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Lipschitz Continuous Solutions to the Cauchy Problem for Quasi-linear Hyperbolic Systems

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Abstract Lipschitz continuous solutions to the Cauchy problem for 1-D first order quasi-linear hyperbolic systems are considered. Based on the methods of approximation and integral equations, the author gives two definitions of Lipschitz solutions to the Cauchy problem and proves the existence and uniqueness of solutions.

Keywords First order quasi-linear hyperbolic systems, Lipschitz continuous solution, Cauchy problem, Existence and uniqueness
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1 Introduction

In this paper, we consider the following Cauchy problem for 1-D first order quasi-linear hyperbolic systems:

$$\begin{cases} \dfrac{\partial u}{\partial t} + A(u)\dfrac{\partial u}{\partial x} = F(u), \\ t = 0, \quad u = \phi(x), \end{cases} \tag{1.1}$$

where $u(t,x) = (u_1(t,x), \cdots, u_n(t,x))^{\mathrm{T}}$ is the unknown vector function and $A(u) = (a_{ij}(u))$ is a given $n \times n$ matrix function of $u \in \mathbb{R}^n$, $F(u)$ is a given vector function of $u \in \mathbb{R}^n$, and $\phi(x)$ is a vector function of $x \in \mathbb{R}$. All the given functions have certain regularity to be mentioned.

The Cauchy problem as a fundamental problem of quasi-linear hyperbolic systems was investigated rather completely in the sense of C^1 classical solution (cf. [4–6, 9, 13, 15, 19] and references therein). Generally speaking, in the nonlinear case, classical solutions to the Cauchy problem exist only locally in time (cf. [13] and references therein).

So far, the corresponding results on the Lipschitz continuous solution are few (cf. [16–17, 1, 7, 10–11]). Douglis [6] conjectured that the Lipschitz continuous solution should be unique, when he considered classical solutions. Moreover, Douglis [7] proved the uniqueness in a function space stronger than Lipschitz continuous space. The later paper discussed the condition to ensure the chain rule in the Lipschitz continuous space. By a method of approximation, Wang and Wu [19] proved the existence of Lipschitz continuous solution in the sense that Lipschitz continuous solution satisfies problem (1.1) almost everywhere.

Myshkis and Filimonov [16–17] considered the solvability of Lipschitz continuous solution to quasi-linear hyperbolic systems of the diagonal form. They first defined the Lipschitz continuous

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solution in the sense that the solution satisfies the corresponding integral system of equations, and then studied the Cauchy problem and the mixed initial-boundary value problem.

Using the framework of conservation laws, Hoff considered the Cauchy problem in several independent variables. Hoff [10] established a condition which is both necessary and sufficient for Lipschitz solvability based on the work of Kruzkov [12]. Hoff [11] also considered the life-span problem. There still exist works on continuous solutions weaker than Lipschitz continuous solutions (cf. [1]).

More recently, Peng and Yang [18] showed that in one space dimension, Lipschitz continuous solutions to generalized extremal surface equations are equivalent to entropy solutions in L^∞ to a non-strictly hyperbolic system of conservation laws. They obtained an explicit representation formula and the uniqueness of entropy solution to the Cauchy problem of the system. Based on this formula, they also considered the long-time behavior and L^1 stability.

In this paper, we consider the general first order quasi-linear hyperbolic system (1.1). Since any given Lipschitz continuous function is differential almost everywhere, one can define the Lipschitz continuous solution in the sense that it satisfies problem (1.1) almost everywhere, and, by regarding [19], the Lipschitz continuous solution under consideration can be obtained by approximation of C^1 solutions to more regular models.

For simplicity, we only consider the Cauchy problem (1.1) under the assumptions that the coefficients $A(u), F(u)$ are C^1 with respect to $u \in \mathbb{R}^n$, and $\phi(x)$ is Lipschitz continuous with respect to $x \in \mathbb{R}$. All the results obtained are still valid for the case that $A(u), F(u)$ are also Lipschitz continuous.

This paper is organized as follows. In Section 2, we first recall some basic results about Lipschitz continuous functions and ordinary differential equations, and give some extensions of classical theorems. Section 3 is devoted to studying the existence and uniqueness of Lipschitz continuous solution obtained by approximation. Finally, the existence and uniqueness of Lipschitz continuous solution in the sense of satisfying the corresponding system of integral equations is considered in Section 4.

2 Preliminaries

In this section, we recall some results about Lipschitz continuous functions, ordinary differential equations and basic knowledge about first order quasi-linear hyperbolic systems in two independent variables.

Firstly, we recall some basic results about Lipschitz continuous functions (cf. [8, 20]).

(1) Let E be a domain in $\mathbb{R}^n$. A function $g : E \to \mathbb{R}^m$ is called Lipschitz continuous, provided that

$$|g(x) - g(y)| \le C|x - y|$$

for some non-negative constant C and all $x, y \in E$. The smallest constant C, such that the above inequality holds for all x, y, is denoted by

$$\mathrm{Lip}(g) = \sup\Big\{\frac{|g(x) - g(y)|}{|x - y|}\ \Big|\ x, y \in E, x \neq y\Big\}.$$

(2) A function $g : E \to \mathbb{R}^m$ is called locally Lipschitz continuous if for each compact set

$K \subset E$, there exists a non-negative constant C_K, such that

$$|g(x) - g(y)| \le C_K |x - y| \quad \text{for all}\, x, y \in K.$$

Lemma 2.1 (cf. [8]) *Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz continuous function. Then g is differentiable a.e. with respect to the Lebesgue measure in $\mathbb{R}^n$.*

For the function of one variable, a Lipschitz continuous function is also absolutely continuous. Thus, the formula of integration by parts for absolutely continuous functions (cf. [20]) also holds for Lipschitz continuous functions, namely, we have the following lemma.

Lemma 2.2 *Let g and h be Lipschitz continuous functions on $[c, d]$. Then*

$$\int_c^d g'(x)h(x)\mathrm{d}x = g(x)h(x)|_c^d - \int_c^d g(x)h'(x)\mathrm{d}x.$$

According to the properties of absolutely continuous functions, the derivatives of Lipschitz continuous functions are locally Lebesgue integrable.

Any Lipschitz continuous function can be approximated by continuously differentiable functions. We have the following result.

Lemma 2.3 *Let $g(x)$ be a Lipschitz continuous function on $[c, d]$ $(-\infty < c < d < \infty)$, $|g(x)| \le M < \infty$, $\forall x \in [c, d]$ and $L = \mathrm{Lip}(g) < \infty$, where M and L are non-negative constants. There exist $g^m \in C^1[c, d]$, $m = 1, 2, \cdots$, such that as $m \to \infty$, g^m converges to g uniformly on $[c, d]$, and $|g^m(x)| \le M$, $|{g^m}'(x)| \le L$, $\forall x \in [c, d]$, $m = 1, 2, \cdots$.*

Proof Let

$$\widetilde{g}(x) = \begin{cases} g(c), & x < c, \\ g(x), & x \in [c, d], \\ g(d), & x > d. \end{cases}$$

Then $|\widetilde{g}(x)| \le M$, $\forall x \in \mathbb{R}$ and $\mathrm{Lip}(\widetilde{g}) \le L$.

Let $J(x)$ be a mollifier function, $J(x) \in C_c^\infty[-1, 1]$, $J(x) \ge 0$ and $\int_{-\infty}^{\infty} J(x)\,\mathrm{d}x = 1$. Let $J_\varepsilon(x) = \frac{1}{\varepsilon} J(\frac{x}{\varepsilon})$, $\varepsilon > 0$. We define

$$g^m(x) = \int_{-\infty}^{\infty} J_{\frac{1}{m}}(x - y)\widetilde{g}(y)\,\mathrm{d}y.$$

Then g^m are C^1 functions on $\mathbb{R}$ and

$$\begin{aligned} &|g^m(x)| \le M, \\ &|{g^m}'(x)| \le \int_{-\infty}^{\infty} J_{\frac{1}{m}}(x - y)|\widetilde{g}'(y)|\,\mathrm{d}y \le L. \end{aligned}$$

The last formula is obtained by the formula of integration by parts given in Lemma 2.2.

Since $\widetilde{g}$ is a Lipschitz continuous function on $\mathbb{R}$, for any given $\varepsilon > 0$, there exists a $\delta > 0$, such that $|\widetilde{g}(x) - \widetilde{g}(y)| \le \varepsilon$, provided that $|x - y| \le \delta$. Hence, when $m > [\frac{1}{\delta}] + 1$, we have

$$|g^m(x) - g(x)| = \Big|\int_{-\infty}^{\infty} J_{\frac{1}{m}}(x - y)(\widetilde{g}(y) - g(x))\,\mathrm{d}y\Big| \le \varepsilon, \quad \forall x \in [c, d].$$

This completes the proof of Lemma 2.3.

Remark 2.1 When the interval $[c,d]$ is replaced by $(-\infty,\infty)$, Lemma 2.3 still holds.

Secondly, in what follows, we will extend the classical theorems about continuous dependence on parameters of solutions to ordinary differential equations (cf. [16]). Let D be a bounded domain in $\mathbb{R}^{1+n}$, $g(t,x)$ a continuous function on D and Lipschitz continuous with respect to x: $K=\mathrm{Lip}_x(g)<\infty$, where K is a non-negative constant.

For the following ordinary differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=g(t,x), \tag{2.1}$$

there exists a time interval I, such that (2.1) has a C^1 solution $x=\psi(t)$ on I. Suppose that $\tau\in I$ and the distance between (τ,ξ) and $(\tau,\psi(\tau))$ is sufficiently small. Then there exists a unique C^1 solution $x=\varphi(t;\tau,\xi)$ passing through (τ,ξ), and φ is continuously dependent on parameters τ and ξ. More precisely, we have the result as follows.

Theorem 2.1 (cf. [3, Chapter 7.1]) *Let $g(t,x)$ be a continuous function on D and Lipschitz continuous with respect to x: $K=\mathrm{Lip}_x(g)<\infty$. Suppose that $x=\psi(t)$ is a C^1 solution to* (2.1) *on the interval $I: c\le t\le d$. Then, there exists a $\delta>0$, such that for any $(\tau,\xi)\in U=\{(\tau,\xi)\,|\,c<\tau<d,\ |\xi-\psi(\tau)|<\delta\}$, there exists a unique C^1 solution $x=\varphi(t;\tau,\xi)$ to* (2.1) *on I with $\varphi(\tau;\tau,\xi)=\xi$, and φ is continuous on*

$$V=\{(t,\tau,\xi)\mid c<t<d,\ (\tau,\xi)\in U\}.$$

Furthermore, we can prove that φ is Lipschitz continuous with respect to $(\tau,\xi)\in U$. We need only to prove this claim in $\mathbb{R}^{1+1}$.

Lemma 2.4 *Suppose that g,ψ,φ are the functions given in Theorem* 2.1. *Then $\varphi(t;\tau,\xi)$ is Lipschitz continuous with respect to $(\tau,\xi)\in U$.*

Proof We need only to prove the conclusion for the variable ξ, and the case for τ is similar. According to the classical theory of ODE (cf. [3]), we can construct the solution $x=\varphi(t;\tau,\xi)$ by Picard's iterative scheme, and the convergence of iterative sequence is uniform.

We construct the following iterative sequences:

$$\varphi_0=\psi(t)-\psi(\tau)+\xi=\xi+\int_\tau^t g(s,\psi(s))\mathrm{d}s,$$
$$\varphi_{j+1}=\xi+\int_\tau^t g(s,\varphi_j(s;\tau,\xi))\mathrm{d}s,\quad j=0,1,\cdots.$$

Then

$$|\varphi_1-\varphi_0|\le\Big|\int_\tau^t(g(s,\psi(s)-\psi(\tau)+\xi)-g(s,\psi(s)))\mathrm{d}s\Big|\le K|\xi-\psi(\tau)||t-\tau|,$$

and by induction, we have

$$|\varphi_{j+1}-\varphi_j|\le\frac{K^{j+1}|t-\tau|^{j+1}}{(j+1)!}|\xi-\psi(\tau)|,\quad j=0,1,\cdots.$$

For any given $(\tau,\xi_1),(\tau,\xi_2)\in U$, without loss of generality, we suppose that $c<\tau<t<d$. Then

$$\begin{aligned}|\varphi_0(t;\tau,\xi_1)-\varphi_0(t;\tau,\xi_2)| &\le |\xi_1-\xi_2|,\\ |\varphi_1(t;\tau,\xi_1)-\varphi_1(t;\tau,\xi_2)| &\le |\xi_1-\xi_2|+\int_\tau^t |g(s,\varphi_0(s;\tau,\xi_1))-g(s,\varphi_0(s;\tau,\xi_2))|\mathrm{d}s\\ &\le |\xi_1-\xi_2|+K|\xi_1-\xi_2|(t-\tau),\\ &\quad\cdots\cdots\\ |\varphi_j(t;\tau,\xi_1)-\varphi_j(t;\tau,\xi_2)| &\le \sum_{i=0}^{j}\frac{(K(t-\tau))^i}{i!}|\xi_1-\xi_2|\\ &\le \mathrm{e}^{K(d-c)}|\xi_1-\xi_2|,\quad j=0,1,\cdots.\end{aligned}$$

Thus, all $\varphi_j(t;\tau,\xi)$ $(j=0,1,\cdots)$ are Lipschitz continuous with respect to ξ, and the corresponding Lipschitz constants are bounded by $\mathrm{e}^{K(d-c)}$. Because of the property of uniform convergence, φ is Lipschitz continuous with respect to ξ, and $\mathrm{Lip}_\xi(\varphi)\le \mathrm{e}^{K(d-c)}$.

For any given function g satisfying the conditions in Theorem 2.1, there exists a unique solution $x=\varphi(t;\tau,\xi;g)$.

Let

$$D=\{(t,x)\mid 0\le t\le\delta,\ |x|\le M-\Lambda t\},$$

where δ,Λ,M $(\Lambda\delta\le M)$ are positive constants. For any given $t\in[0,\delta]$, let

$$D(t)=\{(\tau,x)\mid 0\le\tau\le t,\ |x|\le M-\Lambda\tau\}.$$

Lemma 2.5 *Suppose that g_1 and g_2 are continuous functions on $D(\delta)$, $|g_i|\le\Lambda$, $\mathrm{Lip}_x(g_i)\le K$ $(i=1,2)$, and $x=\varphi(t;\tau,\xi;g_1)$ and $x=\varphi(t;\tau,\xi;g_2)$ are solutions to (2.1), passing through point $(\tau,\xi)\in D(\tau)$ and corresponding to g_1,g_2, respectively. For any given t $(0\le t\le\delta)$, define*

$$|g_1-g_2|(t)=\max_{(s,\xi)\in D(t)}|g_1(s,\xi)-g_2(s,\xi)|.$$

Then, there exists a positive constant C depending on Λ,K and δ, such that

$$|\varphi(t;\tau,\xi;g_1)-\varphi(t;\tau,\xi;g_2)|\le C\tau|g_1-g_2|(\tau),\quad \forall(\tau,\xi)\in D(\delta),\ 0\le t\le\tau. \tag{2.2}$$

Proof According to the classical theory of ODEs, $x=\varphi(t;\tau,\xi;g_1)$ and $x=\varphi(t;\tau,\xi;g_2)$ satisfy the corresponding integral equations

$$\begin{cases}\varphi(t;\tau,\xi;g_1)=\xi-\displaystyle\int_t^\tau g_1(s,\varphi(s;\tau,\xi;g_1))\mathrm{d}s,\\ \varphi(t;\tau,\xi;g_2)=\xi-\displaystyle\int_t^\tau g_2(s,\varphi(s;\tau,\xi;g_2))\mathrm{d}s,\end{cases}$$

respectively. Since $(\tau,\xi)\in D(\tau)$, $(t,\varphi(t;\tau,\xi;g_1)),(t,\varphi(t;\tau,\xi;g_2))\in D(\tau)$ for any given $t\in$

$[0, \tau]$. Hence

$$\begin{aligned}
&|\varphi(t;\tau,\xi;g_1) - \varphi(t;\tau,\xi;g_2)| \\
\le& \int_t^\tau |g_1(s,\varphi(s;\tau,\xi;g_1)) - g_2(s,\varphi(s;\tau,\xi;g_2))|\mathrm{d}s \\
\le& \int_t^\tau |g_1(s,\varphi(s;\tau,\xi;g_1)) - g_2(s,\varphi(s;\tau,\xi;g_1))|\mathrm{d}s \\
&+ \int_t^\tau |g_2(s,\varphi(s;\tau,\xi;g_1)) - g_2(s,\varphi(s;\tau,\xi;g_2))|\mathrm{d}s \\
\le& (\tau - t)|g_1 - g_2|(\tau) + K\int_t^\tau |\varphi(s;\tau,\xi;g_1) - \varphi(s;\tau,\xi;g_2)|\mathrm{d}s.
\end{aligned}$$

Using the Gronwall's inequality, we get (2.2).

Remark 2.2 Suppose that g^m $(m = 1, 2, \cdots)$ satisfy the conditions given in Lemma 2.5, and as $m \to \infty$, g^m converges to g uniformly on $D(\delta)$. Then as $m \to \infty$, $\varphi(t;\tau,\xi;g^m)$ corresponding to g^m converges to $\varphi(t;\tau,\xi;g)$ uniformly on $\{(t,\tau,\xi) \mid (\tau,\xi) \in D(\delta),\ t \in [0,\tau]\}$.

We now consider another metric on g. Let g_1, g_2 be functions satisfying the conditions in Lemma 2.5 and

$$|g_1 - g_2|(t) = \max_{|x| \le M - \Lambda t} |g_1(t,x) - g_2(t,x)|, \quad \forall\, t \in [0,\delta].$$

Similarly, we have the next lemma.

Lemma 2.6 *Suppose that g_1 and g_2 are Lipschitz continuous functions on $D(\delta)$: $|g_i| \le \Lambda$, $\mathrm{Lip}_x(g_i) \le K$ $(i = 1, 2)$, and $\varphi(t;\tau,\xi;g_1)$ and $\varphi(t;\tau,\xi;g_2)$ are solutions to* (2.1)*, passing through point $(\tau,\xi) \in D(\delta)$, and corresponding to g_1, g_2, respectively. There exists a positive constant C depending on Λ, K and δ, such that*

$$|\varphi(t;\tau,\xi;g_1) - \varphi(t;\tau,\xi;g_2)| \le C\int_0^\tau |g_1 - g_2|(s)\mathrm{d}s, \quad 0 \le t \le \tau \le \delta,\ (\tau,\xi) \in D(\delta). \tag{2.3}$$

Finally, we recall some basic knowledge about first order quasi-linear hyperbolic systems in two independent variables. Let $\lambda_i(u)$ $(i = 1, \cdots, n)$ be the eigenvalues of $A(u)$, and $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ be the left eigenvector corresponding to $\lambda_i(u)$ $(i = 1, \cdots, n)$. Cauchy problem (1.1) can be equivalently reduced to the following Cauchy problem for first order quasi-linear hyperbolic systems of the characteristic form (cf. [15, 19]):

$$\begin{cases}\displaystyle\sum_{j=1}^n l_{ij}(u)(\partial_t u_j + \lambda_i(u)\partial_x u_j) = f_i(u), \quad i = 1, \cdots, n, \\ t = 0: \quad u = \phi(x),\end{cases} \tag{2.4}$$

where $l_{ij},\ \lambda_i,\ f_i = \sum_{j=1}^n l_{ij}F_j$ $(i, j = 1, \cdots, n)$ are C^1 functions of $u \in \mathbb{R}^n$, $\det(l_{ij}) \ne 0$, and $\phi_i(x)$ $(i = 1, \cdots, n)$ are Lipschitz continuous functions on $[a, b]$ $(-\infty \le a < b \le \infty)$.

In what follows, the absolute value of a vector is defined as the maximum absolute value of all the components of this vector. Let

$$\|\phi\|_{C^0} \triangleq \max_{x \in [a,b]} |\phi(x)| \le \frac{\Omega}{2},$$

where Ω is a positive constant,

$$\Gamma \triangleq \{l_{ij}, \lambda_i, f_i,\ i,j = 1, \cdots, n\},$$
$$B_\Omega(0) \triangleq \{u \in \mathbb{R}^n \,|\, |u| \le \Omega\}.$$

A domain

$$R(\delta) = \{(t,x) \mid 0 \le t \le \delta,\ x_1(t) \le x \le x_2(t)\}$$

is called a strong determinate domain of $[a,b]$, if

(1) $x_i(t)$ $(i = 1,2)$ are C^1 functions on $[0,\delta]$,

(2) $x_1(0) = a,\ x_2(0) = b$,

(3) for any given Lipschitz continuous function $u(t,x)$ on $R(\delta)$ with $|u| \le \Omega$ and $u(0,x) = \phi(x),\ x \in [a,b]$, we have

$$\begin{cases} x_1'(t) \ge \sup\limits_{i=1,\cdots,n} \lambda_i(u(t,x_1(t))), \\ x_2'(t) \le \inf\limits_{i=1,\cdots,n} \lambda_i(u(t,x_2(t))). \end{cases}$$

Obviously, if $R(\delta)$ is a strong determinate domain of $[a,b]$, so is $R(\tau)$ for any given $\tau \in [0,\delta]$. We define

$$L_0 \triangleq \mathrm{Lip}(\phi),$$
$$\Lambda \triangleq \max_{\substack{i=1,\cdots,n \\ |u|\le\Omega}} |\lambda_i(u)| < \infty,$$

and assume that $\det(l_{ij}) \ge \alpha > 0$ for $|u| \le \Omega$. Obviously, for any given $t \in (0,\delta]$, the domain $D(t)$ in Lemma 2.5 is a strong determinate domain of $[-M,M]$.

3 Lipschitz Continuous Solution Defined by Approximation, Its Existence and Uniqueness

We now consider Cauchy problem (2.4) for first order quasi-linear hyperbolic systems of the characteristic form under the assumptions that all the coefficients are C^1 and ϕ is Lipschitz continuous.

We first recall the existence of Lipschitz continuous solution (cf. [19]). From Lemma 2.3, there exist

$$\phi^{(m)} \in C^1(\mathbb{R}), \quad |\phi^{(m)}| \le \frac{\Omega}{2}, \quad |{\phi^{(m)}}'(x)| \le L_0, \quad \forall x \in \mathbb{R},\ m = 1,2,\cdots,$$

such that as $m \to \infty$, $\phi^{(m)}$ converges to ϕ uniformly in $\mathbb{R}$. Taking $\phi^{(m)}$ as the initial value, we get the following Cauchy problem:

$$\begin{cases} \sum\limits_{j=1}^{n} l_{ij}(u^{(m)})(\partial_t u_j^{(m)} + \lambda_i(u^{(m)})\partial_x u_j^{(m)}) = f_i(u^{(m)}), \quad i = 1,\cdots,n, \\ t = 0: \quad u^{(m)} = \phi^{(m)}(x), \end{cases} \tag{3.1}$$

where $u^{(m)} = u^{(m)}(t,x)$ is the C^1 classic solution.

According to the classical local theory of quasi-linear hyperbolic systems (cf. [4–6, 9, 15, 19] and references therein), there exists a common number $\delta_0 > 0$, such that Cauchy problem (3.1) admits a unique C^1 solution $u^{(m)} = u^{(m)}(t,x)$ on $[0,\delta_0]$, and the C^1 norm of $u^{(m)}$ is uniformly bounded,

$$\|u^{(m)}\|_{C^1} \le C_1.$$

Here and hereafter, C_i $(i = 1, 2, \cdots)$ are positive constants depending possibly on Ω, L_0, α and the C^1 norm of Γ on $B_\Omega(0)$.

Furthermore, under the assumption that $f(0) = 0$, we have (cf. [2, 14])

$$\|u^{(m)}\|_{C^1} \le C_2 \|\phi^{(m)}\|_{C^1}.$$

According to Ascoli-Arzela lemma, there exists a uniformly convergent subsequence $\{u^{(m_k)}\}_{k=1}^{\infty}$ on $D(\delta_0)$ $(\forall M > 0)$. Denoting the limit function as u, u is Lipschitz continuous on $D(\delta_0)$ with Lipschitz constant depending possibly on Ω, L_0, α and the C^1 norm of Γ on $B_\Omega(0)$. Moreover, it is obvious that

$$t = 0: \quad u = \phi(x), \quad \forall x \in \mathbb{R}.$$

Theorem 3.1 *The limit function u mentioned above satisfies* (2.4) *almost everywhere.*

Proof By the uniform convergence, we have

$$\begin{aligned} &\partial_t u^{(m_k)} \to \partial_t u, \quad k \to \infty, \\ &\partial_x u^{(m_k)} \to \partial_x u, \quad k \to \infty, \end{aligned} \tag{3.2}$$

in the sense of distribution. We now show that for any given $g(t,x) \in C_c^\infty(D(\delta_0))$, we have

$$\lim_{k\to\infty} \iint_{D(\delta_0)} l_{ij}(u^{(m_k)}) \partial_t u_j^{(m_k)} g \mathrm{d}t\mathrm{d}x = \iint_{D(\delta_0)} l_{ij}(u) \partial_t u_j g \mathrm{d}t\mathrm{d}x, \quad i = 1, \cdots, n.$$

In fact,

$$\begin{aligned} &\iint_{D(\delta_0)} l_{ij}(u^{(m_k)}) \partial_t u_j^{(m_k)} g \mathrm{d}t\mathrm{d}x - \iint_{D(\delta_0)} l_{ij}(u) \partial_t u_j g \mathrm{d}t\mathrm{d}x \\ = &\iint_{D(\delta_0)} (l_{ij}(u^{(m_k)}) - l_{ij}(u)) \partial_t u_j^{(m_k)} g \mathrm{d}t\mathrm{d}x \\ &+ \iint_{D(\delta_0)} l_{ij}(u) g (\partial_t u_j^{(m_k)} - \partial_t u_j) \mathrm{d}t\mathrm{d}x. \end{aligned}$$

By the uniform convergence of $\{u^{(m_k)}\}$ and the uniform boundedness of the C^1 norm of $\{u^{(m)}\}$, the first term on the right-hand side of the above equation converges to 0 as $k \to \infty$. By noting (3.2), the second term on the right-hand side of the above equation also converges to 0 as $k \to \infty$.

Similarly, we get

$$\lim_{k\to\infty} \iint_{D(\delta_0)} l_{ij}(u^{(m_k)}) \lambda_i(u^{(m_k)}) \partial_t u_j^{(m_k)} g \mathrm{d}t\mathrm{d}x = \iint_{D(\delta_0)} l_{ij}(u) \lambda_i(u) \partial_t u_j g \mathrm{d}t\mathrm{d}x.$$

Then, multiplying both sides of (3.1) by $g(t,x)$, integrating on $D(\delta_0)$ and taking $m = m_k \to \infty$, we have

$$\sum_{j=1}^{n} l_{ij}(u)(\partial_t u_j + \lambda_i(u) \partial_x u_j) = f_i(u), \quad i = 1, \cdots, n,$$

almost everywhere on $D(\delta_0)$. Furthermore, as we said before, $u = u(t,x)$ obviously satisfies the initial condition $t = 0$: $u = \phi(x)$, $\forall x \in \mathbb{R}$.

Now $u = u(t,x)$ is a Lipschitz continuous solution to Cauchy problem (2.4) (namely (1.1)) in the sense that (2.4) is satisfied almost everywhere.

The above results are in the spirit of [19], the authors of which first established the boundedness of the C^1 norm of $\{u^{(m)}\}$, and then got the existence of the Lipschitz continuous solution by means of Ascoli-Arzela lemma. But they did not mention the uniqueness of the Lipschitz continuous solution and whether the Lipschitz continuous solution they obtained depends on the choice of subsequence or not.

In what follows, we study the uniqueness of the Lipschitz continuous solution. For this purpose, we prove that the sequence $\{u^{(m)}\}$ itself is convergent.

Theorem 3.2 *The whole sequence $\{u^{(m)}\}$ of solutions to Cauchy problem* (3.1) *is uniformly convergent on $[0,\delta_0]$. Then, the Lipschitz continuous solution to Cauchy problem* (2.4) *defined by approximation not only exists but is also unique.*

Proof Let (l^{ij}) denote the inverse matrix of (l_{ij}), and

$$\frac{\mathrm{d}}{\mathrm{d}_i^m t} = \frac{\partial}{\partial t} + \lambda_i(u^{(m)})\frac{\partial}{\partial x}.$$

By (3.1), we have

$$\begin{aligned}\sum_{j=1}^n l_{ij}(u^{(m)})\frac{\mathrm{d}}{\mathrm{d}_i^m t}(u_j^{(m)} - u_j^{(k)}) &= f_i(u^{(m)}) - f_i(u^{(k)}) - \sum_{j=1}^n (l_{ij}(u^{(m)}) - l_{ij}(u^{(k)}))\frac{\partial u_j^{(k)}}{\partial t}\\ &\quad - \sum_{j=1}^n (l_{ij}(u^{(m)})\lambda_i(u^{(m)}) - l_{ij}(u^{(k)})\lambda_i(u^{(k)}))\frac{\partial u_j^{(k)}}{\partial x}.\end{aligned}$$

Hence

$$\begin{aligned}&\sum_{j=1}^n \frac{\mathrm{d}}{\mathrm{d}_i^m t}[l_{ij}(u^{(m)})(u_j^{(m)} - u_j^{(k)})]\\ &= f_i(u^{(m)}) - f_i(u^{(k)}) - \sum_{j=1}^n (l_{ij}(u^{(m)}) - l_{ij}(u^{(k)}))\frac{\partial u_j^{(k)}}{\partial t}\\ &\quad - \sum_{j=1}^n (l_{ij}(u^{(m)})\lambda_i(u^{(m)}) - l_{ij}(u^{(k)})\lambda_i(u^{(k)}))\frac{\partial u_j^{(k)}}{\partial x} + \sum_{j=1}^n \frac{\mathrm{d}}{\mathrm{d}_i^m t} l_{ij}(u^{(m)})(u_j^{(m)} - u_j^{(k)}).\end{aligned}$$

Integrating it from 0 to t, we get

$$\begin{aligned}&\sum_{j=1}^n l_{ij}(u^{(m)}(t,x))(u_j^{(m)}(t,x) - u_j^{(k)}(t,x))\\ &= \sum_{j=1}^n l_{ij}(\phi^{(m)}(\xi_i(0;t,x;u^{(m)})))(\phi_j^{(m)}(\xi_i(0;t,x;u^{(m)})) - \phi_j^{(k)}(\xi_i(0;t,x;u^{(m)})))\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \Big[f_i(u^{(m)})-f_i(u^{(k)})-\sum_{j=1}^n (l_{ij}(u^{(m)})-l_{ij}(u^{(k)}))\frac{\partial u_j^{(k)}}{\partial t}\Big]\Big|_{(\tau,\xi_i(\tau;t,x;u^{(m)}))}\mathrm{d}\tau\\
&-\sum_{j=1}^n\int_0^t (l_{ij}(u^{(m)})\lambda_i(u^{(m)})-l_{ij}(u^{(k)})\lambda_i(u^{(k)}))\frac{\partial u_j^{(k)}}{\partial x}\Big|_{(\tau,\xi_i(\tau;t,x;u^{(m)}))}\mathrm{d}\tau\\
&+\sum_{j=1}^n\int_0^t \frac{\mathrm{d}}{\mathrm{d}_i^m\tau} l_{ij}(u^{(m)})(u_j^{(m)}-u_j^{(k)})\Big|_{(\tau,\xi_i(\tau;t,x;u^{(m)}))}\mathrm{d}\tau.
\end{aligned}
$$

Multiplying both sides of the above formula by $l^{hi}(u^{(m)}(t,x))$ and summing up from $i=1$ to n, we have

$$
\begin{aligned}
&u_h^{(m)}(t,x)-u_h^{(k)}(t,x)\\
=&\sum_{i,j=1}^n l^{hi}(u^{(m)}(t,x))l_{ij}(\phi^{(m)}(\xi_i(0;t,x;u^{(m)})))(\phi_j^{(m)}(\xi_i(0;t,x;u^{(m)}))-\phi_j^{(k)}(\xi_i(0;t,x;u^{(m)})))\\
&+\sum_{i=1}^n l^{hi}(u^{(m)}(t,x))\int_0^t \Big[f_i(u^{(m)})-f_i(u^{(k)})\\
&-\sum_{j=1}^n (l_{ij}(u^{(m)})-l_{ij}(u^{(k)}))\frac{\partial u_j^{(k)}}{\partial t}\Big]\Big|_{(\tau,\xi_i(\tau;t,x;u^{(m)}))}\mathrm{d}\tau\\
&-\sum_{i,j=1}^n l^{hi}(u^{(m)}(t,x))\int_0^t (l_{ij}(u^{(m)})\lambda_i(u^{(m)})-l_{ij}(u^{(k)})\lambda_i(u^{(k)}))\frac{\partial u_j^{(k)}}{\partial x}\Big|_{(\tau,\xi_i(\tau;t,x;u^{(m)}))}\mathrm{d}\tau\\
&+\sum_{i,j=1}^n l^{hi}(u^{(m)}(t,x))\int_0^t \frac{\mathrm{d}}{\mathrm{d}_i^m\tau} l_{ij}(u^{(m)})(u_j^{(m)}-u_j^{(k)})\Big|_{(\tau,\xi_i(\tau;t,x;u^{(m)}))}\mathrm{d}\tau.
\end{aligned}
$$

Noting $\|u^{(m)}\|_{C^1}\le C_1$, it is easy to see that

$$
|w^{(m,k)}|(t)\le C_3\|\phi^{(m)}-\phi^{(k)}\|_{C^0}+C_4\int_0^t |w^{(m,k)}|(\tau)\mathrm{d}\tau,
$$

where

$$
\begin{aligned}
w^{(m,k)}(t,x)&=u^{(m)}(t,x)-u^{(k)}(t,x),\\
|w^{(m,k)}|(t)&=\max_{\substack{x\in\mathbb{R}\\ h=1,\cdots,n}}|w_h^{(m,k)}(t,x)|.
\end{aligned}
$$

Then, from Gronwall's inequality, we get

$$
|w^{(m,k)}|(t)\le C_5\|\phi^{(m)}-\phi^{(k)}\|_{C^0},\quad \forall\, t\in[0,\delta_0].
$$

Hence, the corresponding sequence $\{u^{(m)}\}$ of solutions is convergent uniformly on $[0,\delta_0]$ for the given approximation sequences $\{\phi^{(m)}(x)\}$ of the initial data $\phi(x)$.

For two approximation sequences $\{\phi_1^{(m)}\},\{\phi_2^{(m)}\}$ of ϕ, $\{\phi_1^{(1)},\phi_2^{(1)},\phi_1^{(2)},\phi_2^{(2)},\cdots\}$ is also a approximation sequence of ϕ. Thus, we conclude that the limit function u is independent of the choice of the approximation sequence of ϕ.

Hence, the Lipschitz continuous solution to Cauchy problem (2.4) defined by approximation is unique.

4 Lipschitz Continuous Solution Defined by System of Integral Equations, Its Existence and Uniqueness

It is well-known that Cauchy problem (2.4) for quasi-linear hyperbolic systems is equivalent to the corresponding system of integral equations in the sense of classical solutions (cf. [6, 9, 15, 19] and references therein). In the framework of Lipschitz continuous solutions, it is not certain whether there is still this kind of equivalence. In this section, by justifying this equivalence for Lipschitz continuous solutions, we give another definition of Lipschitz continuous solution to Cauchy problem (2.4) and prove its existence and uniqueness under certain assumptions.

Now, we show that the limit function u obtained in the previous section satisfies a system of integral equations, which is independent of the choice of approximation sequence.

Let $x = \xi_i(t;\tau,\xi;u^{(m)})$ be the i-th characteristic of problem (3.1) with respect to $u^{(m)}$ $(i = 1,\cdots,n;\ m = 1,2,\cdots)$, satisfying

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \lambda_i(u^{(m)}(t,x)), \\ t=\tau: \quad x=\xi, \end{cases}$$

where $u^{(m)}$ is the solution sequence in Section 3.

Let $x = \xi_i(t;\tau;\xi;u)$ be the i-th characteristic of problem (2.4) with respect to u $(i = 1,\cdots,n)$, satisfying

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \lambda_i(u(t,x)), \\ t=\tau: \quad x=\xi, \end{cases}$$

where u is the limit function of the solution sequence in Section 3.

By Remark 2.2, as $m\to\infty$, $\xi_i(t;\tau,\xi;u^{(m)})$ converges to $\xi_i(t;\tau,\xi;u)$ uniformly on $\{(t,\tau,\xi) \mid (\tau,\xi)\in D(\delta_0),\ t\in[0,\tau]\}$ $(i=1,\cdots,n)$, and moreover, $\xi_i(t;\tau,\xi;u^{(m)})$ and $\xi_i(t;\tau,\xi;u)$ satisfy the properties given in Lemmas 2.4 and 2.5.

By (3.1), we have

$$\begin{aligned} &\sum_{j=1}^{n} l_{ij}(u^{(m)}(t,\xi_i(t;\tau,\xi;u^{(m)})))\frac{\mathrm{d}}{\mathrm{d}_i^m t}u_j^{(m)}(t,\xi_i(t;\tau,\xi;u^{(m)})) \\ &= f_i(u^{(m)}(t,\xi_i(t;\tau,\xi;u^{(m)}))), \quad i=1,\cdots,n. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{j=1}^{n} \frac{\mathrm{d}}{\mathrm{d}_i^m t}[l_{ij}(u^{(m)}(t,\xi_i(t;\tau,\xi;u^{(m)})))u_j^{(m)}(t,\xi_i(t;\tau,\xi;u^{(m)}))] \\ &= f_i(u^{(m)}(t,\xi_i(t;\tau,\xi;u^{(m)}))) \\ &\quad + \sum_{j=1}^{n} \frac{\mathrm{d}}{\mathrm{d}_i^m t} l_{ij}(u^{(m)}(t,\xi_i(t;\tau,\xi;u^{(m)})))u_j^{(m)}(t,\xi_i(t;\tau,\xi;u^{(m)})), \quad i=1,\cdots,n. \end{aligned}$$

Interchanging (t,x) and (τ,ξ), and integrating the above equation with respect to τ from 0 to t, we get

$$\begin{aligned}&\sum_{j=1}^{n} l_{ij}(u^{(m)}(t,x))u_j^{(m)}(t,x)\\&=\sum_{j=1}^{n} l_{ij}(\phi^{(m)}(\xi_i(0;t,x;u^{(m)})))\phi_j^{(m)}(\xi_i(0;t,x;u^{(m)}))\\&\quad+\int_0^t \Big[f_i(u^{(m)})+\sum_{j=1}^{n}\frac{\mathrm{d}}{\mathrm{d}_i^m\tau}l_{ij}(u^{(m)})u_j^{(m)}\Big]\Big|_{(\tau,\xi_i(\tau;t,x;u^{(m)}))}\mathrm{d}\tau,\quad i=1,\cdots,n.\end{aligned}\tag{4.1}$$

Theorem 4.1 *The limit function u satisfies the following system of integral equations*:

$$\begin{aligned}\sum_{j=1}^{n} l_{ij}(u(t,x))u_j(t,x)&=\sum_{j=1}^{n} l_{ij}(\phi(\xi_i(0;t,x;u)))\phi_j(\xi_i(0;t,x;u))\\&\quad+\int_0^t \Big[f_i(u)+\sum_{j=1}^{n}\frac{\mathrm{d}}{\mathrm{d}_i\tau}l_{ij}(u)u_j\Big]\Big|_{(\tau,\xi_i(\tau;t,x;u))}\mathrm{d}\tau,\quad i=1,\cdots,n,\end{aligned}\tag{4.2}$$

which is independent of the choice of approximation sequence.

Proof Without loss of generality, we suppose that $u^{(m)}$ converges to u uniformly as $m\to\infty$. By the uniform convergence of $u^{(m)}$ and $\xi_i(\tau;t,x;u^{(m)})$, noting Remark 2.2 and taking $m\to\infty$ in (4.1), it is easy to see that on $D(\delta_0)$ we have that

$$\begin{aligned}&\sum_{j=1}^{n} l_{ij}(u^{(m)}(t,x))u_j^{(m)}(t,x)\ \text{ converges uniformly to }\ \sum_{j=1}^{n} l_{ij}(u(t,x))u_j(t,x),\\&\sum_{j=1}^{n} l_{ij}(\phi^{(m)}(\xi_i(0;t,x;u^{(m)})))\phi_j^{(m)}(\xi_i(0;t,x;u^{(m)}))\ \text{ converges uniformly to}\\&\sum_{j=1}^{n} l_{ij}(\phi(\xi_i(0;t,x;u)))\phi_j(\xi_i(0;t,x;u))\end{aligned}$$

and

$$\int_0^t f_i(u^{(m)}(\tau,\xi_i(\tau;t,x;u^{(m)})))\mathrm{d}\tau\to\int_0^t f_i(u(\tau,\xi_i(\tau;t,x;u)))\mathrm{d}\tau.$$

To justify (4.2), we need to prove that as $m\to\infty$, we have

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}_i^m\tau}l_{ij}(u^{(m)})u_j^{(m)}(\tau,\xi_i(\tau;t,x;u^{(m)}))\mathrm{d}\tau\to\int_0^t\frac{\mathrm{d}}{\mathrm{d}_i\tau}l_{ij}(u)u_j(\tau,\xi_i(\tau;t,x;u))\mathrm{d}\tau.$$

In fact,

$$\begin{aligned}&\int_0^t \frac{\mathrm{d}}{\mathrm{d}_i^m\tau}l_{ij}(u^{(m)})u_j^{(m)}(\tau,\xi_i(\tau;t,x;u^{(m)}))-\frac{\mathrm{d}}{\mathrm{d}_i\tau}l_{ij}(u)u_j(\tau,\xi_i(\tau;t,x;u))\mathrm{d}\tau\\&=\int_0^t\Big[\frac{\mathrm{d}}{\mathrm{d}_i^m\tau}l_{ij}(u^{(m)}(\tau,\xi_i(\tau;t,x;u^{(m)})))-\frac{\mathrm{d}}{\mathrm{d}_i\tau}l_{ij}(u(\tau,\xi_i(\tau;t,x;u)))\Big]u_j^{(m)}(\tau,\xi_i(\tau;t,x;u^{(m)}))\mathrm{d}\tau\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \frac{\mathrm{d}}{\mathrm{d}_i\tau} l_{ij}(u(\tau,\xi_i(\tau;t,x;u)))[u_j^{(m)}(\tau,\xi_i(\tau;t,x;u^{(m)}))-u_j(\tau,\xi_i(\tau;t,x;u))]\mathrm{d}\tau\\
&=[l_{ij}(u^{(m)}(\tau,\xi_i(\tau;t,x;u^{(m)})))-l_{ij}(u(\tau,\xi_i(\tau;t,x;u)))]u_j^{(m)}(\tau,\xi_i(\tau;t,x;u^{(m)}))\big|_0^t\\
&\quad-\int_0^t [l_{ij}(u^{(m)}(\tau,\xi_i(\tau;t,x;u^{(m)})))-l_{ij}(u(\tau,\xi_i(\tau;t,x;u)))]\frac{\mathrm{d}}{\mathrm{d}_i^m\tau}u_j^{(m)}(\tau,\xi_i(\tau;t,x;u^{(m)}))\mathrm{d}\tau\\
&\quad+\int_0^t \frac{\mathrm{d}}{\mathrm{d}_i\tau} l_{ij}(u(\tau,\xi_i(\tau;t,x;u)))[u_j^{(m)}(\tau,\xi_i(\tau;t,x;u^{(m)}))-u_j(\tau,\xi_i(\tau;t,x;u))]\mathrm{d}\tau\\
&=\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3.
\end{aligned}
$$

Since l_{ij} are C^1 functions, $u^{(m)}$ converges to u uniformly, $\xi_i(t;\tau,\xi;u^{(m)})$ converges to $\xi_i(t;\tau,\xi;u)$ uniformly and $u^{(m)}$ is uniformly bounded, it is easy to see that $\mathrm{I}_1\to 0$ as $m\to\infty$.

Since the C^1 norm of $\{u^{(m)}\}$ is uniformly bounded, u is bounded Lipschitz continuous, and both $u(\tau,\xi_i(\tau;t,x;u))$ and $\xi_i(\tau;t,x;u^{(m)})$ $(m=1,2,\cdots)$ satisfy the properties given in Lemmas 2.4 and 2.5, the Lipschitz constant of $u^{(m)}(\tau,\xi_i(\tau;t,x;u^{(m)}))$ $(m=1,2,\cdots)$ and $u(\tau,\xi_i(\tau;t,x;u))$ with respect to τ is uniformly bounded. Therefore

$$
\begin{aligned}
|\mathrm{I}_2| &\le C_6\int_0^t |u^{(m)}(\tau,\xi_i(\tau;t,x;u^{(m)}))-u(\tau,\xi_i(\tau;t,x;u))|\mathrm{d}\tau\\
&\le C_6\Big\{\int_0^t |\xi_i(\tau;t,x;u^{(m)})-\xi_i(\tau;t,x;u)|\mathrm{d}\tau\\
&\quad+\int_0^t |u^{(m)}(\tau,\xi_i(\tau;t,x;u))-u(\tau,\xi_i(\tau;t,x;u))|\mathrm{d}\tau\Big\}.
\end{aligned}
$$

By the uniform convergence of $\xi_i(\tau;t,x;u^{(m)})$ and $u^{(m)}$, it is obvious that $\mathrm{I}_2\to 0$ as $m\to\infty$. Similarly, $\mathrm{I}_3\to 0$ as $m\to\infty$.

Thus, the limit function u satisfies system (4.2) of integral equations. Obviously, system (4.2) of integral equations is independent of the choice of approximation sequence.

Therefore, system (4.2) of integral equations can be used to define the Lipschitz continuous solution. From the above discussion, we have the existence of Lipschitz continuous solution defined by the system of integral equations.

In the last part, we want to show that the bounded Lipschitz continuous solution to systems (4.2) of integral equations is also unique.

Theorem 4.2 *The bounded Lipschitz continuous solution u to system* (4.2) *of integral equations* (4.2) *is unique.*

Proof Suppose that u,v are bounded Lipschitz continuous solutions to system (4.2) of integral equations on the interval $[0,\delta]$. We will show $u\equiv v$ on $D(\delta)$ for any given $M>0$. To this end, we define

$$
\alpha(t)=\max_{\substack{|x|\le M-\Lambda t\\ i=1,\cdots,n}}|u_i(t,x)-v_i(t,x)|,\quad 0\le t\le\delta.
$$

By (4.2), we have

$$
\sum_{j=1}^n l_{ij}(u)(u_j(t,x)-v_j(t,x))=\mathrm{I}_4+\mathrm{I}_5,
$$

where

$$\begin{aligned}\mathrm{I}_4 = &\sum_{j=1}^{n}(l_{ij}(\phi(\xi_i(0;t,x;u))) - l_{ij}(\phi(\xi_i(0;t,x;v))))\phi_j(\xi_i(0;t,x;u)) \\ &+ \sum_{j=1}^{n} l_{ij}(\phi(\xi_i(0;t,x;v)))(\phi_j(\xi_i(0;t,x;u)) - \phi_j(\xi_i(0;t,x;v))).\end{aligned}$$

Then, by Lemma 2.6, we have

$$|\mathrm{I}_4| \le (C_7\theta + C_8)\int_0^t \alpha(s)\mathrm{d}s, \tag{4.3}$$

in which θ is the C^0 norm of ϕ, while

$$\begin{aligned}\mathrm{I}_5 = &\int_0^t [f_i(u(\tau,\xi_i(\tau;t,x;u))) - f_i(v(\tau,\xi_i(\tau;t,x;v)))]\mathrm{d}\tau \\ &+ \sum_{j=1}^{n}\int_0^t \frac{\mathrm{d}}{\mathrm{d}_i\tau} l_{ij}(u)(u_j(\tau,\xi_i(\tau;t,x;u)) - v_j(\tau,\xi_i(\tau;t,x;v)))\mathrm{d}\tau \\ &+ \sum_{j=1}^{n}\int_0^t \Big(\frac{\mathrm{d}}{\mathrm{d}_i\tau} l_{ij}(u) - \frac{\mathrm{d}}{\mathrm{d}_i\tau} l_{ij}(v)\Big) v_j(\tau,\xi_i(\tau;t,x;v))\mathrm{d}\tau \\ &+ \sum_{j=1}^{n}(l_{ij}(v) - l_{ij}(u))v_j(t,x) \\ = &\,\mathrm{I}_6 + \mathrm{I}_7 + \mathrm{I}_8,\end{aligned}$$

in which I_6 denotes the first term on the right-hand side, I_7 the second term and I_8 the last two terms.

We have

$$|\mathrm{I}_6| \le (C_9 + C_{10}\delta)\int_0^t \alpha(s)\mathrm{d}s, \tag{4.4}$$

$$|\mathrm{I}_7| \le (C_{11} + C_{12}\delta)\int_0^t \alpha(s)\mathrm{d}s. \tag{4.5}$$

Moreover, by using integration by parts, it is easy to see that

$$\begin{aligned}\mathrm{I}_8 = &-\sum_{j=1}^{n}(l_{ij}(\phi(\xi_i(0;t,x;u))) - l_{ij}(\phi(\xi_i(0;t,x;v))))\phi_j(\xi_i(0;t,x;v)) \\ &- \sum_{j=1}^{n}\int_0^t (l_{ij}(u) - l_{ij}(v))\frac{\mathrm{d}}{\mathrm{d}_i\tau} v_j(\tau,\xi_i(\tau;t,x;v))\mathrm{d}\tau.\end{aligned}$$

Then

$$|\mathrm{I}_8| \le (C_{13}\theta + C_{14} + C_{15}\delta)\int_0^t \alpha(s)\mathrm{d}s. \tag{4.6}$$

From the above arguments, we have

$$\Big|\sum_{j=1}^{n} l_{ij}(u)(u_j(t,x) - v_j(t,x))\Big| \le C_{16}(1 + \theta + \delta)\int_0^t \alpha(s)\mathrm{d}s, \quad i = 1,\cdots,n.$$

Then

$$\alpha(t) \le C_{17}(1+\theta+\delta)\int_0^t \alpha(s)\mathrm{d}s. \tag{4.7}$$

Hence, from Gronwall's inequality, we get

$$\alpha(t) \equiv 0, \quad 0 \le t \le \delta,$$

which shows the uniqueness of the Lipschitz continuous solution.

Remark 4.1 Suppose that u is a bounded Lipschitz solution to system (4.2) of integral equations on the interval $[0,\delta]$. Under the additional assumption $f(0)=0$, we have

$$|u(t,x)| \le C_{18}\theta, \quad \forall\, t \in [0,\delta],\ x \in \mathbb{R}, \tag{4.8}$$

where θ is the C^0 norm of ϕ. Similar results for the C^1 solution can be found in [2, 14].

Remark 4.2 We can get the existence of Lipschitz continuous solution to system (4.2) of integral equations through a fixed point theorem.

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