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Asymptotic Results for Tail Probabilities of Sums of Dependent and Heavy-Tailed Random Variables*

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Abstract Let $X_1, X_2, \cdots$ be a sequence of dependent and heavy-tailed random variables with distributions $F_1, F_2, \cdots$ on $(-\infty, \infty)$, and let τ be a nonnegative integer-valued random variable independent of the sequence $\{X_k, k \geq 1\}$. In this framework, the asymptotic behavior of the tail probabilities of the quantities $S_n = \sum\limits_{k=1}^{n} X_k$ and $S_{(n)} = \max\limits_{1\leq k\leq n} S_k$ for $n > 1$, and their randomized versions S_τ and $S_{(\tau)}$ are studied. Some applications to the risk theory are presented.

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1 Introduction

Like many research works in the fields of applied probability and risk theory, this paper focuses on the study of heavy-tailed distributions. Therefore, at the beginning of this section, some heavy-tailed distributions will be introduced. A random variable X (or its distribution F) is said to be heavy-tailed to the right if $E\exp(\alpha X) = \infty$ for all $\alpha > 0$. Denote by $\mathcal{K}$ the class of heavy-tailed distributions. One of the most important classes of heavy-tailed distributions is the subexponential class. By definition, a distribution F on $[0, \infty)$ is said to be subexponential ($F \in \mathcal{S}$) if the relation $\overline{F^{*2}}(x) \sim 2\overline{F}(x)$ $(x \to \infty)$ holds, where the symbol $\sim$ means that the ratio of the two sides tends to 1. More generally, a distribution function F on $(-\infty, \infty)$ belongs to the subexponential class $\mathcal{S}$ if $F^+(x) = F(x)1\ (x \geq 0)$ holds, where $1(\cdot)$ is the indicator function. The classes closely related to the class $\mathcal{S}$ include the class $\mathcal{S}^*$, the class $\mathcal{D}$ of distributions with dominatedly varying tails, and the class $\mathcal{L}$ of distributions with long tails. By definition, a distribution function F on $\mathbb{R}$ with a finite mean belongs to the class $\mathcal{S}^*$ if and only if $\overline{F}(x) > 0$ for all x and $\int_0^x \overline{F}(x-y)\overline{F}(y)\mathrm{d}y \sim 2m_{F^+}\overline{F}(x)$, as $x \to \infty$, where $m_{F^+} = \int_0^\infty \overline{F}(x)\mathrm{d}x$ is the mean of F^+. It is known that if $F \in \mathcal{S}^*$, then both F and F_I are subexponential, where F_I

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is defined by $\overline{F_I}(x) = \min(1, \int_x^\infty \overline{F}(t)\mathrm{d}t)$ (see [15]). A distribution function F with support on $(-\infty, \infty)$ belongs to the class $\mathcal{D}$ if $\limsup\limits_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty$ holds for some (or, equivalently, for all) $0 < y < 1$. Obviously, if $F \in \mathcal{D}$, then for any $y > 0$, $\overline{F}(xy)$ and $\overline{F}(x)$ are of the same order as $x \to \infty$ in the sense that

$$0 < \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \le \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty.$$

A distribution function F is said to belong to the class $\mathcal{L}$ if $\lim\limits_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1$ holds for some (or, equivalently, for all) $y \ne 0$. One can easily check that for a distribution $F \in \mathcal{L}$, there exists a positive function $h(x) \to \infty$ such that $\overline{F}(x + h(x)) \sim \overline{F}(x)$. The class $\mathcal{S}^*$ and the intersection $\mathcal{D} \cap \mathcal{L}$ are two well-known subclasses of subexponential distribution functions. For details of these classes of heavy-tailed distributions and their applications, the reader is referred to Asmussen [2], Bingham et al. [3], Embrechts et al. [9], and Embrechts et al. [10]. Furthermore, a distribution F is said to be strongly subexponential, denoted by $F \in \mathcal{S}_*$, if $\overline{F_h^{*2}}(x) \sim 2\overline{F_h}(x)$, uniformly in $h \in [1, \infty)$, where the distribution F_h is defined as

$$F_h(x) = \min\Big(1, \int_x^{x+h} F(t)\mathrm{d}t\Big), \quad x > 0.$$

See Korshunov [17] for sufficient conditions for a distribution belonging to the class $\mathcal{S}_*$. Kaas and Tang [14] proved that $\mathcal{S}_*$ is a subclass of $\mathcal{S}$ while Denisov et al. [5] showed that $\mathcal{S}^*$ is a subclass of $\mathcal{S}_*$. It is well-known that these distribution classes fulfill the inclusions $\mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{K}$. Moreover, if the underlying distribution function F has a finite mean, then $F \in \mathcal{D} \cap \mathcal{L}$ implies $F \in \mathcal{S}^* \subset \mathcal{S}_* \subset \mathcal{K}$.

Throughout this paper, let $X_1, X_2, \cdots$ be a sequence of random variables with distributions $F_1, F_2, \cdots$ supported on $\mathbb{R} := (-\infty, \infty)$ satisfying $\overline{F_k}(x) = 1 - F_k(x) > 0$ for all x. For $n \ge 1$, we write

$$S_n = \sum_{k=1}^n X_k, \quad S_{(n)} = \max_{1\le k\le n} S_k.$$

Let τ be a counting random variable independent of $\{X_k, k \ge 1\}$. Then, the randomized versions of S_n and $S_{(n)}$ are given by S_τ and $S_{(\tau)}$. Tail probabilities of the quantities S_n, $S_{(n)}$, S_τ, $S_{(\tau)}$ with heavy-tailed random variables are of great interest in finance, insurance and many other disciplines. Since accurate distributions for these quantities are not available in most cases, the study of asymptotic relationships for their tail probabilities becomes important. Many results have been derived under different degrees of generality in the literature. In particular, most of the results are for independent $X_1, \cdots, X_n$ with distributions belonging to the class of subexponential distributions. A recent result on tail asymptotic results for the sum of two independent random variables can be found in Foss and Korshunov [11]. They proved that, for two distributions F_1 and F_2 on $[0, \infty)$, if one of F_1 and F_2 is heavy-tailed, then

$$\liminf_{x\to\infty} \frac{\overline{F_1 * F_2}(x)}{\overline{F_1}(x) + \overline{F_2}(x)} = 1, \tag{1.1}$$

and that, for any heavy-tailed distribution F,

$$\liminf_{x\to\infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = 2. \tag{1.2}$$

Denisov et al. [6] extended (1.2) to

$$\liminf_{x\to\infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} = E\tau \tag{1.3}$$

with τ being a light-tailed random variable. Furthermore, for any heavy-tailed distribution F on $\mathbb{R}^+$ with a finite mean, Denisov et al. [7] showed that if $P(c\tau > x) = o(\overline{F}(x))$ for some $c > EX$ as $x \to \infty$, then (1.3) holds. Also, if F is subexponential and τ is light-tailed and independent of the summands, then

$$P(S_\tau > x) \sim \overline{F}(x)E\tau, \quad x \to \infty. \tag{1.4}$$

Note that all the above mentioned results were established for independent nonnegative random variables. The recent work of Foss and Richards [12] treats the asymptotics of nonnegative heavy-tailed random variables under a conditional independence assumption. In addition, the distributions of random variables are asymptotically equivalent to multiples of a given reference subexponential distribution. When the random variables are possibly negative and dependent according to certain structures, the validity of these results remains to be studied.

We now consider three examples in which some of the above relations do not hold. The first comes from [26], while the last two are extracted from [1].

Example 1.1 Assume that X is a discrete random variable with masses $p_n = P(X = 2^{n+1} - 1) = 2^{-n-1}$, $n \geq 0$. Denote its distribution by ρ. Then, for any $0 < q < 1$, define $F = q\rho + (1-q)\sigma$, where σ is a non-degenerate distribution on a subset of $(-\infty, 0]$. Without loss of generality, we assume that σ has support on $[-3, 0)$, and that $\sigma(-2) - \sigma(-3) = \delta > 0$. Then, F is heavy-tailed but does not belong to the class $\mathcal{L}$ of distributions with long tails ($F \notin \mathcal{L}$). Also, it satisfies

$$\liminf_{x\to\infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} < 2.$$

Example 1.2 Let X_1 and X_2 have a common distribution function F belonging to the subexponential class $\mathcal{S}$. Then, there exists a copula for X_1 and X_2 such that

$$\lim_{x\to\infty} \frac{P(X_1 + X_2 > x)}{P(X_1 > x)} = \infty.$$

Example 1.3 Assume that random variables X_1 and X_2 are comonotonic with a common distribution $F = 1 - \overline{F}$, where $\overline{F} \in \mathcal{R}_{-\alpha}$ is regularly varying at infinity with an index $\alpha > 0$. Then, we have

$$\lim_{x\to\infty} \frac{P(X_1 + X_2 > x)}{P(X_1 > x)} = 2^\alpha.$$

These examples indicate that relations (1.1)–(1.4) may not hold for heavy-tailed distributions supported on $[a, \infty)$ with $a < 0$ or for dependent and heavy-tailed distributions. More examples can be found in Albrecher et al. [1].

The purpose of this paper is to find sufficient conditions under which relations (1.1)–(1.4) hold for possibly negative, non-identically distributed, and dependent heavy-tailed random variables. This paper is organized as follows. Section 2 presents several classes of heavy-tailed distributions and the dependence assumptions used in later sections. Section 3 is devoted to the tail behaviors of S_n and $S_{(n)}$. Section 4 investigates the tail behaviors of S_τ and $S_{(\tau)}$. Section 5 presents some applications of the main results to the risk theory.

2 Preliminaries

Recall that $X_1, \cdots, X_n$ are n real-valued random variables with distributions $F_1, \cdots, F_n$, respectively. Here, we assume that these random variables are dependent. To model the dependence of a multivariate distribution with non-identical marginals, one may use the theory of copulas (see [19]). A copula is a multivariate joint distribution defined on the n-dimensional unit cube $[0,1]^n$ such that every marginal distribution is uniform on the interval [0,1]. By Sklar's theorem, for a multivariate joint distribution F of a random vector $(X_1, \cdots, X_n)$ with marginals $F_1, \cdots, F_n$, there exists a copula C such that

$$F(x_1, \cdots, x_n) = C(F_1(x_1), \cdots, F_n(x_n)), \quad (x_1, \cdots, x_n) \in \mathbb{R}^n. \tag{2.1}$$

If $F_1, \cdots, F_n$ are all continuous, then C is unique and can be written as

$$C(u_1, \cdots, u_n) = P(F_1(X_1) \le u_1, \cdots, F_n(X_n) \le u_n) = F(F_1^{-1}(u_1), \cdots, F_n^{-1}(u_n))$$

for any $(u_1, \cdots, u_n) \in [0,1]^n$. Conversely, if C is a copula and $F_1, \cdots, F_n$ are distribution functions, then F defined in (2.1) is a multivariate joint distribution with marginals $F_1, \cdots, F_n$.

For notational convenience, we state the following three assumptions regarding the random variables $X_1, \cdots, X_n$.

(H1) Assume that

$$\widehat{\lambda}_{ij} = \lim_{x_i \wedge x_j \to \infty} P(|X_i| > x_i \mid X_j > x_j) = 0 \tag{2.2}$$

holds for all $1 \le i \ne j \le n$, where $x_i \wedge x_j = \min(x_i, x_j)$. This concept is related to the so-called asymptotic independence (see [23]). Note that the asymptotic independence means that a large value in one component is unlikely to be accompanied by a large value in another one.

(H2) Assume that there exist positive constants x_0 and c_0 such that the inequality

$$P(X_i > x_i \mid X_j = x_j, j \in J) \le c_0 \overline{F_i}(x_i)$$

holds for all $1 \le i \le n$, $\varnothing \ne J \subset \{1, 2, \cdots, n\} \backslash \{i\}$, $x_i > x_0$, and $x_j > x_0$ with $j \in J$. When x_j is not a possible value of X_j, the conditional probability above is simply understood as 0. Note that this dependence assumption was used in Geluk and Tang [13].

(H3) Let $X_1, \cdots, X_n$ be dependent. Assume that the dependent structure is governed by an absolutely continuous copula $C(u_1, \cdots, u_n)$ such that there exist positive constants m, M with $m \le c(u_1, \cdots, u_n) \le M$ for all $(u_1, \cdots, u_n) \in [0,1]^n$, where c is the copula density given by

$$c(u_1, \cdots, u_n) = \frac{\partial^n C(u_1, \cdots, u_n)}{\partial u_1 \cdots \partial u_n}.$$

Remark 2.1 It can be shown that (H3) $\Rightarrow$ (H2) $\Rightarrow$ (H1). Some related interesting discussions can be found in Geluk and Tang [13] and Ko and Tang [16].

To end the section, we present an example in which the three assumptions are satisfied.

Example 2.1 A joint n-dimensional distribution is called a Farlie-Gumbel-Morgenstern (FGM) distribution if it has the form

$$F(x_1, \cdots, x_n) = C(F_1(x_1), \cdots, F_n(x_n)), \quad (x_1, \cdots, x_n) \in \mathbb{R}^n, \tag{2.3}$$

where $F_1, \cdots, F_n$ are the one-dimensional marginal distributions, and the copula C is given by

$$C(u_1, \cdots, u_n) = \prod_{k=1}^{n} u_k \Big(1 + \sum_{1 \le i < j \le n} a_{ij}(1-u_i)(1-u_j)\Big), \quad (u_1, \cdots, u_n) \in [0,1]^n, \tag{2.4}$$

where a_{ij} are real numbers fulfilling certain requirements so that $F(x_1, \cdots, x_n)$ is a proper n-dimensional distribution. For details of FGM distributions, see Kotz et al. [18]. It is easy to check that if the random variables $X_1, \cdots, X_n$ follow a joint n-dimensional FGM distribution defined in (2.3) and (2.4) whose marginal distributions F_k $(1 \le k \le n)$ are absolutely continuous and satisfy $F_k(-x) = o(\overline{F_k}(x))$, then the three assumptions (H1)–(H3) are fulfilled.

3 Results for Finite Sums

In this section, we present our main results for finite sums. Let $(X_1^*, \cdots, X_n^*)$ be an independent copy of $(X_1, \cdots, X_n)$, that is, $(X_1^*, \cdots, X_n^*)$ and $(X_1, \cdots, X_n)$ are two independent random vectors with the same marginal distributions and the components of $(X_1^*, \cdots, X_n^*)$ are independent. We define $S_{n,k} = S_n - X_k$, $S_n^* = \sum\limits_{k=1}^{n} X_k^*$, and $S_{n,k}^* = S_n^* - X_k^*$.

Theorem 3.1 *Let $X_1, \cdots, X_n$ be heavy-tailed random variables with distributions $F_1, \cdots, F_n$, respectively.*

(1) *If $F_1, F_2, \cdots, F_n \in \mathcal{L}$, then under the assumption* (H1)*, we have*

$$\liminf_{x \to \infty} \frac{P(S_{(n)} > x)}{\sum\limits_{k=1}^{n} \overline{F_k}(x)} \ge \liminf_{x \to \infty} \frac{P(S_n > x)}{\sum\limits_{k=1}^{n} \overline{F_k}(x)} \ge 1. \tag{3.1}$$

(2) *If $F_1, F_2, \cdots, F_n \in \mathcal{L}$ such that the assumption* (H2) *holds and if additionally*

$$P\Big(\sum_{k=1}^{n} X_k^{*+} > x\Big) \sim \sum_{k=1}^{n} \overline{F_k}(x), \tag{3.2}$$

then

$$\lim_{x\to\infty}\frac{P(S_n>x)}{\sum_{k=1}^{n}\overline{F_k}(x)}=\lim_{x\to\infty}\frac{P(S_{(n)}>x)}{\sum_{k=1}^{n}\overline{F_k}(x)}=1. \tag{3.3}$$

Proof We only prove the result for S_n since the result for $S_{(n)}$ follows from the fact that

$$P\Big(\sum_{k=1}^{n}X_k^+>x\Big)\geq P(S_{(n)}>x)\geq P(S_n>x).$$

It follows from the definition of the class $\mathcal{L}$ that there exists a function $a(x)$ such that $a(x)\to\infty$ as $x\to\infty$, $2a(x)\leq x$, and

$$\overline{F_k}(x+a(x))\sim\overline{F_k}(x),\quad x\to\infty,\quad k=1,2,\cdots,n.$$

Note that

$$\begin{aligned}P(S_n>x)&\geq P(S_n>x,X_{(n)}>x+a(x))\\&\geq\sum_{k=1}^{n}P(S_n>x,X_k>x+a(x))\\&\quad-\sum_{1\leq i<j\leq n}P(X_i>x+a(x),X_j>x+a(x))\\&\equiv \mathrm{I}_1(x)+\mathrm{I}_2(x).\end{aligned}$$

The assumption (H1) implies that

$$\mathrm{I}_2(x)=o\Big(\sum_{k=1}^{n}\overline{F_k}(x)\Big).$$

Recall that $S_{n,k}=S_n-X_k$ for $1\leq k\leq n$. Then

$$\begin{aligned}\mathrm{I}_1(x)&\geq\sum_{k=1}^{n}P(S_{n,k}>-a(x),X_k>x+a(x))\\&=\sum_{k=1}^{n}P(X_k>x+a(x))-\sum_{k=1}^{n}P(S_{n,k}\leq -a(x),X_k>x+a(x)).\end{aligned}$$

It follows from the assumption (H1) that

$$\sum_{k=1}^{n}P(S_{n,k}\leq -a(x),X_k>x+a(x))=o\Big(\sum_{k=1}^{n}\overline{F_k}(x+a(x))\Big).$$

Thus

$$\begin{aligned}\mathrm{I}_1(x)&\geq\sum_{k=1}^{n}P(X_k>x+a(x))-o\Big(\sum_{k=1}^{n}\overline{F_k}(x+a(x))\Big)\\&\sim\sum_{k=1}^{n}\overline{F_k}(x).\end{aligned}$$

This proves (3.1).

We next show that if $F_1, F_2, \cdots, F_n \in \mathcal{L}$, (3.2) and the assumption (H2) hold, then

$$\limsup_{x\to\infty} \frac{P(S_n > x)}{\sum\limits_{k=1}^{n} \overline{F_k}(x)} \le 1. \tag{3.4}$$

It is clear that the inequality

$$P(X_{(n)}^{+} > x) \le \sum_{k=1}^{n} \overline{F_k}(x) \tag{3.5}$$

holds. The conditions $F_1, F_2, \cdots, F_n \in \mathcal{L}$ and the assumption (H2) imply that there exist positive constants x_0 and d_n such that

$$P(S_{n,k} > x \mid X_k = x_x) \le d_n P(S_{n,k}^{*} > x)$$

holds for all $1 \le k \le n,\ x > x_0$ and $x_k > x_0$ (see [13]). It can be shown that for every function $a(\cdot) : (0, \infty) \to (x_0, \infty)$ and for every $1 \le k \le n$, and $x > x_0$,

$$P(S_n > x, a(x) < X_k \le x) \le d_n P(S_n^{*} > x, a(x) < X_k^{*} \le x). \tag{3.6}$$

Furthermore, following the proof of Lemma 5.2 in Geluk and Tang [13], we get for every function $a(\cdot) : [0, \infty) \to [0, \infty)$ with $a(x) \to \infty$ and for every $1 \le j \le n$,

$$P(S_n^{*} > x, a(x) < X_k^{*} \le x) = o(1) \sum_{k=1}^{n} \overline{F_k}(x). \tag{3.7}$$

Hence, (3.5)–(3.7) imply that

$$\begin{aligned} P(S_n > x) &\le P\Big(\bigcup_{k=1}^{n} (X_k^{+} > x)\Big) + P\Big(S_n^{+} > x, \bigcap_{k=1}^{n} (X_k^{+} \le x)\Big) \\ &\le P(X_{(n)}^{+} > x) + \sum_{k=1}^{n} P\Big(S_n^{+} > x, \frac{x}{n} < X_k^{+} \le x\Big) \\ &\le P(X_{(n)}^{+} > x) + d_n \sum_{k=1}^{n} P\Big(S_n^{+*} > x, \frac{x}{n} < X_k^{+*} \le x\Big) \\ &\sim \sum_{k=1}^{n} \overline{F_k}(x). \end{aligned}$$

This proves (3.4). The result for S_n in (3.3) immediately follows from (3.1) and (3.4).

Corollary 3.1 *Let $X_1, \cdots, X_n$ be heavy-tailed random variables with a common distribution F.*

(1) *Under the assumption* (H1), *if $F \in \mathcal{L}$, then*

$$\liminf_{x\to\infty} \frac{P(S_{(n)} > x)}{\overline{F}(x)} \ge \liminf_{x\to\infty} \frac{P(S_n > x)}{\overline{F}(x)} \ge n. \tag{3.8}$$

(2) *If $F \in \mathcal{L}$ such that the assumption* (H2) *holds and if additionally*

$$P\Big(\sum_{k=1}^{n} X_k^{*+} > x\Big) \sim n\overline{F}(x), \tag{3.9}$$

then

$$\lim_{x\to\infty} \frac{P(S_n > x)}{\overline{F}(x)} = \lim_{x\to\infty} \frac{P(S_{(n)} > x)}{\overline{F}(x)} = n. \tag{3.10}$$

Remark 3.1 For heavy-tailed random variables $X_1, \cdots, X_n$ with distributions $F_1, \cdots, F_n$, if $F_1, F_2, \cdots, F_n \in \mathcal{S}$, $F_i * F_j \in \mathcal{S}$ for all $1 \le i \ne j \le n$, and the assumption (H2) holds, then the result of Geluk and Tang [13, Theorem 3.2] gives

$$\lim_{x\to\infty} \frac{P(S_n > x)}{\sum\limits_{k=1}^{n} \overline{F_k}(x)} = 1.$$

It is well-known that (see [25, Lemma 2.1]) if $F_1 \in \mathcal{S}$, $F_1 \in \mathcal{L}$, and $\overline{F_2}(x) = O(\overline{F_1}(x))$, then $F_1 * F_2 \in \mathcal{S}$ and $\overline{F_1 * F_2}(x) \sim \overline{F_1}(x) + \overline{F_2}(x)$. Therefore, the condition (3.2) in Theorem 3.1 is slightly more general than the above conditions of Geluk and Tang [13, Theorem 3.2].

4 Results for Random Sums

In this section, we extend the results of Denisov et al. [7] to the case of dependent and heavy-tailed random variables. Note that their asymptotic results are for random sums of i.i.d. nonnegative and heavy-tailed random variables with the random size τ following a light-tailed or heavy-tailed distribution.

Theorem 4.1 *Let $X_1, \cdots, X_n$ be heavy-tailed random variables with a common distribution F and a finite mean, and τ be a counting random variable independent of the sequence $\{X_k\}$ with a finite mean $E\tau$. Under the assumption* (H3)*, we have the following results:*

(i) *Assume that $EX_1 < 0$. If $F \in \mathcal{S}_*$, then*

$$\lim_{x\to\infty} \frac{P(S_\tau > x)}{\overline{F}(x)} = \lim_{x\to\infty} \frac{P(S_{(\tau)} > x)}{\overline{F}(x)} = E\tau. \tag{4.1}$$

(ii) *Assume that $EX_1 \ge 0$ and that there exists a $c > EX_1$ such that $P(c\tau > x) = o(\overline{F}(x))$ as $x \to \infty$. If $F \in \mathcal{S}^*$, then*

$$\lim_{x\to\infty} \frac{P(S_\tau > x)}{\overline{F}(x)} = \lim_{x\to\infty} \frac{P(S_{(\tau)} > x)}{\overline{F}(x)} = E\tau. \tag{4.2}$$

Proof If $EX_1 < 0$ and $F \in \mathcal{S}_*$, it follows from the result of Korshunov [17] that

$$P(S^*_{(n)} > x) \sim \frac{1}{|EX_1|} \int_x^{x+n|EX_1|} \overline{F}(y)\mathrm{d}y \tag{4.3}$$

uniformly in $n \geq 1$. Consider the relation

$$P(S_{(\tau)} > x) \sim E\tau\overline{F}(x). \tag{4.4}$$

Since $c(u_1, \cdots, u_n) < M$ for all $(u_1, \cdots, u_n) \in [0,1]^n$, then we have

$$P(S_{(n)} > x) \leq MP(S^*_{(n)} > x) \tag{4.5}$$

for any x. Thus, from (4.3), we have

$$P(S_{(n)} > x) \leq M(1 + o(1))n|EX_1|\overline{F}(x)$$

for all $n \geq 1$. Applying the dominated convergence theorem and (3.10), we obtain

$$\lim_{x\to\infty} \frac{P(S_{(\tau)} > x)}{\overline{F}(x)} = \sum_{k=1}^{\infty} \Big(\lim_{x\to\infty} \frac{P(S_{(n)} > x)}{\overline{F}(x)}\Big) P(\tau = n) = E\tau, \tag{4.6}$$

which proves (4.4). Furthermore, Fatou's lemma gives

$$\liminf_{x\to\infty} \frac{P(S_\tau > x)}{\overline{F}(x)} \geq E\tau, \tag{4.7}$$

without any restriction on the sign of EX_1. Since $P(S_\tau > x) \leq P(S_{(\tau)} > x)$ for all x, (4.1) follows from (4.6) and (4.7).

To prove (ii), it is sufficient to prove (4.4). Since $F \in \mathcal{S}^*$, it follows from (3.10) that

$$P(S_{(n)} > x) \sim n\overline{F}(x).$$

Thus, there exists an increasing function $N(x) \to \infty$ such that

$$P_1(x) := P(S_{(\tau)} > x, \tau \leq N(x)) \sim E\tau\overline{F}(x).$$

Let $\varepsilon = \frac{c - EX_1}{2} > 0$ and $b = \frac{EX_1 + c}{2}$. Put $\widetilde{X_i} = X_i - b$ and $\widetilde{S_n} = \widetilde{X_1} + \cdots + \widetilde{X_n}$. Then, $E\widetilde{X_i} = -\varepsilon < 0$. By using (4.3) and (4.5), we have

$$P(S_{(n)} > x) \leq MP(S^*_{(n)} > x) \leq MP(\widetilde{S}^*_{(n)} > x - bn).$$

Following the steps of the proof of Denisov et al. [8, Theorem 1(ii)], one gets

$$P_2(x) := P\Big(S_{(\tau)} > x, \tau \in \Big(N(x), \frac{x}{c}\Big]\Big) = o(\overline{F}(x)).$$

Finally, the condition $P(c\tau > x) = o(\overline{F}(x))$ gives

$$P_3(x) := P(S_{(\tau)} > x, c\tau > x) = o(\overline{F}(x)).$$

Thus,

$$P(S_{(\tau)} > x) \equiv P_1(x) + P_2(x) + P_3(x) \sim E\tau\overline{F}(x), \quad \text{as } x \to \infty.$$

Hence, the proof of (ii) is complete.

Remark 4.1 Theorem 4.1 partially extends (3.10) to the case of random sums. Under the independence assumption and for $F \in \mathcal{S}^*$, (4.1) was established in Denisov et al. [8, Theorem 1(i)]. Here, (4.2) generalizes the result of Denisov et al. [8, Theorem 1(ii)] to the dependent case. For related works, we refer the readers to Ng et al. [22], and Ng and Tang [21].

5 Applications to Risk Theory

In this section, we present two examples to illustrate some applications of our main results.

Example 5.1 Consider the following discrete-time insurance risk model with a constant interest rate (see [24]):

$$U(0)=x,\ \ U(n)=x(1+r)^n-\sum_{k=1}^{n}X_k(1+r)^{n-k},\quad n=1,2.\cdots, \tag{5.1}$$

where $x\geq 0$ is the initial surplus, $r\geq 0$ is the constant interest rate, X_n denotes the gross loss (i.e., the total claim amount minus the total incoming premium) during the nth year, and $\{X_k, k\geq 1\}$ constitute a sequence of random variables which are not necessarily independent or identically distributed.

Let $V_k=X_k(1+r)^{-k}$ and $S_n=\sum\limits_{k=1}^{n}V_k$. Then, we can rewrite (5.1) as $U(n)=(1+r)^n(x-S_n)$. Define the n-period finite-time ruin probability as

$$\psi_n(x)=P\Big(\min_{1\leq k\leq n}U(k)<0\,\Big|\,U(0)=x\Big)=P\Big(\max_{1\leq k\leq n}S_k>x\Big).$$

From Theorem 3.1, we obtain

Corollary 5.1 *Assume that X_k follows a distribution function F_k for each $k=1,2,\cdots,n$. If $F_1,F_2,\cdots,F_n\in\mathcal{L}$ such that the assumption* (H2) *holds and if additionally*

$$P\Big(\sum_{k=1}^{n}V_k^{*+}>x\Big)\sim\sum_{k=1}^{n}\overline{F_k}(x),$$

then

$$\lim_{x\to\infty}\frac{\psi_n(x)}{\sum\limits_{k=1}^{n}\overline{F_k}(x(1+r)^k)}=1. \tag{5.2}$$

Example 5.2 Consider the following customer-arrival-based insurance risk model (see [20]):

$$U(t)=x-\sum_{k=1}^{N(t)}(Z_k-(1+\rho)\mu)\equiv x-S_{N(t)}, \tag{5.3}$$

where $\{N(t),t\geq 0\}$ is the individual customer-arrival process with a mean function $\lambda(t)=EN(t)$ for $t\geq 0$, $\{Z_k,k\geq 1\}$ is a sequence of nonnegative random variables with a common distribution function F and a finite mean μ, and the positive constant ρ can be interpreted as the safety loading. We assume that $\{N(t),t\geq 0\}$ and $\{Z_k,k\geq 1\}$ are mutually independent. The ruin probability within a finite horizon T is defined by

$$\psi(x;T)=P\Big(\min_{0\leq t\leq T}U(t)<0\Big)=P(S_{N(T)}>x).$$

From Theorem 4.1, we obtain

Corollary 5.2 *Assume that $F \in \mathcal{S}^*$ and there exists a constant $c > EX_1$ such that $P(cN(T) > x) = o(\overline{F}(x))$ as $x \to \infty$. Then, under the assumption* (H3)*, we have*

$$\lim_{x\to\infty} \frac{\psi(x;T)}{\overline{F}(x)} = \lambda(T).$$

Remark 5.1 Under the independent setting of $\{Z_k, k \geq 1\}$ and other conditions, (5.2) has appeared in several papers. For example, Ng et al. [22] obtained (5.2) under the conditions that $F \in \mathcal{L} \cap \mathcal{D}$ and $P(N(T) > x) = o(\overline{F}(x))$; Ng et al. [20] obtained (5.2) under the conditions that $\{Z_k, k \geq 1\}$ is a sequence of i.i.d. subexponential random variables and $E(1+\varepsilon)^{N(T)} < \infty$ for some $\varepsilon > 0$; and Kaas and Tang [14] weakened the condition on $N(\cdot)$ and obtained (5.2) under the condition that F is strongly subexponential, that is, $F \in \mathcal{S}_*$.

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