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Double Subordination Preserving Properties for a New Generalized Srivastava-Attiya Integral Operator

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Abstract The authors obtain subordination and superordination preserving properties for a new defined generalized operator involving the Srivastava-Attiya integral operator. Differential sandwich-type theorems for these univalent functions, and some consequences involving well-known special functions are also presented.

Keywords Differential subordination and superordination, Analytic functions, Univalent functions, Starlike functions, Convex functions, Srivastava-Attiya integral operator, Hypergeometric functions

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1 Introduction

Let $\mathcal{H}(\mathrm{U})$ represent the space of analytic functions in the open unit disk $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(\mathrm{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}.$$

We denote by $\mathcal{A}$ the subclass of the functions $f \in \mathcal{H}[a, 1]$ normalized with the conditions $f(0) = f'(0) - 1 = 0$. Denote also by $\mathcal{K}$ the subclass of $\mathcal{A}$ consisting of all those functions that are convex (univalent) and normalized in U, i.e.,

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathrm{U}.$$

If f and g are two members of $\mathcal{H}(\mathrm{U})$, then the function f is said to be subordinate to g, and we write $f(z) \prec g(z)$, if there exists a function w analytic in U with $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \mathrm{U}$, such that $f(z) = g(w(z))$, $z \in \mathrm{U}$. Furthermore, if the function g is univalent in U, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathrm{U}) \subset g(\mathrm{U}). \tag{1.1}$$

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Definition 1.1 (see [16]) *Let $\psi : \mathbb{C}^2 \to \mathbb{C}$, and let h be univalent in* U. *If p is analytic in* U *and satisfies the following differential subordination*:

$$\psi(p(z), zp'(z)) \prec h(z), \tag{1.2}$$

then p is called a solution to the differential subordination (1.2). *A univalent function q is called a dominant of the solutions to the differential subordination* (1.2), *or, more simply, a dominant if $p(z) \prec q(z)$ for all p satisfying* (1.2). *A dominant $\widetilde{q}$ that satisfies $\widetilde{q}(z) \prec q(z)$ for all dominants q of* (1.2) *is said to be the best dominant of* (1.2).

Recently, Miller and Mocanu [17] introduced the notion of differential superordinations, as the dual concept of differential subordinations.

Definition 1.2 (see [17]) *Let $\phi : \mathbb{C}^2 \to \mathbb{C}$, and let h be analytic in* U. *If p and $\phi(p(z), zp'(z))$ are univalent in* U *and satisfy the differential superordination*

$$h(z) \prec \phi(p(z), zp'(z)), \tag{1.3}$$

then p is called a solution to the differential superordination (1.3). *An analytic function q is called a subordinant of the solutions to the differential superordination* (1.3), *or, more simply, a subordinant if $q(z) \prec p(z)$ for all p satisfying* (1.3). *A univalent subordinant $\widetilde{q}$ that satisfies $q(z) \prec \widetilde{q}(z)$ for all subordinants q of* (1.3) *is said to be the best subordinant of* (1.3).

Definition 1.3 (see [16, p. 21, Definition 2.2b]) *We denote by $\mathcal{Q}$ the class of functions f that are analytic and injective on $\overline{\mathrm{U}} \setminus \mathrm{E}(f)$, where*

$$\mathrm{E}(f) = \left\{\zeta \in \partial\mathrm{U} : \lim_{z \to \zeta} f(z) = \infty\right\},$$

such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathrm{U} \setminus \mathrm{E}(f)$.

Definition 1.4 (i) *The generalized hypergeometric function ${}_qF_s$ is defined by*

$${}_qF_s(z) = {}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^n}{n!}, \quad z \in \mathrm{U},$$

where $\alpha_j \in \mathbb{C}$ $(j = 1, \cdots, q)$, $\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $\mathbb{Z}_0^- = \{0, -1, \cdots\}$ $(j = 1, \cdots, s)$, $q \leq s + 1$, $q, s \in \mathbb{N}_0$, where $(\alpha)_k$ is the Pochhammer symbol defined by

$$(\alpha)_0 = 1, \ (\alpha)_k = \alpha(\alpha + 1) \cdots (\alpha + k - 1), \quad k \in \mathbb{N}.$$

(ii) *The general Hurwitz-Lerch Zeta function $\phi(z, s, a)$ is defined by* (*see, e.g.,* [25, *p.* 21 *et seq.*])

$$\phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(a + n)^s} = \frac{1}{a^s} + \frac{z}{(1 + a)^s} + \frac{z^2}{(2 + a)^s} + \cdots \tag{1.4}$$

with $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $s \in \mathbb{C}$ when $|z| < 1$, and $\mathrm{Re}(s) > 1$ when $|z| = 1$.

This general Hurwitz-Lerch Zeta function $\phi(z,s,a)$ also contains its special cases, the well-known functions such as the Riemann and Hurwitz (or generalized) Zeta function, the Lerch Zeta function, the Polylogarithmic function and the Lipschitz-Lerch Zeta function. One may refer to the Srivastava and Choi [25] (see also [24]) for further details and references about these functions.

A generalization of the above defined Hurwitz-Lerch Zeta function $\phi(z,s,b)$ was studied by Garg et al. [9, p. 27, equation (1.4)] in the following form (see also [26]):

$$\Phi_{\lambda,\mu;\nu}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu)_n}{(\nu)_n n!}\frac{z^n}{(n+a)^s} \tag{1.5}$$

with $\lambda,\mu,s\in\mathbb{C}$, $\nu,a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ when $|z|<1$, and $\mathrm{Re}(s+\nu-\lambda-\mu)>1$ when $|z|=1$.

Various properties and integral representations for $\Phi_{\lambda,\mu;\nu}(z,s,a)$ can be found in the works of Garg et al. [9] and Srivastava et al. [26]. The following interesting special cases of the function $\Phi_{\lambda,\mu;\nu}(z,s,a)$ are worthy of mentioning here:

(i) For $\lambda=\nu$, we find that (1.5) reduces to the function $\phi_\mu^*(z,s,a)$ studied by Goyal and Laddha [10, p. 100, equation (1.5)] (see also [11]);

(ii) If we set $\lambda=\nu$ and $\mu=1$, then (1.5) yields the general Hurwitz-Lerch Zeta function $\phi(z,s,a)$ defined by (1.4).

Motivated by the earlier investigation due to Srivastava and Attiya [24], Prajapat and Goyal [22], we introduced the linear operator

$$\mathcal{J}_{\lambda,\mu;\nu}^{s,a}:\mathcal{A}\to\mathcal{A},$$

which is defined by means of the following Hadamard (or convolution) product, that is

$$\mathcal{J}_{\lambda,\mu;\nu}^{s,a}(f)(z)=\mathcal{G}_{\lambda,\mu;\nu}^{s,a}(z)*f(z),\quad z\in\mathrm{U}, \tag{1.6}$$

where $\lambda,\mu,s\in\mathbb{C}$, $\nu,a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ and $f\in\mathcal{A}$, while the function $\mathcal{G}_{\lambda,\mu;\nu}^{s,a}$ is defined by

$$\begin{aligned}\mathcal{G}_{\lambda,\mu;\nu}^{s,a}(z)&=\frac{\nu(1+a)^s}{\lambda\,\mu}[\Phi_{\lambda,\mu;\nu}(z,s,a)-a^{-s}]\\&=z+\sum_{n=2}^{\infty}\frac{(\lambda+1)_{n-1}(\mu+1)_{n-1}}{(\nu+1)_{n-1}\,n!}\Big(\frac{1+a}{n+a}\Big)^s z^n,\quad z\in\mathrm{U}.\end{aligned} \tag{1.7}$$

Now, by using (1.7) in (1.6), we get

$$\mathcal{J}_{\lambda,\mu;\nu}^{s,a}f(z)=z+\sum_{n=2}^{\infty}\frac{(\lambda+1)_{n-1}(\mu+1)_{n-1}}{(\nu+1)_{n-1}\,n!}\Big(\frac{1+a}{n+a}\Big)^s a_n z^n,\quad z\in\mathrm{U}. \tag{1.8}$$

Note that the above last form of the operator $\mathcal{J}_{\lambda,\mu;\nu}^{s,a}$ is well-defined for $\lambda,\mu,s\in\mathbb{C}$ and $\nu,a\in\mathbb{C}\setminus\mathbb{Z}^-$.

We observe that the operator $\mathcal{J}_{\lambda,\mu;\nu}^{s,a}$ generalizes several previously studied familiar operators, and we show some of the interesting particular cases as follows:

(i) $\mathcal{J}_{\gamma-1,1;\nu}^{s,a}\equiv\mathrm{I}_{a,\nu,\gamma}^{s}$, where $\mathrm{I}_{a,\nu,\gamma}^{s}$ is the generalized operator studied recently by Noor and Bukhari [19, p. 2, equation (1.3)];

(ii) $\mathcal{J}^{0,0}_{\gamma-1,1;\nu} \equiv \mathrm{I}_{\nu,\gamma}$, where $\mathrm{I}_{\nu,\gamma}$ is the Choi-Saigo-Srivastava operator (see [8]);

(iii) $\mathcal{J}^{s,a}_{\lambda,1;\lambda} \equiv \mathrm{J}_{s,a}$, where $\mathrm{J}_{s,a}$ is the Srivastava-Attiya operator (see [22, 24]);

(iv) $\mathcal{J}^{-r,a}_{\lambda,1;\lambda} \equiv \mathrm{I}(r,a)$ $(a \geq 0,\ r \in \mathbb{Z})$, where the operator $\mathrm{I}(r,a)$ was studied by Cho and Srivastava [7];

(v) $\mathcal{J}^{0,a}_{\beta,1;\alpha+\beta} \equiv \mathcal{Q}^{\alpha}_{\beta}$ $(\alpha \geq 0,\ \beta > -1)$, where the operator $\mathcal{Q}^{\alpha}_{\beta}$ was studied by Jung et al. [13];

(vi) $\mathcal{J}^{\sigma,1}_{\lambda,1;\lambda} \equiv \mathrm{I}^{\sigma}$ $(\sigma > 0)$, where I^{σ} is the Jung-Kim-Srivastava integral operator (see [13]);

(vii) $\mathcal{J}^{1,a}_{\lambda,1;\lambda} = \mathcal{J}_a$ $(a > -1)$, where $\mathcal{J}_a$ is the Bernardi operator (see [3]);

(viii) $\mathcal{J}^{0,0}_{\lambda,1;\nu} \equiv \mathcal{L}(\lambda,\nu)$, where $\mathcal{L}(\lambda,\nu)$ is the well-known Carlson-Shaffer operator (see [6]);

(ix) $\mathcal{J}^{0,0}_{2,1;2-\lambda} \equiv \Omega^{\lambda}_{z}$ $(0 \leq \lambda < 1)$, where Ω^{λ}_{z} is the fractional differ-integral operator introduced by [20] (see also [23]).

It is readily verified from (1.8) that

$$z(\mathcal{J}^{s+1,a}_{\lambda,\mu;\nu} f(z))' = (a+1)\mathcal{J}^{s,a}_{\lambda,\mu;\nu} f(z) - a\mathcal{J}^{s+1,a}_{\lambda,\mu;\nu} f(z), \tag{1.9}$$

$$z(\mathcal{J}^{s,a}_{\lambda,\mu;\nu} f(z))' = (\lambda+1)\mathcal{J}^{s,a}_{\lambda+1,\mu;\nu} f(z) - \lambda\mathcal{J}^{s,a}_{\lambda,\mu;\nu} f(z), \tag{1.10}$$

$$z(\mathcal{J}^{s,a}_{\lambda,\mu;\nu+1} f(z))' = (\nu+1)\mathcal{J}^{s,a}_{\lambda,\mu;\nu} f(z) - \nu\mathcal{J}^{s,a}_{\lambda,\mu;\nu+1} f(z). \tag{1.11}$$

Using the principle of subordination, Miller and Mocanu and Reade [18] obtained different subordination-preserving theorems for certain integral operators for analytic functions in the unit disk. Moreover, in [4–5], the author investigated the subordination and superordination preserving properties of integral operators, while some other interesting developments involving subordination and superordination were considered in [1–2, 27]. In the present paper, by a sandwich-type theorem, we obtain subordination- and superordination-preserving properties of the differ-integral operator $\mathcal{J}^{s,a}_{\lambda,\mu;\nu}$ defined by (1.8).

The following lemmas will be required in our present investigation.

Lemma 1.1 (see [14]) *Suppose that the function $H : \mathbb{C}^2 \to \mathbb{C}$ satisfies the condition*

$$\operatorname{Re} H(\mathrm{i}s, t) \leq 0$$

for all $s, t \in \mathbb{R}$ with $t \leq -\frac{n(1+s^2)}{2}$, where n is a positive integer. If the function $p(z) = 1 + p_n z^n + \cdots$ is analytic in U and

$$\operatorname{Re} H(p(z), zp'(z)) > 0, \quad z \in \mathrm{U},$$

then $\operatorname{Re} p(z) > 0$, $z \in \mathrm{U}$.

Lemma 1.2 (see [15]) *Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$, and let $h \in \mathcal{H}(\mathrm{U})$ with $h(0) = c$. If $\operatorname{Re}[\beta h(z) + \gamma] > 0$ for $z \in \mathrm{U}$, then the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z), \quad q(0) = c$$

has an analytic solution in U, which satisfies $\operatorname{Re}[\beta q(z) + \gamma] > 0$, $z \in \mathrm{U}$.

Lemma 1.3 (see [16, p. 24, Lemma 2.2d]) *Let $q \in \mathcal{Q}$ with $q(0) = a$, and let $p(z) = a + a_n z^n + \cdots$ be analytic in* U *with $p(z) \not\equiv a$ and $n \geq 1$. If p is not subordinate to q, then there exist the points $z_0 = r_0 \mathrm{e}^{\mathrm{i}\theta} \in \mathrm{U}$ and $\zeta_0 \in \partial\mathrm{U} \setminus E(f)$, and an $m \geq n \geq 1$ for which $p(\mathrm{U}_{r_0}) \subset q(\mathrm{U})$, such that*

$$p(z_0) = q(\zeta_0) \quad \textit{and} \quad z_0 p'(z_0) = m\zeta_0 q'(\zeta_0),$$

where $\mathrm{U}_{r_0} = \{z \in \mathbb{C} : |z| < r_0\}$.

A function $L(z,t) : \mathrm{U} \times [0, +\infty) \to \mathbb{C}$ is called a subordination (or a Loewner) chain if $L(\,\cdot\,, t)$ is analytic and univalent in U for all $t \geq 0$, and $L(z,s) \prec L(z,t)$ when $0 \leq s \leq t$.

Lemma 1.4 (see [17, p. 822, Theorem 7]) *Let $q \in \mathcal{H}[a,1]$, $\phi : \mathbb{C}^2 \to \mathbb{C}$, and let $\phi(q(z), zq'(z)) \equiv h(z)$. If $L(z,t) = \phi(q(z), tzq'(z))$ is a subordination chain and $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$, then*

$$h(z) \prec \phi(p(z), zp'(z))$$

implies that

$$q(z) \prec p(z).$$

Furthermore, if the differential equation $\phi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in \mathcal{Q}$, then q is the best subordinant.

The next well-known lemma gives a sufficient condition, so that the $L(z,t)$ function will be a subordination chain.

Lemma 1.5 (see [21, p. 159]) *Let $L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots$ with $a_1(t) \neq 0$ for all $t \geq 0$ and $\lim_{t\to+\infty} |a_1(t)| = +\infty$. Suppose that $L(\,\cdot\,, t)$ is analytic in* U *for all $t \geq 0$, and $L(z, \,\cdot\,)$ is continuously differentiable on $[0, +\infty)$ for all $z \in \mathrm{U}$. If $L(z,t)$ satisfies*

$$\mathrm{Re}\left[z \frac{\frac{\partial L}{\partial z}}{\frac{\partial L}{\partial t}}\right] > 0, \quad z \in \mathrm{U},\ t \geq 0$$

and

$$|L(z,t)| \leq K_0 |a_1(t)|, \quad |z| < r_0 < 1,\ t \geq 0$$

for some positive constants K_0 and r_0, then $L(z,t)$ is a subordination chain.

2 Main Results

We first prove the following subordination theorem involving the operator $\mathcal{J}^{s,a}_{\lambda,\mu;\nu}$.

Theorem 2.1 *Let $f, g \in \mathcal{A}$ and $a \geq 0$. Suppose that*

$$\mathrm{Re}\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > -\rho, \quad z \in \mathrm{U} \tag{2.1}$$

with $\varphi(z) = \mathcal{J}^{s,a}_{\lambda,\mu,\nu} g(z)$, where $\rho = 0$ if $a = 0$ and

$$\rho = \rho(a) = \begin{cases} \dfrac{a}{2}, & \textit{if } \ 0 < a \leq 1, \\ \dfrac{1}{2a}, & \textit{if } \ a > 1. \end{cases} \tag{2.2}$$

Then, the subordination condition

$$\mathcal{J}^{s,a}_{\lambda,\mu,\nu} f(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu} g(z) \tag{2.3}$$

implies

$$\mathcal{J}^{s+1,a}_{\lambda,\mu,\nu} f(z) \prec \mathcal{J}^{s+1,a}_{\lambda,\mu,\nu} g(z).$$

Moreover, the function $\mathcal{J}^{s+1,a}_{\lambda,\mu,\nu} g$ *is the best dominant of* (2.3).

Proof If we define the functions F and G by

$$F(z) = \mathcal{J}^{s+1,a}_{\lambda,\mu,\nu} f(z) \quad \text{and} \quad G(z) = \mathcal{J}^{s+1,a}_{\lambda,\mu,\nu} g(z), \tag{2.4}$$

then $F, G \in \mathcal{A}$. We first show that, if the function q is defined by

$$q(z) = 1 + \frac{zG''(z)}{G'(z)}, \tag{2.5}$$

then

$$\operatorname{Re} q(z) > 0, \quad z \in \mathrm{U}.$$

Differentiating both sides of the second equation in (2.4) and using (1.9) for $g \in \mathcal{A}$, we have

$$(1+a)\varphi(z) = aG(z) + zG'(z).$$

Hence, it follows that

$$1 + \frac{z\varphi''(z)}{\varphi'(z)} = q(z) + \frac{zq'(z)}{q(z)+a} \equiv h(z). \tag{2.6}$$

From (2.1) and (2.6), we have

$$\operatorname{Re}[h(z)+a] > 0, \quad z \in \mathrm{U}.$$

By using Lemma 1.2, we deduce that the differential equation (2.6) has a solution $q \in \mathcal{H}(\mathrm{U})$ with $q(0) = h(0) = 1$.

Let us define the function

$$H(u,v) = u + \frac{v}{u+a} + \rho, \tag{2.7}$$

where ρ is given by (2.2). From (2.1) and (2.6)–(2.7), we obtain

$$\operatorname{Re} H(q(z), zq'(z)) > 0, \quad z \in \mathrm{U}.$$

Now we will show that $\operatorname{Re} H(\mathrm{i}s, t) \le 0$ for all $s \in \mathbb{R}$ and $t \le -\frac{1+s^2}{2}$. From (2.7), we have

$$\operatorname{Re} H(\mathrm{i}s, t) = \operatorname{Re}\left(\mathrm{i}s + \frac{t}{\mathrm{i}s+a} + \rho\right) = \frac{at}{|a+\mathrm{i}s|^2} + \rho \le -\frac{E_\rho(s)}{2|a+\mathrm{i}s|^2}, \tag{2.8}$$

where

$$E_\rho(s) = (a-2\rho)s^2 + a(1-2\rho a). \tag{2.9}$$

For ρ given by (2.2), the coefficient of s^2 in the $E_\rho(s)$ given by (2.9) is positive or equal to zero, and $E_\rho(0) \geq 0$. Hence, we deduce that $E_\rho(s) \geq 0$ for all $s \in \mathbb{R}$. Now, from (2.8), we see that $\operatorname{Re} H(\mathrm{i}s, t) \leq 0$ for all $s \in \mathbb{R}$ and $t \leq -\frac{1+s^2}{2}$. Thus, by using Lemma 1.1, we conclude that $\operatorname{Re} q(z) > 0$ for all $z \in \mathrm{U}$, i.e., the function G defined by (2.4) is convex (univalent) in U.

Next we will prove that the subordination condition (2.3) implies

$$F(z) \prec G(z) \tag{2.10}$$

for the functions F and G defined by (2.4). Without loss of generality, we can assume that G is analytic and univalent on $\overline{\mathrm{U}}$ and $G'(\zeta) \neq 0$ for $|\zeta| = 1$. Otherwise, we replace F and G by $F_r(z) = F(rz)$ and $G_r(z) = G(rz)$, respectively, where $r \in (0,1)$. These functions satisfy the conditions of the theorem on $\overline{\mathrm{U}}$, and we need to prove that $F_r(z) \prec G_r(z)$ for all $r \in (0,1)$, which enables us to obtain (2.10) by letting $r \to 1^-$.

Let us define the function $L(z,t)$ by

$$L(z,t) \equiv \frac{a}{a+1} G(z) + \frac{t+1}{a+1} zG'(z), \quad z \in \mathrm{U},\ t \geq 0. \tag{2.11}$$

Then,

$$\frac{\partial L(z,t))}{\partial z}\Big|_{z=0} = G'(0)\Big(1 + \frac{t}{a+1}\Big) = 1 + \frac{t}{a+1} \neq 0, \quad t \geq 0,$$

and this shows that the function $L(z,t) = a_1(t)z + \cdots$ satisfies the conditions $a_1(t) \neq 0$ for all $t \geq 0$ and $\lim\limits_{t\to+\infty} |a_1(t)| = +\infty$.

From the definition (2.11) and for all $t \geq 0$, we have

$$\frac{|L(z,t)|}{|a_1(t)|} = \frac{|\frac{a}{a+1}G(z) + \frac{t+1}{a+1}zG'(z)|}{1+\frac{t}{a+1}} \leq \frac{\frac{a}{a+1}|G(z)| + \frac{t+1}{a+1}|zG'(z)|}{1+\frac{t}{a+1}}. \tag{2.12}$$

Since the function G is convex and normalized in the unit disk, i.e., $G \in \mathcal{K}$, the following well-known growth and distortion sharp inequalities (see [12]) are true:

$$\frac{r}{1+r} \leq |G(z)| \leq \frac{r}{1-r}, \qquad \text{if } |z| \leq r < 1, \tag{2.13}$$

$$\frac{1}{(1+r)^2} \leq |G'(z)| \leq \frac{1}{(1-r)^2}, \quad \text{if } |z| \leq r < 1. \tag{2.14}$$

Using the right-hand sides of these inequalities in (2.12), we deduce that

$$\frac{|L(z,t)|}{|a_1(t)|} \leq \frac{r}{(1-r)^2}\, \frac{t+1+a(1-r)}{a+t+1} \leq \frac{r}{(1-r)^2}, \quad |z| \leq r,\ t \geq 0,$$

and thus, the second assumption of Lemma 1.5 holds.

Furthermore,

$$\operatorname{Re}\left[z\frac{\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right] = a + (1+t)\operatorname{Re}\Big(1 + \frac{zG''(z)}{G'(z)}\Big) > 0, \quad z \in \mathrm{U},\ t \geq 0,$$

and according to Lemma 1.5, the function $L(z,t)$ is a subordination chain. From the definition of the subordination chain combined with (2.11), we obtain

$$L(\zeta,t) \notin L(\mathrm{U},0) = \varphi(\mathrm{U}), \quad \text{whenever } \zeta \in \partial\mathrm{U},\ t \geq 0. \tag{2.15}$$

Suppose that F is not subordinate to G. Then by Lemma 1.3, there exist points $z_0 \in \mathrm{U}$ and $\zeta_0 \in \partial\mathrm{U}$, and a number $t \geq 0$, such that

$$F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0 F'(z_0) = t\zeta_0 G'(\zeta_0).$$

From these two relations, and by virtue of the subordination condition (2.3), we deduce that

$$\begin{aligned} L(\zeta_0, t) &= \frac{a}{a+1} G(\zeta_0) + \frac{t+1}{a+1} \zeta_0 G'(\zeta_0) \\ &= \frac{a}{a+1} F(z_0) + \frac{1}{a+1} z_0 F'(z_0) \\ &= \mathcal{J}^{s,a}_{\lambda,\mu,\nu} f(z) \in \varphi(\mathrm{U}), \end{aligned}$$

which contradicts the above observation (2.15) that $L(\zeta_0, t) \notin \varphi(\mathrm{U})$. Therefore, the subordination condition (2.3) must imply the subordination given by (2.10). Considering $F(z) = G(z)$, we see that the function G is the best dominant, which completes our proof.

Applying the similar method used in the proof of Theorem 2.1, as well as the relations (1.10)–(1.11), we easily get the following results.

Corollary 2.1 *Let $f, g \in \mathcal{A}$ and $\lambda \geq 0$. Suppose that*

$$\mathrm{Re}\left(1 + \frac{z\psi''(z)}{\psi'(z)}\right) > -\tau, \quad z \in \mathrm{U}$$

with $\psi(z) = \mathcal{J}^{s,a}_{\lambda+1,\mu,\nu} g(z)$, where $\tau = 0$ if $\lambda = 0$, and

$$\tau = \tau(\lambda) = \begin{cases} \dfrac{\lambda}{2}, & \text{if } 0 < \lambda \leq 1, \\ \dfrac{1}{2\lambda}, & \text{if } \lambda > 1. \end{cases} \tag{2.16}$$

Then, the subordination condition

$$\mathcal{J}^{s,a}_{\lambda+1,\mu,\nu} f(z) \prec \mathcal{J}^{s,a}_{\lambda+1,\mu,\nu} g(z) \tag{2.17}$$

implies

$$\mathcal{J}^{s,a}_{\lambda,\mu,\nu} f(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu} g(z).$$

Moreover, the function $\mathcal{J}^{s,a}_{\lambda,\mu,\nu} g$ is the best dominant of (2.17).

Corollary 2.2 *Let $f, g \in \mathcal{A}$ and $\lambda \geq 0$. Suppose that*

$$\mathrm{Re}\left(1 + \frac{z\vartheta''(z)}{\vartheta'(z)}\right) > -\sigma, \quad z \in \mathrm{U}$$

with $\vartheta(z) = \mathcal{J}^{s,a}_{\lambda,\mu,\nu} g(z)$, where $\sigma = 0$ if $\nu = 0$, and

$$\sigma = \sigma(\nu) = \begin{cases} \dfrac{\nu}{2}, & \text{if } 0 < \nu \leq 1, \\ \dfrac{1}{2\nu}, & \text{if } \nu > 1. \end{cases} \tag{2.18}$$

Then, the subordination condition

$$\mathcal{J}^{s,a}_{\lambda,\mu,\nu}f(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu}g(z) \tag{2.19}$$

implies

$$\mathcal{J}^{s,a}_{\lambda,\mu,\nu+1}f(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu+1}g(z).$$

Moreover, the function $\mathcal{J}^{s,a}_{\lambda,\mu,\nu+1}g$ is the best dominant of (2.19).

We next prove the dual result of Theorem 2.1, in the sense that subordinations are replaced by superordinations.

Theorem 2.2 *Let $f, g \in \mathcal{A}$, $a > 0$ and*

$$\operatorname{Re}\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > -\rho, \quad z \in \mathrm{U}$$

with $\varphi(z) = \mathcal{J}^{s,a}_{\lambda,\mu,\nu}g(z)$, where ρ is given by (2.2)*. Suppose that the function $\mathcal{J}^{s,a}_{\lambda,\mu,\nu}f$ is univalent in* U*, and $\mathcal{J}^{s+1,a}_{\lambda,\mu,\nu}f \in \mathcal{H}[0,1] \cap \mathcal{Q}$.*

Then, the subordination condition

$$\mathcal{J}^{s,a}_{\lambda,\mu,\nu}g(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu}f(z) \tag{2.20}$$

implies

$$\mathcal{J}^{s+1,a}_{\lambda,\mu,\nu}g(z) \prec \mathcal{J}^{s+1,a}_{\lambda,\mu,\nu}f(z). \tag{2.21}$$

Moreover, the function $\mathcal{J}^{s+1,a}_{\lambda,\mu,\nu}g$ is the best subordinant of (2.20).

Proof Since the proof has similar parts as that of the previous theorem, we will use the same notations as in the proof of Theorem 2.1.

Let F and G be two functions defined by (2.4). If the function q is defined by (2.5), similarly, as in the proof of Theorem 2.1, we obtain that

$$\varphi(z) = \frac{a}{a+1}G(z) + \frac{1}{a+1}zG'(z) \equiv \phi(G(z), zG'(z)).$$

Using the same method as in the proof of the above theorem, we may prove that $\operatorname{Re} q(z) > 0$ for all $z \in \mathrm{U}$, i.e., the function G defined by (2.4) is convex (univalent) in U.

Next we prove that the subordination condition (2.20) implies $G(z) \prec F(z)$. Considering the function $L(z,t)$ defined by

$$L(z,t) \equiv \frac{a}{a+1}G(z) + \frac{t}{a+1}zG'(z), \quad z \in \mathrm{U},\ t \geq 0, \tag{2.22}$$

we have

$$\left.\frac{\partial L(z,t))}{\partial z}\right|_{z=0} = \frac{a+t}{a+1}G'(0) = \frac{a+t}{a+1} \neq 0, \quad t \geq 0.$$

Hence, the function $L(z,t) = a_1(t)z + \cdots$ satisfies the conditions $a_1(t) \neq 0$ for all $t \geq 0$ and $\lim\limits_{t\to+\infty} |a_1(t)| = +\infty$.

From the definition (2.22), for all $t \geq 0$, we have

$$\frac{|L(z,t)|}{|a_1(t)|} = \frac{|\frac{a}{a+1}G(z) + \frac{t}{a+1}zG'(z)|}{\frac{a+t}{a+1}} \leq \frac{\frac{a}{a+1}|G(z)| + \frac{t}{a+1}|zG'(z)|}{\frac{a+t}{a+1}}. \tag{2.23}$$

Since G is convex and normalized, using the right-hand sides of the inequalities (2.13) in (2.23), we deduce that

$$\frac{|L(z,t)|}{|a_1(t)|} \leq \frac{r}{(1-r)^2} \frac{t + a(1-r)}{a+t} \leq \frac{r}{(1-r)^2}, \quad |z| \leq r, \ t \geq 0.$$

Hence, the second assumption of Lemma 1.5 holds. Moreover,

$$\mathrm{Re}\left[z\frac{\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right] = a + t\,\mathrm{Re}\left(1 + \frac{zG''(z)}{G'(z)}\right) > 0, \quad z \in \mathrm{U}, \ t \geq 0,$$

and according to Lemma 1.5, the function $L(z,t)$ is a subordination chain. Therefore, according to Lemma 1.4, we conclude that the superordination condition (2.20) implies the superordination (2.21). Furthermore, since the differential equation (2.21) has the univalent solution G, it is the best subordinant of the given differential superordination, which completes the proof of the theorem.

Applying a similar method as in the proof of Theorem 2.2 and using (1.10)–(1.11), we easily obtain the following results.

Corollary 2.3 *Let $f, g \in \mathcal{A}$, $\lambda \geq 0$ and*

$$\mathrm{Re}\left(1 + \frac{z\psi''(z)}{\psi'(z)}\right) > -\tau, \quad z \in \mathrm{U}$$

with $\psi(z) = \mathcal{J}^{s,a}_{\lambda+1,\mu,\nu}g(z)$, where τ is given by (2.16)*. Suppose that the function $\mathcal{J}^{s,a}_{\lambda+1,\mu,\nu}f$ is univalent in* U*, and $\mathcal{J}^{s,a}_{\lambda,\mu,\nu}f \in \mathcal{H}[0,1] \cap \mathcal{Q}$.*

Then, the subordination condition

$$\mathcal{J}^{s,a}_{\lambda+1,\mu,\nu}g(z) \prec \mathcal{J}^{s,a}_{\lambda+1,\mu,\nu}f(z) \tag{2.24}$$

implies

$$\mathcal{J}^{s,a}_{\lambda,\mu,\nu}g(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu}f(z).$$

Moreover, the function $\mathcal{J}^{s,a}_{\lambda,\mu,\nu}g$ is the best subordinant of (2.24)*.*

Corollary 2.4 *Let $f, g \in \mathcal{A}$, $\nu \geq 0$ and*

$$\mathrm{Re}\left(1 + \frac{z\vartheta''(z)}{\vartheta'(z)}\right) > -\sigma, \quad z \in \mathrm{U}$$

with $\vartheta(z) = \mathcal{J}^{s,a}_{\lambda,\mu,\nu}g(z)$, where σ is given by (2.18)*. Further, suppose that the function $\mathcal{J}^{s,a}_{\lambda,\mu,\nu}f$ is univalent in* U*, and $\mathcal{J}^{s,a}_{\lambda,\mu,\nu+1}f \in \mathcal{H}[0,1] \cap \mathcal{Q}$.*

Then, the subordination condition

$$\mathcal{J}^{s,a}_{\lambda,\mu,\nu}g(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu}f(z) \tag{2.25}$$

implies

$$\mathcal{J}^{s,a}_{\lambda,\mu,\nu+1}g(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu+1}f(z).$$

Moreover, the function $\mathcal{J}^{s,a}_{\lambda,\mu,\nu+1}g$ *is the best subordinant of* (2.25).

By combining the above mentioned subordination and superordination results involving the operator $\mathcal{J}^{s,a}_{\lambda,\mu,\nu}$, the following three sandwich-type results are derived.

Theorem 2.3 *Let* $f, g_k \in \mathcal{A}$ $(k = 1, 2)$, $a \geq 0$, *and*

$$\mathrm{Re}\left(1 + \frac{z\varphi_k''(z)}{\varphi_k'(z)}\right) > -\rho, \quad z \in \mathrm{U}$$

with $\varphi_k(z) = \mathcal{J}^{s,a}_{\lambda,\mu,\nu}g_k(z)$ $(k = 1, 2)$, *where* ρ *is given by* (2.2). *Suppose that the function* $\mathcal{J}^{s,a}_{\lambda,\mu,\nu}f$ *is univalent in* U, *and* $\mathcal{J}^{s+1,a}_{\lambda,\mu,\nu}f \in \mathcal{H}[0, 1] \cap \mathcal{Q}$.

Then, the double subordination

$$\mathcal{J}^{s,a}_{\lambda,\mu,\nu}g_1(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu}f(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu}g_2(z) \tag{2.26}$$

implies

$$\mathcal{J}^{s+1,a}_{\lambda,\mu,\nu}g_1(z) \prec \mathcal{J}^{s+1,a}_{\lambda,\mu,\nu}f(z) \prec \mathcal{J}^{s+1,a}_{\lambda,\mu,\nu}g_2(z).$$

Moreover, the functions $\mathcal{J}^{s+1,a}_{\lambda,\mu,\nu}g_1$ *and* $\mathcal{J}^{s+1,a}_{\lambda,\mu,\nu}g_2$ *are the best subordinant and the best dominant of* (2.26), *respectively.*

Theorem 2.4 *Let* $f, g_k \in \mathcal{A}$ $(k = 1, 2)$, $\lambda > 0$ *and*

$$\mathrm{Re}\left(1 + \frac{z\psi_k''(z)}{\psi_k'(z)}\right) > -\tau, \quad z \in \mathrm{U}$$

with $\psi_k(z) = \mathcal{J}^{s,a}_{\lambda+1,\mu,\nu}g_k(z)$ $(k = 1, 2)$, *where* τ *is given by* (2.16). *Suppose that the function* $\mathcal{J}^{s,a}_{\lambda+1,\mu,\nu}f$ *is univalent in* U, *and* $\mathcal{J}^{s,a}_{\lambda,\mu,\nu}f \in \mathcal{H}[0, 1] \cap \mathcal{Q}$.

Then, the double subordination

$$\mathcal{J}^{s,a}_{\lambda+1,\mu,\nu}g_1(z) \prec \mathcal{J}^{s,a}_{\lambda+1,\mu,\nu}f(z) \prec \mathcal{J}^{s,a}_{\lambda+1,\mu,\nu}g_2(z) \tag{2.27}$$

implies

$$\mathcal{J}^{s,a}_{\lambda,\mu,\nu}g_1(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu}f(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu}g_2(z).$$

Moreover, the functions $\mathcal{J}^{s,a}_{\lambda,\mu,\nu}g_1$ *and* $\mathcal{J}^{s,a}_{\lambda,\mu,\nu}g_2$ *are the best subordinant and the best dominant of* (2.27), *respectively.*

Theorem 2.5 *Let* $f, g_k \in \mathcal{A}$ $(k = 1, 2)$, $\nu > 0$ *and*

$$\mathrm{Re}\left(1 + \frac{z\vartheta_k''(z)}{\vartheta_k'(z)}\right) > -\sigma, \quad z \in \mathrm{U}$$

with $\vartheta_k(z) = \mathcal{J}^{s,a}_{\lambda,\mu,\nu}g_k(z)$ $(k = 1, 2)$, *where* σ *is given by* (2.18). *Suppose that the function* $\mathcal{J}^{s,a}_{\lambda,\mu,\nu}f$ *is univalent in* U, *and* $\mathcal{J}^{s,a}_{\lambda,\mu,\nu+1}f \in \mathcal{H}[0, 1] \cap \mathcal{Q}$.

Then, the double subordination

$$\mathcal{J}^{s,a}_{\lambda,\mu,\nu}g_1(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu}f(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu}g_2(z) \tag{2.28}$$

implies

$$\mathcal{J}^{s,a}_{\lambda,\mu,\nu+1}g_1(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu+1}f(z) \prec \mathcal{J}^{s,a}_{\lambda,\mu,\nu+1}g_2(z).$$

Moreover, the functions $\mathcal{J}^{s,a}_{\lambda,\mu,\nu+1}g_1$ *and* $\mathcal{J}^{s,a}_{\lambda,\mu,\nu+1}g_2$ *are the best subordinant and the best dominant of* (2.28), *respectively.*

3 Applications and Concluding Remarks

As an interesting application, let us define a linear operator $\mathcal{S}^m_a f : \mathcal{A} \to \mathcal{A}$ $(m \in \mathbb{N}_0,\ a \geq 0)$ by

$$\mathcal{S}^0_a f(z) = f(z), \quad \mathcal{S}^{m+1}_a(z) = \frac{1}{a+1}[a\mathcal{S}^m_a(z) + z(\mathcal{S}^m_a(z))'], \quad m \in \mathbb{N}.$$

For

$$\mathrm{I}_d(z) = \frac{z}{1-z},$$

denote $s_{m,a}(z) \equiv \mathcal{S}^m_a \, \mathrm{I}_d(z)$. Then the explicit form of the function $s_{m,a}$ is given by

$$s_{m,a}(z) = z + \sum_{n=2}^{\infty} \left(\frac{n+a}{1+a}\right)^m z^n, \quad z \in \mathrm{U}. \tag{3.1}$$

Taking $s = m$ $(m = \mathbb{N}_0)$ and $g(z) = z(s_{m,a}(z))'$ in Theorem 2.1, we obtain the following special case.

Theorem 3.1 *Let* $f \in \mathcal{A}$, $a \geq 0$ *and* $m \in \mathbb{N}_0$. *Suppose that*

$$\operatorname{Re} \frac{{}_4F_3(\lambda+1, \mu+1, 2, 2; \nu+1, 1, 1; z)}{{}_3F_2(\lambda+1, \mu+1, 2; \nu+1, 1; z)} > -\rho, \quad z \in \mathrm{U},$$

where $\rho = 0$ *if* $a = 0$, *and* ρ *is given by* (2.1) *if* $a > 0$. *Then the subordination condition*

$$\mathcal{J}^{m,a}_{\lambda,\mu,\nu}f(z) \prec z\,{}_2F_1(\lambda+1, \mu+1; \nu+1; z) \tag{3.2}$$

implies

$$\mathcal{J}^{m+1,a}_{\lambda,\mu,\nu}f(z) \prec z\,{}_3F_2(\lambda+1, \mu+1, a+1; \nu+1, a+2; z).$$

Moreover, the function $z\,{}_3F_2(\lambda+1, \mu+1, a+1; \nu+1, a+2; z)$ *is the best dominant of* (3.2).

Further, setting $\lambda = \nu$ and $\mu = 1$ in the above theorem, we get the result below.

Corollary 3.1 *Let* $f \in \mathcal{A}$, $a \geq 0$ *and* $m \in \mathbb{N}_0$. *Suppose that*

$$\operatorname{Re} \frac{{}_3F_2(2, 2, 2; 1, 1; z)}{{}_2F_1(2, 2; 1; z)} > -\rho, \quad z \in \mathrm{U},$$

where $\rho = 0$ *if* $a = 0$, *and* ρ *is given by* (2.1) *if* $a > 0$.

Then, the subordination condition

$$J_{m,a}f(z) \prec \frac{z}{(1-z)^2} \tag{3.3}$$

implies

$$J_{m+1,a}f(z) \prec {}_2F_1(a+1, 2; a+2; z).$$

Moreover, the function ${}_2F_1(a+1, 2; a+2; z)$ *is the best dominant of* (3.3). *Here,* $J_{m,a} \equiv \mathcal{J}^{m,a}_{\lambda,1;\lambda}$ *is the Srivastava-Attiya integral operator already mentioned.*

We conclude this paper by remarking that in view of the generalized operator defined by (1.8) and expressed in terms of convolution (1.6) involving arbitrary coefficients, the main results would lead to additional new results. In fact, by appropriately selecting the arbitrary parameters in (1.8), the results presented in this paper would find further applications which incorporate the generalized form of linear operators. These considerations can be fruitfully worked out and we skip the details in this regard.

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