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Hausdorff Operators on Function Spaces*

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Abstract The authors mainly study the Hausdorff operators on Euclidean space $\mathbb{R}^n$. They establish boundedness of the Hausdorff operators in various function spaces, such as Lebesgue spaces, Hardy spaces, local Hardy spaces and Herz type spaces. The results reveal that the Hausdorff operators have better performance on the Herz type Hardy spaces $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ than their performance on the Hardy spaces $H^p(\mathbb{R}^n)$ when $0 < p < 1$. Also, the authors obtain some new results and reprove or generalize some known results for the high dimensional Hardy operator and adjoint Hardy operator.

Keywords Hausdorff operator, Hardy operator, Hardy space, Herz space
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1 Introduction

We begin this article by recalling the 1-dimensional Hausdorff operator

$$h_\Phi f(x) = \int_0^\infty \frac{\Phi(t)}{t} f\Big(\frac{x}{t}\Big) \mathrm{d}t, \tag{1.1}$$

where $\Phi(t)$ is a locally integrable function in $(0, \infty)$. The operator h_Φ has a deep root in the study of the 1-dimensional Fourier analysis. Particularly, it is closely related to the summability of the classical Fourier series. The reader can see [9–12] to find details of the background and the historical development of the Hausdorff operator. An easy computation involving the Minkowski inequality and scaling shows that, for all $1 \le p \le \infty$,

$$\|h_\Phi f\|_{L^p(\mathbb{R})} \le \int_0^\infty \frac{|\Phi(t)|}{t} t^{\frac{1}{p}} \mathrm{d}t \|f\|_{L^p(\mathbb{R})}.$$

Thus, the Hausdorff operator is bounded in the Lebesgue space $L^p(\mathbb{R})$, if

$$\int_0^\infty |\Phi(t)| t^{-1+\frac{1}{p}} \mathrm{d}t < \infty.$$

Another important function space is the Hardy space $H^p(\mathbb{R})$ for $0 < p \le 1$. It is known that $L^p = H^p$ for $1 < p < \infty$ and H^1 is a proper subspace imbedding in the space L^1. In [12],

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Liflyand and Móricz proved that the Hausdorff operator has the same bound on the Hardy space $H^1(\mathbb{R})$ as that in the Lebesgue space $L^1(\mathbb{R})$,

$$\|h_\Phi(f)\|_{H^1(\mathbb{R})} \preceq \Big(\int_0^\infty |\Phi(t)|\mathrm{d}t\Big)\|f\|_{H^1(\mathbb{R})}.$$

However, the boundedness of $h_\Phi(f)$ in the $H^p(\mathbb{R})$, when $0<p<1$, is significantly different from the case of $H^1(\mathbb{R})$. This phenomena was discovered by Liflyand and Miyachi. In fact, Liflyand and Miyachi [11] found a bounded function Φ whose support is a compact set in $(0,\infty)$, such that the operator h_Φ is not bounded in $H^p(\mathbb{R})$ for any $0<p<1$. So some smoothness condition on Φ may be needed to ensure the boundedness of h_Φ in the Hardy space $H^p(\mathbb{R})$ if $0<p<1$. By this observation, as a corollary of main results in [11], Liflyand and Miyachi established the following theorem.

Theorem 1.1 (see [11]) Let $0<p<1$ *and* $M=[\frac{1}{p}-\frac{1}{2}]+1$. *If* $\Phi\in C^M$ *and its support is a compact set in* $(0,\infty)$, *then*

$$\|h_\Phi(f)\|_{H^p(\mathbb{R})} \preceq \|f\|_{H^p(\mathbb{R})}.$$

Using the same idea, Zhong obtained a similar theorem on the Triebel-Lizorkin space $\dot{F}^\alpha_{p,q}$ (see [5–6, 8, 15–16] for the definition of this space).

Theorem 1.2 (see [17]) Let $0<p\le 1$, $1<q<\infty$, $\alpha\in\mathbb{R}$, $J=\frac{1}{\min\{p,q\}}$, $[\alpha]_+=\max\{0,[\alpha]\}$. *Suppose that* $\Phi\in C^{L+2+[\alpha]_+}$, *where* L *is the smallest integer that is larger than*

$$\max\{J-2,[J-1-\alpha]\}.$$

Also, assume that the support of Φ *is a compact set in* $(0,\infty)$. *Then*

$$\|h_\Phi(f)\|_{\dot{F}^\alpha_{p,q}(\mathbb{R})} \preceq \|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R})}.$$

Our next observation is that the operator h_Φ has an important feature: the Hardy operator and its adjoint operator actually are special cases of the Hausdorff operator if one chooses suitable functions Φ. To see this fact, if $x>0$, by a change of variables, one has

$$h_\Phi f(x)=\int_0^\infty \frac{\Phi(\frac{x}{t})}{t}f(t)\mathrm{d}t.$$

So if one chooses

$$\Phi_1(t)=\chi_{(0,1)}(t) \quad \text{and} \quad \Phi_2(t)=\frac{\chi_{(1,\infty)}(t)}{t},$$

then one can obtain the adjoint Hardy operator

$$h_{\Phi_1}f(x):=H^*f(x)=\int_x^\infty \frac{f(t)}{t}\mathrm{d}t$$

and the Hardy operator

$$h_{\Phi_2}f(x):=Hf(x)=\frac{1}{x}\int_0^x f(t)\mathrm{d}t,$$

respectively. It is well-known that Hardy operators, particularly in high dimension, are important operators in harmonic analysis and they were attracted extensive research by many authors, for instance, see [1, 4, 7], among many references. These observations motivate us to study the high dimensional Hausdorff operators and their boundedness in various function spaces, while the Hardy operators are our model case. With this purpose, we study three Hausdorff operators on the Euclidean space $\mathbb{R}^n$,

$$H_\Phi f(x) = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} f\Big(\frac{x}{|y|}\Big)\mathrm{d}y, \tag{1.2}$$

$$H_{\Phi,A}(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} f(A(y)x)\mathrm{d}y, \tag{1.3}$$

$$\widetilde{H}_{\Phi,\Omega} f(x) = \int_{\mathbb{R}^n} \frac{\Phi(x|y|^{-1})}{|y|^n} \Omega(y') f(y)\mathrm{d}y, \tag{1.4}$$

where $A(y)$ is an $n\times n$ matrix and we assume $\det A(y) \neq 0$ almost everywhere in the support of Φ, and $\Omega(y')$ is an integrable function defined on the unit sphere S^{n-1}. We denote $\widetilde{H}_{\Phi,\Omega} = \widetilde{H}_\Phi$ if $\Omega = 1$.

The operator $H_{\Phi,A}$ was defined and studied by Lerner and Liflyand [9], and the operator H_Φ in (1.2) is a direct extension of the 1-dimensional Hausdorff operator h_Φ and it is a special case of $H_{\Phi,A}$, since

$$H_{\Phi,A}(f)(x) = H_\Phi(f)(x)$$

if

$$A(y) = \mathrm{diag}\Big[\frac{1}{|y|}, \cdots, \frac{1}{|y|}\Big].$$

Also, we obtain the n-dimensional adjoint Hardy operator

$$H^* f(x) = \int_{|y|\geq |x|} \frac{f(y)}{|y|^n}\mathrm{d}y$$

and the n-dimensional Hardy operator

$$Hf(x) = \frac{1}{|x|^n} \int_{|y|\leq |x|} f(y)\mathrm{d}y,$$

if we choose $\Phi(y) = \chi_{\{|y|\leq 1\}}(y)$ and $\Phi(y) = \frac{\chi_{\{|y|\geq 1\}}(y)}{|y|^n}$, respectively, in the operator $\widetilde{H}_\Phi$.

We also define a discrete Hausdorff operator

$$H_{\Phi,\mathrm{dis}} f(x) = \sum_{k\in\mathbb{Z}} \Phi(\beta^k) f(A(\beta^k)x)$$

with a positive number β $(\beta \neq 1)$.

The aim of this paper is to establish the boundedness of Hausdorff operators in various function spaces. In Section 2, we study the boundedness of $H_{\Phi,A}$ in the Hardy space $H^1(\mathbb{R}^n)$ and on the local Hardy space $h^1(\mathbb{R}^n)$. Our theorems and methods are different from those in [9] and this method allows us to establish the boundedness of $H_{\Phi,A}$ and H_Φ in the Herz-type Hardy space $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and in the Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, respectively, for all $0 < p < \infty$. We show that the operator $H_{\Phi,A}$ is bounded in the space $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ for all $0 < p < \infty$, only assuming some size condition on Φ (no smoothness condition). Comparing our results with Theorems 1.1–1.2, we find that the performance of the Hausdorff operators in the space $H\dot{K}_q^{\alpha,p}$ is much better than that in the Hardy space H^p or in the Trieble-Lizorkin space $\dot{F}_{p,q}^{\alpha}$ when $0 < p < 1$. In the third section, we study the boundedness of the operator $\widetilde{H}_\Phi f(x)$ in the Lebesgue space L^p for all $1 < p < \infty$, and in the Hardy type Herz space $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$. As an application, we reprove the L^p boundedness of the Hardy operator and the adjoint Hardy operator and obtain the sharp L^p bounds that are known in [4]. We also obtain the boundedness of the adjoint Hardy operator in the Hardy space $H^1(\mathbb{R}^n)$. In Section 4, we show that the discrete Hausdorff operator $H_{\Phi,\mathrm{dis}}f$ is bounded on H^p for all $0 < p \leq 1$, if the function Φ satisfies some size condition depending on the index p. In addition, for some further extensions to multilinear or multiparameter cases, one can see [2–3, 18].

In this paper, we use the notation $A \preceq B$ to mean that there is a positive constant C independent of all essential variables such that $A \leq CB$.

2 Operator $H_{\Phi,A}$

2.1 $H^1(\mathbb{R}^n)$ boundedness

We first observe that it is trivial to obtain the L^p boundedness of $H_{\Phi,A}$. In fact, if

$$\Big(\int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |\det A(y)|^{-\frac{1}{p}} \mathrm{d}y\Big) < \infty,$$

then by the Minkowski inequality and a change of variables, for any $1 \leq p \leq \infty$,

$$\|H_{\Phi,A}(f)\|_{L^p(\mathbb{R}^n)} \leq \Big(\int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |\det A(y)|^{-\frac{1}{p}} \mathrm{d}y\Big) \|f\|_{L^p(\mathbb{R}^n)}.$$

On the other hand, the same argument involving scaling and Minkowski inequality gives the boundedness of the operator H_Φ in the Triebel-Lizorkin space $\dot{F}_{p,q}^{\alpha}(\mathbb{R}^n)$ for any $p, q \geq 1$,

$$\|H_\Phi(f)\|_{\dot{F}_{p,q}^{\alpha}(\mathbb{R}^n)} \leq \Big(\int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |y|^{-\alpha+\frac{n}{p}} \mathrm{d}y\Big) \|f\|_{\dot{F}_{p,q}^{\alpha}(\mathbb{R}^n)}.$$

It is well-known that

$$\dot{F}_{p,2}^{0}(\mathbb{R}^n) = H^p(\mathbb{R}^n)$$

for all $0 < p < \infty$. However, the scaling argument fails for the more general operator $H_{\Phi,A}$. Thus, to establish the H^1 boundedness of $H_{\Phi,A}$, we need to use the atomic characterization of the Hardy space H^p. Let $\Psi \in S(\mathbb{R}^n)$ satisfy

$$\int_{\mathbb{R}^n} \Psi(y) \mathrm{d}y \neq 0.$$

Denote

$$\Psi_s(y) = \frac{1}{s^n} \Psi\Big(\frac{y}{s}\Big).$$

The Hardy space $H^p(\mathbb{R}^n)$ is the space of all distributions f satisfying

$$\|f\|_{H^p(\mathbb{R}^n)} := \Big\| \sup_{0<s<\infty} |\Psi_s * f| \Big\|_{L^p(\mathbb{R}^n)} < \infty.$$

The local Hardy space $h^p(\mathbb{R}^n)$ is the space of all distributions f satisfying

$$\|f\|_{h^p(\mathbb{R}^n)} := \Big\| \sup_{0<s\leq 1} |\Psi_s * f| \Big\|_{L^p(\mathbb{R}^n)} < \infty.$$

It is known that the definitions of $h^p(\mathbb{R}^n)$ and $H^p(\mathbb{R}^n)$ are flexible on the choices of the function Ψ. It is also clear that

$$\|f\|_{L^p(\mathbb{R}^n)} \preceq \|f\|_{h^p(\mathbb{R}^n)} \leq \|f\|_{H^p(\mathbb{R}^n)},$$

so that we have the imbedding

$$H^p \subset h^p \subset L^p.$$

We remark that $H^p = h^p = L^p$ when $\infty > p > 1$, while H^p and h^p are merely quasi-norm spaces if $0 < p < 1$. More details of Hardy spaces can be found in [13].

We recall that Lerner and Liflyand [9] established $H^1(\mathbb{R}^n)$-boundedness of the operator $H_{\Phi,A}$ under the following condition:

$$\int_{\mathbb{R}^n} |\Phi(u)| \|A^{-1}(u)\|^n \mathrm{d}u < \infty.$$

In the following, we will obtain another criterion of the $H^1(\mathbb{R}^n)$ boundedness. Our method is different from that in [9], and it allows us to obtain a further criterion in the local Hardy spaces and in the Herz type Hardy spaces.

Theorem 2.1 *For any* $1 < q \leq \infty$,

$$\|H_{\Phi,A}(f)\|_{H^1(\mathbb{R}^n)} \preceq \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} B(y)^{-n+\frac{n}{q}} \; |\det A^{-1}(y)|^{\frac{1}{q}} \mathrm{d}y \; \|f\|_{H^1(\mathbb{R}^n)},$$

where

$$B(y) = \|A^{-1}(y)\|^{-1}, \quad \|A^{-1}(y)\| = \sup_{x\neq 0} |A^{-1}(y)x||x|^{-1}.$$

To compare the theorem to a result in [9], we note that $|\det A| \leq \|A\|^n$.

Proof We prove the theorem by using the atomic characterization of the Hardy spaces. Any $f \in H^p(\mathbb{R}^n)$ has the atomic decomposition

$$f = \sum \lambda_j a_j,$$

where

$$\sum |\lambda_j|^p \simeq \|f\|_{H^p}^p,$$

and each a_j is a (p, q) atom. Here, a (p, q)-atom a, $1 < q \leq \infty$, is a function satisfying

(i)

$$\operatorname{supp} a \subset B(x_0, \rho);$$

(ii)

$$\int_{\mathbb{R}^n} P_k(y)a(y)\mathrm{d}y = 0$$

for any polynomial of degree $k \le n\big[\frac{1}{p} - 1\big]$;

(iii)

$$\|a\|_q \le \rho^{n(\frac{1}{q}-\frac{1}{p})}.$$

Thus,

$$H_{\Phi,A}(f) = \sum \lambda_j H_{\Phi,A}(a_j).$$

By the Minkowski inequality,

$$\|H_{\Phi,A}(f)\|_{H^1(\mathbb{R}^n)} \le \sum |\lambda_j| \|H_{\Phi,A}(a_j)\|_{H^1(\mathbb{R}^n)}.$$

Thus it suffices to show that

$$\|H_{\Phi,A}(a)\|_{H^1(\mathbb{R}^n)} \preceq 1$$

uniformly for all $(1,q)$-atom a. Using the Minkowski inequality again, and noticing that $|A(y)x| \ge B(y)|x|$, we obtain

$$\|H_{\Phi,A}(a)\|_{H^1(\mathbb{R}^n)} \le \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} B(y)^{-n+\frac{n}{q}} \ |\det A^{-1}(y)|^{\frac{1}{q}} \|a_y(\,\cdot\,)\|_{H^1(\mathbb{R}^n)} \mathrm{d}y,$$

where

$$a_y(x) = a(A(y)x)B(y)^{n-\frac{n}{q}} \ |\det A^{-1}(y)|^{-\frac{1}{q}}.$$

It remains to show that

$$\|a_y(\,\cdot\,)\|_{H^1(\mathbb{R}^n)} \preceq 1$$

for any atom a and any fixed $y \in \mathbb{R}^n$. To this end, we only need to show that a_y is also a $(1,q)$-atom. Also, by a shift argument, we may assume that the support of a is $B(0,\rho)$. It is obvious that a_y satisfies the cancellation condition (ii). Next, we check that the support of $a_y(x)$ is contained in $B(0, B(y)^{-1}\rho)$. In fact, if $|x| \ge B(y)^{-1}\rho$, then

$$|A(y)x| \ge B(y)|x| \ge \rho,$$

which leads to $a(A(y)x) = 0$. Finally,

$$\begin{aligned}\|a_y\|_{L^q(\mathbb{R}^n)} &= B(y)^{n-\frac{n}{q}} \Big(\int_{\mathbb{R}^n} |\det A^{-1}(y)|^{-1} |a(A(y)x)|^q \mathrm{d}x \Big)^{\frac{1}{q}} \\ &\le (\rho B(y)^{-1})^{\frac{n}{q}-n}.\end{aligned}$$

Thus $a_y(x)$ is a $(1,q)$-atom. The theorem is proved.

2.2 $h^1(\mathbb{R}^n)$ boundedness

The following theorem is a modification of Theorem 2.1.

Theorem 2.2 *Let $B(y)$ be the same as in Theorem 2.1. We have*

$$\|H_{\Phi,A}(f)\|_{h^1(\mathbb{R}^n)} \le C\|f\|_{h^1(\mathbb{R}^n)},$$

where

$$C \simeq \int_{B(y)^{-1}>1} \frac{|\Phi(y)|}{|y|^n} B(y)^{-n+\frac{n}{q}} \ |\det A^{-1}(y)|^{\frac{1}{q}} \mathrm{d}y + \int_{B(y)^{-1}\le 1} \frac{|\Phi(y)|}{|y|^n} B(y)^{-n+\frac{n}{q}} \ |\det A^{-1}(y)|^{\frac{1}{q}} (1+\log_2 B(y))\mathrm{d}y.$$

Proof As same as an H^1 function, an $f \in h^1(\mathbb{R}^n)$ has the atomic characterization with atoms and big atoms. We say that a function $a^{\#}$ is a big (p,q)-atom if $a^{\#}$ satisfies (i) and (iii) in the definition of (p,q)-atom, and the diameter 2ρ of the support of $a^{\#}$ is larger than 1. We can write

$$f = f_1 + f_2.$$

Here

$$f_1 = \sum_j \mu_j a_j,$$

where each a_j is a $(1,q)$-atom.

$$f_2 = \sum_j \lambda_j a_j^{\#},$$

where each $a_j^{\#}$ is a big $(1,q)$-atom. They satisfy

$$\sum_j (|\lambda_j| + |\mu_j|) \simeq \|f\|_{h^1(\mathbb{R}^n)}.$$

By the proof for H^1 boundedness (see Theorem 2.1), it suffices to show that

$$\|H_{\Phi,A}(a^{\#})\|_{h^1(\mathbb{R}^n)} \preceq 1$$

uniformly for all big $(1,q)$-atoms $a^{\#}$. By the Minkowski inequality, we have

$$\|H_{\Phi,A}(a^{\#})\|_{h^1(\mathbb{R}^n)} \le \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} B(y)^{-n+\frac{n}{q}} \ |\det A^{-1}(y)|^{\frac{1}{q}} \|a_y^{\#}(\,\cdot\,)\|_{h^1(\mathbb{R}^n)}\mathrm{d}y,$$

where

$$a_y^{\#}(x) = a^{\#}(A(y)x)B(y)^{n-\frac{n}{q}} \ |\det A^{-1}(y)|^{-\frac{1}{q}}.$$

Since $a_y^{\#}(x)$ is again a big atom if $B(y)^{-1} > 1$ and

$$\|a\|_{h^1(\mathbb{R}^n)} \preceq 1$$

uniformly for all big atoms, we have

$$\begin{aligned}&\int_{B(y)^{-1}>1}\frac{|\Phi(y)|}{|y|^n}B(y)^{-n+\frac{n}{q}}\ |\det A^{-1}(y)|^{\frac{1}{q}}\|a_y^{\#}(\,\cdot\,)\|_{h^1(\mathbb{R}^n)}\mathrm{d}y\\ \preceq&\int_{B(y)^{-1}>1}\frac{|\Phi(y)|}{|y|^n}B(y)^{-n+\frac{n}{q}}\ |\det A^{-1}(y)|^{\frac{1}{q}}\mathrm{d}y\end{aligned}$$

uniformly on $a^{\#}$. To finish the proof of the theorem, it remains to estimate

$$\int_{B(y)^{-1}\leq 1}\frac{|\Phi(y)|}{|y|^n}B(y)^{-n+\frac{n}{q}}\ |\det A^{-1}(y)|^{\frac{1}{q}}\|a_y^{\#}(\,\cdot\,)\|_{h^1(\mathbb{R}^n)}\mathrm{d}y.$$

To this end, we only need to show

$$\|a_y^{\#}(\,\cdot\,)\|_{h^1(\mathbb{R}^n)}\preceq(1+\log_2 B(y)).$$

Since we can assume that the support of $a^{\#}$ is contained in the ball $B(0,\rho)$ with $\rho\geq 1$, the support of $a_y^{\#}$ is contained in the ball $B(0,B(y)^{-1}\rho)$. Let

$$\Psi_s(x)=s^{-\frac{n}{2}}\mathrm{e}^{-\frac{|x|^2}{s}}.$$

Without loss of generality, we may assume

$$4B(y)^{-1}\rho<100\rho.$$

We have

$$\begin{aligned}\|a_y^{\#}(\,\cdot\,)\|_{h^1(\mathbb{R}^n)}=&\int_{|x|\leq 4B(y)^{-1}\rho}\sup_{0<s\leq 1}\Big|\int_{\mathbb{R}^n}\Psi_s(x-z)a_y^{\#}(z)\mathrm{d}z\Big|\mathrm{d}x\\ &+\int_{4B(y)^{-1}\rho\leq|x|\leq 100\rho}\sup_{0<s\leq 1}\Big|\int_{\mathbb{R}^n}\Psi_s(x-z)a_y^{\#}(z)\mathrm{d}z\Big|\mathrm{d}x\\ &+\int_{100\rho\leq|x|}\sup_{0<s\leq 1}\Big|\int_{\mathbb{R}^n}\Psi_s(x-z)a_y^{\#}(z)\mathrm{d}z\Big|\mathrm{d}x\\ =&\ \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3.\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}\mathrm{I}_1\preceq&\Big(\int_{|x|\leq 4B(y)^{-1}\rho}\sup_{0<s\leq 1}\Big|\int_{\mathbb{R}^n}\Psi_s(x-z)a_y^{\#}(z)\mathrm{d}z\Big|^q\mathrm{d}x\Big)^{\frac{1}{q}}(B(y)^{-1}\rho)^{\frac{n}{q'}}\\ \preceq&(B(y)^{-1}\rho)^{\frac{n}{q'}}\|M(a_y^{\#})\|_{L^q(\mathbb{R}^n)},\end{aligned}$$

where $M(a_y^{\#})$ is the Hardy-Littlewood maximal function of $a_y^{\#}$. Thus

$$\mathrm{I}_1\preceq(B(y)^{-1}\rho)^{\frac{n}{q'}}\|a_y^{\#}\|_{L^q(\mathbb{R}^n)}\preceq 1.$$

Note in I_3,

$$|x-z|\geq 90\rho.$$

Thus, for $0<s\leq 1$, we have

$$|\Psi_s(x-z)|=\Big|\frac{1}{s^n}\Psi\Big(\frac{x-z}{s}\Big)\Big|\preceq|x-z|^{-n-1}s^n\preceq|x|^{-n-1}.$$

This gives

$$\mathrm{I}_3 \preceq \int_{|x|>90\rho} |x|^{-n-1}\Big(\int_{\mathbb{R}^n} |a_y^{\#}(z)|\mathrm{d}z\Big)\mathrm{d}x \preceq 1,$$

since $\rho \geq 1$.

Also, for any fixed x, the maximal value of

$$s^{-\frac{n}{2}}\mathrm{e}^{-\frac{|x|^2}{s}}$$

is achieved at $s \simeq |x|^2$. Thus

$$\mathrm{I}_2 \preceq \int_{4B(y)^{-1}\rho\leq|x|\leq100\rho} \frac{1}{|x|^n}\mathrm{d}x\Big(\int_{\mathbb{R}^n} |a_y^{\#}(z)|\mathrm{d}z\Big) \preceq 1 + |\log_2 B(y)|.$$

The theorem is proved. By this theorem, we obtain the following corollary for the operator H_Φ.

Corollary 2.1

$$\|H_\Phi(f)\|_{h^1(\mathbb{R}^n)} \preceq \Big(\int_{|y|\geq1} |\Phi(y)|\mathrm{d}y + \int_{|y|\leq1} |\Phi(y)|\Big(1+\log_2\Big(\frac{1}{|y|}\Big)\Big)\mathrm{d}y\Big)\|f\|_{h^1(\mathbb{R}^n)}.$$

By a direct proof using the maximum function definition of $h^1(\mathbb{R}^n)$, we can obtain the following result for the operator H_Φ.

Theorem 2.3 *If*

$$\int_{|y|\leq1} |\Phi(y)|\Big(1+\log_2\Big(\frac{1}{|y|}\Big)\Big)\mathrm{d}y + \int_{\mathbb{R}^n} |\Phi(y)|\mathrm{d}y < \infty,$$

then

$$\|H_\Phi(f)\|_{h^1} \preceq \int_{\mathbb{R}^n} |\Phi(y)|\mathrm{d}y\|f\|_{h^1} + \int_{|y|\leq1} |\Phi(y)|\Big(1+\log_2\Big(\frac{1}{|y|}\Big)\Big)\mathrm{d}y\|f\|_{L^1}.$$

Proof Let $\Psi \in S(\mathbb{R}^n)$ be a radial function satisfying that $\Psi(t)$ is decreasing in the interval $(0,\infty)$. By Minkowski inequality, we have

$$\begin{aligned}\|H_\Phi(f)\|_{h^1(\mathbb{R}^n)} &\leq \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n}\Big\|f\Big(\frac{\cdot}{|y|}\Big)\Big\|_{h^1(\mathbb{R}^n)}\mathrm{d}y \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \sup_{0<s\leq1}\Big|\int_{\mathbb{R}^n} \Psi_s(x-u)f\Big(\frac{u}{|y|}\Big)\mathrm{d}u\Big|\mathrm{d}y\mathrm{d}x.\end{aligned}$$

By changing variables,

$$x \to |y|x, \quad u \to |y|u,$$

we have

$$\begin{aligned}\|H_\Phi(f)\|_{h^1} &\le \int_{\mathbb{R}^n}|\Phi(y)|\int_{\mathbb{R}^n}\sup_{0<s\le 1}\Big|\int_{\mathbb{R}^n}\frac{1}{(\frac{s}{|y|})^n}\Psi\Big(\frac{x-u}{\frac{s}{|y|}}\Big)f(u)\mathrm{d}u\Big|\mathrm{d}x\mathrm{d}y\\ &= \int_{\mathbb{R}^n}|\Phi(y)|\int_{\mathbb{R}^n}\sup_{0<s\le\frac{1}{|y|}}\Big|\int_{\mathbb{R}^n}\Psi_s(x-u)f(u)\mathrm{d}u\Big|\mathrm{d}x\mathrm{d}y\\ &\le \int_{|y|\ge 1}|\Phi(y)|\mathrm{d}y\Big\|\sup_{0<s\le 1}\Psi_s*f\Big\|_{L^1}\\ &\quad+\int_{|y|\le 1}|\Phi(y)|\int_{\mathbb{R}^n}\sup_{0<s\le\frac{1}{|y|}}|\Psi_s*f(x)|\mathrm{d}x\mathrm{d}y\\ &\simeq \int_{|y|\ge 1}|\Phi(y)|\mathrm{d}y\|f\|_{h^1(\mathbb{R}^n)}+\mathrm{I}_2.\end{aligned}$$

To estimate I_2, we let

$$N=\Big[\log_2\frac{1}{|y|}\Big]+1.$$

Then

$$\begin{aligned}\sup_{0<s\le\frac{1}{|y|}}|\Psi_s*f(x)| &\le \sup_{0<s\le 1}|\Psi_s*f(x)|+\sum_{k=0}^{N}\sup_{2^k<s\le 2^{k+1}}|\Psi_s*f(x)|\\ &\le \sup_{0<s\le 1}|\Psi_s*f(x)|+\sum_{k=0}^{N}(\widetilde{\Psi}_k*|f|)(x),\end{aligned}$$

where

$$\widetilde{\Psi}_k(y)=\frac{1}{2^{kn}}\Psi\Big(\frac{y}{2^{k+1}}\Big).$$

This gives

$$\begin{aligned}&\int_{|y|\le 1}|\Phi(y)|\int_{\mathbb{R}^n}\sup_{0<s\le\frac{1}{|y|}}|\Psi_s*f(x)|\mathrm{d}x\mathrm{d}y\\ &\le \int_{|y|\le 1}|\Phi(y)|\Big(\Big\|\sup_{0<s\le 1}|\Psi_s*f|\Big\|_{L^1(\mathbb{R}^n)}+\sum_{k=0}^{N}\Big\|\widetilde{\Psi}_k*|f|\Big\|_{L^1(\mathbb{R}^n)}\Big)\mathrm{d}y.\end{aligned}$$

It is easy to check that there is a constant C independent of k such that

$$\|\widetilde{\Psi}_k*|f|\|_{L^1(\mathbb{R}^n)}\le C\|f\|_{L^1(\mathbb{R}^n)}.$$

Therefore,

$$\mathrm{I}_2\preceq\|f\|_{L^1(\mathbb{R}^n)}\int_{|y|\le 1}|\Phi(y)|\Big(1+\log_2\frac{1}{|y|}\Big)\mathrm{d}y+\|f\|_{h^1(\mathbb{R}^n)}\int_{|y|\le 1}|\Phi(y)|\mathrm{d}y.$$

The theorem is proved.

2.3 Boundedness on Herz spaces

Herz type spaces are important function spaces in harmonic analysis. It should be pointed out that Lu and Yang made tremendous contributions to these spaces. Their book (jointly with Hu) [14] is the unique research book on this topic. Below, we briefly recall the definition of the Herz type spaces.

Suppose

$$B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}, \quad E_k = B_k \backslash B_{k-1}$$

and $\chi_k = \chi_{E_k}$ for $k \in \mathbb{Z}$, where χ_{E_k} is the characteristic function of E_k.

(a) Let $\alpha \in \mathbb{R}$, $0 < p, q < \infty$. The homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L^q_{\rm loc}(\mathbb{R}^n\backslash\{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \Big\{ \sum_{k=-\infty}^{+\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^q(\mathbb{R}^n)} \Big\}^{\frac{1}{p}}.$$

(b) Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $1 < q < \infty$. The homogeneous Herz-type Hardy space $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in S'(\mathbb{R}^n) : \ \mathrm{Gf} \in \dot{\mathrm{K}}_{\mathrm{q}}^{\alpha,\mathrm{p}}(\mathbb{R}^{\mathrm{n}})\},$$

where

$$\|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \|\mathrm{Gf}\|_{\dot{\mathrm{K}}_{\mathrm{q}}^{\alpha,\mathrm{p}}(\mathbb{R}^{\mathrm{n}})}$$

and Gf is the grand maximal function of f. Similar to the Hardy spaces, the space $H\dot{K}_q^{\alpha,p}$ has atomic decompositions. Suppose $1 < q < \infty$, $n\big(1 - \frac{1}{q}\big) \leq \alpha < \infty$, and $s \geq \big[\alpha + n\big(\frac{1}{q} - 1\big)\big]$. A function $a(x)$ on $\mathbb{R}^n$ is said to be a central (α, q) atom if

(i) $\operatorname{supp} a \subset B(0, \rho) = \{x \in \mathbb{R}^n : |x| < \rho\}$;

(ii) $\|a\|_{L^q(\mathbb{R}^n)} \preceq |B(0,\rho)|^{-\frac{\alpha}{n}}$;

(iii) $\int_{\mathbb{R}^n} a(x)x^\beta \mathrm{d}x = 0$ for a multi-index β with $|\beta| \leq s$.

It is known that, for $0 < p < \infty$, $1 < q < \infty$ and $n\big(1 - \frac{1}{q}\big) \leq \alpha < \infty$, $f \in H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ if and only if there exist a sequence of numbers $\{\lambda_k\}$ and a sequence of central (α, q)-atoms $\{a_k\}$ with the support in B_k and $\sum_{k\in\mathbb{Z}} |\lambda_k|^p < \infty$, such that

$$f = \sum_k \lambda_k a_k,$$

in S'. Moreover,

$$\|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \simeq \inf \Big\{ \Big(\sum_k |\lambda_k|^p \Big)^{\frac{1}{p}} : \text{all possible representations } f = \sum_k \lambda_k a_k \Big\}.$$

Theorem 2.4 *If* $p, q \geq 1$, *then*

$$\|H_\Phi f\|_{K_q^{\alpha,p}} \preceq \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |y|^{\frac{n}{q}} |y|^\alpha \mathrm{d}y \ \|f\|_{\dot{K}_q^{\alpha,p}}.$$

Proof By the definition and Minkowski inequality,

$$\begin{aligned}\|H_\Phi f\|^p_{\dot K_q^{\alpha,p}} &= \sum_{k\in\mathbb{Z}} 2^{\alpha kp}\Big(\int_{E_k} |H_\Phi f(x)|^q \mathrm{d}x\Big)^{\frac{p}{q}} \\ &= \sum_{k\in\mathbb{Z}} 2^{\alpha kp}\Big(\int_{E_k} \Big|\sum_{j\in\mathbb{Z}}\int_{E_j} \frac{\Phi(y)}{|y|^n} f\Big(\frac{x}{|y|}\Big)\mathrm{d}y\Big|^q \mathrm{d}x\Big)^{\frac{p}{q}} \\ &\le \sum_{k\in\mathbb{Z}} 2^{\alpha kp}\Big(\sum_{j\in\mathbb{Z}}\Big(\int_{E_k} \Big|\int_{E_j} \frac{\Phi(y)}{|y|^n} f\Big(\frac{x}{|y|}\Big)\mathrm{d}y\Big|^q \mathrm{d}x\Big)^{\frac{1}{q}}\Big)^p \\ &\le \sum_{k\in\mathbb{Z}} 2^{\alpha kp}\Big(\sum_{j\in\mathbb{Z}}\Big(\int_{E_j} \frac{|\Phi(y)|}{|y|^n} \Big\|f\Big(\frac{\cdot}{|y|}\Big)\Big\|_{L^q(E_k)}\mathrm{d}y\Big)\Big)^p.\end{aligned}$$

Note that for $y \in E_j$,

$$\begin{aligned}\Big\|f\Big(\frac{\cdot}{|y|}\Big)\Big\|_{L^q(E_k)} &= \Big(\int_{E_k}\Big|f\Big(\frac{x}{|y|}\Big)\Big|^q \mathrm{d}x\Big)^{\frac{1}{q}} \\ &\simeq |y|^{\frac{n}{q}}\Big(\int_{E_{k-j}} |f(x)|^q\mathrm{d}x\Big)^{\frac{1}{q}} = |y|^{\frac{n}{q}}\|f\chi_{E_{k-j}}\|_{L^q}.\end{aligned}$$

Thus by Minkowski inequality,

$$\begin{aligned}\|H_\Phi f\|_{K_q^{\alpha,p}} &\preceq \sum_{j\in\mathbb{Z}}\Big(\int_{E_j}\frac{|\Phi(y)|}{|y|^n}|y|^{\frac{n}{q}}|y|^\alpha\Big\{\sum_{k\in\mathbb{Z}} 2^{\alpha(k-j)p}\|f\chi_{E_{k+j}}\|^p_{L^q}\Big\}^{\frac{1}{p}}\mathrm{d}y\Big) \\ &\preceq \Big\{\sum_{k\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}}\Big(2^{\alpha k}\int_{E_j}\frac{|\Phi(y)|}{|y|^n}|y|^{\frac{n}{q}}\|f\chi_{E_{k-j}}\|_{L^q}\mathrm{d}y\Big)\Big)^p\Big\}^{\frac{1}{p}} \\ &= \int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|y|^{\frac{n}{q}}|y|^\alpha\mathrm{d}y\|f\|_{\dot K_q^{\alpha,p}}.\end{aligned}$$

The theorem is proved.

Note that

$$\|f\|_{\dot K_p^{\alpha,p}} \simeq \|f\|_{L^p(|x|^\alpha \mathrm{d}x)}.$$

We obtain the boundedness of H_Φ in the power weighted space.

Corollary 2.2 *If* $p \ge 1$ *then*

$$\|H_\Phi f\|_{L^p(|x|^\alpha\mathrm{d}x)} \preceq \int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|y|^{\frac{n}{p}}|y|^{\frac{\alpha}{p}}\mathrm{d}y\ \|f\|_{L^p(|x|^\alpha\mathrm{d}x)}.$$

Theorem 2.5 *Let* $0 < p \le 1 < q < \infty$, $n\big(1-\frac{1}{q}\big) \le \alpha < \infty$, *and* $B(y)$ *be the same as in Theorem* 2.1*. Then we have*

$$\|H_{\Phi},_A(f)\|_{H\dot K_q^{\alpha,1}(\mathbf{R}^n)} \preceq \int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}B(y)^{-\alpha}\ |\det A^{-1}(y)|^{\frac{1}{q}}\mathrm{d}y\ \|f\|_{H\dot K_q^{\alpha,1}(\mathbf{R}^n)},$$

and for $0 < p < 1$,

$$\begin{aligned}&\|H_{\Phi},_A(f)\|_{H\dot K_q^{\alpha,p}(\mathbf{R}^n)} \\ &\preceq \|f\|_{H\dot K_q^{\alpha,p}(\mathbf{R}^n)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}B(y)^{-\alpha}\ |\det A^{-1}(y)|^{\frac{1}{q}}(1+\log B(y))^\sigma\mathrm{d}y\end{aligned}$$

with $\sigma > \frac{1-p}{p}$.

Proof We only prove the case $0 < p < 1$, since the proof for $p = 1$ is the same as that for Theorem 2.2. By the central atomic decomposition, for $f \in H\dot{K}_q^{\alpha,p}$, we write

$$f = \sum_k \lambda_k a_k,$$

where

$$\sum_{k\in\mathbb{Z}} |\lambda_k|^p \simeq \|f\|^p_{H\dot{K}_q^{\alpha,p}}.$$

Also, following the discussion of [14, Chapter 6], we may assume that each central (α, q)-atom a_k is a regular function supported in $B(0, \rho)$. Now

$$H_{\Phi,A}(f) = \sum_{k\in\mathbb{Z}} \lambda_k H_{\Phi,A}(a_k).$$

To prove the theorem, it suffices to show that

$$H_{\Phi,A}(a_k) = \sum_{j\in\mathbb{Z}} c_{k,j} a_{k,j},$$

where each $a_{k,j}$ again is a central (α, q)-atom and

$$\sum_{j\in\mathbb{Z}} |c_{k,j}|^p \preceq 1$$

uniformly on $k \in \mathbb{Z}$.

We write

$$b_{k,j}(x) = \int_{2^j\le B(y)\le 2^{j+1}} \frac{\Phi(y)}{|y|^n} a_k(A(y)x)\mathrm{d}y, \quad j \in \mathbb{Z}.$$

So

$$H_{\Phi,A}(a_k)(x) = \sum_{j\in\mathbb{Z}} b_{k,j}(x).$$

It is easy to check that each $b_{k,j}$ satisfies the same cancellation condition as a_k. Also the size of $b_{k,j}$ is

$$\begin{aligned}\|b_{k,j}\|_{L^q} &\le \int_{2^j\le B(y)\le 2^{j+1}} \frac{|\Phi(y)|}{|y|^n} \|a_k(A(y)\,\cdot\,)\|_{L^q}\mathrm{d}y \\ &\preceq \rho^{-\alpha} \int_{2^j\le B(y)\le 2^{j+1}} \frac{|\Phi(y)|}{|y|^n} \,|\det A^{-1}(y)|^{\frac{1}{q}}\mathrm{d}y.\end{aligned}$$

Now we check that $\mathrm{supp}(a_{k,j}) \subset B(0, 2^{-j}\rho)$. In fact, if $|x| > 2^{-j}\rho$, then

$$|A(y)x| \ge B(y)|x| > \rho,$$

which leads to $a_k(A(y)x) = 0$.

Now we write

$$H_{\Phi,A}(a_k)(x)=\sum_{j\in\mathbb{Z}} c_{k,j}a_{k,j}$$

with

$$c_{k,j}=2^{-j\alpha}\int_{2^j\le B(y)\le 2^{j+1}}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{\frac{1}{q}}\mathrm{d}y$$

and

$$a_{k,j}=c_j^{-1}b_{k,j}.$$

It is easy to check that $a_{k,j}$ is a central (α,q)-atom and

$$\begin{aligned}\sum_{j\in\mathbb{Z}}|c_{k,j}|^p &\preceq \sum_{j\in\mathbb{Z}}\Big(\int_{2^j\le B(y)\le 2^{j+1}}\frac{|\Phi(y)|}{|y|^n}B(y)^{-\alpha}\ |\det A^{-1}(y)|^{\frac{1}{q}}\mathrm{d}y\Big)^p\\ &\preceq \int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}B(y)^{-\alpha}\ |\det A^{-1}(y)|^{\frac{1}{q}}(1+\log B(y))^{\sigma}\mathrm{d}y.\end{aligned}$$

The theorem is proved.

3 $\widetilde{H}_{\Phi,\Omega}$ and Hardy Operator

3.1 L^p boundedness

In this section, we study the operator

$$\widetilde{H}_{\Phi,\Omega}f(x)=\int_{\mathbb{R}^n}\frac{\Phi(x|y|^{-1})}{|y|^n}\Omega(y')f(y)\mathrm{d}y$$

and

$$\widetilde{H}_{\Phi}f(x)=\int_{\mathbb{R}^n}\frac{\Phi(x|y|^{-1})}{|y|^n}f(y)\mathrm{d}y.$$

Theorem 3.1 *Assume that Φ is a radial function. Let S^{n-1} be the unit sphere in $\mathbb{R}^n$. We have, for $1\le p\le\infty$,*

$$\|\widetilde{H}_{\Phi,\Omega}f\|_{L^p(\mathbb{R}^n)}\le\|\Omega\|_{L^{p'}(S^{n-1})}|S^{n-1}|^{\frac{1}{p}}\int_0^\infty\frac{|\Phi(t)|}{t}t^{\frac{n}{p}}\mathrm{d}t\ \|f\|_{L^p(\mathbb{R}^n)}.$$

Particularly, we have

$$\|\widetilde{H}_{\Phi}f\|_{L^p(\mathbb{R}^n)}\le|S^{n-1}|\int_0^\infty\frac{|\Phi(t)|}{t}t^{\frac{n}{p}}\mathrm{d}t\ \|f\|_{L^p(\mathbb{R}^n)},$$

where $|S^{n-1}|$ denotes the volume of the unit sphere S^{n-1}.

Proof Clearly, we only need to show the result for $\widetilde{H}_{\Phi,\Omega}f$, since the operator $\widetilde{H}_{\Phi}$ is a special case of $\widetilde{H}_{\Phi,\Omega}$ by letting $\Omega=1$. Using polar coordinates and changing variables, we obtain

$$\begin{aligned}\widetilde{H}_{\Phi,\Omega}f(x)&=\int_{\mathbb{R}^n}\frac{\Phi(|x||y|^{-1})}{|y|^n}\Omega(y')f(y)\mathrm{d}y\\ &=\int_0^\infty\frac{\Phi(t)}{t}\int_{S^{n-1}}f(|x|t^{-1}y')\Omega(y')\mathrm{d}\sigma(y')\mathrm{d}t.\end{aligned}$$

Thus, by Minkowski inequality and scaling,

$$\begin{aligned}\|\widetilde{H}_{\Phi,\Omega}f\|_{L^p(\mathbb{R}^n)} &\le \int_0^\infty \frac{|\Phi(t)|}{t}\Big\|\int_{S^{n-1}} f(|\cdot|t^{-1}y')\Omega(y')\mathrm{d}\sigma(y')\Big\|_{L^p(\mathbb{R}^n)}\mathrm{d}t\\ &= \int_0^\infty \frac{|\Phi(t)|}{t}t^{\frac{n}{p}}\Big\|\int_{S^{n-1}} f(|\cdot|y')\Omega(y')\mathrm{d}\sigma(y')\Big\|_{L^p(\mathbb{R}^n)}\mathrm{d}t.\end{aligned}$$

Here, by Hölder inequality,

$$\Big|\int_{S^{n-1}} f(|\cdot|y')\Omega(y')\mathrm{d}\sigma(y')\Big| \le \Big(\int_{S^{n-1}} |f(|\cdot|y')|^p\mathrm{d}\sigma(y')\Big)^{\frac{1}{p}}\|\Omega\|_{L^{p'}(S^{n-1})}.$$

This shows

$$\begin{aligned}\|\widetilde{H}_{\Phi,\Omega}f\|_{L^p(\mathbb{R}^n)} &\le \|\Omega\|_{L^{p'}(S^{n-1})}|S^{n-1}|^{\frac{1}{p}}\int_0^\infty \frac{|\Phi(t)|}{t}t^{\frac{n}{p}}\\ &\quad\cdot\Big(\int_0^\infty r^{n-1}\int_{S^{n-1}} |f(ry')|^p\mathrm{d}\sigma(y')\mathrm{d}r\Big)^{\frac{1}{p}}\mathrm{d}t\\ &= \|\Omega\|_{L^{p'}(S^{n-1})}|S^{n-1}|^{\frac{1}{p}}\|f\|_{L^p(\mathbb{R}^n)}\int_0^\infty \frac{|\Phi(t)|}{t}t^{\frac{n}{p}}\mathrm{d}t.\end{aligned}$$

The theorem is proved.

Corollary 3.1 *The Hardy operator*

$$Hf(x) = \frac{1}{|x|^n}\int_{|y|\le|x|} f(y)\mathrm{d}y$$

satisfies, for all $1 < p \le \infty$,

$$\|Hf\|_{L^p(\mathbb{R}^n)} \le \frac{|S^{n-1}|}{n}\frac{p}{p-1}\|f\|_{L^p(\mathbb{R}^n)}.$$

Proof In the operator $\widetilde{H}_\Phi f(x)$, choose

$$\Phi(t) = \chi_{(1,\infty)}(t)t^{-n}.$$

Then we obtain the Hardy operator

$$\widetilde{H}_\Phi f(x) = Hf(x).$$

Thus, by Theorem 3.1,

$$\|Hf\|_{L^p(\mathbb{R}^n)} \le |S^{n-1}|\int_1^\infty t^{\frac{n}{p}-n-1}\mathrm{d}t\ \|f\|_{L^p(\mathbb{R}^n)} = \frac{|S^{n-1}|}{n}\frac{p}{p-1}\ \|f\|_{L^p(\mathbb{R}^n)}.$$

Corollary 3.2 *For the adjoint Hardy operator*

$$\int_{|y|>|x|}\frac{f(y)}{|y|^n}\mathrm{d}y = H^*f(x),$$

we have that, for all $1 \le p < \infty$,

$$\|H^*f\|_{L^p(\mathbb{R}^n)} \le p\frac{|S_{n-1}|}{n}\|f\|_{L^p(\mathbb{R}^n)}.$$

Proof In the operator $\widetilde{H}_\Phi f(x)$, choose

$$\Phi(t) = \chi_{(0,1)}(t).$$

So, by Theorem 3.1,

$$\|H^* f\|_{L^p(\mathbb{R}^n)} \le |S^{n-1}| \int_0^1 t^{\frac{n}{p}-1}\mathrm{d}t \ \|f\|_{L^p(\mathbb{R}^n)} = p\|f\|_{L^p(\mathbb{R}^n)} \frac{|S_{n-1}|}{n}.$$

The corollary is proved.

It is known that the bounds in Corollary 3.1 and Corollary 3.2 are sharp (see [4]).

3.2 $H\dot{K}_q^{\alpha,p}$ and H^1 Boundedness

Theorem 3.2 *Suppose* $1 < q < \infty$, $n\big(1-\frac{1}{q}\big) \le \alpha < \infty$, *and* $\big[\alpha + n\big(\frac{1}{q}-1\big)\big] < 1$. *Let* Φ *be a radial function. Then*

$$\|\widetilde{H}_\Phi(f)\|_{H\dot{K}_q^{\alpha,1}(\mathbb{R}^n)} \preceq \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n}|y|^{\alpha+\frac{n}{q}}\mathrm{d}y\|f\|_{H\dot{K}_q^{\alpha,1}(\mathbb{R}^n)},$$

and for all $0 < p < 1$,

$$\|\widetilde{H}_\Phi(f)\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \preceq \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n}|y|^{\alpha+\frac{n}{q}}(1+\log|y|)^\sigma \mathrm{d}y \ \|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)},$$

where $\sigma > \frac{1-p}{p}$.

Proof Checking the proof of Theorem 2.5, it suffices to show that for any central (α, q)-atom a,

$$\widetilde{H}_\Phi a(x) = \sum_{k\in\mathbb{Z}} c_k A_k(x),$$

where each $A_k(x)$ is again a central (α, q)-atom and

$$\sum_{k\in\mathbb{Z}} |c_k|^p \preceq 1$$

uniformly for atoms a.

Write

$$\widetilde{H}_\Phi a(x) = \sum_{k\in\mathbb{Z}} b_k(x),$$

where

$$b_k(x) = \int_{2^k}^{2^{k+1}} \frac{\Phi(t)}{t} \int_{S^{n-1}} a(|x|t^{-1}y')\mathrm{d}\sigma(y')\mathrm{d}t.$$

By the Minkowski inequality,

$$\|b_k\|_{L^q(\mathbb{R}^n)} \le \int_{2^k}^{2^{k+1}} \frac{|\Phi(t)|}{t} \int_{S^{n-1}} \|a(|\cdot|t^{-1}y')\|_{L^q}\mathrm{d}\sigma(y')\mathrm{d}t.$$

Here, an easy computation shows

$$\int_{S^{n-1}} \|a(|\cdot|t^{-1}y')\|_{L^q}\mathrm{d}\sigma(y') \leq t^{\frac{n}{q}}|B(0,\rho)|^{-\frac{\alpha}{n}}.$$

Therefore,

$$\begin{aligned}\|b_k\|_{L^q(\mathbb{R}^n)} &\leq |B(0,\rho)|^{-\frac{\alpha}{n}} \int_{2^k}^{2^{k+1}} \frac{|\Phi(t)|}{t} t^{\frac{n}{q}}\mathrm{d}t \\ &\simeq |B(0,\rho)|^{-\frac{\alpha}{n}} \int_{2^k\leq|y|<2^{k+1}} \frac{|\Phi(y)|}{|y|^n}|y|^{\frac{n}{q}}\mathrm{d}y.\end{aligned}$$

By polar coordinates and the Fubini theorem,

$$\begin{aligned}\int_{\mathbb{R}^n} b_k(x)\mathrm{d}x &= |S^{n-1}|\int_{2^k}^{2^{k+1}} \frac{\Phi(t)}{t}\int_0^\infty r^{n-1}\int_{S^{n-1}} a(rt^{-1}y')\mathrm{d}\sigma(y')\mathrm{d}r\mathrm{d}t \\ &= |S^{n-1}|\int_{2^k}^{2^{k+1}} \Phi(t)t^{n-2}\int_0^\infty s^{n-1}\int_{S^{n-1}} a(sy')\mathrm{d}\sigma(y')\mathrm{d}s\mathrm{d}t \\ &= |S^{n-1}|\int_{2^k}^{2^{k+1}} \Phi(t)t^{n-2}\mathrm{d}t\int_{\mathbb{R}^n} a(y)\mathrm{d}y = 0.\end{aligned}$$

Also, the support of b_k is contained in $B(0,2^{k+1}\rho)$. Thus we write

$$b_k(x) = c_kA_k(x)$$

with

$$c_k = 2^{(k+1)\alpha}\int_{2^k\leq|y|<2^{k+1}} \frac{|\Phi(y)|}{|y|^n}|y|^{\frac{n}{q}}\mathrm{d}y \simeq \int_{2^k\leq|y|<2^{k+1}} \frac{|\Phi(y)|}{|y|^n}|y|^{\alpha+\frac{n}{q}}\mathrm{d}y$$

and

$$A_k(x) = c_k^{-1}b_k(x).$$

We now have

$$\|A_k\|_{L^q(\mathbb{R}^n)} \preceq c_k^{-1}\|b_k\|_{L^q(\mathbb{R}^n)} \preceq |B(0,2^{k+1}\rho)|^{-\frac{\alpha}{n}}.$$

Combining the support and cancellation conditions, we know that A_k is really a central (α,q)-atom. Now following the same proof for Theorem 2.5, it is easy to check that

$$\sum_{k\in\mathbb{Z}}|c_k|^p \preceq \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n}|y|^{\alpha+\frac{n}{q}}(1+\log|y|)^\sigma\mathrm{d}y,$$

where $\sigma = 0$ if $p = 1$ and $\sigma > \frac{1-p}{p}$ if $0 < p < 1$. The theorem is proved.

Theorem 3.3 *Let* Φ *be a radial function. We have*

$$\|\widetilde{H}_\Phi(f)\|_{H^1(\mathbb{R}^n)} \leq \|f\|_{H^1(\mathbb{R}^n)}\int_{\mathbb{R}^n}|\Phi(y)|\mathrm{d}y.$$

Proof By the atomic decomposition, it suffices to show that

$$\|\widetilde{H}_\Phi(a)\|_{H^1(\mathbb{R}^n)} \preceq 1$$

uniformly on all $(1,q)$-atoms a, where $q>1$. For a $(1,q)$-atoms a with support in the ball $B(x_0,\rho)$, using polar coordinates and changing variables, we write

$$\widetilde{H}_\Phi a(x)\simeq\int_0^\infty\frac{\Phi(t)}{t}\int_{S^{n-1}}a(|x|t^{-1}y')\mathrm{d}\sigma(y')\mathrm{d}t.$$

By the Minkowski inequality,

$$\|\widetilde{H}_\Phi(a)\|_{H^1(\mathbb{R}^n)}\leq\int_0^\infty\frac{|\Phi(t)|}{t}\Big\|\int_{S^{n-1}}a(|\cdot|t^{-1}y')\mathrm{d}\sigma(y')\Big\|_{H^1(\mathbb{R}^n)}\mathrm{d}t.$$

So, by a shift we may assume that a is a central $(1,q)$-atom with support in $B(0,\rho)$. Denote

$$A(x)=\int_{S^{n-1}}a(|x|y')\mathrm{d}\sigma(y').$$

We now claim that $\frac{1}{t^n}A(\frac{x}{t})$ is again a central $(1,q)$-atom.

First, we check the support condition of $\frac{1}{t^n}A(\frac{x}{t})$. We see that

$$\Big|\frac{|x|}{t}y'\Big|\geq\rho$$

if and only if

$$|x|\geq t\rho.$$

This implies that $\frac{1}{t^n}A(\frac{x}{t})$ is supported in $B(0,t\rho)$.

Second, an easy computation shows

$$\begin{aligned}\Big\|\frac{1}{t^n}A\Big(\frac{\cdot}{t}\Big)\Big\|_{L^q}&\leq\int_{S^{n-1}}\|a(|\cdot|t^{-1}y')\|_{L^q}\mathrm{d}\sigma(y')\\&=\frac{1}{t^n}\int_{S^{n-1}}\Big(\int_0^\infty r^{n-1}|a(rt^{-1}y')|^q\mathrm{d}r\Big)^{\frac{1}{q}}\mathrm{d}\sigma(y')|S^{n-1}|^{\frac{1}{q}}\\&\leq\frac{1}{t^n}|S^{n-1}|\Big(\int_{S^{n-1}}\int_0^\infty r^{n-1}|a(rt^{-1}y')|^q\mathrm{d}rd\sigma(y')\Big)^{\frac{1}{q}}\\&=\frac{1}{t^n}|S^{n-1}|\;t^{\frac{n}{q}}\|a\|_{L^q}\preceq(t\rho)^{\frac{n}{q}-n}.\end{aligned}$$

Finally, by polar coordinates and the Fubini theorem,

$$\begin{aligned}\int_{\mathbb{R}^n}A\Big(\frac{x}{t}\Big)\mathrm{d}x&=|S^{n-1}|\int_0^\infty r^{n-1}\int_{S^{n-1}}a(rt^{-1}y')\mathrm{d}\sigma(y')\mathrm{d}r\\&=|S^{n-1}|t^n\int_0^\infty s^{n-1}\int_{S^{n-1}}a(sy')\mathrm{d}\sigma(y')\mathrm{d}s\\&=|S^{n-1}|t^n\int_{\mathbb{R}^n}a(y)\mathrm{d}y=0.\end{aligned}$$

So, we show that $\frac{1}{t^n}A(\frac{x}{t})$ is a central $(1,q)$-atom. As a consequence, we obtain that

$$\|\widetilde{H}_\Phi(a)\|_{H^1(\mathbb{R}^n)}\preceq\int_0^\infty\frac{|\Phi(t)|}{t}t^n\mathrm{d}t\simeq\int_{\mathbb{R}^n}|\Phi(y)|\mathrm{d}y.$$

The theorem is proved.

Corollary 3.3 *The adjoint Hardy operator* H^* *is bounded in* $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ *for all* $0<p\le 1$ *if* $1<q<\infty$, $n\big(1-\frac{1}{q}\big)\le\alpha<\infty$, *and* $\big[\alpha+n\big(\frac{1}{q}-1\big)\big]<1$.

Corollary 3.4 *The adjoint Hardy operator* H^* *is bounded in the Hardy space* $H^1(\mathbb{R}^n)$.

Corollary 3.5 *For the modified Hardy operator*

$$H^\delta f(x)=\frac{1}{|x|^\delta}\int_{|y|\le|x|}\frac{f(y)}{|y|^{n-\delta}}\mathrm{d}y,$$

H^δ *is bounded in* $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ *for all* $0<p\le 1$ *if* $1<q<\infty$, $n\big(1-\frac{1}{q}\big)\le\alpha<\infty$, *and* $\big[\alpha+n\big(\frac{1}{q}-1\big)\big]<1$, *provided that*

$$\delta>\alpha+\frac{n}{q}.$$

Proof In the operator

$$\widetilde{H}_\Phi f(x)=\int_{\mathbb{R}^n}\frac{\Phi(x|y|^{-1})}{|y|^n}f(y)\mathrm{d}y,$$

we choose

$$\Phi(y)=\frac{\chi_{\{|y|\ge 1\}}(y)}{|y|^\delta}.$$

Then

$$\widetilde{H}_\Phi f(x)=\frac{1}{|x|^\delta}\int_{|y|\le|x|}\frac{f(y)}{|y|^{n-\delta}}\mathrm{d}y$$

and

$$\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|y|^{-\delta+\alpha+\frac{n}{q}}(1+\log|y|)^\sigma\mathrm{d}y<\infty.$$

So the corollary is proved.

We remark that in Corollary 3.4, we need $\delta>n$, while $H^nf(x)$ is the Hardy operator $Hf(x)$.

4 Discrete Hausdorff Operator

Recall that the discrete Hausdorff operator is defined by

$$H_{\Phi,\mathrm{dis}}f(x)=\sum_{k\in\mathbb{Z}}\Phi(\beta^k)f(A(\beta^k)x),$$

where β is a positive number and $\beta\ne 1$.

Theorem 4.1 *Assume*

$$\sum_{k\in\mathbb{Z}}(|\Phi(\beta^k)|B(\beta^k)^{-\frac{n}{p}+\frac{n}{q}}|\det A^{-1}(\beta^k)|^{\frac{1}{q}})^p<\infty,$$

where $B(y)$ *is the same as in Theorem* 2.1. *Then*

$$\|H_{\Phi,\mathrm{dis}}f\|_{H^p(\mathbb{R}^n)} \preceq \|f\|_{H^p(\mathbb{R}^n)}$$

for all $0 < p \le 1$.

Proof By the atomic decomposition, for $f \in H^p(\mathbb{R}^n)$, we write

$$f = \sum_j \lambda_j a_j.$$

Thus

$$\|H_{\Phi,\mathrm{dis}}f\|^p_{H^p(\mathbb{R}^n)} \preceq \sum_j |\lambda_j|^p \|H_{\Phi,\mathrm{dis}}(a_j)\|^p_{H^p(\mathbb{R}^n)}.$$

By a shift, we may assume that each atom a_j has support in $B(0,\rho)$. The rest of the proof is similar to the proof of Theorem 2.5. We leave the details to the reader.

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