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H^2-Stabilization of the Isothermal Euler Equations: a Lyapunov Function Approach*

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Abstract The authors consider the problem of boundary feedback stabilization of the 1D Euler gas dynamics locally around stationary states and prove the exponential stability with respect to the H^2-norm. To this end, an explicit Lyapunov function as a weighted and squared H^2-norm of a small perturbation of the stationary solution is constructed. The authors show that by a suitable choice of the boundary feedback conditions, the H^2-exponential stability of the stationary solution follows. Due to this fact, the system is stabilized over an infinite time interval. Furthermore, exponential estimates for the C^1-norm are derived.

Keywords Boundary control, Feedback stabilization, Quasilinear hyperbolic system, Balance law, Gas dynamics, Isothermal Euler equations, Exponential stability, Lyapunov function, H^2-norm, Stationary state, Characteristic variable
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1 Introduction

Hyperbolic systems of balance laws are used to model the flow dynamics on networks, whether we speak about the water flow in hydraulic networks, traffic flow or supply chains, etc. Another example is the isothermal Euler equations with friction which describe the gas flow through pipelines (see, for example, [2–3, 6, 10, 18, 27, 22] and the references therein) that we analyze in the present paper. The equations mentioned above form a 2×2 system of hyperbolic quasilinear PDEs. Questions related to controllability, observability and stabilizability of such systems have recently been the objects of intensive research (see, for example, [25, 31, 33]). The problem of stabilization of 2×2 systems of conservation laws (i.e., balance laws where the source terms are zero) was considered in the literature for over two decades. Among the first results were those published in [13] and [29], where the stabilization of quasilinear wave equations was discussed. In [28], global C^1 solutions to the dissipative boundary value problem for first order quasilinear hyperbolic systems are studied, and global classical solutions to general quasilinear hyperbolic systems were discussed in [24]. The present paper extends upon [8–9,

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11]. In [8], the authors construct a strict H^2-Lyapunov function to stabilize solutions to a system of two hyperbolic conservation laws around zero equilibrium. Later, the stabilization of one-dimensional $n \times n$ nonlinear hyperbolic systems was considered in [9]. In these two papers, the results are shown for systems without source terms. A strict Lyapunov function is constructed in [11] for boundary feedback stabilization of gas dynamics with respect to the H^1-norm on a finite time interval. In [21], the H^2-stabilization of a second-order quasilinear hyperbolic equation is established.

As in [11], in the present paper, we consider the stabilization of the 1D isothermal Euler equations. However, here we discuss the problem of boundary feedback stabilization around stationary states in the H^2-norm. In this framework, we show the existence and the exponential decay of the solutions on an infinite time interval. In this paper, we study H^2-solutions that require more regularity than C^1-solutions and this allows us to prove the existence of solutions on an infinite time interval, in contrast to semi-global classical solutions obtained for example by Li [25] and by Wang [33]. Moreover, the system that we consider is not covered by the results of Coron et al. mentioned before, since they consider systems with constant stationary solutions, whereas in our case, due to the friction term (see (2.6)) the stationary solutions are not constant. In this paper, the Lyapunov function is extended to the case of space-dependent eigenvalues of the system matrix, and the first-order term of the Taylor expansion of the source term at the equilibrium does not vanish. Therefore, for our system, the results presented by Li [24] for systems with quadratic source terms are not applicable.

Our analysis relies on the construction of a strict Lyapunov function, given as a weighted and squared H^2-norm of a small perturbation around a given equilibrium. Roughly speaking, the time derivative of this Lyapunov function satisfies a certain inequality that implies, under a suitable choice of the boundary conditions, the exponential decay of the Lyapunov function and therefore the exponential decay of the H^2-norm of a perturbation around an equilibrium.

This paper has the following structure. In Section 2, we describe the isothermal Euler equations with the friction in physical variables represented by density and mass flux. We then present our main result followed by a brief analysis of the diagonalization of the system with Riemann invariants. We also show the existence and uniqueness of C^2-stationary solutions (see Lemma 2.1). Moreover, we provide the equations describing the dynamics of non-stationary solutions, see systems (2.31), (2.36)–(2.37). In Section 3, the purpose is to prove the well-posedness of the quasilinear hyperbolic system considered in the previous section. The result is stated in Proposition 3.1 and it is shown that for initial data with a sufficiently small H^2-norm, the system has a unique continuous H^2-solution. In Section 4, we present the Lyapunov function used for the stability analysis and our main result about stability in the H^2-norm is stated in Theorem 4.1. Furthermore, an estimate of the exponential decay in the C^1-norm is given. In Section 5, we prove Theorem 4.1 about the existence and exponential decay of the solution on an infinite time interval. This proof is divided into two steps, according to the time intervals on which the solution is defined, finite and infinite time intervals, respectively. Finally, Section 6 concludes the paper with a summary and an outlook on future work.

2 Isothermal Euler Equations with Friction

In this section, we set up the mathematical framework which is represented by the isothermal

Euler equations with friction. We analyze the system in physical variables as well as the diagonalization with Riemann invariants. Moreover, we discuss stationary and non-stationary states of the gas flow.

2.1 Isothermal Euler equations with friction

The Euler equations for a compressible non-viscous gas in Lagrangian coordinates are given as a system of three conservation laws: conservation of mass, momentum and energy, respectively. An assumption often made in the gas pipeline simulation is constant temperature in the gas along the pipelines. Taking also the friction forces into account, we are led to the isothermal Euler equations with friction.

Considering one-dimensional flow, which is appropriate for physical models, where the lengths of pipelines are much larger in comparison to their diameters, for a single pipe, the isothermal Euler equations with friction have the following form:

$$\rho_t + q_x = 0, \tag{2.1a}$$

$$q_t + \Big(\frac{q^2}{\rho} + a^2\rho\Big)_x = -\frac{f_g}{2D}\frac{q|q|}{\rho}, \tag{2.1b}$$

where $\rho = \rho(x,t) > 0$ (we assume no vacuum states) and $q = q(x,t) \neq 0$ are dependent physical variables denoting the density of the gas and the mass flux through the pipe, respectively. The time t and the space coordinate x are independent variables and we consider the system over the time horizon $[0,+\infty)$, and the pipe length parameterized by $[0,L]$ with $L > 0$. The parameter f_g is a friction factor, $D > 0$ is the diameter of the pipe and $a > 0$ represents the sonic speed in the gas. The term $-\frac{f_g}{2D}\frac{q|q|}{\rho}$ accounts for the momentum loss due to viscous friction between the gas and the pipe walls.

Equations (2.1a) and (2.1b) form a hyperbolic balance law. (2.1a) is the continuity equation expressing conservation of mass, whereas (2.1b) is the momentum equation describing the physical forces acting on the gas particles.

We consider the system (2.1) together with the following boundary conditions in the feedback form:

$$\begin{aligned} x = 0&\colon \frac{q(0,t)}{\rho(0,t)} + \frac{1+k_0}{1-k_0} a \ln \rho(0,t) = \frac{\overline{q}(0)}{\overline{\rho}(0)} + \frac{1+k_0}{1-k_0} a \ln \overline{\rho}(0), \\ x = L&\colon \frac{q(L,t)}{\rho(L,t)} - \frac{1+k_L}{1-k_L} a \ln \rho(L,t) = \frac{\overline{q}(L)}{\overline{\rho}(L)} - \frac{1+k_L}{1-k_L} a \ln \overline{\rho}(L), \end{aligned} \tag{2.2}$$

where by $\overline{q} = \overline{q}(x)$ and $\overline{\rho} = \overline{\rho}(x)$, $x \in [0,L]$ we denote the physical variables flux and the density in the stationary case (see Section 2.3), with feedback parameters $0 < k_0 < 1$ and $0 < k_L < 1$. The physical relevance of (2.2) will be explained in Section 3. In contrast to (2.2), a kind of Neumann boundary feedback is considered in [21].

The pipe walls exert a frictional force on the gas flow and the flow speed tends to the sonic speed. Throughout this paper, we consider subsonic flow, that is, when the wave speed does not exceed the sonic speed, and in our notation this is

$$\frac{|q|}{\rho} < a. \tag{2.3}$$

We assume that the mass flux is positive, that is to say, that gas flows from the end $x = 0$ of the pipe to the end $x = L$. However, without loss of generality, it is hereby introduced.

Our main result reads as follows:

There exists an H^2-solution on the time interval $[0, +\infty)$ to the problem consisting in system (2.1) with the boundary conditions (2.2) and an initial datum that is in an H^2-neighborhood of the stationary state. The H^2-norm of the difference between this solution and the stationary state decays exponentially.

The details are given in Theorem 4.1 from Section 4.

2.2 Characteristic system: Riemann invariants

In what follows, we consider the transformation of the system equations to a diagonal form. This is done by a change of coordinates, to Riemann invariants. The isothermal Euler equations (2.1a) and (2.1b) are rewritten in the quasilinear format as

$$\partial_t \begin{pmatrix} \rho \\ q \end{pmatrix} + F(\rho, q)\, \partial_x \begin{pmatrix} \rho \\ q \end{pmatrix} = \sigma(\rho, q) \tag{2.4}$$

with the flux matrix defined as

$$F(\rho, q) := \begin{pmatrix} 0 & 1 \\ a^2 - \frac{q^2}{\rho^2} & 2\frac{q}{\rho} \end{pmatrix} \tag{2.5}$$

and the source term

$$\sigma(\rho, q) := \begin{pmatrix} 0 \\ -\frac{\theta}{2}\frac{q^2}{\rho} \end{pmatrix}, \tag{2.6}$$

where for the simplicity of notations, we introduce the parameter

$$\theta = \frac{f_g}{D}. \tag{2.7}$$

The system (2.4) is strictly hyperbolic, which means that $F(\rho, q)$ has two real distinct eigenvalues

$$\lambda_\pm = \lambda_\pm(\rho, q) = \frac{q}{\rho} \pm a. \tag{2.8}$$

For subsonic states, the eigenvalues have opposite signs, $\lambda_- < 0 < \lambda_+$. The system can be diagonalized with Riemann invariants

$$P_+ = P_+(\rho, q) = -\frac{q}{\rho} - a\ln(\rho), \tag{2.9}$$

$$P_- = P_-(\rho, q) = -\frac{q}{\rho} + a\ln(\rho). \tag{2.10}$$

The physical variables ρ, q and the system eigenvalues $\lambda_\pm$ are expressed with respect to the Riemann invariants $P_\pm$ as

$$\lambda_\pm = -\frac{P_+ + P_-}{2} \pm a, \tag{2.11}$$

$$\rho = \exp\Big(\frac{P_- - P_+}{2a}\Big), \tag{2.12}$$

$$q = -\frac{P_+ + P_-}{2} \exp\Big(\frac{P_- - P_+}{2a}\Big). \tag{2.13}$$

Assuming the mass flux is positive, the system adopts then the following diagonal structure:

$$\partial_t \begin{pmatrix} P_+ \\ P_- \end{pmatrix} + D(P_+, P_-)\partial_x \begin{pmatrix} P_+ \\ P_- \end{pmatrix} = S(P_+, P_-) \tag{2.14}$$

with the diagonal system matrix

$$D(P_+, P_-) = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} = \begin{pmatrix} -\frac{P_+ + P_-}{2} + a & 0 \\ 0 & -\frac{P_+ + P_-}{2} - a \end{pmatrix} \tag{2.15}$$

and the source term

$$S(P_+, P_-) = \frac{\theta}{8} (P_+ + P_-)^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{2.16}$$

2.3 System at equilibrium

In this section, we consider stationary subsonic C^2-solutions to (2.1). Stationary flow refers to the condition that the state of the system does not change in time, i.e., the time derivative is zero. Stationary states exist as the C^2-solutions only on a finite interval, until a critical length (see (2.17)) is reached. At this length, a blow-up in the derivatives occurs and the solution cannot be extended as a C^2-solution beyond this length, nor as a subsonic state.

For a detailed description of C^1-stationary solutions, we refer to [18]. In what follows, $\overline{\rho}(x)$ and $\overline{q}(x)$ denote the stationary density and the stationary flux, respectively, and by $\overline{\lambda}_\pm(x)$ we denote the corresponding eigenvalues and by $\overline{P}_\pm(x)$ the Riemann invariants. We have the following result on the existence and the uniqueness of a stationary C^2-solution (for the C^1-case, the result is given in [10, 18]).

Lemma 2.1 (Existence of Unique Stationary Subsonic C^2-Solutions) *Let constants $\overline{\rho}_0 > 0$ and $\overline{q}_0 > 0$ be given, such that the condition for subsonic flow is fulfilled, i.e., $\frac{\overline{q}_0}{\overline{\rho}_0} < a$. Define the quantity*

$$x_0 = \frac{1}{\theta}\Big(a^2 \frac{\overline{\rho}_0^2}{\overline{q}_0^2} - 2\ln(\overline{\rho}_0) - 1 + 2\ln\Big(\frac{\overline{q}_0}{a}\Big)\Big) > 0. \tag{2.17}$$

Then there exists a unique C^2-stationary subsonic solution $(\overline{\rho}(x), \overline{q}(x))$ on the interval $[0, x_0)$ that verifies the boundary conditions

$$(\overline{\rho}(0), \overline{q}(0)) = (\overline{\rho}_0, \overline{q}_0). \tag{2.18}$$

The interval $[0, x_0)$ is the maximal interval of existence of this C^2-solution, in the sense that

$$\lim_{x\to x_0} \frac{\mathrm{d}}{\mathrm{d}x}\overline{\rho}(x) = -\infty, \tag{2.19a}$$

$$\lim_{x\to x_0} \frac{\mathrm{d}^2}{\mathrm{d}x^2}\overline{\rho}(x) = -\infty. \tag{2.19b}$$

We sketch below the proof of this result. For a detailed proof in the case of C^1-stationary solutions, we refer to [18].

Proof of Lemma 2.1 Given the source term in (2.1), $\overline{q}(x)$ is constant with

$$\overline{q}(x) \equiv \overline{q}_0 \tag{2.20}$$

and $\overline{\rho}(x)$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(a^2\overline{\rho}+\frac{\overline{q}^2}{\overline{\rho}}\Big)=-\frac{\theta}{2}\frac{\overline{q}^2}{\overline{\rho}}. \tag{2.21}$$

As in [18], we define the function

$$\Phi(z)=\frac{a^2}{\overline{q}^2}z^2-2\ln(z) \tag{2.22}$$

on the interval $I=[\frac{\overline{q}}{a},\infty)$. The function Φ is well-defined, strictly increasing and strictly convex on I. Moreover, $\Phi'(\frac{\overline{q}}{a})=0$ and $\lim\limits_{z\to\infty}\Phi(z)=\infty$. Therefore, its inverse, Φ^{-1} is well-defined, strictly increasing and strictly concave on $[\Phi(\frac{\overline{q}}{a}),\infty)$. Hence, $\Phi''>0$.

We obtain that the density is given by the following equation ($x\in[0,L]$)

$$\overline{\rho}(x)=\Phi^{-1}\left(\Phi(\overline{\rho}(0))-\theta x\right). \tag{2.23}$$

As Φ^{-1} is strictly increasing, (2.23) implies that the density $\overline{\rho}$ is strictly decreasing along the pipe. Define the critical length

$$x_0=\frac{1}{\theta}\Big(\Phi(\overline{\rho}_0)-\Phi\Big(\frac{\overline{q}}{a}\Big)\Big)>0. \tag{2.24}$$

We have that

$$\lim_{x\to x_0}\overline{\rho}(x)=\frac{\overline{q}}{a}, \tag{2.25}$$

and therefore, as x approaches the critical length x_0, the state becomes critical. Straightforward calculations give that

$$\frac{\mathrm{d}}{\mathrm{d}x}\overline{\rho}(x)=-\frac{\theta}{\Phi'(\overline{\rho}(x))}<0, \tag{2.26}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\overline{\rho}(x)=\frac{\theta}{[\Phi'(\overline{\rho}(x))]^2}\Phi''(\overline{\rho}(x))\frac{\mathrm{d}}{\mathrm{d}x}\overline{\rho}(x)<0. \tag{2.27}$$

As

$$\lim_{t\to\Phi(\frac{\overline{q}}{a})^+}(\Phi^{-1}(t))'=\lim_{z\to\frac{\overline{q}}{a}}\frac{1}{\Phi'(z)}=\infty,$$

the strict convexity of Φ and (2.26) directly imply (2.19a). As $\lim\limits_{z\to\frac{\overline{q}}{a}}\Phi''(z)$ is finite and strictly positive, we obtain (2.19b) for the second-order derivative of the stationary density.

Hence, for the stationary solutions, after the critical length (see (2.17)) a blow-up in the derivative occurs. We conclude that C^2-solutions exist on the whole pipe if and only if the pipe length is less than the critical length.

We give two remarks below, the first one concerning the eigenvalues $\overline{\lambda}_\pm$ corresponding to stationary states and the second one about stationary Riemann invariants $\overline{P}_\pm$. These properties will be used later in the stability analysis (see Section 5).

Remark 2.1 Direct calculations show that the stationary eigenvalues $\overline{\lambda}_\pm(x)$ ($x\in[0,x_0)$) satisfy

$$\frac{\mathrm{d}}{\mathrm{d}x}\overline{\lambda}_\pm(x)>0, \tag{2.28}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\overline{\lambda}_\pm(x)>0. \tag{2.29}$$

Remark 2.2 Lemma 2.1 together with (2.9), (2.10) directly implies that

$$\overline{P}_{\pm} \in C^2([0, L]). \tag{2.30}$$

2.4 Nonstationary system

While in the previous section we are concerned with the system equations in stationary case, in this section, we analyze the behaviour of the system undergoing a time-dependent perturbation in a neighborhood of equilibrium solutions. More precisely, let $\overline{P}_{\pm} = \overline{P}_{\pm}(x) \in C^2[0, L]$ be a given stationary solution with corresponding eigenvalues $\overline{\lambda}_{\pm} = \overline{\lambda}_{\pm}(x)$. We perturb the system by $p_{\pm} = p_{\pm}(x, t)$, so we look at non-stationary solutions $\overline{P}_{+} + p_{+}$, $\overline{P}_{-} + p_{-}$ locally around $\overline{P}_{+}$ and $\overline{P}_{-}$, respectively. Straightforward calculations involving (2.14) yield the following equations describing the dynamics of $p_{\pm}$:

$$\begin{aligned} \partial_t p_{+} + \Big(\overline{\lambda}_{+} - \frac{p_{+} + p_{-}}{2}\Big)\partial_x p_{+} &= -(p_{+} + p_{-})\Big(K_{+} - \frac{\theta}{8}(p_{+} + p_{-})\Big), \\ \partial_t p_{-} + \Big(\overline{\lambda}_{-} - \frac{p_{+} + p_{-}}{2}\Big)\partial_x p_{-} &= -(p_{+} + p_{-})\Big(K_{-} - \frac{\theta}{8}(p_{+} + p_{-})\Big), \end{aligned} \tag{2.31}$$

where the functions $K_{\pm}$ are defined as

$$K_{\pm} \;=\; K_{\pm}(x) \;=\; \frac{\theta}{8}\,|\overline{P}_{+} + \overline{P}_{-}|\,\frac{4a \mp (\overline{P}_{+} + \overline{P}_{-})}{2a \mp (\overline{P}_{+} + \overline{P}_{-})}. \tag{2.32}$$

Note that the subsonic state condition in Riemann invariants, i.e.,

$$-a < \frac{P_{+} + P_{-}}{2} < 0, \tag{2.33}$$

implies that the functions $K_{\pm}$ are strictly positive, i.e.,

$$K_{\pm}(x) > 0 \tag{2.34}$$

for $x \in [0, L]$.

Denote the first and the second order time derivatives of p_{+}, p_{-} by

$$\begin{aligned} r_{\pm} &:= \partial_t p_{\pm}, \\ s_{\pm} &:= \partial_t r_{\pm} = \partial_{tt} p_{\pm}. \end{aligned} \tag{2.35}$$

The dynamics for the new variables $r_{\pm} = r_{\pm}(x, t)$ and $s_{\pm} = s_{\pm}(x, t)$ obtained from (2.31) by differentiation with respect to time are governed by the following system equations:

$$\begin{aligned} \partial_t r_{+} &= \Big(\frac{p_{+} + p_{-}}{2} - \overline{\lambda}_{+}\Big)\partial_x r_{+} + \Big(\frac{r_{+} + r_{-}}{2}\Big)\partial_x p_{+} \\ &\quad + \Big(\frac{\theta}{4}(p_{+} + p_{-}) - K_{+}\Big)(r_{+} + r_{-}), \\ \partial_t r_{-} &= \Big(\frac{p_{+} + p_{-}}{2} - \overline{\lambda}_{-}\Big)\partial_x r_{-} + \Big(\frac{r_{+} + r_{-}}{2}\Big)\partial_x p_{-} \\ &\quad + \Big(\frac{\theta}{4}(p_{+} + p_{-}) - K_{-}\Big)(r_{+} + r_{-}) \end{aligned} \tag{2.36}$$

and

$$\begin{aligned}\partial_t s_+ &= \Big(\frac{p_+ + p_-}{2} - \overline{\lambda}_+\Big)\partial_x s_+ + (r_+ + r_-)\partial_x r_+ + \Big(\frac{s_+ + s_-}{2}\Big)\partial_x p_+ \\ &\quad + \frac{\theta}{4}(r_+ + r_-)^2 + \Big(\frac{\theta}{4}(p_+ + p_-) - K_+\Big)(s_+ + s_-), \\ \partial_t s_- &= \Big(\frac{p_+ + p_-}{2} - \overline{\lambda}_-\Big)\partial_x s_- + (r_+ + r_-)\partial_x r_- + \Big(\frac{s_+ + s_-}{2}\Big)\partial_x p_- \\ &\quad + \frac{\theta}{4}(r_+ + r_-)^2 + \Big(\frac{\theta}{4}(p_+ + p_-) - K_-\Big)(s_+ + s_-),\end{aligned} \tag{2.37}$$

respectively.

For $t \geq 0$, define the norms $\|p_+(\cdot,t), p_-(\cdot,t)\|_{H^1(0,L)}$ and $\|p_+(\cdot,t), p_-(\cdot,t)\|_{H^2(0,L)}$ as

$$\|(p_+, p_-)\|_{H^1(0,L)} = \Big(\int_0^L (p_+^2 + p_-^2 + (\partial_x p_+)^2 + (\partial_x p_-)^2)\mathrm{d}x\Big)^{\frac{1}{2}}, \tag{2.38}$$

$$\begin{aligned}\|(p_+, p_-)\|_{H^2(0,L)} &= (\|(p_+, p_-)\|_{H^1(0,L)}^2 + \|(\partial_{xx} p_+, \partial_{xx} p_-)\|_{L^2(0,L)}^2)^{\frac{1}{2}} \\ &= \Big(\int_0^L (p_+^2 + p_-^2 + (\partial_x p_+)^2 + (\partial_x p_-)^2 \\ &\qquad + (\partial_{xx} p_+)^2 + (\partial_{xx} p_-)^2)\mathrm{d}x\Big)^{\frac{1}{2}}.\end{aligned} \tag{2.39}$$

In order to simplify the computations, when we analyze the $H^1(0,L)$-norm and the $H^2(0,L)$-norm, we take time derivatives instead of space derivatives. In order to relate them to the space derivatives, we give without proof the following technical lemma mentioned in [32] to which we will refer in the sequel.

Lemma 2.2 *Let a finite time $T > 0$ be given and the space-time domain $\Omega := [0,L] \times [0,T]$. There exist positive constants $C_i > 0$ $(i = 1, \cdots, 5)$ and a real number $\delta > 0$, such that for $p_\pm = (p_+, p_-)^{\mathrm{T}} \in (C^1(\Omega))^2$ satisfying* (2.31) *and $\|p_\pm\|_{C^1(\Omega)} \leq \delta$, the following inequalities hold*:

$$\|\partial_t p_\pm\|_{L^2(0,L)} \leq C_1(\|\partial_x p_\pm\|_{L^2(0,L)} + \|p_\pm\|_{L^2(0,L)}), \tag{2.40}$$

$$\|\partial_x p_\pm\|_{L^2(0,L)} \leq C_2(\|\partial_t p_\pm\|_{L^2(0,L)} + \|p_\pm\|_{L^2(0,L)}), \tag{2.41}$$

$$\|\partial_{tt} p_\pm\|_{L^2(0,L)} \leq C_3(\|\partial_{xx} p_\pm\|_{L^2(0,L)} + \|\partial_x p_\pm\|_{L^2(0,L)} + \|p_\pm\|_{L^2(0,L)}), \tag{2.42}$$

$$\|\partial_{xx} p_\pm\|_{L^2(0,L)} \leq C_4(\|\partial_{tt} p_\pm\|_{L^2(0,L)} + \|\partial_t p_\pm\|_{L^2(0,L)} + \|p_\pm\|_{L^2(0,L)}), \tag{2.43}$$

$$\|\partial_{xt} p_\pm\|_{L^2(0,L)} \leq C_5(\|\partial_{tt} p_\pm\|_{L^2(0,L)} + \|\partial_t p_\pm\|_{L^2(0,L)} + \|p_\pm\|_{L^2(0,L)}). \tag{2.44}$$

3 Well-Posedness of the Quasilinear Hyperbolic System

We are concerned with the asymptotic behaviour of solutions to the system (2.31), under the following boundary feedback conditions ($t \in [0, +\infty)$):

$$x = 0\colon p_+(0,t) = k_0 p_-(0,t), \tag{3.1}$$

$$x = L\colon p_-(L,t) = k_L p_+(L,t) \tag{3.2}$$

with feedback constants $0 < k_0 < 1$ and $0 < k_L < 1$ and initial conditions ($x \in [0,L]$)

$$t = 0\colon p_\pm(x,0) = p_\pm^0(x). \tag{3.3}$$

Remark 3.1 For the mixed initial-boundary value problem (2.31), (3.1)–(3.3), at the points $(x,t)=(0,0)$ and $(x,t)=(L,0)$, the initial data and the boundary conditions should satisfy the following C^1-compatibility conditions:

$$\begin{aligned} p_+^0(0) &= k_0 p_-^0(0),\\ \partial_t p_+^0(0) &= -\Big(\overline{\lambda}_+(0) - \frac{1+k_0}{2}p_-^0(0)\Big)\partial_x p_+^0(0)\\ &\quad - (1+k_0)p_-^0(0)\Big(K_+(0) - \frac{\theta}{8}(1+k_0)p_-^0(0)\Big) \end{aligned} \tag{3.4}$$

at the point $(x,t)=(0,0)$ and

$$\begin{aligned} p_-^0(L) &= k_L p_+^0(L),\\ \partial_t p_-^0(L) &= -\Big(\overline{\lambda}_-(L) - \frac{1+k_L}{2}p_+^0(L)\Big)\partial_x p_-^0(L)\\ &\quad - (1+k_L)p_+^0(L)\Big(K_-(L) - \frac{\theta}{8}(1+k_L)p_+^0(L)\Big) \end{aligned} \tag{3.5}$$

at the point $(x,t)=(L,0)$, respectively.

Questions related to the well-posedness of quasilinear hyperbolic systems have been intensively studied in the corresponding literature (see [8–9, 25, 30, 33]). In proving the well-posedness of our system, we also use a result from [33], about the existence of a C^1-solution of a quasilinear hyperbolic system on a finite time-interval $[0,T]$ $(T>0)$ for appropriate initial and boundary conditions. This result is recalled in [11, Lemma 4.1]. The following lemma from [23] states a result about the existence of solutions for a Cauchy problem for general quasi-linear symmetric hyperbolic systems on a finite time interval $[0,T]$ and a space interval $\mathbb{R}^m$, $m\geq 1$. Consider the quasilinear symmetric hyperbolic system

$$a_0(x,t,u)\frac{\partial u}{\partial t} + \sum_{j=1}^m a_j(x,t,u)\frac{\partial u}{\partial x_j} = f(x,t,u) \tag{3.6}$$

with an initial condition

$$t=0\colon u(x,0)=u_0(x), \tag{3.7}$$

where $u=(u_1,\cdots,u_N)$ and $x=(x_1,\cdots,x_m)\in\mathbb{R}^m$, $a_j=a_j(x,t,u)$ are bounded linear operators on $\mathbb{R}^N$, $f=f(x,t,u)$ is an N-vector valued function, which satisfy the following conditions:

(i) For $\Lambda\subseteq\mathbb{R}^m\times\mathbb{R}^N$,

$$\begin{aligned} &a_j\in C([0,T];C^s(\Lambda,\mathbb{R}^m)),\quad 0\leq j\leq m,\ s>\frac{m}{2}+1,\\ &a_0\in \mathrm{Lip}([0,T];C^{s-1}(\Lambda,\mathbb{R}^N)),\\ &f\in C([0,T];C^{s+1}(\Lambda,\mathbb{R}^N)),\\ &f_0\in L^\infty([0,T];H^s(\mathbb{R}^m,\mathbb{R}^N))\cap C([0,T];H^0(\mathbb{R}^m,\mathbb{R}^N)) \end{aligned}$$

with $f_0(x,t)=f(x,t,u_0(x))$.

(ii) The values of the operators (a_j) are symmetric operators, $\langle a_j(x,t,u),v\rangle=\langle u,a_j(x,t,v)\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^m$, and a_0 is uniformly positive definite.

Lemma 3.1 *Assume that $s > \frac{m}{2} + 1$ and that conditions* (i), (ii) *hold. Then the Cauchy problem* (3.6)–(3.7) *is well-posed in the following sense. Given $u_0 \in H^s(\mathbb{R}^m, \mathbb{R}^N)$, there exists a neighborhood V of u_0 in H^s and a positive number $T > 0$, such that for any initial data in V, the Cauchy problem* (3.6)–(3.7) *admits a unique solution $u(x,t)$ on $[0,T]$ such that $u \in C^0([0,T]; H^s(\mathbb{R}^m, \mathbb{R}^N)) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^m, \mathbb{R}^N))$.*

Remark 3.2 Lemma 3.1 can be easily adapted to the Cauchy problem (2.31), (3.3) in the case when $N = m = 1$ and $s = 2$.

For the initial-boundary value problem (2.31), (3.1)–(3.3), a similar regularity result on the well-posedness holds. Below we sketch the proof of this result, by using fixed point methods.

Define the set

$$X = L^\infty((0,\mathrm{T}); H^2(0,L))^2 \cap W^{1,\infty}((0,\mathrm{T}); H^1(0,L))^2 \cap W^{2,\infty}((0,T); L^2(0,L))^2$$

and for a radius $r > 0$ and for $\mathbf{p} = (p_+, p_-)^{\mathrm{T}}$, $\mathbf{p}^0 = (p_+^0, p_-^0)^{\mathrm{T}}$, define the set

$$\begin{aligned} B_r(\mathbf{p}^0) = \{\mathbf{p} \in X,\ &\|\mathbf{p}\|_{L^\infty((0,T);H^2(0,L))} \le r,\ \|\mathbf{p}\|_{W^{1,\infty}((0,T);H^1(0,L))} \le r, \\ &\|\mathbf{p}\|_{W^{2,\infty}((0,T);L^2(0,L))} \le r,\ \mathbf{p}(L,\cdot) \in H^2(0,T),\ \|\mathbf{p}(L,\cdot)\|_{H^2(0,T)} \le r^2, \\ &\mathbf{p}(0,\cdot) \in H^2(0,T),\ \|\mathbf{p}(0,\cdot)\|_{H^2(0,T)} \le r^2,\ \mathbf{p}(\cdot,0) = \mathbf{p}^0\}. \end{aligned}$$

Define the function $\mathfrak{F} : B_r(\mathbf{p}^0) \to X$, by $\mathfrak{F}(\widehat{\mathbf{p}}) = \mathbf{p}$, where $\mathbf{p}$ is solution to the following linear hyperbolic problem:

$$\begin{aligned} &\partial_t p_+ + \Big(\overline{\lambda}_+ - \frac{\widehat{p}_+ + \widehat{p}_-}{2}\Big)\partial_x p_+ = (\widehat{p}_+ + \widehat{p}_-)\Big(K_+ - \frac{\theta(\widehat{p}_+ + \widehat{p}_-)}{8}\Big), \\ &\partial_t p_+ - \Big(\overline{\lambda}_- - \frac{\widehat{p}_+ + \widehat{p}_-}{2}\Big)\partial_x p_- = (\widehat{p}_+ + \widehat{p}_-)\Big(K_- - \frac{\theta(\widehat{p}_+ + \widehat{p}_-)}{8}\Big), \\ &t = 0\colon \mathbf{p}(x,0) = \mathbf{p}^0(x), \quad x \in [0,L], \\ &x = 0\colon p_+(0,t) = k_0 \widehat{p}_-(0,t), \quad t \in [0,T], \\ &x = L\colon p_-(L,t) = k_L \widehat{p}_+(L,t), \quad t \in [0,T]. \end{aligned} \tag{3.8}$$

If the initial datum $\mathbf{p}^0$ is small enough in the sense of (3.9) below, the following can be proved:

(1) $B_r(\mathbf{p}^0)$ is a closed subset of $L^\infty((0,T); L^2(0,L))$.

(2) For a given $r > 0$, $B_r(\mathbf{p}^0) \neq \varnothing$.

(3) Energy estimates for the linear system imply

$$\mathfrak{F}(B_r(\mathbf{p}^0)) \subset B_r(\mathbf{p}^0)$$

and

$$\begin{aligned} &\|\mathfrak{F}(\widehat{\mathbf{p}}_2) - \mathfrak{F}(\widehat{\mathbf{p}}_1)\|_{L^\infty([0,T),L^2(0,L))} + M\|\mathfrak{F}(\widehat{\mathbf{p}}_2(L,\cdot)) - \mathfrak{F}(\widehat{\mathbf{p}}_1(L,\cdot))\|_{L^2(0,T)} \\ &+ M\|\mathfrak{F}(\widehat{\mathbf{p}}_2(0,\cdot)) - \mathfrak{F}(\widehat{\mathbf{p}}_1(0,\cdot))\|_{L^2(0,T)} \\ \le{}& \frac{1}{2}(\|\widehat{\mathbf{p}}_2 - \widehat{\mathbf{p}}_1\|_{L^\infty([0,T);L^2(0,L))} + M\|\widehat{\mathbf{p}}_2(L,\cdot) - \widehat{\mathbf{p}}_1(L,\cdot)\|_{L^2(0,T)} \\ &+ M\|\widehat{\mathbf{p}}_2(0,\cdot) - \widehat{\mathbf{p}}_1(0,\cdot)\|_{L^2(0,T)}) \end{aligned}$$

for all $\widehat{\mathbf{p}}_1, \widehat{\mathbf{p}}_2 \in B_r(\mathbf{p}^0)$. Banach's fixed point theorem ([1, p. 160, Theorem 6]) implies now the regularity of the solution to the mixed initial-boundary value problem (2.31), (3.1)–(3.3). For the Cauchy problem for conservation laws, a similar analysis is done in [9] and for the first order hyperbolic systems in Section 7.3 from [12]. We have thus proved the following proposition.

Proposition 3.1 *Let a length $L > 0$ and a real number $\varepsilon_0 > 0$ be given. Then there exists a constant $\varepsilon_1 \in (0, \varepsilon_0]$, and a finite time $T > 0$, such that for every initial data $p^0_\pm \in H^2(0, L)$ satisfying*

$$\|p^0_\pm\|_{H^2(0,L)} \le \varepsilon_1, \tag{3.9}$$

and such that the C^1-compatibility conditions are satisfied at the points $(x, t) = (0, 0)$ and $(x, t) = (L, 0)$ (see (3.4)–(3.5)), the mixed initial-boundary value problem (2.31), (3.1)–(3.3) admits a unique solution

$$(p_+, p_-)^{\mathrm{T}} \in C^0([0, T]; H^2(0, L))^2 \cap C^1([0, T]; H^1(0, L))^2,$$

which satisfies

$$\|p_\pm\|_{C^1(\Omega)} \le \varepsilon_0. \tag{3.10}$$

Remark 3.3 Inequality (3.10) follows from [33, Theorem 2.1], as $\|p^0_\pm\|_{H^2(0,L)} \le \varepsilon_1$ implies $\|p^0_\pm\|_{C^1([0,L])} \le \varepsilon_1$. [33, Theorem 2.1] implies that the unique solution $p_\pm$ to the initial-boundary value problem (2.31), (3.1)–(3.3) satisfies $\|p_\pm\|_{C^1(\Omega)} \le \varepsilon_0$.

Remark 3.4 In fact, later we will show that the solution exists on the time interval $[0, +\infty)$. This follows from the exponential decay.

4 Main Result: H^2-Stability

In this section, we are concerned with the exponential stability of system (2.31). For the stability analysis, we introduce a strict H^2-Lyapunov function, given as a weighted and squared L^2-norm of the perturbation $p_\pm$. In the previous section, we have proved that under appropriate initial and boundary feedback conditions, the mixed initial-boundary value problem (2.31), (3.1)–(3.3) admits a unique $C^0([0, T]; H^2(0, L))^2 \cap C^1([0, T]; H^1(0, L))^2$ solution. In this section we will show that the H^2-norm of this solution decays exponentially with time $t \ge 0$.

To do so, we use a Lyapunov function approach and introduce the following Lyapunov function ($t \ge 0$):

$$\widetilde{L}(t) = \widetilde{L}_1(t) + \widetilde{L}_2(t) + \widetilde{L}_3(t), \tag{4.1}$$

where

$$\widetilde{L}_1(t) = \int_0^L \Big(\frac{A_1}{\overline{\lambda}_+(x)} h_+(x) p_+^2(x, t) + \frac{A_2}{|\overline{\lambda}_-(x)|} h_-(x) p_-^2(x, t)\Big) \mathrm{d}x, \tag{4.2}$$

$$\widetilde{L}_2(t) = \int_0^L (A_3 h_+(x) r_+^2(x, t) + A_4 h_-(x) r_-^2(x, t)) \mathrm{d}x, \tag{4.3}$$

$$\widetilde{L}_3(t) = \int_0^L (A_3 h_+(x) s_+^2(x, t) + A_4 h_-(x) s_-^2(x, t)) \mathrm{d}x \tag{4.4}$$

with exponential weights $h_\pm = h_\pm(x)$ ($x \in [0, L]$) defined by

$$h_\pm(x) = \exp\Big(-\mu \int_0^x \frac{1}{\overline{\lambda}_\pm(s)} \mathrm{d}s\Big) \tag{4.5}$$

and with $\mu > 0$ defined by

$$\frac{1}{\mu} = \int_0^L \Big(\frac{1}{\overline{\lambda}_+(x)} + \frac{1}{|\overline{\lambda}_-(x)|}\Big)\mathrm{d}x. \tag{4.6}$$

Define the intervals

$$\begin{aligned}
I_1 &:= \Big[\mathrm{e} \max_{x\in[0,L]} \Big|\frac{\overline{\lambda}_+(x)}{\overline{\lambda}_-(x)}\Big| \frac{K_-(x)}{K_+(x)}, \min_{x\in[0,L]} \Big|\frac{\overline{\lambda}_+(x)}{\overline{\lambda}_-(x)}\Big| \frac{K_-(x)}{K_+(x)}\Big(1 + \frac{\mu}{4\,K_-(x)}\Big)\Big],\\
I_2 &:= \Big[\max_{x\in[0,L]} \Big|\frac{\overline{\lambda}_-(x)}{\overline{\lambda}_+(x)}\Big| \frac{K_+(x)}{K_-(x)}, \mathrm{e}^{-1} \min_{x\in[0,L]} \Big|\frac{\overline{\lambda}_-(x)}{\overline{\lambda}_+(x)}\Big| \frac{K_+(x)}{K_-(x)}\Big(1 + \frac{\mu}{4\,K_+(x)}\Big)\Big],\\
I_3 &:= \Big[\mathrm{e} \max_{x\in[0,L]} \frac{K_-(x)}{K_+(x)}, \min_{x\in[0,L]} \frac{K_-(x)}{K_+(x)}\Big(1 + \frac{\mu}{4K_-(x)}\Big)\Big],\\
I_4 &:= \Big[\max_{x\in[0,L]} \frac{K_+(x)}{K_-(x)}, \mathrm{e}^{-1} \min_{x\in[0,L]} \frac{K_+(x)}{K_-(x)}\Big(1 + \frac{\mu}{4K_+(x)}\Big)\Big]
\end{aligned} \tag{4.7}$$

with $\mathrm{e} = \exp(1)$, μ given in (4.6) and $K_\pm$ defined by (2.32). The numbers $A_i > 0$ ($i = 1, \cdots, 4$) are chosen to satisfy

$$\frac{A_1}{A_2} \in I_1 \quad \text{or} \quad \frac{A_2}{A_1} \in I_2, \tag{4.8}$$

$$\frac{A_3}{A_4} \in I_3 \quad \text{or} \quad \frac{A_4}{A_3} \in I_4. \tag{4.9}$$

Assume that the number μ defined by (4.6) is large enough (see Remark 4.1 below) such that the following inequality holds

$$-\partial_x\Big(\frac{\overline{P}_+ + \overline{P}_-}{2}\Big) + \varepsilon_0 \le \frac{\mu}{4}. \tag{4.10}$$

Moreover, assume that $k_0 \in (0, 1)$ and $k_L \in (0, 1)$ satisfy

$$\begin{aligned}
A_1 k_0^2 &< A_2,\\
2A_3 k_0^2(\overline{\lambda}_+(0) + \varepsilon_0) &\le A_4(|\overline{\lambda}_-(0)| - \varepsilon_0)
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
\mathrm{e}A_2 k_L^2 &< A_1,\\
2\mathrm{e}A_4 k_L^2(|\overline{\lambda}_-(L)| + \varepsilon_0) &\le A_3(\overline{\lambda}_+(L) - \varepsilon_0).
\end{aligned} \tag{4.12}$$

In the stability analysis, we will use the following property of the functions $h_\pm$:

$$1 = \frac{h_-(0)}{h_+(0)} \le \frac{h_-(x)}{h_+(x)} \le \frac{h_-(L)}{h_+(L)} = \mathrm{e} \tag{4.13}$$

for $x \in [0, L]$. This follows immediately from (4.5).

Remark 4.1 In particular, the intervals I_i $(i = 1, \cdots, 4)$ in (4.7) above must be nonempty. This is the case when the number μ defined in (4.6) is sufficiently large, which holds when the pipe length L is sufficiently small. For more details, we refer to [18, Section 5].

Our main result is stated in the following theorem.

Theorem 4.1 (H^2-Exponential Stability) *Let a stationary subsonic state* $\overline{P}_\pm \in C^2([0,L])^2$ *with corresponding eigenvalues* $\overline{\lambda}_\pm(x)$ *and a constant* $\varepsilon_0 > 0$ *be given. Define the positive real number* μ *as in* (4.6) *and assume that constants* $A_i > 0$ *are chosen to satisfy the conditions* (4.8)–(4.10) *stated above. Assume that* $k_0 \in (0,1)$, $k_L \in (0,1)$ *satisfy* (4.11) *and* (4.12). *There exists a constant* $\varepsilon_1 \in (0, \varepsilon_0]$, *such that if the initial data satisfy*

$$\|p^0_\pm\|_{H^2(0,L)} \le \varepsilon_1 \tag{4.14}$$

and the compatibility conditions (3.4)–(3.5) *hold at the points* $(x,t) = (0,0)$ *and* $(x,t) = (L,0)$, *respectively, then the mixed initial-boundary value problem* (2.31), (3.1)–(3.3) *admits a unique* $C^0([0,T]; H^2(0,L))^2 \cap C^1([0,T]; H^1(0,L))^2$ *solution* $(p_+, p_-)^{\mathrm{T}}$ *which satisfies the inequality*

$$\widetilde{L}(t) \le \widetilde{L}(0) \exp\Big(-\frac{\mu}{2}t\Big) \quad \text{for } t \in [0, +\infty) \tag{4.15}$$

with the Lyapunov function $\widetilde{L}(t)$ *defined as in* (4.1).

The proof of Theorem 4.1 is given in Section 5.

Corollary 4.1 (Exponential Decay of the H^2-Norm and the C^1-Norm) *Under the assumptions of Theorem* 4.1, *for the solution* $p_\pm$ *to the mixed initial-boundary value problem* (2.31), (3.1)–(3.3), *the* H^2*-norm of* $p_\pm(x,t)$ (*see* (2.39)) *decays exponentially with time on* $[0, +\infty)$. *More precisely, there exists a constant* $\eta_1 > 0$, *such that for any* $t \in [0, +\infty)$, *the inequality*

$$\|p_+(\cdot,t), p_-(\cdot,t)\|_{H^2(0,L)} \le \eta_1(\|p^0_+\|_{H^2(0,L)} + \|p^0_-\|_{H^2(0,L)}) \exp\Big(-\frac{\mu}{4}t\Big) \tag{4.16}$$

holds. Furthermore, there exists a constant $\eta_2 > 0$ *such that for any* $t \in [0, +\infty)$, *the* C^1*-norm of the solution satisfies*

$$\|p_+(\cdot,t), p_-(\cdot,t)\|_{C^1([0,L])} \le \eta_2(\|p^0_+\|_{H^2(0,L)} + \|p^0_-\|_{H^2(0,L)}) \exp\Big(-\frac{\mu}{4}t\Big). \tag{4.17}$$

Proof From the definition of the Lyapunov function $\widetilde{L}(t)$, we obtain the following inequalities:

$$\sqrt{\tau_1}\|p_\pm(\cdot,t)\|_{H^2(0,L)} \le \sqrt{\widetilde{L}(t)}, \tag{4.18}$$

$$\sqrt{\tau_2}\|p^0_\pm\|_{H^2(0,L)} \ge \sqrt{\widetilde{L}(0)} \tag{4.19}$$

with positive constants

$$\tau_1 = \min\Big\{\min_{x\in[0,L]} \frac{A_1}{\overline{\lambda}_+}h_+, \min_{x\in[0,L]} \frac{A_2}{|\overline{\lambda}_-|}h_-, \min_{x\in[0,L]} A_3 h_+, \min_{x\in[0,L]} A_4 h_-\Big\}, \tag{4.20}$$

$$\tau_2 = \max\Big\{\max_{x\in[0,L]} \frac{A_1}{\overline{\lambda}_+}h_+, \max_{x\in[0,L]} \frac{A_2}{|\overline{\lambda}_-|}h_-, \max_{x\in[0,L]} A_3 h_+, \max_{x\in[0,L]} A_4 h_-\Big\}. \tag{4.21}$$

For

$$\eta_1 = \sqrt{\frac{\tau_2}{\tau_1}}, \tag{4.22}$$

the inequalities (4.15), (4.18)–(4.19) imply (4.16).

The inequality (4.17) follows from (4.16) and the Sobolev embedding theorem for $H^2(0,L) \hookrightarrow C^1([0,L])$ (see, for instance, [12, Section 5.2]).

5 Proof of the Main Result

In this section, we prove Theorem 4.1 from Section 4. The proof is based on an estimate of the time derivative of the Lyapunov function $\widetilde{L}(t)$ constructed in the previous section.

Proof of Theorem 4.1 We divide the proof of this result into two parts. In the first part, we consider the unique solution to the mixed initial-boundary value problem (2.31), (3.1)–(3.3) on a finite time interval $[0,T)$ with $T > 0$ and prove that for this solution, the Lyapunov function $\widetilde{L}(t)$ decays exponentially with time $t \in [0,T)$. Based upon this result, we show in the second part that this solution is defined and decays exponentially for all times $t \geq 0$. This completes also the proof of Proposition 3.1.

5.1 Step 1: solution and exponential decay on a finite time interval

We analyze the time derivative of the Lyapunov function $\widetilde{L}(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{L}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{L}_1(t) + \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{L}_2(t) + \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{L}_3(t) \tag{5.1}$$

with $\widetilde{L}_1(t)$, $\widetilde{L}_2(t)$ and $\widetilde{L}_3(t)$ defined in (4.2), (4.3) and (4.4), respectively. We will show that the time derivative of the Lyapunov function $\widetilde{L}(t)$ ($t \in [0,T]$) satisfies the following estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{L}(t) \leq -\frac{\mu}{2}\widetilde{L}(t) + B_0(t) + B_L(t), \tag{5.2}$$

where $B_0(t)$ and $B_L(t)$ ($t \in [0,T]$) are boundary terms given by

$$\begin{aligned} B_0(t) = {} & A_1 p_+^2(0,t) - A_2 p_-^2(0,t) \\ & + A_3\Big(\overline{\lambda}_+(0) - \frac{p_+(0,t)+p_-(0,t)}{2}\Big)(r_+^2(0,t) + s_+^2(0,t)) \\ & - A_4\Big(|\overline{\lambda}_-(0)| + \frac{p_+(0,t)+p_-(0,t)}{2}\Big)(r_-^2(0,t) + s_-^2(0,t)) \end{aligned} \tag{5.3}$$

at the end $x = 0$ and

$$\begin{aligned} B_L(t) = {} & A_2 h_-(L) p_-^2(L,t) - A_1 h_+(L) p_+^2(L,t) \\ & + A_4 h_-(L)\Big(|\overline{\lambda}_-(L)| - \frac{p_+(L,t)+p_-(L,t)}{2}\Big)(r_-^2(L,t) + s_-^2(L,t)) \\ & - A_3 h_+(L)\Big(\overline{\lambda}_+(L) + \frac{p_+(L,t)+p_-(L,t)}{2}\Big)(r_+^2(L,t) + s_+^2(L,t)) \end{aligned} \tag{5.4}$$

at the end $x = L$.

5.1.1 Time derivative of $\widetilde{L}_1$

Let us compute in what follows the time derivative $(\frac{\mathrm{d}}{\mathrm{d}t})\widetilde{L}_1$ along the solutions to (2.31) together with boundary conditions (3.1)–(3.2). (2.31) and straightforward calculations yield (see [11])

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{L}_1(t) = \widetilde{L}_{11} + \widetilde{L}_{12} + \widetilde{L}_{13} + \widetilde{L}_{14}, \tag{5.5}$$

where the terms above are

$$\widetilde{L}_{11} = \int_0^L (A_2\, h_-\, \partial_x\,(p_-^2) \;-\; A_1\, h_+\, \partial_x\,(p_+^2))\mathrm{d}x, \tag{5.6}$$

$$\widetilde{L}_{12} = -2\int_0^L \Big(\frac{A_1}{\overline{\lambda}_+}\, h_+\, K_+\, p_+ + \frac{A_2}{|\overline{\lambda}_-|}\, h_-\, K_-\, p_-\Big)(p_+ + p_-)\mathrm{d}x, \tag{5.7}$$

$$\widetilde{L}_{13} = \frac{\theta}{4}\int_0^L \Big(\frac{A_1}{\overline{\lambda}_+}\, h_+\, p_+ + \frac{A_2}{|\overline{\lambda}_-|}\, h_-\, p_-\Big)(p_+ + p_-)^2\mathrm{d}x, \tag{5.8}$$

$$\widetilde{L}_{14} = \int_0^L \Big(\frac{A_1}{\overline{\lambda}_+}\, h_+\, p_+\, \partial_x p_+ + \frac{A_2}{|\overline{\lambda}_-|}\, h_-\, p_-\, \partial_x p_-\Big)(p_+ + p_-)\mathrm{d}x. \tag{5.9}$$

The integrals above are estimated exactly as in [11]. Integration by parts yields for $\widetilde{L}_{11}$,

$$\widetilde{L}_{11} = \big[A_2 h_- p_-^2 - A_1 h_+ p_+^2\big]_{x=0}^{x=L} - \mu\widetilde{L}_1. \tag{5.10}$$

For $\widetilde{L}_{12}$, Young's inequality together with the positiveness of the functions $K_\pm$ (see (2.34)) implies

$$\widetilde{L}_{12} \le \int_0^L \Big(\Big(\frac{A_2}{|\overline{\lambda}_-|}h_-K_- - \frac{A_1}{\overline{\lambda}_+}h_+K_+\Big)r_+^2 + \Big(\frac{A_1}{\overline{\lambda}_+}h_+K_+ - \frac{A_2}{|\overline{\lambda}_-|}h_-K_-\Big)r_-^2\Big)\mathrm{d}x. \tag{5.11}$$

By using the assumption (4.8), we obtain (see [11] for details)

$$\widetilde{L}_{12} \le \frac{\mu}{4}\widetilde{L}_1. \tag{5.12}$$

Inequality (3.10) and the monotonicity of $\overline{\lambda}_\pm$ (see (2.28)) yield

$$\widetilde{L}_{13} \le \varepsilon_0\frac{\theta}{4}\Big(3 + \max\Big\{\mathrm{e}\frac{A_2}{A_1}\frac{\overline{\lambda}_+(L)}{|\overline{\lambda}_-(L)|}, \frac{A_1}{A_2}\frac{|\overline{\lambda}_-(0)|}{\overline{\lambda}_+(0)}\Big\}\Big)\widetilde{L}_1. \tag{5.13}$$

In the same manner, we obtain

$$\widetilde{L}_{14} \le \frac{\varepsilon_0}{2}\Big(3 + \max\Big\{\mathrm{e}\frac{A_2}{A_1}\frac{\overline{\lambda}_+(L)}{|\overline{\lambda}_-(L)|}, \frac{A_1}{A_2}\frac{|\overline{\lambda}_-(0)|}{\overline{\lambda}_+(0)}\Big\}\Big)\widetilde{L}_1. \tag{5.14}$$

5.1.2 Time derivative of $\widetilde{L}_2$

Now we analyze the time derivative $(\frac{\mathrm{d}}{\mathrm{d}t})\widetilde{L}_2$ along the solutions to (2.36) together with boundary conditions (3.1)–(3.2). (2.36) and direct calculations yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{L}_2(t) = \widetilde{L}_{21} + \widetilde{L}_{22} + \widetilde{L}_{23} \tag{5.15}$$

with the integrals

$$\begin{aligned}
\widetilde{L}_{21} &= \int_0^L \Big(A_3 h_+\Big(\frac{p_+ + p_-}{2} - \overline{\lambda}_+\Big)\partial_x(r_+^2) + A_4 h_-\Big(\frac{p_+ + p_-}{2} - \overline{\lambda}_-\Big)\partial_x(r_-^2)\Big)\mathrm{d}x,\\
\widetilde{L}_{22} &= \int_0^L (A_3 h_+ r_+ \partial_x p_+ + A_4 h_- r_- \partial_x p_-)(r_+ + r_-)\mathrm{d}x,\\
\widetilde{L}_{23} &= 2\int_0^L \Big(A_3 h_+ r_+\Big(\frac{\theta}{4}(p_+ + p_-) - K_+\Big)(r_+ + r_-)\\
&\quad + A_4 h_- r_-\Big(\frac{\theta}{4}(p_+ + p_-) - K_-\Big)(r_+ + r_-)\Big)\mathrm{d}x.
\end{aligned} \tag{5.16}$$

In what follows, we estimate the terms $\widetilde{L}_{21}$, $\widetilde{L}_{22}$, $\widetilde{L}_{23}$ above. Integration by parts yields

$$\begin{aligned}
\widetilde{L}_{21} &= \int_0^L \Big(A_4 h_-\Big(|\overline{\lambda}_-| + \frac{p_+ + p_-}{2}\Big)\partial_x(r_-^2) - A_3 h_+\Big(\overline{\lambda}_+ - \frac{p_+ + p_-}{2}\Big)\partial_x(r_+^2)\Big)\mathrm{d}x\\
&= \Big[A_4 h_-\Big(|\overline{\lambda}_-| + \frac{p_+ + p_-}{2}\Big)r_-^2 - A_3 h_+\Big(\overline{\lambda}_+ - \frac{p_+ + p_-}{2}\Big)r_+^2\Big]_{x=0}^L\\
&\quad - \mu\int_0^L \Big(A_4 h_-\Big(1 + \frac{p_+ + p_-}{2|\overline{\lambda}_-|}\Big)r_-^2 + A_3 h_+\Big(1 - \frac{p_+ + p_-}{2\overline{\lambda}_+}\Big)r_+^2\Big)\mathrm{d}x\\
&\quad - \int_0^L \Big(A_4 h_-\Big(-\frac{\mathrm{d}}{\mathrm{d}x}\overline{\lambda}_- + \frac{\partial_x(p_+ + p_-)}{2}\Big)r_-^2 - A_3 h_+\Big(\frac{\mathrm{d}}{\mathrm{d}x}\overline{\lambda}_+ - \frac{\partial_x(p_+ + p_-)}{2}\Big)r_+^2\Big)\mathrm{d}x.
\end{aligned}$$

Inequality (3.10) together with the monotonicity of $\overline{\lambda}_\pm$ implies

$$\begin{aligned}
\widetilde{L}_{21} &\le \Big[A_4 h_-\Big(|\overline{\lambda}_-| + \frac{p_+ + p_-}{2}\Big)r_-^2 - A_3 h_+\Big(\overline{\lambda}_+ - \frac{p_+ + p_-}{2}\Big)r_+^2\Big]_{x=0}^L\\
&\quad - \mu\int_0^L \Big(A_4 h_-\Big(1 - \frac{\varepsilon_0}{|\overline{\lambda}_-(L)|}\Big)r_-^2 + A_3 h_+\Big(1 - \frac{\varepsilon_0}{\overline{\lambda}_+(0)}\Big)r_+^2\Big)\mathrm{d}x\\
&\quad + \int_0^L \Big(A_3 h_+\Big(\frac{\mathrm{d}}{\mathrm{d}x}\overline{\lambda}_+ - \frac{\partial_x(p_+ + p_-)}{2}\Big)r_+^2 - A_4 h_-\Big(-\frac{\mathrm{d}}{\mathrm{d}x}\overline{\lambda}_- + \frac{\partial_x(p_+ + p_-)}{2}\Big)r_-^2\Big)\mathrm{d}x.
\end{aligned}$$

Applying (4.10) now, we obtain

$$\begin{aligned}
\widetilde{L}_{21} &\le \Big[A_4 h_-\Big(|\overline{\lambda}_-| + \frac{p_+ + p_-}{2}\Big)r_-^2 - A_3 h_+\Big(\overline{\lambda}_+ - \frac{p_+ + p_-}{2}\Big)r_+^2\Big]_{x=0}^L\\
&\quad - \mu\widetilde{L}_2 + \frac{\mu}{4}\widetilde{L}_2 - \mu\Big(-\frac{\varepsilon_0}{|\overline{\lambda}_-(L)|} - \frac{\varepsilon_0}{\overline{\lambda}_+(0)}\Big)\widetilde{L}_2.
\end{aligned} \tag{5.17}$$

For $\widetilde{L}_{22}$, inequality (3.10) and Young's inequality imply

$$\begin{aligned}
\widetilde{L}_{22} &= \int_0^L (A_3 h_+ r_+ \partial_x p_+ + A_4 h_- r_- \partial_x p_-)(r_+ + r_-)\mathrm{d}x\\
&\le \frac{\varepsilon_0}{2}\int_0^L ((3A_3 h_+ + A_4 h_-)r_+^2 + (3A_4 h_- + A_3 h_+)r_-^2)\mathrm{d}x\\
&\le \frac{\varepsilon_0}{2}\Big(3 + \max\Big(\mathrm{e}\frac{A_4}{A_3}, \frac{A_3}{A_4}\Big)\Big)\widetilde{L}_2.
\end{aligned} \tag{5.18}$$

By using inequality (3.10), Young's inequality and the positiveness of functions $K_{\pm}$, we obtain the following estimate:

$$\begin{aligned}\widetilde{L}_{23} &= 2\int_0^L \Big(A_3h_+r_+\Big(\frac{\theta}{4}(p_+ + p_-) - K_+\Big)(r_+ + r_-)\\ &\quad + A_4h_-r_-\Big(\frac{\theta}{4}(p_+ + p_-) - K_-\Big)(r_+ + r_-)\Big)\mathrm{d}x\\ &\le \varepsilon_0\frac{\theta}{2}\int_0^L ((3A_3h_+ + A_4h_-)r_+^2 + (3A_4h_- + A_3h_+)r_-^2)\mathrm{d}x\\ &\quad + \int_0^L ((A_4h_-K_- - A_3h_+K_+)r_+^2 + (A_3h_+K_+ - A_4h_-K_-)r_-^2)\mathrm{d}x.\end{aligned}$$

By a similar analysis as done for the term $\widetilde{L}_{12}$, using (4.9), we obtain the following bound on the term $\widetilde{L}_{23}$:

$$\widetilde{L}_{23} \le \varepsilon_0\frac{\theta}{2}\Big(3 + \max\Big(\mathrm{e}\frac{A_4}{A_3}, \frac{A_3}{A_4}\Big)\Big)\widetilde{L}_2 + \frac{\mu}{4}\widetilde{L}_2. \tag{5.19}$$

5.1.3 Time derivative of $\widetilde{L}_3$

Next, we analyze the time derivative $(\frac{\mathrm{d}}{\mathrm{d}t})\widetilde{L}_3$ along the solutions to (2.37) together with boundary conditions (3.1)–(3.2). We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{L}_3(t) = \widetilde{L}_{31} + \widetilde{L}_{32} + \widetilde{L}_{33} + \widetilde{L}_{34} + \widetilde{L}_{35} \tag{5.20}$$

with

$$\begin{aligned}\widetilde{L}_{31} &= \int_0^L \Big(A_3h_+\Big(\frac{p_+ + p_-}{2} - \overline{\lambda}_+\Big)\partial_x(s_+^2) + A_4h_-\Big(\frac{p_+ + p_-}{2} - \overline{\lambda}_-\Big)\partial_x(s_-^2)\Big)\mathrm{d}x,\\ \widetilde{L}_{32} &= 2\int_0^L (A_3h_+s_+(r_+ + r_-)\partial_x r_+ + A_4h_-s_-(r_+ + r_-)\partial_x r_-)\mathrm{d}x,\\ \widetilde{L}_{33} &= 2\int_0^L \Big(\Big(A_3h_+s_+\Big[\frac{\theta}{4}(p_+ + p_-) - K_+\Big]\\ &\quad + A_4h_-s_-\Big[\frac{\theta}{4}(p_+ + p_-) - K_-\Big]\Big)(s_+ + s_-)\Big)\mathrm{d}x,\\ \widetilde{L}_{34} &= \frac{\theta}{2}\int_0^L (A_3h_+s_+(r_+ + r_-)^2 + A_4h_-s_-(r_+ + r_-)^2)\mathrm{d}x,\\ \widetilde{L}_{35} &= \int_0^L (A_3h_+s_+(s_+ + s_-)\partial_x p_+ + A_4h_-s_-(s_+ + s_-)\partial_x p_-)\mathrm{d}x.\end{aligned}$$

Let us now estimate the integrals $\widetilde{L}_{31}, \cdots, \widetilde{L}_{35}$ above. For $\widetilde{L}_{31}$, integration by parts gives

$$\begin{aligned}\widetilde{L}_{31} &= \int_0^L \Big(A_3h_+\Big(\frac{p_+ + p_-}{2} - \overline{\lambda}_+\Big)\partial_x(s_+^2) + A_4h_-\Big(\frac{p_+ + p_-}{2} - \overline{\lambda}_-\Big)\partial_x(s_-^2)\Big)\mathrm{d}x\\ &= \Big[A_4h_-\Big(|\overline{\lambda}_-| + \frac{p_+ + p_-}{2}\Big)s_-^2 - A_3h_+\Big(\overline{\lambda}_+ - \frac{p_+ + p_-}{2}\Big)s_+^2\Big]_{x=0}^L\\ &\quad - \mu\int_0^L \Big(A_4h_-\Big(1 + \frac{p_+ + p_-}{2|\overline{\lambda}_-|}\Big)s_-^2 + A_3h_+\Big(1 - \frac{p_+ + p_-}{2\overline{\lambda}_+}\Big)s_+^2\Big)\mathrm{d}x\\ &\quad + \int_0^L \Big(A_3h_+\Big(\frac{\mathrm{d}}{\mathrm{d}x}\overline{\lambda}_+ - \frac{\partial_x(p_+ + p_-)}{2}\Big)s_+^2 - A_4h_-\Big(-\frac{\mathrm{d}}{\mathrm{d}x}\overline{\lambda}_- + \frac{\partial_x(p_+ + p_-)}{2}\Big)s_-^2\Big)\mathrm{d}x.\end{aligned}$$

Inequality (3.10) and the monotonicity of $\overline{\lambda}_\pm$ (see (2.28)) further imply

$$\begin{aligned}\widetilde{L}_{31} \le& \Big[A_4 h_-\Big(|\overline{\lambda}_-| + \frac{p_+ + p_-}{2}\Big)s_-^2 - A_3 h_+\Big(\overline{\lambda}_+ - \frac{p_+ + p_-}{2}\Big)s_+^2\Big]_{x=0}^L \\ &- \mu\int_0^L \Big(A_4 h_-\Big(1 - \frac{\varepsilon_0}{|\overline{\lambda}_-(L)|}\Big)s_-^2 + A_3 h_+\Big(1 - \frac{\varepsilon_0}{\overline{\lambda}_+(0)}\Big)s_+^2\Big)\mathrm{d}x \\ &+ \int_0^L \Big(A_3 h_+\Big(\frac{\mathrm{d}}{\mathrm{d}x}\overline{\lambda}_+ - \frac{\partial_x(p_+ + p_-)}{2}\Big)s_+^2 - A_4 h_-\Big(-\frac{\mathrm{d}}{\mathrm{d}x}\overline{\lambda}_- + \frac{\partial_x(p_+ + p_-)}{2}\Big)s_-^2\Big)\mathrm{d}x.\end{aligned}$$

By using the assumption (4.10), we conclude

$$\begin{aligned}\widetilde{L}_{31} \le& \Big[A_4 h_-\Big(|\overline{\lambda}_-| + \frac{p_+ + p_-}{2}\Big)s_-^2 - A_3 h_+\Big(\overline{\lambda}_+ - \frac{p_+ + p_-}{2}\Big)s_+^2\Big]_{x=0}^L \\ &- \mu\widetilde{L}_3 + \frac{\mu}{4}\widetilde{L}_3 - \mu\Big(-\frac{\varepsilon_0}{|\overline{\lambda}_-(L)|} - \frac{\varepsilon_0}{\overline{\lambda}_+(0)}\Big)\widetilde{L}_3. \end{aligned}\tag{5.21}$$

By using (3.10) together with Young's inequality, we estimate the integral $\widetilde{L}_{32}$ as follows:

$$\begin{aligned}\widetilde{L}_{32} &= 2\int_0^L (A_3 h_+ s_+(r_+ + r_-)\partial_x r_+ + A_4 h_- s_-(r_+ + r_-)\partial_x r_-)\mathrm{d}x \\ &\le 2\varepsilon_0\int_0^L (A_3 h_+(s_+^2 + (\partial_x r_+)^2) + A_4 h_-(s_-^2 + (\partial_x r_-)^2))\mathrm{d}x.\end{aligned}$$

Now, applying (2.44) from Lemma 2.2, we obtain the following bound on the term $\widetilde{L}_{32}$:

$$\widetilde{L}_{32} \le \varepsilon_0\alpha_1(\widetilde{L}_1 + \widetilde{L}_2 + \widetilde{L}_3) \tag{5.22}$$

with a positive constant $\alpha_1 > 0$. For $\widetilde{L}_{33}$ we proceed analogously as in the case of $\widetilde{L}_{23}$ and estimate as follows:

$$\begin{aligned}\widetilde{L}_{33} &= 2\int_0^L \Big(A_3 h_+ s_+\Big[\frac{\theta}{4}(p_+ + p_-) - K_+\Big] + A_4 h_- s_-\Big[\frac{\theta}{4}(p_+ + p_-) - K_-\Big]\Big)(s_+ + s_-)\mathrm{d}x \\ &\le \varepsilon_0\frac{\theta}{2}\int_0^L ((3A_3 h_+ + A_4 h_-)s_+^2 + (3A_4 h_- + A_3 h_+)s_-^2)\mathrm{d}x \\ &\quad + \int_0^L ((A_4 h_- K_- - A_3 h_+ K_+)s_+^2 + (A_3 h_+ K_+ - A_4 h_- K_-)s_-^2)\mathrm{d}x \\ &\le \varepsilon_0\frac{\theta}{2}\Big(3 + \max\Big(\mathrm{e}\frac{A_4}{A_3}, \frac{A_3}{A_4}\Big)\Big)\widetilde{L}_3 + \frac{\mu}{4}\widetilde{L}_3.\end{aligned}\tag{5.23}$$

Considering $\widetilde{L}_{34}$ now, by using (3.10) and Young's inequality, we obtain

$$\begin{aligned}\widetilde{L}_{34} &= \frac{\theta}{2}\int_0^L (A_3 h_+ s_+ + A_4 h_- s_-)(r_+ + r_-)(r_+ + r_-)\mathrm{d}x \\ &\le \varepsilon_0\frac{\theta}{2}\int_0^L (A_3 h_+(2s_+^2 + r_+^2 + r_-^2) + A_4 h_-(2s_-^2 + r_-^2 + r_+^2))\mathrm{d}x \\ &\le \varepsilon_0\frac{\theta}{2}\Big(\Big(1 + \max\Big(\mathrm{e}\frac{A_4}{A_3}, \frac{A_3}{A_4}\Big)\Big)\widetilde{L}_2 + 2\widetilde{L}_3\Big).\end{aligned}\tag{5.24}$$

By using (3.10) and Young's inequality, we obtain the following estimate:

$$\begin{aligned}\widetilde{L}_{35} &= \int_0^L (A_3 h_+ s_+ (s_+ + s_-)\partial_x p_+ + A_4 h_- s_- (s_+ + s_-)\partial_x p_-)\mathrm{d}x \\ &\le \frac{\varepsilon_0}{2}\int_0^L ((3A_3 h_+ + A_4 h_-)s_+^2 + (3A_4 h_- + A_3 h_+)s_-^2)\mathrm{d}x \\ &\le \frac{\varepsilon_0}{2}\Big(3 + \max\Big(\mathrm{e}\frac{A_4}{A_3}, \frac{A_3}{A_4}\Big)\Big)\widetilde{L}_3. \end{aligned} \tag{5.25}$$

5.1.4 Boundary terms

We now consider the boundary terms $B_0(t)$ and $B_L(t)$ and show that if the constants $A_i > 0$ $(i = 1, \cdots, 4)$ and the parameters k_0 and k_L satisfy (4.11) and (4.12), then the boundary terms are negative.

By (3.1)–(3.2), the boundary terms are rewritten as ($t \in [0, T)$)

$$B_0(t) = B_{01} p_-^2(0, t) + B_{02}(r_-^2(0, t) + s_-^2(0, t)), \tag{5.26}$$

where

$$B_{01} = A_1 k_0^2 - A_2, \tag{5.27}$$

$$B_{02} = A_3 k_0^2 \Big(\overline{\lambda}_+(0) - \frac{1 + k_0}{2} p_-(0, t)\Big) - A_4\Big(|\overline{\lambda}_-(0)| + \frac{1 + k_0}{2} p_-(0, t)\Big) \tag{5.28}$$

at the end $x = 0$, and on the other hand at the end $x = L$,

$$B_L(t) = B_{L1} p_+^2(L, t) + B_{L2}(r_+^2(L, t) + s_+^2(L, t)) \tag{5.29}$$

with

$$B_{L1} = A_2 k_L^2 h_-(L) - A_1 h_+(L), \tag{5.30}$$

$$\begin{aligned} B_{L2} = {} & A_4 k_L^2 h_-(L)\Big(|\overline{\lambda}_-(L)| + \frac{1 + k_L}{2} p_+(L, t)\Big) \\ & - A_3 h_+(L)\Big(\overline{\lambda}_+(L) - \frac{1 + k_L}{2} p_+(L, t)\Big). \end{aligned} \tag{5.31}$$

Inequalities (4.11) and (4.12) imply for $t \in [0, T)$,

$$B_0(t) \le 0, \quad B_L(t) \le 0. \tag{5.32}$$

Choosing $\varepsilon_0 > 0$ small enough, the estimates (5.5)–(5.25) together with (5.32) readily imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{L}(t) \le -\frac{\mu}{2}\widetilde{L}(t), \quad \forall t \in [0, T), \tag{5.33}$$

which yields

$$\widetilde{L}(t) \le \widetilde{L}(0)\exp\Big(-\frac{\mu}{2}t\Big), \quad \forall t \in [0, T). \tag{5.34}$$

5.2 Step 2: extension to infinite time

Concerning the existence and the decay of solutions to the mixed initial-boundary value problem (2.31), (3.1)–(3.3) for all times $t \geq 0$, we give the following lemma.

Lemma 5.1 *Let $T > 0$ be given. Under the assumptions of Proposition* 3.1*, and moreover if*

$$\|p_\pm(\cdot, t)\|_{H^2(0,L)} \leq \varepsilon_1 \quad \text{for every } t \in [0, T), \tag{5.35}$$

then the unique solution $(p_+, p_-)^{\mathrm{T}} \in C^0([0,T]; H^2(0,L))^2 \cap C^1([0,T]; H^1(0,L))^2$ to the mixed initial-boundary value problem (2.31)*,* (3.1)*–*(3.3) *exists for all $t \in [0, +\infty)$ and the H^2-norm decays exponentially.*

Proof Let a finite time $T > 0$ be given. We apply Theorem 4.1 on the finite time interval $[0, T)$ and obtain for H^2-initial data with a sufficiently small H^2-norm (see (4.14)), the existence and the uniqueness of a continuous H^2-solution which satisfies (4.15) for $t \in [0, T)$. Corollary 4.1 further implies that the H^2-norm of this solution stays small on $[0, T)$ (see (4.16)). In particular,

$$\|p_\pm(\cdot, T)\|_{H^2(0,L)} \leq \varepsilon_1. \tag{5.36}$$

Apply again Theorem 4.1 to obtain that the H^2-solution extends to a solution $p_\pm$ to (2.31), which belongs to $C^0([0, 2T); H^2(0,L))^2 \cap C^1([0, 2T); H^1(0,L))^2$.

The procedure can be repeated iteratively to obtain a solution defined on arbitrarily large time intervals.

Concerning the stabilization of the system (2.31), we proceed by an induction argument, as sketched below:

Step 1 For finite time $T > 0$, we apply Theorem 4.1 on the time interval $[0, T)$. For H^2-initial data with a sufficiently small H^2-norm (see (4.14)), we obtain from (5.34) the exponential decay of the Lyapunov function on $[0, T)$

$$\widetilde{L}(t) \leq \widetilde{L}(0) \exp\Big(-\frac{\mu}{2} t\Big), \quad \forall t \in [0, T). \tag{5.37}$$

Step i ($i > 1$) Assume that for H^2-initial data $p_\pm(\cdot, (i-1)T)$ with a sufficiently small H^2-norm (see (4.14)), the Lyapunov function decays exponentially, i.e.,

$$\widetilde{L}(t) \leq \widetilde{L}(0) \exp\Big(-\frac{\mu}{2} t\Big), \quad \forall t \in [(i-1)T, iT) \tag{5.38}$$

holds.

Step $i + 1$ Prove that

$$\widetilde{L}(t) \leq \widetilde{L}(0) \exp\Big(-\frac{\mu}{2} t\Big), \quad \forall t \in [iT, (i+1)T) \tag{5.39}$$

holds. Assumption (5.38) implies in particular that the H^2-norm of the solution $p_\pm$ is still small on $[(i-1)T, iT)$ (see (4.18)–(4.19)). In particular, $\|p_\pm(\cdot, iT)\|_{H^2(0,L)}$ is small. We obtain thus the existence of a solution for which (5.39) holds.

The procedure can be repeated iteratively to obtain exponential decay over an arbitrarily large time interval. This completes the proof of the lemma.

Lemma 5.1 implies inequality (4.15). We have thus shown that the unique solution $(p_+, p_-)^{\mathrm{T}}$ to the mixed initial-boundary value problem (2.31), (3.1)–(3.3) exists for all time $t \geq 0$ and moreover, the H^2-norm decays exponentially. The proof of Theorem 4.1 is now complete.

6 Summary and Outlook

We have addressed the issue of boundary feedback stabilization of the isothermal Euler equations with friction, in the H^2-norm, in a neighborhood of stationary states. We have proved that under certain assumptions on the initial data and by a suitable choice of boundary conditions, the H^2-norm of perturbation around the stationary solution decays exponentially global in time. This result was proved by means of a Lyapunov function approach. As a direct consequence, the exponential decay in the C^1-norm has been derived.

In this paper, the analysis was restricted to the case of a single space interval $[0, L]$. A natural continuation of the work would be the extension to networks, like star graphs or trees, where conditions at junction nodes have to be taken into consideration (see, for example, [17–19]). Another interesting point is to take into account the time delay in the feedback stabilization process as in [15–16]. Moreover, in this framework also exact controllability to a given demand will be studied (see [20]). We will consider these questions in forthcoming publications.

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