$|\mathrm{J}_7| \le \|\mathbf{n}_r\|_{L^\infty}\|\mathbf{n}_{rr}\|_{L^\infty}\|r^{\frac{m}{2}}\mathbf{n}_t\|_{L^2}\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2} \le C_4\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2}^2 + C_4,$$
$$|\mathrm{J}_8| + |\mathrm{J}_9| \le C\|\mathbf{n}_r\|_{L^\infty}^2\|r^{\frac{m}{2}}\mathbf{n}_{rt}\|_{L^2}\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2} \le C_2\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2}^2 + C_2\|r^{\frac{m}{2}}\mathbf{n}_{rt}\|_{L^2}^2,$$
$$|\mathrm{J}_{10}| \le C(a)\|r^{\frac{m}{2}}\mathbf{n}_{rt}\|_{L^2}\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2} \le C(a)\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2}^2 + C(a)\|r^{\frac{m}{2}}\mathbf{n}_{rt}\|_{L^2}^2,$$
$$|\mathrm{J}_{11}| \le C(a)\|r^{\frac{m}{2}}\mathbf{n}_{rrt}\|_{L^2}\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2} \le \tau\|r^{\frac{m}{2}}\mathbf{n}_{rrt}\|_{L^2}^2 + C(\tau,a)\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2}^2.$$

Thus integrating (3.58) over $(0,t)$, choosing τ small enough, and then using the estimates above and the previous lemmas, we get

$$\begin{aligned}
&\int_a^b r^m|\mathbf{n}_{rrr}|^2 + \int_0^t\int_a^b r^m|\mathbf{n}_{rrt}|^2 \\
&\le \int_a^b r^m\Big[(u\mathbf{n}_r)_r - (|\mathbf{n}_r|^2\mathbf{n})_r - \Big(\frac{m}{r}\mathbf{n}_r\Big)_r\Big]\cdot\mathbf{n}_{rrr} \\
&\quad - \int_a^b r^m\Big[(u_0\mathbf{n}_{0r})_r - (|\mathbf{n}_{0r}|^2\mathbf{n}_0)_r - \Big(\frac{m}{r}\mathbf{n}_{0r}\Big)_r\Big]\cdot\mathbf{n}_{0rrr} + C_6\int_0^t(\|r^{\frac{m}{2}}\mathbf{n}_{rt}\|_{L^2}^2+1)\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2}^2 \\
&\quad + \int_a^b r^m|\mathbf{n}_{0rrr}|^2 + C_6\int_0^t\Big[\|r^{\frac{m}{2}}\mathbf{u}_{rt}\|_{L^2}^2 + \|r^{\frac{m}{2}}\mathbf{n}_{rt}\|_{L^2}^2 + \int_a^b r^m\Big(u_{rt}^2 + m\frac{u_t^2}{r^2}\Big)\Big] \\
&\le \int_a^b r^m\Big[(u\mathbf{n}_r)_r - (|\mathbf{n}_r|^2\mathbf{n})_r - \Big(\frac{m}{r}\mathbf{n}_r\Big)_r\Big]\cdot\mathbf{n}_{rrr} \\
&\quad + C_6\int_0^t(\|r^{\frac{m}{2}}\mathbf{n}_{rt}\|_{L^2}^2+1)\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2}^2 + C_7.
\end{aligned}\tag{3.60}$$

For the first term of the right-hand side of (3.60), we have

$$\begin{aligned}
&\int_a^b r^m\Big[(u\mathbf{n}_r)_r - (|\mathbf{n}_r|^2\mathbf{n})_r - \Big(\frac{m}{r}\mathbf{n}_r\Big)_r\Big]\cdot\mathbf{n}_{rrr} \\
&= \int_a^b r^m\Big[u_r\mathbf{n}_r + u\mathbf{n}_{rr} - 2(\mathbf{n}_r\cdot\mathbf{n}_{rr})\mathbf{n} - |\mathbf{n}_r|^2\mathbf{n}_r + \frac{m}{r^2}\mathbf{n}_r - \frac{m}{r}\mathbf{n}_{rr}\Big]\cdot\mathbf{n}_{rrr} \\
&\le C(a)(\|r^{\frac{m}{2}}u_r\|_{L^2}\|\mathbf{n}_r\|_{L^\infty} + \|u\|_{L^\infty}\|r^{\frac{m}{2}}\mathbf{n}_{rr}\|_{L^2} + \|\mathbf{n}_r\|_{L^\infty}\|r^{\frac{m}{2}}\mathbf{n}_{rr}\|_{L^2} \\
&\quad + \|\mathbf{n}_r\|_{L^\infty}^2\|r^{\frac{m}{2}}\mathbf{n}_r\|_{L^2} + \|r^{\frac{m}{2}}\mathbf{n}_{rr}\|_{L^2})\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2} \\
&\le \frac{1}{2}\|r^{\frac{m}{2}}\mathbf{n}_{rrr}\|_{L^2}^2 + C_6.
\end{aligned}\tag{3.61}$$

Then (3.60) and the Gronwall inequality imply that

$$\int_a^b r^m|\mathbf{n}_{rrr}|^2 + \int_0^t\int_a^b r^m|\mathbf{n}_{rrt}|^2 \le C_7 \quad \text{for } 0 < t < T. \tag{3.62}$$

Finally, (1.5c) and the previous lemmas imply that $\int_0^t\int_a^b r^m|\mathbf{n}_{rrrr}|^2 \le C_7$ for $0 < t < T$.

In order to get the higher regularity about ρ, we need the following lemma. The idea is given by Fan, Jiang and Ni [7].

Lemma 3.10 *For any $0 \le t < T$, it holds that*

$$\int_a^b r^m|(\rho^{\frac{\gamma}{2}})_r|^4 \le C_8 \tag{3.63}$$

for some C_8 depending only on $\|(\rho_0^{\frac{\gamma}{2}})_r\|_{L^4}$ and the parameters of C_7.

Proof Multiplying (1.5a) by $\rho^{\frac{\gamma}{2}-1}$, we obtain

$$(\rho^{\frac{\gamma}{2}})_t + (\rho^{\frac{\gamma}{2}})_r u + \frac{\gamma}{2}\rho^{\frac{\gamma}{2}} u_r + \frac{\gamma}{2}\frac{m}{r}\rho^{\frac{\gamma}{2}} u = 0. \tag{3.64}$$

Differentiating (3.64) with respect to r, we get

$$\begin{aligned}&(\rho^{\frac{\gamma}{2}})_{rt} + (\rho^{\frac{\gamma}{2}})_{rr} u + \Big(1+\frac{\gamma}{2}\Big)(\rho^{\frac{\gamma}{2}})_r u_r + \frac{\gamma}{2}\rho^{\frac{\gamma}{2}} u_{rr}\\ &- \frac{\gamma}{2}\frac{m}{r^2}\rho^{\frac{\gamma}{2}} u + \frac{\gamma}{2}\frac{m}{r}(\rho^{\frac{\gamma}{2}})_r u + \frac{\gamma}{2}\frac{m}{r}\rho^{\frac{\gamma}{2}} u_r = 0.\end{aligned} \tag{3.65}$$

Thus, we have

$$(\rho^{\frac{\gamma}{2}})_{rt} + (\rho^{\frac{\gamma}{2}})_{rr} u + \Big(1+\frac{\gamma}{2}\Big)(\rho^{\frac{\gamma}{2}})_r u_r + \frac{\gamma}{2}\frac{m}{r}(\rho^{\frac{\gamma}{2}})_r u + \frac{\gamma}{2}\rho^{\frac{\gamma}{2}}(G_r + P_r) = 0. \tag{3.66}$$

Multiplying (3.66) by $r^m|(\rho^{\frac{\gamma}{2}})_r|^2(\rho^{\frac{\gamma}{2}})_r$, and then integrating over (a,b), we have

$$\begin{aligned}&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b r^m|(\rho^{\frac{\gamma}{2}})_r|^4\\ &= \frac{m}{4}\int_a^b r^{m-1}|(\rho^{\frac{\gamma}{2}})_r|^4 u - \Big(\frac{3}{4}+\frac{\gamma}{2}\Big)\int_a^b r^m|(\rho^{\frac{\gamma}{2}})_r|^4 u_r - \frac{m\gamma}{2}\int_a^b r^{m-1}|(\rho^{\frac{\gamma}{2}})_r|^4 u\\ &\quad - \frac{\gamma}{2}\int_a^b r^m((\rho^{\frac{\gamma}{2}})_r)^3\rho^{\frac{\gamma}{2}}(G_r + P_r)\\ &\le \Big[\Big(\frac{m}{4a}+\frac{m\gamma}{2a}\Big)\|u\|_{L^\infty} + \Big(\frac{3}{4}+\frac{\gamma}{2}\Big)\|u_r\|_{L^\infty}\Big]\int_a^b r^m|(\rho^{\frac{\gamma}{2}})_r|^4\\ &\quad - \frac{\gamma}{2}\int_a^b r^m((\rho^{\frac{\gamma}{2}})_r)^3\rho^{\frac{\gamma}{2}} G_r - \frac{\gamma}{2}\int_a^b r^m((\rho^{\frac{\gamma}{2}})_r)^3\rho^{\frac{\gamma}{2}}(\rho^\gamma)_r\\ &\le C_6\int_a^b r^m|(\rho^{\frac{\gamma}{2}})_r|^4 + \|\rho\|_{L^\infty}^{2\gamma}\|G_r\|_{L^\infty}^2\int_a^b r^m G_r^2,\end{aligned} \tag{3.67}$$

where we have used the Cauchy inequality and the following fact:

$$\frac{\gamma}{2}\int_a^b r^m((\rho^{\frac{\gamma}{2}})_r)^3\rho^{\frac{\gamma}{2}}(\rho^\gamma)_r = \gamma\int_a^b r^m|(\rho^{\frac{\gamma}{2}})_r|^4\rho^\gamma \ge 0. \tag{3.68}$$

Finally, we can make use of the Sobolev inequalities for radially symmetric functions:

$$\|G_r\|_{L^\infty}^2 \le C(a)\int_a^b r^m(G_r^2 + G_{rr}^2), \tag{3.69}$$

the Gronwall inequality and the previous lemmas to complete the proof.

Lemma 3.11 *For any $0 \le t < T$, it holds that*

$$\int_a^b r^m(\rho_{rr}^2 + \rho_{tr}^2)(t) + \int_0^t\int_a^b r^m u_{rrr}^2 \le C_9 \tag{3.70}$$

for some C_9 depending only on $\|(\rho_0)_{rr}\|_{L^2}$ and the parameters of C_8.

Proof The following fact is the key to the proof of this lemma:

$$P_{rr} = (\rho^\gamma)_{rr} = \frac{4(\gamma-1)}{\gamma}|(\rho^{\frac{\gamma}{2}})_r|^2 + \gamma\rho^{\gamma-1}\rho_{rr}. \tag{3.71}$$

Then we have

$$\int_a^b r^m P_{rr}^2 \le \frac{8(\gamma-1)^2}{\gamma^2}\int_a^b r^m|(\rho^{\frac{\gamma}{2}})_r|^4 + 2\gamma^2\int_a^b r^m\rho^{2\gamma-2}\rho_{rr}^2. \tag{3.72}$$

Now differentiating (1.5a) with respect to r twice, multiplying the resulting equation by $r^m\rho_{rr}$, and then integrating over (a,b), one obtains

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b r^m\rho_{rr}^2\\
&= -\int_a^b r^m(\rho u)_{rrr}\rho_{rr} - \int_a^b r^m\Big(\frac{m}{r}\rho u\Big)_{rr}\rho_{rr}\\
&= -r^m(\rho u)_{rr}\rho_{rr}|_a^b + \int_a^b r^m(\rho u)_{rr}\rho_{rrr} + m\int_a^b r^{m-1}(\rho u)_{rr}\rho_{rr} - \int_a^b r^m\Big(\frac{m}{r}\rho u\Big)_{rr}\rho_{rr}\\
&= -r^m(\rho u)_{rr}\rho_{rr}|_a^b + \int_a^b r^m(\rho_{rr}u + 2\rho_r u_r + \rho u_{rr})\rho_{rrr} + m\int_a^b r^{m-1}(\rho u)_{rr}\rho_{rr}\\
&\quad - \int_a^b r^m\Big(\frac{2m}{r^3}\rho u + \frac{-2m}{r^2}(\rho u)_r + \frac{m}{r}(\rho u)_{rr}\Big)\rho_{rr}\\
&= -\frac{m}{2}\int_a^b r^{m-1}\rho_{rr}^2 u - \frac{5}{2}\int_a^b r^m\rho_{rr}^2 u_r - 2m\int_a^b r^{m-1}\rho_r\rho_{rr}u_r - 3\int_a^b r^m\rho_r\rho_{rr}u_{rr}\\
&\quad - m\int_a^b r^{m-1}\rho\rho_{rr}u_{rr} - \int_a^b r^m\rho\rho_{rr}u_{rrr} - 2m\int_a^b r^{m-3}\rho\rho_{rr}u + 2m\int_a^b r^{m-2}(\rho u)_r\rho_{rr}\\
&= -\frac{m}{2}\int_a^b r^{m-1}\rho_{rr}^2 u - \frac{5}{2}\int_a^b r^m\rho_{rr}^2 u_r - 2m\int_a^b r^{m-1}\rho_r\rho_{rr}u_r - 3\int_a^b r^m\rho_r\rho_{rr}u_{rr}\\
&\quad + 2m\int_a^b r^{m-2}\rho_r\rho_{rr}u - \int_a^b r^m\rho\rho_{rr}(G_{rr} + P_{rr})\\
&\le \Big(\frac{m}{2a}\|u\|_{L^\infty} + \frac{5}{2}\|u_r\|_{L^\infty}^2 + \frac{2m}{a^2} + 5 + \frac{2m}{a^4}\Big)\int_a^b r^m\rho_{rr}^2 + (2m+3)\|\rho_r\|_{L^\infty}^2\int_a^b r^m(u_{rr}^2 + u_r^2)\\
&\quad + 2m\|u\|_{L^\infty}^2\int_a^b r^m\rho_r^2 + \|\rho\|_{L^\infty}^2\int_a^b r^m G_{rr}^2 + \|\rho\|_{L^\infty}^2\int_a^b r^m P_{rr}^2.
\end{aligned} \tag{3.73}$$

Then by the previous lemmas and the Gronwall inequality, we get that

$$\int_a^b r^m\rho_{rr}^2 \le C_9, \tag{3.74}$$

and (3.70) follows by (1.5a) and the definition of G.

Lemma 3.12 *For any $0 \le t < T$, it holds that*

$$\begin{aligned}
&\|\rho\|_{L^1\cap H^1} + \|\rho_t\|_{L^2} + \|\mathbf{u}\|_{D_0^1\cap D^2} + \|\sqrt{\rho}\ \mathbf{u}_t\|_{L^2} + \|\nabla\mathbf{n}\|_{H^2} + \|\mathbf{n}_t\|_{H^1}\\
&+ \int_0^t(\|\mathbf{u}_t\|_{D_0^1}^2 + \|\nabla\mathbf{n}\|_{H^3}^2 + \|\mathbf{n}_t\|_{H^2}^2) \le \widetilde{C}_1
\end{aligned} \tag{3.75}$$

for some $\widetilde{C}_1$ depending only on N, γ, a, T, $\|\rho_0\|_{L^1\cap H^1}$, $\|\mathbf{u}_0\|_{D_0^1}$, $\varphi(\rho_0,\mathbf{u}_0,\mathbf{n}_0)$ and $\|\nabla\mathbf{n}_0\|_{H^2}$, and

$$\begin{aligned}
&\|\rho\|_{L^1\cap H^2} + \|\rho_t\|_{H^1} + \|\mathbf{u}\|_{D_0^1\cap D^2} + \|\sqrt{\rho}\ \mathbf{u}_t\|_{L^2} + \|\nabla\mathbf{n}\|_{H^2} + \|\mathbf{n}_t\|_{H^1}\\
&+ \int_0^t(\|\nabla\mathbf{u}\|_{H^2}^2 + \|\mathbf{u}_t\|_{D_0^1}^2 + \|\nabla\mathbf{n}\|_{H^3}^2 + \|\mathbf{n}_t\|_{H^2}^2) \le \widetilde{C}_2
\end{aligned} \tag{3.76}$$

for some $\widetilde{C}_2$ *depending only on* N, γ, a, T, $\|\rho_0\|_{L^1\cap H^2}$, $\|\nabla\rho_0^{\frac{\gamma}{2}}\|_{L^4}$, $\|\mathbf{u}_0\|_{D_0^1}$, $\varphi(\rho_0,\mathbf{u}_0,\mathbf{n}_0)$ *and* $\|\nabla\mathbf{n}_0\|_{H^2}$.

If, in addition, $\rho_0\ge\varepsilon>0$, *then it holds that*

$$\Big\|\frac{1}{\rho}\Big\|_{L^\infty}+\|\rho\|_{L^1\cap H^1}+\|\rho_t\|_{L^2}+\|\mathbf{u}\|_{H^2\cap H_0^1}+\|\mathbf{u}_t\|_{L^2}+\|\nabla\mathbf{n}\|_{H^2}+\|\mathbf{n}_t\|_{H^1} + \int_0^t(\|\mathbf{u}_t\|_{H_0^1}^2+\|\nabla\mathbf{n}\|_{H^3}^2+\|\mathbf{n}_t\|_{H^2}^2)\le\widetilde{C}_3 \tag{3.77}$$

for some $\widetilde{C}_3$ *depending only on the parameters of* $\widetilde{C}_1$ *and* ε, *and*

$$\Big\|\frac{1}{\rho}\Big\|_{L^\infty}+\|\rho\|_{L^1\cap H^2}+\|\rho_t\|_{H^1}+\|\mathbf{u}\|_{H^2\cap H_0^1}+\|\mathbf{u}_t\|_{L^2}+\|\nabla\mathbf{n}\|_{H^2}+\|\mathbf{n}_t\|_{H^1} + \int_0^t(\|\mathbf{u}\|_{H^3}^2+\|\mathbf{u}_t\|_{H_0^1}^2+\|\nabla\mathbf{n}\|_{H^3}^2+\|\mathbf{n}_t\|_{H^2}^2)\le\widetilde{C}_4 \tag{3.78}$$

for some $\widetilde{C}_2$ *depending only on the parameters of* $\widetilde{C}_2$ *and* ε.

Proof By calculation about the radial symmetric functions defined as before, we have

$$\begin{aligned}
&\nabla\rho=\rho_r\frac{\mathbf{x}}{r},\quad \nabla_i\nabla_j\rho=\rho_{rr}\frac{x_ix_j}{r^2}+\rho_r\frac{\delta_{ij}r^2-x_ix_j}{r^3},\\
&|\nabla\mathbf{u}|^2=u_r^2+m\frac{u^2}{r^2},\quad \Delta\mathbf{u}=\nabla\mathrm{div}u=\Big(u_r+m\frac{u}{r}\Big)_r\frac{\mathbf{x}}{r},\\
&|\nabla\Delta\mathbf{u}|=\sqrt{\sum_{i,j}\Big(\partial_{x_i}\Big(\Big(u_{rr}+\frac{m}{r}u_r-\frac{m}{r^2}u\Big)\frac{x_j}{r}\Big)\Big)^2}\le C(a)\Big(u_{rrr}^2+u_{rr}^2+u_r^2+\frac{u^2}{r^2}\Big),\\
&\nabla\mathbf{n}=\mathbf{n}_r\frac{\mathbf{x}}{r},\quad \nabla_i\nabla_j\mathbf{n}=\mathbf{n}_{rr}\frac{x_ix_j}{r^2}+\mathbf{n}_r\frac{\delta_{ij}r^2-x_ix_j}{r^3},\\
&|\nabla^3\mathbf{n}|^2\le C(a)(\mathbf{n}_{rrr}^2+\mathbf{n}_{rr}^2+\mathbf{n}_r^2),\quad |\nabla^4\mathbf{n}|^2\le C(a)(\mathbf{n}_{rrrr}^2+\mathbf{n}_{rrr}^2+\mathbf{n}_{rr}^2+\mathbf{n}_r^2).
\end{aligned}$$

On the other hand, we get the elliptic estimates for the velocity $\mathbf{u}$ (see [2, Lemma 12]):

$$\begin{aligned}\|\nabla^2\mathbf{u}\|_{L^2}&\le C(\|-\mu\Delta\mathbf{u}-(\lambda+\mu)\nabla\mathrm{div}\mathbf{u}\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2})\\&\le C(\|\Delta\mathbf{u}\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2}).\end{aligned}\tag{3.79}$$

Then the lemma follows from the above calculations and the previous lemmas.

Remark 3.1 In view of Theorem 2.1 and all the lemmas in this section, we conclude that the solutions obtained in Theorem 2.1 exist in $[0,T]$ for any $T>0$.

4 Proofs of Theoroms 1.1 and 1.2

In this section, we establish the existence of global strong solutions for $\rho_0\ge0$ and $b\le\infty$. The proof of Theorem 1.1 is based on several uniform estimates of the approximate solutions. We consider only the case that $b<\infty$, while the remaining case can be obtained by passing to the limit $b\to\infty$ by means of the domain expansion technique, and one may refer to [2, 10] for more details. We need the following proposition.

Lemma 4.1 (see [20]) *Assume that $X \subset E \subset Y$ are Banach spaces and $X \hookrightarrow\hookrightarrow E$. Then the following embeddings are compact:*

(i) $\Big\{\varphi : \varphi \in L^q(0,T;X), \dfrac{\partial\varphi}{\partial t} \in L^1(0,T;Y)\Big\} \hookrightarrow\hookrightarrow L^q(0,T;E)$, if $1 \le q \le \infty$,

(ii) $\Big\{\varphi : \varphi \in L^\infty(0,T;X), \dfrac{\partial\varphi}{\partial t} \in L^r(0,T;Y)\Big\} \hookrightarrow\hookrightarrow C([0,T];E)$, if $1 < r \le \infty$.

Proof of Theorem 1.1 We only consider the case of $b < \infty$. Regularizing the initial data, we construct a sequence ρ_0^ε of smooth radial functions, such that for each $T > 0$,

$$0 < \varepsilon \le \rho_0^\varepsilon, \quad \rho_0^\varepsilon \in H_r^2, \quad \lim_{\epsilon\downarrow 0} \|\rho_0^\epsilon - \rho_0\|_{H_r^1} = 0, \quad \|\rho_0^\varepsilon\|_{L^1\cap H^1} \le \widetilde{C}_1$$

for some $\widetilde{C}_1$ independent of b and ε, and $\rho_0^\varepsilon(\mathbf{x}) = \rho_0^\varepsilon(|\mathbf{x}|)$. Let $u_0^\varepsilon \in H_{0,r}^1 \cap H_r^2$ be the solution to the boundary value problem

$$-\Big(u_{0r}^\epsilon + m\frac{u_0^\epsilon}{r}\Big)_r + ((\rho_0^\epsilon)^\gamma)_r + \frac{1}{2}(|\mathbf{n}_{0r}|^2)_r + \frac{m}{r}|\mathbf{n}_{0r}|^2 = (\rho_0^\epsilon)^{\frac{1}{2}} g. \tag{4.1}$$

Let $(\rho^\epsilon, u^\epsilon, \mathbf{n}^\epsilon)$ be the strong solution to (1.5) along with the initial condition $(\rho_0^\epsilon, u_0^\varepsilon, \mathbf{n}_0)$ and the boundary condition $(u^\epsilon, \frac{\partial \mathbf{n}^\epsilon}{\partial \mathbf{v}})|_{\partial\Omega} = (0,0)$. If we define

$$\rho^\epsilon(t,\mathbf{x}) = \rho^\epsilon(t,|\mathbf{x}|), \quad \mathbf{u}^\epsilon(t,\mathbf{x}) = u^\epsilon(t,|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|} \quad \text{and} \quad \mathbf{n}^\epsilon(t,\mathbf{x}) = \mathbf{n}^\epsilon(t,|\mathbf{x}|),$$

then $(\rho^\epsilon, \mathbf{u}^\epsilon, \mathbf{n}^\epsilon)$ is a global radially symmetric solution to problem (1.1) with the initial data $(\rho_0^\epsilon, \mathbf{u}_0^\epsilon, \mathbf{n}_0)$ and the boundary condition (1.3), where $\mathbf{u}_0(\mathbf{x}) = u_0(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}$. Note that $(\rho_0^\epsilon, \mathbf{u}_0^\epsilon, \mathbf{n}_0)$ satisfies the same compatibility condition as $(\rho_0, \mathbf{u}_0, \mathbf{n}_0)$,

$$-\Delta\mathbf{u}_0^\varepsilon + \nabla(P(\rho_0^\varepsilon)) + \operatorname{div}\Big(\nabla\mathbf{n}_0 \odot \nabla\mathbf{n}_0 - \frac{|\nabla\mathbf{n}_0|^2}{2} I_N\Big) = (\rho_0^\varepsilon)^{\frac{1}{2}}\mathbf{g}. \tag{4.2}$$

We get $\mathbf{u}_0^\epsilon \to \mathbf{u}_0$ in H^2 by elliptic regularity theory. Hence, the previous lemmas infer that $(\rho^\epsilon, \mathbf{u}^\epsilon, \mathbf{n}^\epsilon)$ satisfies the following estimate (uniform in ε and b):

$$\begin{aligned} &\|\rho^\varepsilon\|_{L^1\cap H^1} + \|\rho_t^\varepsilon\|_{L^2} + \|\mathbf{u}^\varepsilon\|_{D_0^1\cap D^2} + \|\sqrt{\rho}\ u_t^\varepsilon\|_{L^2} + \|\nabla\mathbf{n}^\varepsilon\|_{H^2} + \|\mathbf{n}_t^\varepsilon\|_{H^1} \\ &+ \int_0^t (\|\mathbf{u}_t^\varepsilon\|_{D_0^1}^2 + \|\nabla\mathbf{n}^\varepsilon\|_{H^3}^2 + \|\mathbf{n}_t^\varepsilon\|_{H^2}^2) \le \widetilde{C}_1. \end{aligned} \tag{4.3}$$

Thus, by using Lemma 4.1 and the Sobolev embedding for radially symmetric functions, we can directly prove that a subsequence of approximate solution $(\rho^\epsilon, \mathbf{u}^\epsilon, \mathbf{n}^\epsilon)$ converges (weakly or weak$*$ in the corresponding space) to a radially symmetric strong solution $(\rho, \mathbf{u}, \mathbf{n})$ to the problem (1.1)–(1.3) in $\Omega \times [0,T]$. The regularity mentioned in Theorem 1.1 can be obtained by (4.3) and the lower semi-continuity of various norms. The uniqueness is based on that of the local solution.

Furthermore, Theorem 1.2 can also be achieved by Lemma 3.12.

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