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Local Existence of Smooth Solutions to the FENE Dumbbell Model

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Abstract The author proves the local existence of smooth solutions to the finite extensible nonlinear elasticity (FENE) dumbbell model of polymeric flows in some weighted spaces if the non-dimensional parameter $b > 2$.

Keywords Fokker-Planck equation, FENE model, Degenerate parabolic equations, Regularity
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1 Introduction

A coupled microscopic-macroscopic model arises from the kinetic theory of diluted solutions to polymeric liquids. In this model, a polymer is idealized as an elastic dumbbell consisting of two beads joined by a spring that can be modeled by an elongation vector m (see, e.g., [6]). This system usually consists of the incompressible Navier-Stokes equation for the macroscopic velocity $v(x,t)$ of the flow and the Fokker-Planck type equation for the probability distribution function $f(x,m,t)$ of molecule separations

$$\begin{cases} \partial_t v + (v\cdot\nabla_x)v + \nabla_x p = \nabla_x\cdot\tau + \nu\Delta_x v, \\ \nabla_x\cdot v = 0, \\ \partial_t f + (v\cdot\nabla_x)f + \nabla_m\cdot(\nabla_x v m f) = \dfrac{2}{\zeta}\nabla_m\cdot(\nabla_m U f) + \dfrac{2k_B T_a}{\zeta}\Delta_m f, \end{cases} \tag{1.1}$$

where $x\in\mathbb{R}^3$ is the macroscopic Eulerian coordinate, $m\in\mathbb{R}^3$ is the microscopic molecular configuration variable, and ν,ζ,T_a and k_B are some physical and polymeric parameters. The tensor τ represents the polymer microscopic contribution to stress,

$$\tau = \lambda\int_B m\otimes\nabla_m U f \mathrm{d}m,$$

where λ is the polymer density constant. The elastic spring potential U is given by

$$U(m) = -\frac{Hb}{2}\log\Big(1-\frac{m^2}{b}\Big), \quad m\in B$$

with the elasticity constant H. Here $B \stackrel{\text{def}}{=} B(0,\sqrt{b}\,)$ is a ball with radius $\sqrt{b}$ denoting the maximum dumbbell extension. For the background of the FENE model (1.1), we refer to [6, 8, 29–30].

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This model has been intensively studied in the last decade in several aspects. Most results are very closely related to the molecule length, the maximum dumbbell extension which is denoted by $\sqrt{b}$ after scaling. For the local existence of (1.1), see [9, 15, 24, 31, 33]. For the global existence of (1.1), all known results are usually limited to solutions near equilibrium (see [17–18]), or to some 2D simplified models (see [19, 25]). The construction of weak solutions to the coupled system was considered in [2–5]. For the study of long time behavior, see [1, 11, 16, 32]. We also refer to [14] for references on numerical aspects of polymeric fluid models.

It seems that most works on the existence for the FENE dumbbell model are restricted to some weighted Sobolev spaces or lower regularity Sobolev spaces. The difficulty mainly lies in that the elastic spring potential U is of singularity at the boundary ∂B. The singularity requires at least zero Dirichlet boundary condition

$$f|_{\partial B} = 0.$$

However, the above condition is insufficient for well-posedness when $b > 2$. In order to discuss the behavior of solutions near the boundary to the above macro-micro model and the exact formulation of the well-posedness of boundary value problems, Liu et al. [20–21] studied the microscopic FENE model, i.e., the underlying Fokker-Planck equation alone. In view of the Fichera-Criterion in [28], the authors of [20] pointed out that any preassigned distribution on the boundary value of a weighted distribution would become redundant once $b \geq 2$. Liu and Shin [21] gave the least boundary requirement for the well-posedness of the microscopic FENE model when $b > 2$. Recently, Liu and Shin [22] established the local well-posedness for the FENE dumbbell model under a class of Dirichlet-type boundary conditions dictated by the parameter $b > 0$, and Masmoudi [26] proved the global existence of weak solutions to the FENE dumbbell model of polymeric flows by many weak convergence techniques. But no result is concerned with the higher order regularity of solutions to the coupled FENE model near the boundary. In the present paper, our main interest is the following question:

Under what condition, the solution to (1.1) is of higher regularity?

In [13], smooth solutions in some weighted spaces to the Fokker-Planck equation were alone studied. In this paper, we prove the local existence of smooth solutions to the FENE dumbbell model in some weighted Sobolev spaces if $b > 2$.

After a suitable scaling and choice of parameters, we arrive at the following problem for the coupled system:

$$\begin{cases} \partial_t v + (v\cdot\nabla_x)v + \nabla_x p = \nabla_x\cdot\tau + \Delta_x v, \quad x\in\mathbb{R}^3,\ t>0, \\ \nabla_x\cdot v = 0, \\ \partial_t f + (v\cdot\nabla_x)f + \nabla_m\cdot(\nabla_x v m f) = \frac{1}{2}\nabla_m\cdot\Big(\frac{m}{\rho}f\Big) + \frac{1}{2}\Delta_m f, \quad m\in B \end{cases} \tag{1.2}$$

with the initial value

$$v(x,0) = v_0(x), \quad f(x,m,0) = f_0(x,m), \tag{1.3}$$

where $\rho \overset{\text{def}}{=} 1 - \frac{m^2}{b}$ and

$$\tau = \int_B m\otimes\frac{m}{\rho} f\,\mathrm{d}m.$$

To present our main result, we first introduce some notations to be used throughout this paper.

Definition 1.1 *Let Ω be an open set in $\mathbb{R}^n$ and $s \in \mathbb{N}$. Denote by $W^{k,s}(\Omega \times (0,T))$ with $k = 2s$ or $k = 2s+1$ the Sobolev space*

$$\{u;\ \partial_x^\alpha \partial_t^r u \in L^2(\Omega \times (0,T)) \text{ for } \forall \alpha \in \mathbb{N}^n \text{ and } |\alpha| + 2r \le k\}$$

equipped with the norm

$$\|u\|^2_{W^{k,s}(\Omega\times(0,T))} = \sum_{|\alpha|+2r\le k} \|\partial_x^\alpha \partial_t^r u\|^2_{L^2(\Omega\times(0,T))}.$$

For $s \in \mathbb{N}$, H_x^s is the usual Sobolev space with respect to x. Let $L_t^2 H_x^s = L^2(0,T;H_x^s)$, $C_t^s H_x^s = C^s([0,T];H_x^s)$. We have

$$|v|_s^2 = \sum_{|\alpha|\le s} \int_{\mathbb{R}^3} |\partial_x^\alpha v|^2 \mathrm{d}x,$$

$$|v|_{k,s}^2 = \|v\|^2_{W^{k,s}(\mathbb{R}^3\times(0,T))} = \sum_{|\alpha|+2r\le k} \int_0^T \int_{\mathbb{R}^3} |\partial_x^\alpha \partial_t^r v|^2 \mathrm{d}x\mathrm{d}t, \quad k = 2s \text{ or } 2s+1,$$

$$I_s(v) = \sup_{0\le t\le T} \sum_{i=0}^{s} |\partial_t^i v(t)|^2_{2s-2i} + |v|^2_{2s+1,s},$$

$$\begin{aligned} J_s(g) = \sum_{|\alpha|+|\beta|+2r\le 2s} \Big(&\sup_{0\le t\le T} \int_{\mathbb{R}^3\times B} |\rho^{\frac{1}{2}} \partial_x^\alpha \partial_\theta^\beta \partial_t^r g|^2 \mathrm{d}x\mathrm{d}m + \int_0^T \int_{\mathbb{R}^3\times\partial B} |\partial_x^\alpha \partial_\theta^\beta \partial_t^r g|^2 \mathrm{d}x\mathrm{d}S\mathrm{d}t \\ &+ \int_0^T \int_{\mathbb{R}^3\times B} (|\rho^{\frac{1}{2}} \partial_x^\alpha \partial_\theta^\beta \partial_t^r \partial_m g|^2 + |\partial_x^\alpha \partial_\theta^\beta \partial_t^r g|^2) \mathrm{d}x\mathrm{d}m\mathrm{d}t \Big), \end{aligned}$$

where $\partial_\theta = (m_1\partial_{m_2} - m_2\partial_{m_1}, m_2\partial_{m_3} - m_3\partial_{m_2})$.

We now state our main result as follows.

Theorem 1.1 *Suppose that $b > 2$, $v_0 \in H^4(\mathbb{R}^3)$ with $\nabla_x \cdot v_0 = 0$ and $\rho^{-\frac{b}{2}} f_0 \in H_0^4(\mathbb{R}^3 \times B)$. Then there exists a constant T_0 and a unique solution (v,f) to (1.2) with (1.3) in $\mathbb{R}^3\times B\times(0,T_0)$, such that*

$$I_2(v) + J_2(\rho^{-\frac{b}{2}} f) \le C \tag{1.4}$$

for some constants C and T_0 depending only on $|v_0|_4$, $\|\rho^{-\frac{b}{2}} f_0\|_{H^4(\mathbb{R}^3\times B)}$. Moreover, for any integer $s \ge 2$, $v_0 \in H^{2s+2}(\mathbb{R}^3)$ and $\rho^{-\frac{b}{2}} f_0 \in H_0^{2s+2}(\mathbb{R}^3 \times B)$, the solution (v,f) satisfies

$$I_s(v) + \sum_{|\alpha|+|\beta|+2r\le 2s} \|\partial_x^\alpha \partial_m^\beta \partial_t^r (\rho^{-\frac{b}{2}} f)\|^2 \le C_s(|v_0|_{2s+2}, \|\rho^{-\frac{b}{2}} f_0\|_{H^{2s+2}(\mathbb{R}^3\times B)}) \tag{1.5}$$

for all $T \le T_0$.

Remark 1.1 Theorem 1.1 tells us that if $v_0 \in H^\infty(\mathbb{R}^3)$ with $\nabla_x \cdot v_0 = 0$ and $\rho^{-\frac{b}{2}} f_0 \in H_0^\infty(\mathbb{R}^3 \times \overline{B})$, then the solution $(v, \rho^{-\frac{b}{2}} f)$ obtained in Theorem 1.1 is also smooth in $\mathbb{R}^3 \times \overline{B} \times [0,T_0]$.

The present paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3, the Fokker-Planck equation involving variables x is investigated. The local existence of smooth solutions to the coupled system is proved in Section 4.

2 Preliminaries

This section intends to introduce several lemmas for later needs. For the proofs of Lemmas 2.1–2.3 and Remark 2.1, see [13].

Lemma 2.1 *Let $f(s)$ be in $C^k[0,1]$ with $f^l(0)=0$ for all $l=1,\cdots,k-1$. Then*

$$\Big\|\frac{1}{y}\int_0^y f(s)\mathrm{d}s\Big\|_{H^k(0,1)} \le C\|f\|_{H^k(0,1)}.$$

Lemma 2.2 *Suppose that $\psi(y)\in L^2(0,1)$ satisfies*

$$y\psi_y + \alpha\psi = \beta, \tag{2.1}$$

where $\alpha(y)\in C^1[0,1]$ with $\alpha(0)\ge 1$, $\beta(y)\in H^1(0,1)$. Then $\psi=\mathcal{T}(\beta)$ is a linear bounded operator in $H^1(0,1)$. Moreover, if β is replaced by $y\beta$, then $\psi=\mathcal{T}(y\beta)$ is a bounded operator from $L^2(0,1)$ into $H^1(0,1)$.

Remark 2.1 In (2.1), if $\alpha(y,x,t)\in C^s([0,1]\times\mathbb{R}^n\times[0,T])$ with $\alpha(0,x,t)\ge 1, s\in\mathbb{N}$ and α is constant as $|x|\ge 1$ and if $\beta(y,x,t)\in C^s([0,1];H^\infty(\mathbb{R}^n\times(0,T)))$ and $\psi\in L^2((0,1);L^2(\mathbb{R}^n\times(0,T)))$, then it follows that for arbitrary $k,r\in\mathbb{Z}^1_+$ and for any $l\le s$, there holds

$$\|\partial_y^l\partial_x^k\partial_t^r\psi\| = \|\partial_y^l\partial_x^k\partial_t^r(\mathcal{T}(\beta))\| \le C_{lkr}\sum_{\substack{\bar{l}\le l\\ \bar{k}\le k\\ \bar{r}\le r}}\|\partial_y^{\bar{l}}\partial_x^{\bar{k}}\partial_t^{\bar{r}}\beta\|,$$

and if β is replaced by $y\beta$, then

$$\|\partial_y^{l+1}\partial_x^k\partial_t^r\psi\| = \|\partial_y^{l+1}\partial_x^k\partial_t^r(\mathcal{T}(y\beta))\| \le C_{lkr}\sum_{\substack{\bar{l}\le l\\ \bar{k}\le k\\ \bar{r}\le r}}\|\partial_y^{\bar{l}}\partial_x^{\bar{k}}\partial_t^{\bar{r}}\beta\|.$$

Lemma 2.3 *For each $\phi(y)\in C^1(\overline{B}_1)$ with $B_1=B_1(0)$, there holds*

$$\int_{B_1}|\phi(y)|^2\mathrm{d}y \le \epsilon\int_{B_1}(1-y^2)|\nabla_y\phi(y)|^2\mathrm{d}y + \frac{C}{\epsilon}\int_{B_1}(1-y^2)|\phi(y)|^2\mathrm{d}y, \quad \forall\epsilon>0$$

for some universal constant C.

Corollary 2.1 *For each $\phi(m,x)\in C^1(\overline{B};H^s(\mathbb{R}^3))$ with $B=B(0,\sqrt{b}\,)$ and $\rho=1-\frac{m^2}{b}$, $s\in\mathbb{N}$, there holds*

$$\int_B|\phi|_s^2\mathrm{d}m \le \epsilon\int_B\rho|\nabla_m\phi|_s^2\mathrm{d}m + \frac{C_s}{\epsilon}\int_B\rho|\phi|_s^2\mathrm{d}m, \quad \forall\epsilon>0 \tag{2.2}$$

for some constant C_s.

Proof For each $\phi(m,x)\in C^1(\overline{B};H^s(\mathbb{R}^3))$, we have $\partial_x^\alpha\phi(m,x)\in C^1(\overline{B};L^2(\mathbb{R}^3))$, $|\alpha|\le s$. By Lemma 2.3 for each x and any $\epsilon>0$, we get

$$\int_B\|\partial_x^\alpha\phi(m,x)\|^2_{L^2_x}\mathrm{d}m \le \epsilon\int_B\rho\|\partial_x^\alpha\nabla_m\phi(m,x)\|^2_{L^2_x}\mathrm{d}m + \frac{C_s}{\epsilon}\int_B\rho\|\partial_x^\alpha\phi(m,x)\|^2_{L^2_x}\mathrm{d}m.$$

Integration of the above inequality with respect to x soon yields (2.2).

Now we consider the following degenerate parabolic equation:

$$\begin{cases}\widetilde{L}(\psi) \overset{\text{def}}{=} \rho(\Delta_m\psi - 2\partial_t\psi) - (m + 2\rho\kappa m)\cdot\nabla_m\psi + 2\big(m\cdot(\kappa m)\big)\psi = \varphi, \quad m\in B,\ t>0,\\ \psi(m,0) = \psi_0(m), \quad m\in B,\end{cases} \tag{2.3}$$

where $\kappa = \kappa(t)$, $\psi = \psi(m,t)$ and $\varphi = \varphi(m,t)$. In [13], we studied the homogeneous problem of the above equation. For the nonhomogeneous problem, also define

$$R_s(\psi,T) = \sum_{|\alpha|+2l\le 2m}\int_{\partial B}|\partial_\theta^\alpha\partial_t^l\psi(T)|^2\mathrm{d}S + \sum_{|\alpha|+2l\le 2m}\int_0^T\int_{\partial B}|\partial_\theta\partial_\theta^\alpha\partial_t^l\psi|^2\mathrm{d}S\mathrm{d}t,$$

and with $\partial_t^j\psi(0) = \partial_t^j\psi(m,0)$,

$$\begin{aligned}\|\psi\|^2_{\widetilde{W}^{2s+2,s+1}(B\times(0,T))} &= \|\psi\|^2_{W^{2s+2,s+1}(B\times(0,T))} + R_s(\psi,T) + \|\partial_t\psi\|^2_{W^{2s,s}(\partial B\times(0,T))}\\ &\quad + \sum_{j=0}^{s}\|\partial_t^j\psi(0)\|^2_{H^{2(s-j)+1}(B)},\end{aligned}$$

where $\partial_\theta = (m_1\partial_{m_2} - m_2\partial_{m_1}, m_2\partial_{m_3} - m_3\partial_{m_2})$.

In the similar way to the proof of Lemma 2.9 in [13], we can easily get the following lemma.

Lemma 2.4 *Suppose that $b>2$, $\psi_0\in H_0^{2s+2}(B)$, $s\in\mathbb{N}$, $\kappa(t)\in C^{2s+2}[0,T]$ and $\varphi\in\widetilde{W}^{2s+2,s+1}(B\times(0,T))$ satisfying the compatibility condition at $t=0$, $m\in\partial B$. Then* (2.3) *admits a unique solution $\psi\in W^{2s+2,s+1}(B\times(0,T))$ subject to*

$$\begin{aligned}&R_s(\psi,T) + \|\rho\psi\|^2_{W^{2s+3,s+1}(B\times(0,T))} + \|\psi\|^2_{W^{2s+2,s+1}(B\times(0,T))} + \|\psi\|^2_{W^{2s+2,s+1}(\partial B\times(0,T))}\\ &\le C_s(\|\varphi\|^2_{\widetilde{W}^{2s+2,s+1}(B\times(0,T))} + \|\psi_0\|^2_{H^{2s+2}(B)})\end{aligned} \tag{2.4}$$

for some constant $C_s = C_s(|\kappa|_{C_t^{2s+2}})$.

To prove the existence and uniqueness of solutions in Theorem 1.1, with $\rho = 1-\frac{m^2}{b}$, we use the following transformation as is done in [20]:

$$f = \rho^{\frac{b}{2}}g,$$

which reduces (1.2) to the following problem:

$$\begin{cases}\partial_t v + (v\cdot\nabla_x)v + \nabla_x p = \nabla_x\cdot\tau + \Delta_x v,\\ \nabla_x\cdot v = 0,\\ \rho(\Delta_m g - 2\partial_t g - 2(v\cdot\nabla_x)g) - (m + 2\rho\nabla_x v m)\cdot\nabla_m g + 2(m\cdot(\nabla_x v m))g = 0\end{cases} \tag{2.5}$$

with the initial value

$$v(x,0) = v_0(x), \quad g(x,m,0) = g_0(x,m) = \rho^{-\frac{b}{2}}f_0(x,m), \tag{2.6}$$

where

$$\tau = \int_B m\otimes m\rho^{\frac{b}{2}-1}g\mathrm{d}m.$$

We shall prove our main results by the fixed-point theorem. Define

$$\mathbf{M} = \{g : J_2(g)\le A,\ g(x,m,0) = g_0\}$$

for some constants A and T to be fixed. For a given $h \in \mathbf{M}$, we first solve the Navier-Stokes equation:

$$\begin{cases} \partial_t v + (v \cdot \nabla_x) v + \nabla_x p = \nabla_x \cdot \tau + \Delta_x v, \\ \nabla_x \cdot v = 0, \quad v(x,0) = v_0(x), \\ \tau = \displaystyle\int_B m \otimes m \rho^{\frac{b}{2}-1} h \mathrm{d}m. \end{cases} \tag{2.7}$$

Then with the v obtained in (2.7), based on [13], we shall solve the following equation:

$$\begin{cases} L(g) \overset{\text{def}}{=} \rho\big(\Delta_m g - 2\partial_t g - 2(v \cdot \nabla_x) g\big) - (m + 2\rho \nabla_x v m) \cdot \nabla_m g + 2\big(m \cdot (\nabla_x v m)\big) g = 0, \\ g(x,m,0) = g_0(x,m) = \rho^{-\frac{b}{2}} f_0(x,m). \end{cases} \tag{2.8}$$

Therefore, (2.7)–(2.8) define a mapping

$$\mathcal{F}: \ \mathbf{M} \ni h \mapsto g.$$

The existence of the problem (2.5)–(2.6) is equivalent to the existence of a fixed point of this mapping in some Sobolev spaces.

3 The Fokker-Planck Equation

In this section, we study the initial value problem for the Fokker-Planck equation alone. Note that (2.8) is of singularity at the boundary ∂B, to which, applying the tangential operator would not change its essential structure. Hence, in order to improve the regularity of m, we can first deal with estimates of g about x, t and the tangential direction of m. The well-posedness of (2.8) is stated as follows.

Theorem 3.1 *Suppose that $b > 2$, $\nabla_x \cdot v = 0$, $0 \le i \le 2$,* and

$$v \in C^i([0,T]; H^{4-2i}(\mathbb{R}^3)) \cap W^{5,2}(\mathbb{R}^3 \times (0,T)) \quad \text{and} \quad g_0 \in H_0^4(\mathbb{R}^3 \times B).$$

Then (2.8) *admits a unique solution subject to*

$$J_2(g) \le C_1 \mathrm{e}^{C_1(T + I_2(v))}, \tag{3.1}$$

where $C_1 = C_1(|v_0|_4, \|g_0\|_{H^4(\mathbb{R}^3 \times B)})$.

First of all, we shall show the existence of the solution g to (2.8) by the flow map.

Lemma 3.1 *Suppose that $b > 2$, and $v \in C_c^\infty(\mathbb{R}^3 \times [0,T])$ with $\nabla_x \cdot v = 0$ and $g_0 \in C_c^\infty(\mathbb{R}^3 \times B)$. Then* (2.8) *admits a unique solution $g \in C^\infty(\mathbb{R}^3 \times \overline{B} \times [0,T])$.*

Proof Define the flow associated with v, namely $x(y,t)$, such that

$$\partial_t x(y,t) = v(x(y,t),t), \quad x(y,0) = y.$$

Obviously, $x(y,t) \in C^\infty(\mathbb{R}^3 \times [0,T])$ and $\det\big(\frac{\partial x}{\partial y}\big) \equiv 1$, since $v \in C_c^\infty(\mathbb{R}^3 \times [0,T])$ and $\nabla_x \cdot v = 0$ (see [23]). By making the change of variable $\widetilde{g}(y,m,t) = g(x(y,t),m,t)$, we see that $g(x,m,t)$ solves (2.8) if and only if $\widetilde{g}(y,m,t)$ solves

$$\begin{cases} \widetilde{L}(\widetilde{g}) = 0, \\ \widetilde{g}(y,m,0) = g_0(y,m), \end{cases} \tag{3.2}$$

where $\widetilde{L}$ is defined in (2.3) with κ replaced by $\widetilde{\kappa}(y,t) = \nabla_x v(x(y,t),t)$ and y as a parameter. By Lemma 2.4 for each y, there exists a unique solution $\widetilde{g}$, such that, for any $s \in \mathbb{N}$,

$$\begin{aligned}&R_s(\widetilde{g}(y),T) + \|\rho\widetilde{g}(y)\|^2_{W^{2s+3,s+1}(B\times(0,T))} + \|\widetilde{g}(y)\|^2_{W^{2s+2,s+1}(B\times(0,T))}\\&+ \|\widetilde{g}(y)\|^2_{W^{2s+2,s+1}(\partial B\times(0,T))}\\ \le\ & C_s(|\widetilde{\kappa}(y)|_{C_t^{2s+2}})\|g_0(y)\|^2_{H^{2s+2}(B)}.\end{aligned} \tag{3.3}$$

Integration of (3.3) with respect to y, using the Sobolev embedding theorem and $\sup\limits_y |\widetilde{\kappa}| \le C|\widetilde{\kappa}|_2$, gives

$$\begin{aligned}&\int_{\mathbb{R}^3}(R_s(\widetilde{g},T) + \|\rho\widetilde{g}\|^2_{W^{2s+3,s+1}(B\times(0,T))} + \|\widetilde{g}\|^2_{W^{2s+2,s+1}(B\times(0,T))} + \|\widetilde{g}\|^2_{W^{2s+2,s+1}(\partial B\times(0,T))})\mathrm{d}y\\ \le\ & C_s(|\widetilde{\kappa}|_{C_t^{2s+2}H_y^2})\int_{\mathbb{R}^3}\|g_0\|^2_{H^{2s+2}(B)}\mathrm{d}y.\end{aligned} \tag{3.4}$$

To prove the regularity of $\widetilde{g}$ with respect to y, we use a difference quotient method. Define the difference operator in y as

$$\tau\widetilde{g} = \frac{1}{\eta}[\widetilde{g}(y+\eta e_i) - \widetilde{g}(y)], \quad 1 \le i \le 3,$$

where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$. Hence, $\tau\widetilde{g}$ solves

$$\begin{cases}\widetilde{L}(\tau\widetilde{g}) = 2\rho(\tau\widetilde{\kappa}m)\cdot\nabla_m\widetilde{g}(y+\eta e_i) - 2\big(m\cdot(\tau\widetilde{\kappa}m)\big)\widetilde{g}(y+\eta e_i) \overset{\text{def}}{=} \widetilde{h},\\ \tau\widetilde{g}(y,m,0) = \tau g_0(y,m).\end{cases}$$

Obviously, $\widetilde{h}$ satisfies the compatibility condition at $t = 0$ and $m \in \partial B$ if η is very small since $g_0 \in C_c^\infty(\mathbb{R}^3 \times B)$. Applying Lemma 2.4 and integrating in y, by means of the Sobolev embedding theorem $H_y^2(\mathbb{R}^3) \hookrightarrow C(\mathbb{R}^3)$, we can get

$$\begin{aligned}&\int_{\mathbb{R}^3}(R_s(\tau\widetilde{g},T) + \|\rho\tau\widetilde{g}\|^2_{W^{2s+3,s+1}(B\times(0,T))} + \|\tau\widetilde{g}\|^2_{W^{2s+2,s+1}(B\times(0,T))}\\&+ \|\tau\widetilde{g}\|^2_{W^{2s+2,s+1}(\partial B\times(0,T))})\mathrm{d}y\\ \le\ & C_s(|\widetilde{\kappa}|_{C_t^{2s+2}H_y^2})\int_{\mathbb{R}^3}(\|g_0\|^2_{H^{2s+3}(B)} + \|\widetilde{h}\|^2_{\widetilde{W}^{2s+2,s+1}(B\times(0,T))})\mathrm{d}y\end{aligned} \tag{3.5}$$

for some constant $C_s(|\widetilde{\kappa}|_{C_t^{2s+2}H_y^2})$ independent of η. By (3.4), it follows that

$$\begin{aligned}&\int_{\mathbb{R}^3}\|\widetilde{h}\|^2_{\widetilde{W}^{2s+2,s+1}(B\times(0,T))}\mathrm{d}y\\ \le\ & C_s(|\tau\widetilde{\kappa}|_{C_t^{2s+2}L_y^\infty})\int_{\mathbb{R}^3}\Big(\|\rho\widetilde{g}\|^2_{W^{2s+3,s+1}(B\times(0,T))} + \|\widetilde{g}\|^2_{W^{2s+2,s+1}(B\times(0,T))}\\&+ R_s(\widetilde{g},T) + \|\partial_t\widetilde{g}\|^2_{W^{2s,s}(\partial B\times(0,T))} + \sum_{j=0}^{s}\|\partial_t^j\widetilde{g}(0)\|^2_{H^{2(s-j)+1}(B)}\Big)\mathrm{d}y\\ \le\ & C_s(|\widetilde{\kappa}|_{C_t^{2s+2}H_y^3})\int_{\mathbb{R}^3}\|g_0\|^2_{H^{2s+2}(B)}\mathrm{d}y\end{aligned}$$

for another constant $C_s(|\widetilde{\kappa}|_{C_t^{2s+2}H_y^3})$ independent of η. Inserting the above inequality into (3.5) yields

$$\tau\widetilde{g} \in L^2(\mathbb{R}^3, W^{2s+2,s+1}(B\times(0,T))).$$

Hence, passing the limit η to 0, we arrive at $\widetilde{g} \in H^1(\mathbb{R}^3, W^{2s+2,s+1}(B \times (0,T)))$. On the other hand, from

$$\|g(x,m,t)\|_{H^1(\mathbb{R}^3,W^{2s+2,s+1}(B\times(0,T)))} \le C\|\widetilde{g}(y,m,t)\|_{H^1(\mathbb{R}^3,W^{2s+2,s+1}(B\times(0,T)))},$$

there holds

$$g(x,m,t) \in H^1(\mathbb{R}^3, W^{2s+2,s+1}(B \times (0,T))).$$

Moreover, in a similar argument, step by step, we can prove $g \in H^s(\mathbb{R}^3, W^{2s+2,s+1}(B\times(0,T)))$. This completes the present lemma.

The estimates obtained in Lemma 3.1 are not good enough to match (2.7) with (2.8), so we need more precise estimates of g obtained in (2.8). The following inequalities will be useful. For any positive integer $r > 0$ and $u, v \in L_x^\infty \cap H_x^r$,

$$\|uv\|_{H^r} \le C(\|u\|_{L^\infty}\|v\|_{H^r} + \|u\|_{H^r}\|v\|_{L^\infty}), \tag{3.6}$$

$$\sum_{|\alpha|\le r} \|\partial^\alpha(uv) - u\partial^\alpha v\|_{L^2} \le C(\|\nabla u\|_{L^\infty}\|v\|_{H^{r-1}} + \|u\|_{H^r}\|v\|_{L^\infty}). \tag{3.7}$$

For $f \in H^1(\mathbb{R}^3)$, by using the Gagliardo-Nirenberg interpolation inequality (see [10, 12, 27]), we have

$$\|f\|_{L^4(\mathbb{R}^3)} \le C\|f\|_{H^1(\mathbb{R}^3)}. \tag{3.8}$$

In what follows, we shall bound the norms of $g_i = \partial_t^i g(x,m,0)$ in $H^{2s-2i}(\mathbb{R}^3 \times B)$, $i = 1, 2, \cdots, s$. Obviously, g_i are involved in $v_i = \partial_t^i v(x,0)$ which depend on g_{i-1}. Differentiation of (2.7) and (2.8) in t, letting $t = 0$, gives, for all $1 \le i \le s$,

$$\begin{cases} v_i = \Delta_x v_{i-1} + \nabla_x \cdot \tau_{i-1} - \nabla_x p_{i-1} - \displaystyle\sum_{j=0}^{i-1} Q_j (v_j \cdot \nabla_x) v_{i-j-1}, \\ g_i = \dfrac{1}{2}\Delta_m g_{i-1} - \displaystyle\sum_{j=0}^{i-1} \widetilde{Q}_j \Big((v_j \cdot \nabla_x) g_{i-j-1} + (\nabla_x v_j m) \cdot \nabla_m g_{i-j-1} \\ \qquad - \dfrac{1}{2\rho} m \cdot (\nabla_x v_j m) g_{i-j-1} \Big) - \dfrac{1}{2\rho}(m \cdot \nabla_m) g_{i-1}, \end{cases} \tag{3.9}$$

where $p_i = \partial_t^i p|_{t=0}$, $\tau_i = \int_B m \otimes m \rho^{\frac{b}{2}-1} g_i \mathrm{d}m$, Q_j and $\widetilde{Q}_j$ are some constants.

Lemma 3.2 *Suppose that $g_0 \in H_0^{2s}(\mathbb{R}^3 \times B)$, $1 \le i \le s$. Then there holds*

$$|v_i|_{2s-2i}^2 + \|g_i\|_{H^{2s-2i}(\mathbb{R}^3\times B)}^2 \le C_s(|v_0|_{2s}, \|g_0\|_{H^{2s}(\mathbb{R}^3\times B)}). \tag{3.10}$$

Proof We shall prove the present lemma by induction on i. When $i = 1$, by using the Sobolev inequality, we have

$$\begin{aligned} |v_1|_{2s-2}^2 &\le C(|\Delta_x v_0|_{2s-2}^2 + |\nabla_x \cdot \tau_0|_{2s-2}^2 + |(v_0 \cdot \nabla_x) v_0|_{2s-2}^2) \\ &\le C\Big(|v_0|_{2s}^2 + \|g_0\|_{H^{2s-1}(\mathbb{R}^3\times B)}^2 + \sum_{|\alpha|+|\beta|\le 2s-2} \|\partial_x^\alpha v_0\|_{L_x^\infty}^2 \|\partial_x^\beta \nabla_x v_0\|_{L_x^2}^2 \Big) \\ &\le C_s(|v_0|_{2s}, \|g_0\|_{H^{2s}(\mathbb{R}^3\times B)}), \end{aligned} \tag{3.11}$$

$$\begin{aligned} \|g_1\|_{H^{2s-2}(\mathbb{R}^3\times B)}^2 &\le C\Big(\|g_0\|_{H^{2s}(\mathbb{R}^3\times B)}^2 + \|(\nabla_x v_0 m) \cdot \nabla_m g_0\|_{H^{2s-2}(\mathbb{R}^3\times B)}^2 \\ &\quad + \|(v_0 \cdot \nabla_x) g_0\|_{H^{2s-2}(\mathbb{R}^3\times B)}^2 \\ &\quad + \Big\| \frac{1}{\rho}\big((m \cdot \nabla_m) g_0 - 2m \cdot (\nabla_x v_0 m) g_0\big) \Big\|_{H^{2s-2}(\mathbb{R}^3\times B)}^2 \Big). \end{aligned} \tag{3.12}$$

Here the constant C is under control, but may be different from line to line. By (3.8), it follows that

$$\begin{aligned}\|(\nabla_x v_0 m)\cdot\nabla_m g_0\|^2_{H^{2s-2}(\mathbb{R}^3\times B)} &\le C\sum_{|\alpha|+|\beta|+|\gamma|\le 2s-2}\|\partial_x^\alpha\nabla_x v_0\|^2_{L^4_x}\|\partial_x^\beta\partial_m^\gamma\nabla_m g_0\|^2_{L^2_m L^4_x}\\ &\le C\sum_{|\alpha|+|\beta|+|\gamma|\le 2s-2}\|\partial_x^\alpha v_0\|^2_{H^2_x}\|\partial_x^\beta\partial_m^\gamma g_0\|^2_{H^2(\mathbb{R}^3\times B)}\\ &\le C_s(|v_0|_{2s},\|g_0\|_{H^{2s}(\mathbb{R}^3\times B)}).\end{aligned}\tag{3.13}$$

Similarly, we have

$$\|(v_0\cdot\nabla_x)g_0\|^2_{H^{2s-2}(\mathbb{R}^3\times B)}\le C_s(|v_0|_{2s},\|g_0\|_{H^{2s}(\mathbb{R}^3\times B)}).\tag{3.14}$$

By Lemma 2.1, we get

$$\begin{aligned}&\Big\|\frac{1}{\rho}((m\cdot\nabla_m)g_0-2m\cdot(\nabla_x v_0 m)g_0)\Big\|^2_{H^{2s-2}(\mathbb{R}^3\times B)}\\ &\le\|\partial_m((m\cdot\nabla_m)g_0-2m\cdot(\nabla_x v_0 m)g_0)\|^2_{H^{2s-2}(\mathbb{R}^3\times B)}\\ &\le C_s(|v_0|_{2s},\|g_0\|_{H^{2s}(\mathbb{R}^3\times B)}).\end{aligned}\tag{3.15}$$

Inserting (3.13)–(3.15) into (3.12), we can obtain

$$\|g_1\|^2_{H^{2s-2}(\mathbb{R}^3\times B)}\le C_s(|v_0|_{2s},\|g_0\|_{H^{2s}(\mathbb{R}^3\times B)}).$$

Combining it with (3.11) soon yields (3.10) with $i=1$.

Now we assume that Lemma 3.2 is true when $i\le k-1$, where $2\le k\le s$. It is easy to see that

$$|v_k|^2_{2s-2k}\le C\Big(|\Delta_x v_{k-1}|^2_{2s-2k}+|\nabla_x\cdot\tau_{k-1}|^2_{2s-2k}+\sum_{j=0}^{k-1}|(v_j\cdot\nabla_x)v_{k-j-1}|^2_{2s-2k}\Big).\tag{3.16}$$

By the Sobolev embedding theorem and the hypotheses of the induction, the last term of the above estimate is bounded by

$$\begin{aligned}\sum_{j=0}^{k-1}\sum_{|\alpha|+|\beta|\le 2s-2k}\|\partial_x^\alpha v_j\|^2_{L^\infty_x}\|\partial_x^\beta\nabla_x v_{k-j-1}\|^2_{L^2_x}&\le C\sum_{j=0}^{k-1}|v_j|^2_{2s-2k+2}|v_{k-j-1}|^2_{2s-2k+1}\\ &\le C_s(|v_0|_{2s},\|g_0\|_{H^{2s}(\mathbb{R}^3\times B)}),\end{aligned}$$

since $2s-2k+2\le 2(s-j)$ and $2s-2k+1\le 2[s-(k-j-1)]$ as $0\le j\le k-1$. Combining it with (3.16) soon yields

$$|v_k|^2_{2s-2k}\le C_s(|v_0|_{2s},\|g_0\|_{H^{2s}(\mathbb{R}^3\times B)}).$$

Similarly, we get

$$\|g_k\|^2_{H^{2s-2k}(\mathbb{R}^3\times B)}\le C_s(|v_0|_{2s},\|g_0\|_{H^{2s}(\mathbb{R}^3\times B)}).$$

Thus we complete the proof of the present lemma.

Proof of Theorem 3.1 We first assume that $v\in C_c^\infty(\mathbb{R}^3\times[0,T])$ and $g_0\in C_c^\infty(\mathbb{R}^3\times B)$. From Lemma 3.1, we only need to prove that g satisfies (3.1). By means of $|\alpha|$-times differentiations of (2.8) in x, $|\alpha|\le 4$, we can see that

$$\begin{cases}L(\partial_x^\alpha g)=\rho\mathrm{I}_1+\rho\mathrm{I}_2+\mathrm{I}_3,\\ \partial_x^\alpha g(x,m,0)=\partial_x^\alpha g_0(x,m),\end{cases}\tag{3.17}$$

where

$$\mathrm{I}_1 = 2[\partial_x^\alpha, v\cdot\nabla_x]g, \quad \mathrm{I}_2 = 2[\partial_x^\alpha, (\nabla_x vm)\cdot\nabla_m]g, \quad \mathrm{I}_3 = -2[\partial_x^\alpha, m\cdot(\nabla_x vm)]g.$$

Taking the inner product of the first equation of (3.17) with $\partial_x^\alpha g$ over $\mathbb{R}^3\times B$ gives

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3\times B}\rho|\partial_x^\alpha g|^2\mathrm{d}x\mathrm{d}m - \int_{\mathbb{R}^3\times B}\rho(\Delta_m\partial_x^\alpha g)\partial_x^\alpha g\mathrm{d}x\mathrm{d}m + \int_{\mathbb{R}^3\times B}(m\cdot\nabla_m\partial_x^\alpha g)\partial_x^\alpha g\mathrm{d}x\mathrm{d}m\\ &= -2\int_{\mathbb{R}^3\times B}\rho((\nabla_x vm)\cdot\nabla_m\partial_x^\alpha g)\partial_x^\alpha g\mathrm{d}x\mathrm{d}m + 2\int_{\mathbb{R}^3\times B}(m\cdot(\nabla_x vm))|\partial_x^\alpha g|^2\mathrm{d}x\mathrm{d}m\\ &\quad - \int_{\mathbb{R}^3\times B}(\rho\mathrm{I}_1+\rho\mathrm{I}_2+\mathrm{I}_3)\partial_x^\alpha g\mathrm{d}x\mathrm{d}m. \end{aligned}\tag{3.18}$$

Integrating by parts, we can obtain

$$\begin{aligned}&-\int_{\mathbb{R}^3\times B}\rho(\Delta_m\partial_x^\alpha g)\partial_x^\alpha g\mathrm{d}x\mathrm{d}m = \int_{\mathbb{R}^3\times B}\rho|\nabla_m\partial_x^\alpha g|^2\mathrm{d}x\mathrm{d}m - \frac{2}{b}\int_{\mathbb{R}^3\times B}(m\cdot\nabla_m\partial_x^\alpha g)\partial_x^\alpha g\mathrm{d}x\mathrm{d}m,\\ &\int_{\mathbb{R}^3\times B}(m\cdot\nabla_m\partial_x^\alpha g)\partial_x^\alpha g\mathrm{d}x\mathrm{d}m = \frac{\sqrt{b}}{2}\int_{\mathbb{R}^3\times\partial B}|\partial_x^\alpha g|^2\mathrm{d}x\mathrm{d}S - \frac{3}{2}\int_{\mathbb{R}^3\times B}|\partial_x^\alpha g|^2\mathrm{d}x\mathrm{d}m.\end{aligned}$$

By the Cauchy inequality, we have

$$\begin{aligned}&-2\int_{\mathbb{R}^3\times B}\rho((\nabla_x vm)\cdot\nabla_m\partial_x^\alpha g)\partial_x^\alpha g\mathrm{d}x\mathrm{d}m\\ &\le \frac{1}{2}\int_{\mathbb{R}^3\times B}\rho|\nabla_m\partial_x^\alpha g|^2\mathrm{d}x\mathrm{d}m + C\|\nabla_x v\|_{L_x^\infty}^2\int_{\mathbb{R}^3\times B}\rho|\partial_x^\alpha g|^2\mathrm{d}x\mathrm{d}m.\end{aligned}$$

Inserting all the above estimates into (3.18) and using Corollary 2.1, one can get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^{\frac12}\partial_x^\alpha g\|_{L^2(\mathbb{R}^3\times B)}^2 + \|\rho^{\frac12}\nabla_m\partial_x^\alpha g\|_{L^2(\mathbb{R}^3\times B)}^2 + \frac{b-2}{\sqrt{b}}\|\partial_x^\alpha g\|_{L^2(\mathbb{R}^3\times\partial B)}^2\\ &\le C(1+\|\nabla_x v\|_{L_x^\infty}^2)\|\rho^{\frac12}\partial_x^\alpha g\|_{L^2(\mathbb{R}^3\times B)}^2 + C\int_{\mathbb{R}^3\times B}(\rho|\mathrm{I}_1|+\rho|\mathrm{I}_2|+|\mathrm{I}_3|)|\partial_x^\alpha g|\mathrm{d}x\mathrm{d}m.\end{aligned}\tag{3.19}$$

By (3.7) and the Sobolev embedding theorem, for $|\alpha|\le 4$, we have

$$\begin{aligned}\|\rho^{\frac12}\mathrm{I}_1\|_{L^2(\mathbb{R}^3\times B)} &\le C(\|\partial_x v\|_{L_x^\infty}\|\rho^{\frac12}\partial_x g\|_{L_m^2H_x^3} + \|v\|_{H_x^4}\|\rho^{\frac12}\partial_x g\|_{L_m^2L_x^\infty})\\ &\le C\|v\|_{H_x^4}\|\rho^{\frac12}g\|_{L_m^2H_x^4}.\end{aligned}$$

Similarly, we get

$$\begin{aligned}\|\rho^{\frac12}\mathrm{I}_2\|_{L^2(\mathbb{R}^3\times B)} &\le C\|v\|_{H_x^5}\|\rho^{\frac12}\nabla_m g\|_{L_m^2H_x^3},\\ \|\mathrm{I}_3\|_{L^2(\mathbb{R}^3\times B)} &\le C\|v\|_{H_x^5}\|g\|_{L_m^2H_x^4}.\end{aligned}$$

Therefore, by means of the Cauchy inequality, we can obtain

$$\begin{aligned}\int_{\mathbb{R}^3\times B}(\rho|\mathrm{I}_1|+\rho|\mathrm{I}_2|)|\partial_x^\alpha g|\mathrm{d}x\mathrm{d}m &\le C(\|v\|_{H_x^4}\|\rho^{\frac12}g\|_{L_m^2H_x^4}^2 + \|v\|_{H_x^5}\|\rho^{\frac12}\nabla_m g\|_{L_m^2H_x^3}\|\rho^{\frac12}g\|_{L_m^2H_x^4})\\ &\le \frac14\|\rho^{\frac12}\nabla_m g\|_{L_m^2H_x^4}^2 + C(1+\|v\|_{H_x^5}^2)\|\rho^{\frac12}g\|_{L_m^2H_x^4}^2.\end{aligned}\tag{3.20}$$

By (2.2), it follows that

$$\int_{\mathbb{R}^3\times B}|\mathrm{I}_3||\partial_x^\alpha g|\mathrm{d}x\mathrm{d}m\le C\|v\|_{H_x^5}\|g\|^2_{L^2_mH^4_x}\le\frac14\|\rho^{\frac12}\nabla_m g\|^2_{L^2_mH^4_x}+C\|v\|^2_{H^5_x}\|\rho^{\frac12}g\|^2_{L^2_mH^4_x}. \tag{3.21}$$

Inserting (3.20)–(3.21) into (3.19), by the Gronwall inequality and (2.2), we can get

$$\begin{aligned}&\sup_{0\le t\le T}\int_B|\rho^{\frac12}g|_4^2\mathrm{d}m+\int_0^T\int_B(|\rho^{\frac12}\nabla_m g|_4^2+|g|_4^2)\mathrm{d}m\mathrm{d}t+\int_0^T\int_{\partial B}|g|_4^2\mathrm{d}S\,\mathrm{d}t\\&\le \mathrm{e}^{C(T+I_2(v))}\|g_0\|^2_{H^4(\mathbb{R}^3\times B)}.\end{aligned}\tag{3.22}$$

Applying $\partial_x^\beta\partial_t$ with $|\beta|\le 2$ to (2.8), by a similar argument, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^{\frac12}\partial_x^\beta\partial_t g\|^2_{L^2(\mathbb{R}^3\times B)}+\|\rho^{\frac12}\nabla_m\partial_x^\beta\partial_t g\|^2_{L^2(\mathbb{R}^3\times B)}+\frac{b-2}{\sqrt b}\|\partial_x^\beta\partial_t g\|^2_{L^2(\mathbb{R}^3\times\partial B)}\\&\le C(1+\|\nabla_x v\|^2_{L^\infty_x})\|\rho^{\frac12}\partial_x^\beta\partial_t g\|^2_{L^2(\mathbb{R}^3\times B)}+C\int_{\mathbb{R}^3\times B}(\rho|\mathrm{I}_4|+\rho|\mathrm{I}_5|+|\mathrm{I}_6|)|\partial_x^\beta\partial_t g|\mathrm{d}x\mathrm{d}m,\end{aligned}\tag{3.23}$$

where

$$\mathrm{I}_4=2[\partial_x^\beta\partial_t,v\cdot\nabla_x]g,\quad \mathrm{I}_5=2[\partial_x^\beta\partial_t,(\nabla_x vm)\cdot\nabla_m]g,\quad \mathrm{I}_6=-2[\partial_x^\beta\partial_t,m\cdot(\nabla_x vm)]g.$$

By (3.6) and the Sobolev embedding theorem, we have

$$\begin{aligned}\|\rho^{\frac12}\mathrm{I}_4\|_{L^2(\mathbb{R}^3\times B)}&\le C\Big(\|\partial_t v\|_{H^2_x}\|\rho^{\frac12}\partial_x g\|_{L^2_mL^\infty_x}+\|\partial_t v\|_{L^\infty_x}\|\rho^{\frac12}\partial_x g\|_{L^2_mH^2_x}\\&\quad+\sum_{|\gamma|\le|\beta|}\|\partial_x^\gamma v\|_{L^\infty_x}\|\rho^{\frac12}\partial_x\partial_t g\|_{L^2_mH^1_x}\Big)\\&\le C(\|\partial_t v\|_{H^2_x}\|\rho^{\frac12}g\|_{L^2_mH^3_x}+\|v\|_{H^4_x}\|\rho^{\frac12}\partial_t g\|_{L^2_mH^2_x}).\end{aligned}$$

Similarly, we get

$$\begin{aligned}\|\rho^{\frac12}\mathrm{I}_5\|_{L^2(\mathbb{R}^3\times B)}&\le C(\|\partial_t v\|_{H^3_x}\|\rho^{\frac12}\nabla_m g\|_{L^2_mH^2_x}+\|v\|_{H^5_x}\|\rho^{\frac12}\nabla_m\partial_t g\|_{L^2_mH^1_x}),\\ \|\mathrm{I}_6\|_{L^2(\mathbb{R}^3\times B)}&\le C(\|\partial_t v\|_{H^3_x}\|g\|_{L^2_mH^2_x}+\|v\|_{H^5_x}\|\partial_t g\|_{L^2_mH^1_x}).\end{aligned}$$

Therefore, by means of the Cauchy inequality, we can obtain

$$\begin{aligned}&\int_{\mathbb{R}^3\times B}(\rho|\mathrm{I}_4|+\rho|\mathrm{I}_5|)|\partial_x^\beta\partial_t g|\mathrm{d}x\mathrm{d}m\\&\le\frac14(\|\rho^{\frac12}\nabla_m g\|^2_{L^2_mH^2_x}+\|\rho^{\frac12}\nabla_m\partial_t g\|^2_{L^2_mH^2_x})\\&\quad+C(1+\|v\|^2_{H^5_x}+\|\partial_t v\|^2_{H^3_x})(\|\rho^{\frac12}g\|^2_{L^2_mH^4_x}+\|\rho^{\frac12}\partial_t g\|^2_{L^2_mH^2_x}).\end{aligned}\tag{3.24}$$

By (2.2), it follows that

$$\begin{aligned}\int_{\mathbb{R}^3\times B}|\mathrm{I}_6||\partial_x^\beta\partial_t g|\mathrm{d}x\mathrm{d}m&\le C(\|\partial_t v\|_{H^3_x}+\|v\|_{H^5_x})(\|g\|^2_{L^2_mH^2_x}+\|\partial_t g\|^2_{L^2_mH^2_x})\\&\le\frac14(\|\rho^{\frac12}\nabla_m g\|^2_{L^2_mH^2_x}+\|\rho^{\frac12}\nabla_m\partial_t g\|^2_{L^2_mH^2_x})\\&\quad+C(\|v\|^2_{H^5_x}+\|\partial_t v\|^2_{H^3_x})(\|\rho^{\frac12}g\|^2_{L^2_mH^2_x}+\|\rho^{\frac12}\partial_t g\|^2_{L^2_mH^2_x}).\end{aligned}\tag{3.25}$$

Inserting (3.24)–(3.25) into (3.23), by the Gronwall inequality, (2.2), (3.10) and (3.22), we can get

$$\begin{aligned}&\sup_{0\le t\le T}\int_B |\rho^{\frac12}\partial_t g|_2^2\mathrm{d}m+\int_0^T\int_B(|\rho^{\frac12}\nabla_m\partial_t g|_2^2+|\partial_t g|_2^2)\mathrm{d}m\mathrm{d}t+\int_0^T\int_{\partial B}|\partial_t g|_2^2\mathrm{d}S\mathrm{d}t\\&\le C\mathrm{e}^{C(T+I_2(v))}\|g_0\|^2_{H^4(\mathbb{R}^3\times B)}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\sup_{0\le t\le T}\int_B \|\rho^{\frac12}\partial_t^2 g\|_{L_x^2}^2\mathrm{d}m+\int_0^T\int_B(\|\rho^{\frac12}\nabla_m\partial_t^2 g\|_{L_x^2}^2+\|\partial_t^2 g\|_{L_x^2}^2)\mathrm{d}m\mathrm{d}t+\int_0^T\int_{\partial B}\|\partial_t^2 g\|_{L_x^2}^2\mathrm{d}S\mathrm{d}t\\&\le C\mathrm{e}^{C(T+I_2(v))}\|g_0\|^2_{H^4(\mathbb{R}^3\times B)}.\end{aligned}$$

Moreover, it is easy to see that

$$[\partial_\theta,\Delta_m]=0,\quad [\partial_\theta, m\cdot\nabla_m]=0,\quad \partial_\theta\rho=0,$$

where $\partial_\theta=(m_1\partial_{m_2}-m_2\partial_{m_1}, m_2\partial_{m_3}-m_3\partial_{m_2})$. Thus, we have

$$L(\partial_\theta^\beta g)=2\rho[\partial_\theta^\beta,(\nabla_x vm)\cdot\nabla_m]g-2[\partial_\theta^\beta, m\cdot(\nabla_x vm)]g,\quad |\beta|\le 4.$$

Applying $\partial_x^\alpha\partial_t^r$ for $|\alpha|+2r\le 4-|\beta|$ to the above equation, and repeating the previous argument, we can get (3.1).

If $v\in C^i([0,T];H^{4-2i}(\mathbb{R}^3))\cap W^{5,2}(\mathbb{R}^3\times(0,T))$ for $0\le i\le 2$ and $g_0\in H_0^4(\mathbb{R}^3\times B)$, we can approximate to v and g_0 in the corresponding spaces by $v_\delta\in C_c^\infty(\mathbb{R}^3\times[0,T])$ and $(g_0)_\delta\in C_c^\infty(\mathbb{R}^3\times B)$. Then for each δ by Lemma 3.1, (2.8) admits a unique solution $g_\delta\in C^\infty(\mathbb{R}^3\times\overline{B}\times[0,T])$ satisfying (3.1), where the constants C are independent of δ. By passing the limit, we can find g just solving (2.8) for the given data v and g_0. Thus, we complete the proof of Theorem 3.1.

4 Coupled Systems

In this section, we shall prove Theorem 1.1. First, we use the fixed point theorem to prove the existence and uniqueness of (2.5) with (2.6).

Lemma 4.1 *Suppose that $b>2$, $v_0\in H^4(\mathbb{R}^3)$ with $\nabla_x\cdot v_0=0$ and $g_0\in H_0^4(\mathbb{R}^3\times B)$. Then there exist a constant T_0 and a unique solution (v,g) to* (2.5) *with* (2.6)*, such that*

$$I_2(v)\le 2C_0,\quad J_2(g)\le 2C_1\mathrm{e}^{2C_0C_1}\tag{4.1}$$

for some constants C_0, C_1 and T_0 depending only on $|v_0|_4, \|g_0\|_{H^4(\mathbb{R}^3\times B)}$.

Proof Let g_i satisfy (3.9), $1\le i\le 2$. Define

$$\mathbf{M}=\{g: J_2(g)\le A,\ g(x,m,0)=g_0\}$$

for some constants A and T to be fixed.

Assume that $h\in\mathbf{M}$. We first prove that, given $v_0\in H^4(\mathbb{R}^3)$ with $\nabla_x\cdot v_0=0$ and $g_0\in H_0^4(\mathbb{R}^3\times B)$, the operator

$$\mathcal{F}:\ \mathbf{M}\ni h\mapsto g\in\mathbf{M},$$

if T is very small. It is well-known that (2.7) has a unique local solution v satisfying

$$I_2(v) \le C_0(1 + |\tau|_{4,2}^2), \tag{4.2}$$

where $C_0 = C_0(|v_0|_4, \|g_0\|_{H^4(\mathbb{R}^3\times B)})$. We proceed to estimate the stress term

$$|\tau|_{4,2}^2 = \sum_{|\alpha|+2r\le 4} \int_0^T \int_{\mathbb{R}^3} |\partial_x^\alpha \partial_t^r \tau|^2 \mathrm{d}x\mathrm{d}t.$$

From Lemma 2.3 as $b > 2$, for any $(x,t) \in \mathbb{R}^3 \times (0,T)$ and $|\alpha| + 2r \le 4$, there holds

$$\begin{aligned} |\partial_x^\alpha \partial_t^r \tau(x,t)|^2 &= \Big| \int_B m \otimes m \rho^{\frac{b}{2}-1} \partial_x^\alpha \partial_t^r h(x,t) \mathrm{d}m \Big|^2 \\ &\le \epsilon \int_B \rho |\nabla_m \partial_x^\alpha \partial_t^r h(x,t)|^2 \mathrm{d}m + C_\epsilon \int_B \rho |\partial_x^\alpha \partial_t^r h(x,t)|^2 \mathrm{d}m \quad \text{ for any } \epsilon > 0. \end{aligned}$$

Since

$$\int_0^T \int_{\mathbb{R}^3\times B} \rho |\partial_x^\alpha \partial_t^r h|^2 \mathrm{d}x\mathrm{d}m\mathrm{d}t \le T J_2(h) \le TA,$$

we obtain

$$|\tau|_{4,2}^2 \le \sum_{|\alpha|+2r\le 4} (\epsilon \|\rho^{\frac{1}{2}} \nabla_m \partial_x^\alpha \partial_t^r h\|^2 + C_\epsilon \|\rho^{\frac{1}{2}} \partial_x^\alpha \partial_t^r h\|^2) \le \epsilon A + C_\epsilon TA. \tag{4.3}$$

Now we choose

$$A = 2C_1 \mathrm{e}^{2C_0C_1}, \quad \epsilon = \frac{1}{2A} \quad \text{ and } \quad \widetilde{T}_0 = \min\Big\{ \frac{1}{2C_\epsilon A}, \frac{\ln 2}{C_1} \Big\}, \tag{4.4}$$

where C_0 and C_1 are the constants in (4.2) and (3.1), respectively. Hence, for all $T \le \widetilde{T}_0$, we can get

$$\epsilon A + C_\epsilon TA \le 1$$

and

$$I_2(v) \le C_0(1 + |\tau|_{4,2}^2) \le C_0(1 + \epsilon A + C_\epsilon TA) \le 2C_0.$$

Combining it with (3.1) and (4.4) gives

$$J_2(g) \le C_1 \mathrm{e}^{C_1(\widetilde{T}_0 + 2C_0)} \le 2C_1 \mathrm{e}^{2C_0C_1} = A.$$

So we have $\mathcal{F}(\mathbf{M}) \subset \mathbf{M}$ for all $T \le \widetilde{T}_0$, where $\widetilde{T}_0$ depends on $|v_0|_4$, $\|g_0\|_{H^4(\mathbb{R}^3\times B)}$.

Next, we show that $\mathcal{F}$ is a contraction mapping in some weak topology. Define

$$\|g\|_{\mathbf{M}}^2 = \sup_{0\le t\le T} \int_B \|\rho^{\frac{1}{2}} g\|_{L_x^2}^2 \mathrm{d}m + \int_B (|g|_{0,0}^2 + \rho |\nabla_m g|_{0,0}^2) \mathrm{d}m + \int_{\partial B} |g|_{0,0}^2 \mathrm{d}S.$$

Suppose that for an arbitrary $\overline{h}_i \in \mathbf{M}$, $\overline{g}_i = \mathbf{M}(\overline{h}_i)$ and $\overline{v}_i$ are solutions to (2.7), where $\overline{\tau}_i = \int_B m \otimes m \rho^{\frac{b}{2}-1} \overline{h}_i \mathrm{d}m$, $i = 1, 2$. Setting $v = \overline{v}_2 - \overline{v}_1$, $p = \overline{p}_2 - \overline{p}_1$, $\tau = \overline{\tau}_2 - \overline{\tau}_1$ and $h = \overline{h}_2 - \overline{h}_1$, we have

$$\partial_t v + (\overline{v}_2 \cdot \nabla_x) v + (v \cdot \nabla_x) \overline{v}_1 + \nabla_x p = \nabla_x \cdot \tau + \Delta_x v, \quad v(x,0) = 0. \tag{4.5}$$

Multiplication of (4.5) by v and integration with respect to x yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{L_x^2}^2+\int_{\mathbb{R}^3}(v\cdot\nabla_x\overline{v}_1)v\mathrm{d}x=-\int_{\mathbb{R}^3}\tau\nabla_x v\mathrm{d}x-\int_{\mathbb{R}^3}|\nabla_x v|^2\mathrm{d}x.$$

Hence,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{L_x^2}^2+\|\nabla_x v\|_{L_x^2}^2&\le(1+\|\nabla_x\overline{v}_1\|_{L_x^\infty}^2)\|v\|_{L_x^2}^2+\|\tau\|_{L_x^2}^2\\&\le(1+2C_0)\|v\|_{L_x^2}^2+\|\tau\|_{L_x^2}^2.\end{aligned}\tag{4.6}$$

To get the last inequality, we have used $I_2(\overline{v}_1)\le C_0(|v_0|_4,\|g_0\|_{H^4(\mathbb{R}^3\times B)})$. Similarly, $g=\overline{g}_2-\overline{g}_1$ solves

$$\begin{aligned}&\rho(\Delta_m g-2\partial_t g-2(\overline{v}_2\cdot\nabla_x)g)-(m+2\rho\nabla_x\overline{v}_2 m)\cdot\nabla_m g+2(m\cdot(\nabla_x\overline{v}_2 m))g\\&=2\rho((v\cdot\nabla_x)\overline{g}_1+(\nabla_x v m)\cdot\nabla_m\overline{g}_1)-2(m\cdot(\nabla_x v m))\overline{g}_1.\end{aligned}\tag{4.7}$$

We deduce from (4.1) and (4.7) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_B\|\rho^{\frac12}g\|_{L_x^2}^2\mathrm{d}m+\int_B\|\rho^{\frac12}\nabla_m g\|_{L_x^2}^2\mathrm{d}m+\int_{\partial B}\|g\|_{L_x^2}^2\mathrm{d}S\\&\le C(1+\|\nabla_x\overline{v}_2\|_{L_x^\infty}^2)\int_B\|\rho^{\frac12}g\|_{L_x^2}^2\mathrm{d}m+C|v|_1^2\int_B(\|\overline{g}_1\|_{L_x^\infty}^2+\|\rho^{\frac12}\nabla_m\overline{g}_1\|_{L_x^\infty}^2+\|\rho^{\frac12}\partial_x\overline{g}_1\|_{L_x^\infty}^2)\mathrm{d}m\\&\le\widetilde{C}\Big(\int_B\|\rho^{\frac12}g\|_{L_x^2}^2\mathrm{d}m+|v|_1^2\Big)\end{aligned}$$

for some constant $\widetilde{C}$ completely determined by $|v_0|_4,\|g_0\|_{H^4(\mathbb{R}^3\times B)}$. Substitution of the estimates of $\|\nabla_x v\|_{L_x^2}^2$ and $\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{L_x^2}^2$ in (4.6) gives

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|v\|_{L_x^2}^2+\int_B\|\rho^{\frac12}g\|_{L_x^2}^2\mathrm{d}m\Big)+\|\nabla_x v\|_{L_x^2}^2+\int_B\|\rho^{\frac12}\nabla_m g\|_{L_x^2}^2\mathrm{d}m+\int_{\partial B}\|g\|_{L_x^2}^2\mathrm{d}S\\&\le D\Big(\|v\|_{L_x^2}^2+\int_B\|\rho^{\frac12}g\|_{L_x^2}^2\mathrm{d}m+\|\tau\|_{L_x^2}^2\Big),\end{aligned}$$

where D is a large constant depending on $|v_0|_4$, $\|g_0\|_{H^4(\mathbb{R}^3\times B)}$. By (2.2) and the Gronwall inequality, we can get

$$\|g\|_{\mathbf{M}}^2\le D\mathrm{e}^{D\widetilde{T}_0}\int_0^T\|\tau\|_{L_x^2}^2\mathrm{d}t$$

for all $T\le\widetilde{T}_0$. Due to the similar estimate for τ as (4.3), the right-hand side is bounded by

$$D\mathrm{e}^{D\widetilde{T}_0}\Big(\delta\int_0^T\int_B\|\rho^{\frac12}\nabla_m h\|_{L_x^2}^2\mathrm{d}m\mathrm{d}t+C_\delta T\sup_t\int_B\|\rho^{\frac12}h\|_{L_x^2}^2\mathrm{d}m\Big).$$

Thus, we obtain

$$\|\overline{g}_2-\overline{g}_1\|_{\mathbf{M}}^2=\|g\|_{\mathbf{M}}^2\le\frac12\|\overline{h}_2-\overline{h}_1\|_{\mathbf{M}}^2,\quad\text{as }T\le T_0,\tag{4.8}$$

if we choose $\delta=\frac{1}{4D\mathrm{e}^{D\widetilde{T}_0}}$ and $T_0=\frac12\min\big\{\widetilde{T}_0,\frac{1}{(C_\delta+1)D\mathrm{e}^{D\widetilde{T}_0}}\big\}$. This shows that $\mathcal{F}$ has a fixed point g in $\mathbf{M}$, which is a solution to the coupled problem (2.5)–(2.6). The uniqueness is the immediate consequence of (4.8). This completes the proof of Lemma 4.1.

Before proving our main result, we first introduce two useful lemmas.

Lemma 4.2 *Suppose that $b > 2$, any integer $s \geq 2$, $v_0 \in H^{2s}(\mathbb{R}^3)$ with $\nabla_x \cdot v_0 = 0$ and $g_0 \in H_0^{2s}(\mathbb{R}^3 \times B)$. Then the solution (v, g) obtained in Lemma* 4.1 *satisfies*

$$I_s(v) + J_s(g) \leq C_s(|v_0|_{2s}, \|g_0\|_{H^{2s}(\mathbb{R}^3\times B)}) \tag{4.9}$$

for all $T \leq T_0(|v_0|_4, \|g_0\|_{H^4(\mathbb{R}^3\times B)})$, where T_0 is just mentioned in Lemma 4.1.

Proof By means of $|\alpha|$-times differentiations of (2.8) in x, $|\alpha| \leq 2s$, in a similar way to the proof of (3.19), using Lemma 4.1, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^{\frac12}\partial_x^\alpha g\|^2_{L^2(\mathbb{R}^3\times B)} + \|\rho^{\frac12}\nabla_m\partial_x^\alpha g\|^2_{L^2(\mathbb{R}^3\times B)} + \frac{b-2}{\sqrt{b}}\|\partial_x^\alpha g\|^2_{L^2(\mathbb{R}^3\times\partial B)} \\
&\leq C'\Big(\|\rho^{\frac12}\partial_x^\alpha g\|^2_{L^2(\mathbb{R}^3\times B)} + \int_{\mathbb{R}^3\times B}(\rho|\mathrm{I}_1'| + \rho|\mathrm{I}_2'| + |\mathrm{I}_3'|)|\partial_x^\alpha g|\mathrm{d}x\mathrm{d}m\Big)
\end{aligned} \tag{4.10}$$

for some constant $C' = C'(|v_0|_4, \|g_0\|_{H^4(\mathbb{R}^3\times B)})$, where

$$\mathrm{I}_1' = 2[\partial_x^\alpha, v\cdot\nabla_x]g, \quad I_2' = 2[\partial_x^\alpha, (\nabla_x v m)\cdot\nabla_m]g, \quad \mathrm{I}_3' = -2[\partial_x^\alpha, m\cdot(\nabla_x v m)]g.$$

By (3.7) and the Sobolev embedding theorem, for $|\alpha| \leq 2s$, we have

$$\begin{aligned}
\|\rho^{\frac12}\mathrm{I}_1'\|_{L^2(\mathbb{R}^3\times B)} &\leq C(\|\partial_x v\|_{L^\infty_x}\|\rho^{\frac12}\partial_x g\|_{L^2_m H^{2s-1}_x} + \|v\|_{H^{2s}_x}\|\rho^{\frac12}\partial_x g\|_{L^2_m L^\infty_x}) \\
&\leq C(\|v\|_{H^3_x}\|\rho^{\frac12}\partial_x g\|_{L^2_m H^{2s-1}_x} + \|v\|_{H^{2s}_x}\|\rho^{\frac12}\partial_x g\|_{L^2_m H^2_x}).
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\|\rho^{\frac12}\mathrm{I}_2'\|_{L^2(\mathbb{R}^3\times B)} &\leq C(\|v\|_{H^4_x}\|\rho^{\frac12}\nabla_m g\|_{L^2_m H^{2s-1}_x} + \|v\|_{H^{2s+1}_x}\|\rho^{\frac12}\nabla_m g\|_{L^2_m H^2_x}), \\
\|\mathrm{I}_3'\|_{L^2(\mathbb{R}^3\times B)} &\leq C(\|v\|_{H^4_x}\|g\|_{L^2_m H^{2s-1}_x} + \|v\|_{H^{2s+1}_x}\|g\|_{L^2_m H^2_x}).
\end{aligned}$$

Therefore, by the Cauchy inequality and Lemma 4.1, we can obtain

$$\begin{aligned}
\int_{\mathbb{R}^3\times B}(\rho|\mathrm{I}_1'| + \rho|\mathrm{I}_2'|)|\partial_x^\alpha g|\mathrm{d}x\mathrm{d}m &\leq \frac14(\|v\|^2_{H^{2s+1}_x} + \|\rho^{\frac12}\nabla_m g\|^2_{L^2_m H^{2s-1}_x}) \\
&\quad + C'(1 + \|\rho^{\frac12}\nabla_m g\|^2_{L^2_m H^2_x})(\|v\|^2_{H^{2s}_x} + \|\rho^{\frac12}g\|^2_{L^2_m H^{2s}_x})
\end{aligned} \tag{4.11}$$

for another constant $C' = C'(|v_0|_4, \|g_0\|_{H^4(\mathbb{R}^3\times B)})$. On the other hand, by (4.1), it follows that

$$\|g\|_{L^\infty(0,T;L^2_m H^2_x)} \leq \|g_0\|_{L^2_m H^2_x} + CT^{\frac12}\|\partial_t g\|_{L^2(0,T;L^2_m H^2_x)} \leq C'(|v_0|_4, \|g_0\|_{H^4(\mathbb{R}^3\times B)}).$$

By the Cauchy inequality and (2.2), one can get

$$\begin{aligned}
\int_{\mathbb{R}^3\times B}|\mathrm{I}_3'||\partial_x^\alpha g|\mathrm{d}x\mathrm{d}m &\leq \frac14\|v\|^2_{H^{2s+1}_x} + C(\|v\|_{H^4_x} + \|g\|^2_{L^2_m H^2_x})\|g\|^2_{L^2_m H^{2s}_x} \\
&\leq \frac14(\|v\|^2_{H^{2s+1}_x} + \|\rho^{\frac12}\nabla_m g\|^2_{L^2_m H^{2s-1}_x}) + C'\|\rho^{\frac12}g\|^2_{L^2_m H^{2s}_x}
\end{aligned} \tag{4.12}$$

for some constant $C' = C'(|v_0|_4, \|g_0\|_{H^4(\mathbb{R}^3\times B)})$. Inserting (4.11)–(4.12) into (4.10), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^{\frac12}g\|^2_{L^2_m H^{2s}_x} + \frac12\|\rho^{\frac12}\nabla_m g\|^2_{L^2_m H^{2s}_x} + \frac{b-2}{\sqrt{b}}\|g\|^2_{L^2(\partial B;H^{2s}_x)} \\
&\leq \frac12\|v\|^2_{H^{2s+1}_x} + C'(1 + \|\rho^{\frac12}\nabla_m g\|^2_{L^2_m H^2_x})(\|v\|^2_{H^{2s}_x} + \|\rho^{\frac12}g\|^2_{L^2_m H^{2s}_x})
\end{aligned} \tag{4.13}$$

for some constant $C' = C'(|v_0|_4, \|g_0\|_{H^4(\mathbb{R}^3\times B)})$. On the other hand, for the Navier-Stokes equation (2.7), we have the following estimate (see [7]):

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{H^{2s}_x} + \|v\|^2_{H^{2s+1}_x} \le C(\|\partial_x v\|_{L^\infty_x}\|v\|^2_{H^{2s}_x} + \|\tau\|^2_{H^{2s}_x}).$$

Due to the similar estimate for τ as (4.3), by the Sobolev embedding theorem, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{H^{2s}_x} + \|v\|^2_{H^{2s+1}_x} \le \frac{1}{4}\|\rho^{\frac12}\nabla_m g\|^2_{L^2_m H^{2s}_x} + C(\|v\|_{H^3_x}\|v\|^2_{H^{2s}_x} + \|\rho^{\frac12} g\|^2_{L^2_m H^{2s}_x}).$$

Combining it with (4.13), by the Gronwall inequality, (2.2) and (4.1), we can obtain

$$\sup_{0\le t\le T}\Big(|v|^2_{2s} + \int_B |\rho^{\frac12} g|^2_{2s}\mathrm{d}m\Big) + \int_0^T \Big(|v|^2_{2s+1} + \int_B (|\rho^{\frac12}\nabla_m g|^2_{2s} + |g|^2_{2s})\mathrm{d}m + \int_{\partial B}|g|^2_{2s}\mathrm{d}S\Big)\mathrm{d}t \le C_s(|v_0|_{2s}, \|g_0\|_{H^{2s}(\mathbb{R}^3\times B)}) \tag{4.14}$$

for all $T \le T_0(|v_0|_4, \|g_0\|_{H^4(\mathbb{R}^3\times B)})$. Applying $\partial_x^\beta\partial_t$ with $|\beta| \le 2s-2$ to (2.8), by a similar argument, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^{\frac12}\partial_x^\beta\partial_t g\|^2_{L^2(\mathbb{R}^3\times B)} + \|\rho^{\frac12}\nabla_m\partial_x^\beta\partial_t g\|^2_{L^2(\mathbb{R}^3\times B)} + \frac{b-2}{\sqrt{b}}\|\partial_x^\beta\partial_t g\|^2_{L^2(\mathbb{R}^3\times\partial B)}\\ &\le C'\Big(\|\rho^{\frac12}\partial_x^\beta\partial_t g\|^2_{L^2(\mathbb{R}^3\times B)} + \int_{\mathbb{R}^3\times B}(\rho|\mathrm{I}_4'| + \rho|\mathrm{I}_5'| + |\mathrm{I}_6'|)|\partial_x^\beta\partial_t g|\mathrm{d}x\mathrm{d}m\Big)\end{aligned} \tag{4.15}$$

for some constant $C' = C'(|v_0|_4, \|g_0\|_{H^4(\mathbb{R}^3\times B)})$, where

$$\mathrm{I}_4' = 2[\partial_x^\beta\partial_t, v\cdot\nabla_x]g,\quad \mathrm{I}_5' = 2[\partial_x^\beta\partial_t, (\nabla_x v m)\cdot\nabla_m]g,\quad \mathrm{I}_6' = -2[\partial_x^\beta\partial_t, m\cdot(\nabla_x v m)]g.$$

By (3.6) and the Sobolev embedding theorem, we have

$$\begin{aligned}\|\rho^{\frac12}\mathrm{I}_4'\|_{L^2(\mathbb{R}^3\times B)} &\le C\Big(\|\partial_t v\|_{H^{2s-2}_x}\|\rho^{\frac12}\partial_x g\|_{L^2_m L^\infty_x} + \|\partial_t v\|_{L^\infty_x}\|\rho^{\frac12}\partial_x g\|_{L^2_m H^{2s-2}_x}\\ &\quad + \sum_{|\gamma|\le|\beta|}\|\partial_x^\gamma v\|_{L^\infty_x}\|\rho^{\frac12}\partial_x\partial_t g\|_{L^2_m H^{2s-3}_x}\Big)\\ &\le C_s(\|\partial_t v\|_{H^{2s-2}_x} + \|\rho^{\frac12}\partial_t g\|_{L^2_m H^{2s-2}_x})\end{aligned}$$

for some constant $C_s = C_s(|v_0|_{2s}, \|g_0\|_{H^{2s}(\mathbb{R}^3\times B)})$. To get the last inequality, we have used (4.1) and (4.14). Similarly, by (3.8), we get

$$\begin{aligned}\|\rho^{\frac12}\mathrm{I}_5'\|_{L^2(\mathbb{R}^3\times B)} &\le C\Big(\|\partial_t v\|_{H^{2s-1}_x}\|\rho^{\frac12}\nabla_m g\|_{L^2_m L^\infty_x} + \|\partial_t\nabla_x v\|_{L^\infty_x}\|\rho^{\frac12}\nabla_m g\|_{L^2_m H^{2s-2}_x}\\ &\quad + \sum_{1\le|\gamma|\le|\beta|}\|\partial_x^\gamma\nabla_x v\|_{L^4_x}\|\rho^{\frac12}\partial_x^{\beta-\alpha}\nabla_m\partial_t g\|_{L^2_m L^4_x}\Big)\\ &\le C_s(\|\partial_t v\|_{H^{2s-1}_x}\|\rho^{\frac12}\nabla_m g\|_{L^2_m H^{2s-2}_x} + \|\rho^{\frac12}\nabla_m\partial_t g\|_{L^2_m H^{2s-2}_x}),\\ \|\mathrm{I}_6'\|_{L^2(\mathbb{R}^3\times B)} &\le C_s(\|\partial_t v\|_{H^{2s-1}_x} + \|\partial_t v\|_{H^3_x}\|g\|_{L^2_m H^{2s-2}_x} + \|\partial_t g\|_{L^2_m H^{2s-2}_x})\end{aligned}$$

for another constant $C_s = C_s(|v_0|_{2s}, \|g_0\|_{H^{2s}(\mathbb{R}^3\times B)})$. Therefore, by means of the Cauchy inequality, we can obtain

$$\begin{aligned}&\int_{\mathbb{R}^3\times B}(\rho|\mathrm{I}_4'| + \rho|\mathrm{I}_5'|)|\partial_x^\beta\partial_t g|\mathrm{d}x\mathrm{d}m\\ &\le \frac14(\|\partial_t v\|^2_{H^{2s-1}_x} + \|\rho^{\frac12}\nabla_m\partial_t g\|^2_{L^2_m H^{2s-2}_x})\\ &\quad + C_s(1 + \|\rho^{\frac12}\nabla_m g\|^2_{L^2_m H^{2s-2}_x})(\|\partial_t v\|^2_{H^{2s-2}_x} + \|\rho^{\frac12}\partial_t g\|^2_{L^2_m H^{2s-2}_x}).\end{aligned} \tag{4.16}$$

By (2.2), it follows that

$$\int_{\mathbb{R}^3\times B} |\mathrm{I}_6'||\partial_x^\beta \partial_t g|\mathrm{d}x\mathrm{d}m \le \frac{1}{4}(\|\partial_t v\|^2_{H_x^{2s-1}} + \|\rho^{\frac12}\nabla_m g\|^2_{L_m^2 H_x^{2s-2}} + \|\rho^{\frac12}\nabla_m \partial_t g\|^2_{L_m^2 H_x^{2s-2}}) \\ + C_s(1+\|\partial_t v\|^2_{H_x^3})(\|\rho^{\frac12} g\|^2_{L_m^2 H_x^{2s-2}} + \|\rho^{\frac12}\partial_t g\|^2_{L_m^2 H_x^{2s-2}}). \tag{4.17}$$

Inserting (4.16)–(4.17) into (4.15), we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^{\frac12}\partial_x^\beta \partial_t g\|^2_{L^2(\mathbb{R}^3\times B)} + \frac12\|\rho^{\frac12}\nabla_m \partial_x^\beta\partial_t g\|^2_{L^2(\mathbb{R}^3\times B)} + \frac{b-2}{\sqrt{b}}\|\partial_x^\beta\partial_t g\|^2_{L^2(\mathbb{R}^3\times\partial B)} \\ &\le \frac12(\|\partial_t v\|^2_{H_x^{2s-1}} + \|\rho^{\frac12}\nabla_m g\|^2_{L_m^2 H_x^{2s-2}}) \\ &\quad + C_s(1+\|\rho^{\frac12}\nabla_m g\|^2_{L_m^2 H_x^{2s-2}} + \|\partial_t v\|^2_{H_x^3})(\|\rho^{\frac12} g\|^2_{L_m^2 H_x^{2s-2}} + \|\rho^{\frac12}\partial_t g\|^2_{L_m^2 H_x^{2s-2}}). \end{aligned} \tag{4.18}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t v\|^2_{H_x^{2s-2}} + \|\partial_t v\|^2_{H_x^{2s-1}} \le C_s(\|\partial_t v\|^2_{H_x^{2s-2}} + \|\partial_t \tau\|^2_{H_x^{2s-2}}).$$

Combining it with (4.18), by the Gronwall inequality, (2.2), (3.10) and (4.14), we can get

$$\begin{aligned} &\sup_{0\le t\le T}\Big(|\partial_t v|^2_{2s-2} + \int_B |\rho^{\frac12}\partial_t g|^2_{2s-2}\mathrm{d}m\Big) \\ &+ \int_0^T \Big(\int_B (|\rho^{\frac12}\nabla_m\partial_t g|^2_{2s-2} + |\partial_t g|^2_{2s-2})\mathrm{d}m + \int_{\partial B}|\partial_t g|^2_{2s-2}\mathrm{d}S + |\partial_t v|^2_{2s-1}\Big)\mathrm{d}t \\ &\le C_s(|v_0|_{2s}, \|g_0\|_{H^{2s}(\mathbb{R}^3\times B)}) \end{aligned}$$

for all $T \le T_0(|v_0|_4, \|g_0\|_{H^4(\mathbb{R}^3\times B)})$. Repeating the previous argument step by step for $\partial_x^\gamma \partial_t^i g$ with $|\gamma| + 2i \le 2s$ from $i = 2, \cdots, s$, we can get

$$\begin{aligned} &\sup_{0\le t\le T}\Big(|\partial_t^i v|^2_{2s-2i} + \int_B |\rho^{\frac12}\partial_t^i g|^2_{2s-2i}\mathrm{d}m\Big) \\ &+ \int_0^T \Big(\int_B (|\rho^{\frac12}\nabla_m\partial_t^i g|^2_{2s-2i} + |\partial_t^i g|^2_{2s-2i})\mathrm{d}m + \int_{\partial B}|\partial_t^i g|^2_{2s-2i}\mathrm{d}S + |\partial_t^i v|^2_{2s-2i+1}\Big)\mathrm{d}t \\ &\le C_s(|v_0|_{2s}, \|g_0\|_{H^{2s}(\mathbb{R}^3\times B)}) \end{aligned}$$

for all $T \le T_0(|v_0|_4, \|g_0\|_{H^4(\mathbb{R}^3\times B)})$.

Moreover, applying $\partial_x^\alpha \partial_\theta^\beta \partial_t^r$ for $|\alpha| + |\beta| + 2r \le 2s$ to (2.8), where

$$\partial_\theta = (m_1\partial_{m_2} - m_2\partial_{m_1}, m_2\partial_{m_3} - m_3\partial_{m_2}),$$

similarly we can get (4.9). Thus we complete the proof of the present lemma.

Lemma 4.3 *Suppose that $b > 2$, any integer $s \ge 2$, $v_0 \in H^{2s+2}(\mathbb{R}^3)$ with $\nabla_x \cdot v_0 = 0$ and $g_0 \in H_0^{2s+2}(\mathbb{R}^3 \times B)$. Then the solution (v, g) obtained in Lemma 4.1 satisfies*

$$\sum_{|\alpha|+|\beta|+2r\le 2s-1} \|\partial_x^\alpha \partial_\theta^\beta \partial_t^r \partial_m g\|^2 \le C_s(|v_0|_{2s+2}, \|g_0\|_{H^{2s+2}(\mathbb{R}^3\times B)}), \tag{4.19}$$

where $\partial_\theta = (m_1\partial_{m_2} - m_2\partial_{m_1}, m_2\partial_{m_3} - m_3\partial_{m_2})$.

Proof Let us compute with $|\alpha|+|\beta|+2r\le 2s-1$ and $\widetilde{g}=\partial_x^\alpha\partial_\theta^\beta\partial_t^r g$,

$$0=\int_0^T\int_{\mathbb{R}^3\times B}(-m\cdot\nabla_m\widetilde{g})\partial_x^\alpha\partial_\theta^\beta\partial_t^r(L(g))\mathrm{d}x\mathrm{d}m\mathrm{d}t=\mathrm{H}_1+\mathrm{H}_2,\quad T\le T_0,\tag{4.20}$$

where

$$\mathrm{H}_1=-\int_0^T\int_{\mathbb{R}^3\times B}\rho\Delta_m\widetilde{g}(m\cdot\nabla_m\widetilde{g})\mathrm{d}x\mathrm{d}m\mathrm{d}t+\int_0^T\int_{\mathbb{R}^3\times B}(m\cdot\nabla_m\widetilde{g})^2\mathrm{d}x\mathrm{d}m\mathrm{d}t,$$
$$\mathrm{H}_2=2\int_0^T\int_{\mathbb{R}^3\times B}\partial_x^\alpha\partial_\theta^\beta\partial_t^r[\rho\partial_t g+\rho(v\cdot\nabla_x)g+\rho(\nabla_x vm)\cdot\nabla_m g$$
$$-(m\cdot(\nabla_x vm)g)](m\cdot\nabla_m\widetilde{g})\mathrm{d}x\mathrm{d}m\mathrm{d}t.$$

Integrating by parts, we have

$$\mathrm{H}_1=\int_0^T\int_{\mathbb{R}^3\times B}\partial_{m_j}(\rho m_i\partial_{m_i}\widetilde{g})\partial_{m_j}\widetilde{g}\mathrm{d}x\mathrm{d}m\mathrm{d}t+\int_0^T\int_{\mathbb{R}^3\times B}(m\cdot\nabla_m\widetilde{g})^2\mathrm{d}x\mathrm{d}m\mathrm{d}t$$
$$\ge\frac{1}{b}\||m|\nabla_m\widetilde{g}\|^2+\Big(1-\frac{2}{b}\Big)\|m\cdot\nabla_m\widetilde{g}\|^2-C\|\rho^{\frac12}\nabla_m\widetilde{g}\|^2\tag{4.21}$$

and

$$H_2\ge-\delta\|m\cdot\nabla_m\widetilde{g}\|^2-C_\delta(I_{s+1}(v)+J_{s+1}(g))\tag{4.22}$$

for any $\delta>0$, where $C_\delta=C(\delta,|v_0|_{2s+2},\|g_0\|_{H^{2s+2}(\mathbb{R}^3\times B)})$. Inserting (4.21)–(4.22) into (4.20), we have

$$\frac{1}{b}\||m|\nabla_m\widetilde{g}\|^2+\Big(1-\frac{2}{b}-\delta\Big)\|m\cdot\nabla_m\widetilde{g}\|^2\le C_\delta\big(I_{s+1}(v)+J_{s+1}(g)\big)\le C_\delta.$$

To get the last inequality, we have used (4.9). Now fixing $\delta=\frac{b-2}{2b}$ and by means of (4.9), we can get

$$\|\nabla_m\widetilde{g}\|^2\le\||m|\nabla_m\widetilde{g}\|^2+\|\rho^{\frac12}\nabla_m\widetilde{g}\|^2\le C_s(|v_0|_{2s+2},\|g_0\|_{H^{2s+2}(\mathbb{R}^3\times B)}).$$

Combining it with (4.9) soon gives the present lemma.

Proof of Theorem 1.1 Suppose that (v,g) is the solution obtained in Lemma 4.1. In Lemma 4.2, we have proved the regularity of the solution on x and t. Next, we shall improve the regularity on m near ∂B.

Let us first focus our attention on any given point $q\in\partial B$. Without loss of generality, we may assume $q=(\sqrt{b},0,0)$ and localize ∂B at this point by the spherical coordinates,

$$m_1=r\sin\alpha\cos\beta,\quad m_2=r\sin\alpha\sin\beta,\quad m_3=r\cos\alpha,$$

where (r,α,β) is near $(\sqrt{b},\frac{\pi}{2},0)$. Rewrite the first equation of (2.8) in the following form:

$$yg_{yy}+\Big[\frac{b(1-y)}{2-y}+ya_{ij}\partial_{x_j}v^i\Big]g_y=G_1+yG_2,$$

where $y=1-\frac{r}{\sqrt{b}}$, v^i is the i-th component of the macroscopic velocity v,

$$G_1=-d_{ij}\partial_{x_j}v^i g,$$
$$G_2=-\Big[\frac{1}{(1-y)^2}\Delta_{S^2}g-2bg_t-2b(v\cdot\nabla_x)g+b_{ij}\partial_{x_j}v^i g_\alpha+c_{ij}\partial_{x_j}v^i g_\beta\Big],$$

$\Delta_{S^2} g = \frac{1}{\sin\alpha}\partial_\alpha(\sin\alpha g_\alpha) + \frac{1}{\sin^2\alpha} g_{\beta\beta}$, $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ are all smooth functions of (y, α, β) near $y = 0$. From Lemmas 4.1 and 4.3, we have, for all $l, i, j, k, p \in \mathbb{N}$ with $l \leq 1$,

$$\partial_y^l \partial_x^i \partial_\alpha^j \partial_\beta^k \partial_t^p g \in L^2(\mathbb{R}^3 \times B \times (0, T_0)), \qquad \text{where } l + i + j + k \text{ is even.} \tag{4.23}$$

Now we claim that (4.23) being continuous is true for all $l \geq 2$, and more precisely, for any given $l \geq 1$, with $l + i + j + k$ being even,

$$\|\partial_y^l \partial_x^i \partial_\alpha^j \partial_\beta^k \partial_t^p g\|^2 \leq C_s(|v_0|_{2s+2}, \|g_0\|_{H^{2s+2}(\mathbb{R}^3\times B)}). \tag{4.24}$$

We shall prove (4.24) by induction on l. It is evident that (4.24) for $l = 1$ are just (4.19) for all $i+j+k+2p = 2s-1$. Suppose that (4.24) are valid for all $l+i+j+k+2p = 2s$, $l \geq 1$. Now let us consider $l+1$ and arbitrary i, j, k, p with $l+1+i+j+k$ being even. Set $s = \frac{l+1+i+j+k}{2} + p$. By means of Lemma 2.2, we have

$$g_y = \mathcal{T}(G_1) + \mathcal{T}(yG_2),$$
$$\partial_y^{l+1} \partial_x^i \partial_\alpha^j \partial_\beta^k \partial_t^p g = \partial_y^l \partial_x^i \partial_\alpha^j \partial_\beta^k \partial_t^p (\mathcal{T}(G_1) + \mathcal{T}(yG_2)).$$

Obviously, by the hypothesis on induction

$$\|\partial_y^l \partial_x^i \partial_\alpha^j \partial_\beta^k \partial_t^p (\mathcal{T}(G_1))\| \leq C_s \sum_{\substack{\bar{l}\leq l\\ \bar{i}\leq i\\ \bar{j}\leq j\\ \bar{k}\leq k\\ \bar{p}\leq p}} \|\partial_y^{\bar{l}} \partial_x^{\bar{i}} \partial_\alpha^{\bar{j}} \partial_\beta^{\bar{k}} \partial_t^{\bar{p}} g\|,$$

which is controlled by the right-hand side of (4.19). Using Remark 2.1, we also have

$$\|\partial_y^l \partial_x^i \partial_\alpha^j \partial_\beta^k \partial_t^p (\mathcal{T}(yG_2))\| \leq C_s \sum_{\substack{\bar{l}\leq l-1\\ \bar{i}+\bar{j}+\bar{k}+2\bar{p}\leq i+j+k+2p+2}} \|\partial_y^{\bar{l}} \partial_x^{\bar{i}} \partial_\alpha^{\bar{j}} \partial_\beta^{\bar{k}} \partial_t^{\bar{p}} g\|.$$

So the proof for induction on l is completed. From the transformation $f = \rho^{\frac{b}{2}} g$, Lemmas 4.1–4.3, we can get (1.5). Thus Theorem 1.1 is proved.

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