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L^2-Algebraic Decay Rate for Transient Birth-Death Processes*

Lijuan CHENG[1] Yingzhe WANG[2]

Abstract This paper is a continuation of the study of the algebraic speed for Markov processes. The authors concentrate on algebraic decay rate for the transient birth-death processes. According to the classification of the boundaries, a series of the sufficient conditions for algebraic decay is presented. To illustrate the power of the results, some examples are included.

Keywords Transient, Birth-death processes, Algebraic decay
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1 Introduction

Consider a birth-death process on the nonnegative integers $\mathbb{Z}_+$ with birth rates $b_n > 0$ ($n \geq 1$), $b_0 \geq 0$, and death rates $a_n > 0$ ($n \geq 1$). Let $(p_{ij}(t))$ be the corresponding transition probabilities. If $b_0 > 0$, i.e., the process has reflecting boundary at origin, we define

$$\mu_0 = 1, \quad \mu_n = \frac{b_0 \cdots b_{n-1}}{a_1 \cdots a_n}, \quad n \geq 1;$$

or, if $b_0 = 0$, the process has absorbing boundary at origin and we define

$$\mu_1 = 1, \quad \mu_n = \frac{b_1 \cdots b_{n-1}}{a_2 \cdots a_n}, \quad n \geq 2,$$

then, the process is μ-symmetric, $\mu_i p_{ij}(t) = \mu_j p_{ji}(t)$ for all i, j and t.

From now on, we suppose that the following conditions hold:

$$\sum_{n=1}^{\infty} \frac{1}{b_n \mu_n} < \infty, \quad \sum_{n=1}^{\infty} \mu_n = \infty. \tag{1.1}$$

These conditions imply that the process is absorbed at ∞. And in fact they are the criteria for the transient property of the birth-death process with reflecting boundary at origin. It is well-known that in the transient case, the transition probabilities $(p_{ij}(t))$ satisfy

$$\lim_{t \to \infty} p_{ij}(t) = 0 \quad \text{for all } i, j \in \mathbb{Z}_+.$$

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[1]Laboratory of Mathematics and Complex Systems, the Ministry of Education of China; School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China. E-mail: chenglj@mail.bnu.edu.cn
[2]Corresponding author. Laboratory of Mathematics and Complex Systems, the Ministry of Education of China; School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China.
E-mail: wangyz@bnu.edu.cn
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The process corresponds in a natural way to a strongly continuous semigroup (P_t) on $L^2(\mu)$ with generator L and domain $\mathcal{D}(L)$. It is said that the process has algebraic decay in L^2-sense if there exists a functional $V : L^2(\mu) \longrightarrow [0,\infty]$ and constants $C > 0$, $q > 1$, such that

$$\|P_t f\|^2 \le CV(f) \cdot t^{1-q}, \quad t > 0,\ f \in L^2(\mu), \tag{1.2}$$

where $\|\cdot\|$ denotes the L^2-norm with respect to μ.

When the process is ergodic, the problem of the algebraic convergence was well studied (see [1] for instance). For further development in diffusion processes, see [2–3]. Mao [4] used another method to study the algebraic convergence for discrete-time ergodic Markov chain. Wang [5] used functional inequalities to discuss the decay of semigroups. In this paper, we focus our attention on the transient birth-death processes, and work out some explicit criteria for the algebraic L^2-decay of the semigroup with respect to special type of $V(f)$.

The starting point of our study is the following result of Liggett [6], which provides some necessary and sufficient conditions for L^2-algebraic decay.

Theorem 1.1 (see [6]) *Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and let $V : L^2(\mu) \longrightarrow [0,\infty]$ satisfy $V(cf) = c^2 V(f)$ for all constants c. Consider the following two statements*:

(a) *There exists a constant $C' > 0$, such that*

$$\|f\|^2 \le C' D(f)^{\frac{1}{p}} V(f)^{\frac{1}{q}} \quad \text{for all } f \in \mathcal{D}(D), \tag{1.3}$$

where $D(f) := D(f,f)$ is the Dirichlet form of L with domain $\mathcal{D}(D)$;

(b) *There exists a constant $C > 0$, such that* (1.2) *holds.*

We have the following conclusions:

(1) *If* (a) *holds and V satisfies the following contraction*: *there exists some constant $c_1 > 0$, such that*

$$V(P_t f) \le c_1 V(f), \quad f \in L^2(\mu),\ t > 0. \tag{1.4}$$

Then (b) *holds.*

(2) *If* (b) *holds, then so does* (a) *when the process is reversible with respect to μ.*

Remark 1.1 In the original Liggett-Stroock theorem in [6], the contraction property of V is represented as $V(P_t f) \le V(f)$. Here, we use $V(P_t f) \le cV(f)$ instead of $V(P_t f) \le V(f)$. It is easy to check that the Liggett-Stroock theorem still holds.

In fact, the criteria for algebraic decay of this paper are some more explicit conditions for (1.3) of the Liggett-Stroock theorem in the context of the transient birth-death processes.

2 Main Results

For the birth-death process with birth rates b_n $(n \ge 0)$ and death rates a_n $(n \ge 1)$, the corresponding operator is $\Omega f(i) := b_i(f_{i+1} - f_i) + a_i(f_{i-1} - f_i),\ i \in \mathbb{Z}_+$. Denote the Dirichlet form by $(D, \mathcal{D}(D))$,

$$D(f) = \sum_{i \ge 1} \mu_i a_i (f_i - f_{i-1})^2$$

with domain $\mathcal{D}(D) = \{f \in L^2(\mu) : D(f) < \infty\}$. If $b_0 > 0$, we can also use the Dirichlet form as $D(f) = \sum_{i\geq 0} \mu_i b_i (f_{i+1} - f_i)^2$. It is necessary to discuss the regularity of the processes, since the process will not be unique. But it is lucky for us to have already known from Chen [7] that $(D, \mathcal{D}(D))$ is regular under the situation discussed now, i.e., the reversible process is just the minimal process.

In this paper, we construct new types of $V(f)$, which is different from the ergodic case but more convenient for us to discuss. Under these types of $V(f)$, a series of necessary and sufficient criteria for the transient birth-death processes is presented according to the classification of boundaries.

We start from discussing the contraction property of V. Let ϕ be a positive function defined on $\mathbb{Z}_+$. For $\delta = 0$ or 1, define

$$V_\delta(f) = \sup_i \frac{f_i^2 \mu_i^2 b_i}{\phi_i^{2\delta}}. \tag{2.1}$$

Then, we have the following criterion for the contraction of $V_\delta(f)$.

Lemma 2.1 *Let $\psi_\delta(i) = \frac{\phi_i^\delta}{\mu_i\sqrt{b_i}}$, where $\delta = 0$ or 1. If there exists an $i_0 \in \mathbb{Z}_+$, such that*

$$\Omega\left(\psi_\delta\right)(i) \leq 0 \quad \text{for all } i \geq i_0,$$

then we have $V_\delta(P_t f) \leq cV_\delta(f)$ for some constant $c > 0$.

Now, we discuss the birth-death process with absorbing (Dirichlet) boundary at infinity and reflecting or absorbing boundary at origin. Let

$$\sigma(i) := \frac{\phi_i}{\mu_i\sqrt{b_i}} \sum_{k\leq i} \frac{\mu_k}{\phi_k^2}, \quad i \in \mathbb{Z}_+, \quad \widetilde{\sigma}(i) := \frac{\phi_i}{\mu_i\sqrt{a_i}} \sum_{j=i}^{\infty} \frac{\mu_j}{\phi_j^2}, \quad i \in \mathbb{Z}_+ \setminus \{0\}.$$

The main results are presented as follows.

Theorem 2.1 *Suppose that the birth-death process satisfies the conditions* (1.1). *Let that the functional V_δ be defined as* (2.1).

(1) *Under the conditions stated below, the transient birth-death process with reflecting boundary at origin has algebraic decay with respect to $V = V_\delta$ ($\delta = 0$ or 1) and the same q, i.e.,* (1.2) *holds.*

(a) *There exists an $i_0 \in \mathbb{Z}_+$, such that $\sqrt{b_{i-1}}\phi_{i-1}^\delta - \frac{a_i\phi_i^\delta}{\sqrt{b_i}}$ is increasing for all $i \geq i_0$;*

(b) $\sup_{n\in\mathbb{Z}_+} \sigma(n) < \infty$;

(c) $\sum_{n=1}^{\infty} \frac{1}{\mu_n b_n} \phi_n^{2(q+\delta-1)} < \infty$ *for some constant $q > 1$.*

Furthermore, if the condition (b) *is replaced by the condition* (b$'$) *below, then the transient birth-death process with absorbing boundary at origin has algebraic decay with respect to $V = V_\delta$ ($\delta = 0$ or 1) and the same q.*

(b$'$) $\sup_{n\in\mathbb{Z}_+} \sigma(n) < \infty$ *or* $\sup_{n\in\mathbb{Z}_+/\{0\}} \widetilde{\sigma}(n) < \infty$.

(2) *Conversely, suppose that the process has algebraic decay with respect to V_δ ($\delta = 0, 1$). Additionally, suppose the function ϕ in the expression of V_δ has the property that ϕ is increasing, $\phi_0 > 0$, and satisfies one of the following two conditions:*

(i) $0 < \liminf_{N\to\infty} \frac{\phi_N}{\phi_{2N}} \le \limsup_{N\to\infty} \frac{\phi_N}{\phi_{2N}} < 1$;

(ii) *There exists an* $R \in \mathbb{Z}^+$*, such that* $0 < \liminf_{N\to\infty} \frac{\phi_N}{\phi_{N+R}} \le \limsup_{N\to\infty} \frac{\phi_N}{\phi_{N+R}} < 1$.

Moreover, if there exists some constant $\beta > 0$*, such that*

$$c(m) := \sup_i \frac{(a_{i+1}\phi_{i+1}^m - \sqrt{b_i b_{i+1}}\phi_i^m)^2}{b_{i+1}\phi_i^{2m-2\beta}} < \infty \quad \textit{for all } m \in \mathbb{Z}^+,$$

then we have $\sum \frac{1}{\mu_i b_i} \cdot \phi_i^k < \infty$ *for all* $k < 2(q-1)\beta + 2\delta$.

Remark 2.1 (1) The process with absorbing boundary at 0 can be decayed at ∞ (explosive) or at 0 (extinctive). These two criteria in (b$'$) stand for the absorbing conditions on the infinity and the origin respectively. we can use them to evaluate on which side the process is decayed faster. It is convenient for us to calculate these two conditions separately.

(2) For the test function ϕ_i in Theorem 2.1(1), we can get it from condition (c). It may have the form i^n or a^i for some $n > 0,\ a > 0$. Then check the conditions (a) and (b).

(3) A typical choice of ϕ in Theorem 2.1(2) is as follows: There exist constants $\alpha > 0,\ c_1 > 0$ and $c_2 < \infty$, such that $c_1 \le \frac{\phi_i}{i^\alpha} \le c_2$ for all $i \ge 1$. Then the condition (i) holds. Otherwise, let ϕ satisfy the following: There exist constants $\alpha > 1,\ c_1' > 0$ and $c_2' < \infty$, such that $c_1' \le \frac{\phi_i}{\alpha^i} \le c_2'$ for all $i \ge 1$. Then the condition (ii) holds.

To illustrate the power of our results, there will be several examples presented in the last section.

Now, we consider another type of $V(f)$. This construction of V is due to Chen and Wang [1]. Let u_n be a positive sequence and denote $\rho_{ij} = \Big|\sum_{k<j} u_k - \sum_{k<i} u_k\Big|$. Define

$$\widetilde{V}_\delta(f) := \sup_{i\ne j} \frac{|f(i)-f(j)|^2}{\rho_{ij}^{2\delta}} = \sup_{k\ge 0} \frac{|f(k+1)-f(k)|^2}{u_k^{2\delta}}, \quad \delta = 0, 1. \tag{2.2}$$

Going back to the work of Chen and Wang [1], we know that the contraction (1.4) works automatically for $\widetilde{V}_0$ and a sufficient condition for (1.4) with $\widetilde{V}_1$ is the following: There exists a coupling operator $\widetilde{\Omega}$, such that

$$\widetilde{\Omega}\rho(i,j) \le 0 \ \text{ for all } i \ne j \quad \text{and} \quad \widetilde{\Omega}\rho(i,i) = 0 \ \text{ for all } i.$$

See [1] for the proof. Then we have the following theorem.

Theorem 2.2 *Suppose that the birth-death process satisfies the conditions* (1.1)*. Let* $\widetilde{V}_1$ *be defined as* (2.2)*. Denote* $\sigma_n := \frac{\sum_{i=0}^n \mu_i}{\mu_n}$*. Suppose that the following conditions hold:*

(i) $b_n u_n - a_n u_{n-1}$ *is nonincreasing* ($u_{-1} = 0$);

(ii) $\sum_i \mu_i b_i (\frac{\sigma_i^2}{b_i})^q u_i^2 < \infty$ *for some* $q > 1$.

Then the transient birth-death process has algebraic decay with respect to $\widetilde{V}_1$.

Remark 2.2 In fact, Theorem 2.2 is true for the transient birth-death process with reflecting or absorbing boundary at origin, because what we consider is the absorbing rate by the state ∞. The detailed proof will appear in the next section.

It is much easier for us to consider the algebraic decay with respect to $\widetilde{V}_0$, as the contraction is satisfied automatically. Using the similar method as in Theorem 2.2, we have the following result.

Corollary 2.1 Consider the birth-death processes as in Theorem 2.2. If $\sum\limits_{i=0}^{\infty}\mu_i b_i(\frac{\sigma_i^2}{b_i})^q<\infty$ for some $q>1$, then the processes have algebraic decay with $\widetilde{V}_0$ and the same q.

3 Proofs of the Main Results

Proof of Lemma 2.1 Due to $\Omega(\psi_\delta)(i)\le 0$ for all $i\ge i_0$, we have $\mathbb{E}^i\psi_\delta(X_t)\le c\psi_\delta(i)$ for some constant c and for all $i\in\mathbb{Z}_+$. Therefore, we get

$$\begin{aligned}\Big|\frac{P_tf(i)}{\psi_\delta(i)}\Big|^2&=\Big|\frac{\mathbb{E}^if(X_t)}{\psi_\delta(i)}\Big|^2=\Big|\mathbb{E}^i\Big(\frac{f(X_t)}{\psi_\delta(X_t)}\cdot\frac{\psi_\delta(X_t)}{\psi(i)}\Big)\Big|^2\\&\le\sup_{k\in\mathbb{Z}_+}\Big|\frac{f(k)}{\psi_\delta(k)}\Big|^2\Big(\frac{\mathbb{E}^i(\psi_\delta(X_t))}{\psi_\delta(i)}\Big)^2\le c^2\cdot V_\delta(f).\end{aligned}\tag{3.1}$$

Taking the supremum over i on the left-hand side yields $V_\delta(P_tf)\le c^2V_\delta(f)$.

Proof of Theorem 2.1 Some ideas of the proof are taken from [1].

(1) We prove part (1) of the theorem.

(i) Firstly, suppose that the process has reflecting boundary at origin. Let f satisfy $\|f\|^2=1$. As the process satisfies the conditions (1.1), it is easy to know that

$$f_\infty:=\lim_{N\to\infty}f_N=0,\quad\text{as}\ \sum_{n=1}^{\infty}\mu_n=\infty.$$

Then we have

$$\begin{aligned}1=\|f\|^2&=\sum_{i=0}^{\infty}\mu_if_i^2=\sum_{i=0}^{\infty}\mu_i\Big(\frac{f_i}{\phi_i}\Big)^2\phi_i^2\\&\le\Big\{\sum_{i=0}^{\infty}\mu_i\Big(\frac{f_i}{\phi_i}\Big)^2\Big\}^{\frac1p}\Big\{\sum_{i=0}^{\infty}\frac{1}{\mu_ib_i}\Big(\frac{f_i}{\phi_i^\delta}\Big)^2\cdot\mu_i^2b_i\cdot\phi_i^{2(q+\delta-1)}\Big\}^{\frac1q}\\&=:\mathrm{I}^{\frac1p}\mathrm{II}^{\frac1q}.\end{aligned}$$

Now, consider the first part I.

$$\begin{aligned}\mathrm{I}&=\sum_{i=0}^{\infty}\frac{\mu_i}{\phi_i^2}\cdot f_i^2=\sum_{i=0}^{\infty}\frac{\mu_i}{\phi_i^2}\Big(\sum_{j=i}^{\infty}(f_j-f_{j+1})\Big)^2\\&\le2\sum_{i=0}^{\infty}\frac{\mu_i}{\phi_i^2}\sum_{i\le k\le l}(f_k-f_{k+1})(f_l-f_{l+1})\\&=2\sum_{k=0}^{\infty}(f_k-f_{k+1})\sqrt{\mu_kb_k}\sum_{i=0}^{k}\frac{\mu_i}{\phi_i^2\sqrt{\mu_kb_k}}\sum_{l=k}^{\infty}(f_l-f_{l+1})\\&\le2\Big(\sum_{k=0}^{\infty}(f_k-f_{k+1})^2\mu_kb_k\Big)^{\frac12}\Big(\sum_{k=0}^{\infty}\Big(\sum_{i=0}^{k}\frac{\mu_i}{\phi_i^2\sqrt{\mu_kb_k}}\cdot f_k\Big)^2\Big)^{\frac12}.\end{aligned}\tag{3.2}$$

Here, we use the Schwarz inequality in the last step as

$$\sum_{k=0}^{\infty}\Big(\sum_{i=0}^{k}\frac{\mu_i}{\phi_i^2\sqrt{\mu_k b_k}}\cdot f_k\Big)^2=\sum_{k=0}^{\infty}f_k^2\cdot\frac{\mu_k}{\phi_k^2}\cdot\frac{\phi_k^2}{\mu_k}\cdot\frac{1}{\mu_k b_k}\Big(\sum_{i=0}^{k}\frac{\mu_i}{\phi_i^2}\Big)^2$$
$$=\sum_{k=0}^{\infty}f_k^2\cdot\frac{\mu_k}{\phi_k^2}\cdot\sigma^2(k)\le \mathrm{I}\cdot\sup_k\sigma^2(k). \tag{3.3}$$

When $\sup\limits_k\sigma^2(k)<\infty$, combining the above two inequalities (3.2)–(3.3), we see that

$$\mathrm{I}\le 2\sqrt{D(f)}\sqrt{\sup_k\sigma^2(k)\mathrm{I}}\ .$$

Solving the inequality, we get $\mathrm{I}\le C_1D(f)$, where $C_1=4\sup\limits_k\sigma^2(k)<\infty$.

Next, consider the second part II.

$$\mathrm{II}=\sum_i\frac{1}{\mu_i b_i}\Big(\frac{f_i}{\phi_i^\delta}\Big)^2\cdot\mu_i^2b_i\cdot\phi_i^{2(q+\delta-1)}\le V_\delta(f)\sum_i\frac{1}{\mu_i b_i}\phi_i^{2(q+\delta-1)}.$$

Hence,

$$\|f\|^2\le CD(f)^{\frac1p}V_\delta(f)^{\frac1q},$$

where $C=C_1^{\frac1p}\Big(\sum\limits_i\frac{1}{\mu_ib_i}\phi_i^{2(q+\delta-1)}\Big)^{\frac1q}<\infty$ by assumption. By the Liggett-Stroock theorem, the process has algebraic decay.

(ii) Secondly, suppose the process has absorbing boundary at origin. Let f satisfy $\|f\|^2=1, f_0=0$ and $f_\infty=0$. As same as the proof of reflecting boundary case, we have

$$1=\|f\|^2=\sum_{i=1}^{\infty}\mu_if_i^2$$
$$\le\Big\{\sum_{i=1}^{\infty}\mu_i\Big(\frac{f_i}{\phi_i}\Big)^2\Big\}^{\frac1p}\Big\{\sum_{i=1}^{\infty}\frac{1}{\mu_ib_i}\cdot\frac{f_i^2\mu_i^2b_i}{\phi_i^{2\delta}}\cdot\phi_i^{2(q-1+\delta)}\Big\}^{\frac1q}$$
$$=:\mathrm{I}^{\frac1p}\mathrm{II}^{\frac1q},\quad \delta=0,1. \tag{3.4}$$

We can use another way to deal with the first part I.

$$\mathrm{I}=\sum_{i=1}^{\infty}\frac{\mu_i}{\phi_i^2}\Big(\sum_{k=1}^{i}(f_k-f_{k-1})\Big)^2\le2\sum_{i=1}^{\infty}\frac{\mu_i}{\phi_i^2}\Big\{\sum_{k=1}^{i}\sum_{1\le k\le l\le i}(f_k-f_{k-1})(f_l-f_{l-1})\Big\}$$
$$=2\sum_{l=1}^{\infty}(f_l-f_{l-1})\sqrt{\mu_la_l}\cdot\frac{1}{\sqrt{\mu_la_l}}\sum_{i=l}^{\infty}\frac{\mu_i}{\phi_i^2}\sum_{k=1}^{l}(f_k-f_{k-1})$$
$$\le2\Big(\sum_{l=1}^{\infty}|f_l-f_{l-1}|^2\mu_la_l\Big)^{\frac12}\Big(\sum_{l=1}^{\infty}\frac{f_l^2}{\mu_la_l}\Big(\sum_{i=l}^{\infty}\frac{\mu_i}{\phi_i^2}\Big)^2\Big)^{\frac12}$$
$$=2\sqrt{D(f)}\Big(\sum_{l=1}^{\infty}\mu_l\frac{f_l^2}{\phi_l^2}\cdot\frac{\phi_l^2}{\mu_l^2a_l}\Big(\sum_{i=l}^{\infty}\frac{\mu_i}{\phi_i^2}\Big)^2\Big)^{\frac12}$$
$$\le2\sqrt{D(f)}\cdot\Big(\mathrm{I}\cdot\sup_i\widetilde{\sigma}^2(i)\Big)^{\frac12}.$$

The following discussion is similar to (i), and here we omit it.

(2) We now prove part (2) of the theorem. We select a particular function as

$$f(k) = \frac{\phi_k^m}{\mu_k\sqrt{b_k}} \cdot \mathrm{I}_{E_N}(k)$$

for some positive natural number $m > \beta$, where $\mathrm{I}_{E_N}(k)$ is the indicator function of the set $E_N = \{0, 1, \cdots, N\}$. Obviously, we have

$$\|f\|^2 = \sum_{i=0}^{N} \mu_i \frac{\phi^{2m}}{\mu_i^2 b_i} = \sum_{i=0}^{N} \frac{1}{\mu_i b_i}\phi^{2m}. \tag{3.5}$$

Then, by estimating the value of $V_\delta(f)$ and the Dirichlet form, we obtain

$$V_\delta(f) = \max_{0\le i\le N} \frac{\phi_i^{2m}\mu_i^2 b_i}{\mu_i^2 b_i \phi_i^{2\delta}} = \max_{0\le i\le N} \phi_i^{2m-2\delta} \le \phi_N^{2m-2\delta} \tag{3.6}$$

and

$$\begin{aligned}
D(f) &= \sum_{i=0}^{N} \mu_i b_i (f_{i+1} - f_i)^2 = \sum_{i=0}^{N} \mu_i b_i \Big(\frac{\phi_{i+1}^m}{\mu_{i+1}\sqrt{b_{i+1}}} - \frac{\phi_i^m}{\mu_i\sqrt{b_i}}\Big)^2 \\
&= \sum_{i=0}^{N} \frac{1}{\mu_i b_i}\Big(\frac{\mu_i b_i \phi_{i+1}^m}{\mu_{i+1}\sqrt{b_{i+1}}} - \frac{b_i\phi_i^m}{\sqrt{b_i}}\Big)^2 = \sum_{i=0}^{N} \frac{1}{\mu_i b_i}\Big(\frac{a_{i+1}\phi_{i+1}^m}{\sqrt{b_{i+1}}} - \sqrt{b_i}\phi_i^m\Big)^2 \\
&\le \sum_{i=0}^{N} c(m)\frac{1}{\mu_i b_i}\phi_i^{2m-2\beta}.
\end{aligned} \tag{3.7}$$

Assume that the process has algebraic decay. By the Liggett-Stroock theorem, combining (1.3) with (3.5)–(3.7), we get

$$\begin{aligned}
\sum_{i=0}^{N} \frac{1}{\mu_i b_i}\phi_i^{2m} &= \|f\|^2 \le C' D(f)^{\frac{1}{p}} V_\delta(f)^{\frac{1}{q}} \\
&= c(m)^{\frac{1}{p}} \phi_N^{\frac{2(m-\delta)}{q}} \Big(\sum_{i=0}^{N} \frac{1}{\mu_i b_i}\phi_i^{2(m-\beta)}\Big)^{\frac{1}{p}} \\
&\le c(m)^{1/p} \phi_N^{\frac{2(m-\delta)}{q}} \cdot \Big(\sum_{i=0}^{N}\frac{1}{\mu_i b_i}\Big)^{\frac{\beta}{mp}} \cdot \Big(\sum_{i=0}^{N}\frac{1}{\mu_i b_i}\phi_i^{2m}\Big)^{\frac{m-\beta}{mp}}.
\end{aligned}$$

Here, we use the Schwarz inequality in the last step. Solving the inequality, we get

$$\sum_{i=0}^{N} \frac{1}{\mu_i b_i}\phi_i^{2m} \le c_1(m)\phi_N^{\frac{2(m-\delta)}{q}\cdot\frac{mp}{mp-m+\beta}}, \tag{3.8}$$

where $c_1(m) = c(m)^{\frac{m}{mp-m+\beta}} \cdot \Big(\sum\limits_{i=0}^{N} \frac{1}{\mu_i b_i}\Big)^{\frac{\beta}{mp-m+\beta}}$.

Now, we consider separately the two cases (i) and (ii) listed in Theorem 2.1(2).

(a) Firstly, suppose that (i) holds. Then, there exist $c_1 > 0,\ 0 < c_2 < 1$, such that

$$0 < c_1 \le \liminf_{N\to\infty} \frac{\phi_N}{\phi_{2N}} \le \limsup_{N\to\infty} \frac{\phi_N}{\phi_{2N}} \le c_2 < 1.$$

Due to the inequality (3.8) and the increasing property of ϕ, for $k < 2m$, we have

$$\begin{aligned}\sum_{i=N}^{2N}\frac{1}{\mu_i b_i}\phi_i^k &= \sum_{i=N}^{2N}\frac{1}{\mu_i b_i}\phi_i^{k-2m+2m} \le \phi_{2N}^{k-2m}\sum_{i=N}^{2N}\frac{1}{\mu_i b_i}\phi_i^{2m}\\ &\le c_1^{2m-k}\phi_{2N}^{k-2m}\sum_{i=N}^{2N}\frac{1}{\mu_i b_i}\phi_i^{2m} \le c_1^{2m-k}\phi_{2N}^{k-2m}\sum_{i=0}^{2N}\frac{1}{\mu_i b_i}\phi_i^{2m}\\ &\le c_1^{2m-k}\phi_{2N}^{k-2m}\cdot c_1(m)\phi_{2N}^{\frac{2(m-\delta)}{q}\cdot\frac{mp}{mp-m+\beta}}\\ &=: c_2(m)\phi_{2N}^{-\gamma},\end{aligned}\tag{3.9}$$

where $c_2(m) = c_1^{2m-k}c_1(m)$ and $-\gamma = k-2m+\frac{2(m-\delta)mp}{q(mp-m+\beta)}$ is the power of ϕ_N. When $m\to\infty$, $-\gamma$ converges to $k-2(q-1)\beta-2\delta$. Thus, when $k<2(q-1)\beta+2\delta$, we can fix m large enough so that $-\gamma<0$. Combining (3.9) with the fact that $\limsup\limits_{N\to\infty}(\frac{\phi_N}{\phi_{2N}})^\gamma \le c_2^\gamma < 1$ (ratio test), we obtain

$$\sum_{i=0}^{\infty}\frac{1}{\mu_i b_i}\phi_i^k = \sum_{l=0}^{\infty}\sum_{2^l\le j\le 2^{l+1}-1}\phi_j^k\frac{1}{\mu_j b_j} \le C\sum_{l=0}^{\infty}\phi_{2^{l+1}}^{-\gamma} < \infty \quad \text{for all } k<2(q-1)\beta+2\delta,$$

where $C = c_1^{2m-k}c(m)^{\frac{m}{mp-m+\beta}}\cdot\Big(\sum\limits_{i=0}^{\infty}\frac{1}{\mu_i b_i}\Big)^{\frac{\beta}{mp-m+\beta}} < +\infty$.

(b) Secondly, suppose that (ii) holds. Then there exist $c_1'>0,\ c_2'<1$, such that

$$0<c_1'\le\liminf_{N\to\infty}\frac{\phi_N}{\phi_{N+R}}\le\limsup_{N\to\infty}\frac{\phi_N}{\phi_{N+R}}\le c_2'<1.$$

Due to the inequality (3.8) and the increasing property of ϕ, we get

$$\begin{aligned}\sum_{i=NR}^{(N+1)R}\frac{1}{\mu_i b_i}\phi_i^k &= \sum_{NR}^{(N+1)R}\frac{1}{\mu_i b_i}\phi_i^{k-2m+2m} \le \phi_{NR}^{k-2m}\sum_{i=NR}^{(N+1)R}\frac{1}{\mu_i b_i}\phi_i^{2m}\\ &\le {c_1'}^{2m-k}\phi_{(N+1)R}^{k-2m}\sum_{i=NR}^{(N+1)R}\frac{1}{\mu_i b_i}\phi_i^{2m}\\ &\le c_3(m)\phi_{(N+1)R}^{-\gamma},\end{aligned}$$

where $c_3(m) = {c_1'}^{2m-k}c_1(m)$ and $-\gamma$ is the same as in (a). By the same discussion as in (a), for m large enough, we have

$$\sum_{i=0}^{\infty}\frac{1}{\mu_i b_i}\phi_i^k = \sum_{N=0}^{\infty}\sum_{j=NR}^{(N+1)R-1}\frac{1}{\mu_i b_i}\phi_i^k \le C'\sum_{N=0}^{\infty}\phi_{(N+1)R}^{-\gamma} < \infty \quad \text{for all } k<2(q-1)\beta+2\delta,$$

where $C' = {c_1'}^{2m-k}c(m)^{\frac{m}{mp-m+\beta}}\cdot\Big(\sum\limits_{i=0}^{\infty}\frac{1}{\mu_i b_i}\Big)^{\frac{\beta}{mp-m+\beta}} < +\infty$.

Proof of Theorem 2.2 Because the process satisfies conditions (1.1), which implies that the process is absorbed at ∞, the proof is concentrated on the absorbing rate by the state ∞.

Let $f \in L^2(\mu)$, $\|f\| = 1$. Then $f_\infty = \lim_{n\to\infty} f_n = 0$. We have

$$\begin{aligned}
1 = \|f\|^2 &= \sum_{i=0}^{\infty}\mu_i\Big\{\sum_{j=i}^{\infty}(f_j - f_{j+1})\Big\}^2 \le \sum_{i=0}^{\infty}\mu_i\Big\{\sum_{j=i}^{\infty}|f_j - f_{j+1}|\Big\}^2 \\
&\le 2\sum_{i=0}^{\infty}\mu_i \sum_{i\le j\le k}|f_j - f_{j+1}|\cdot|f_k - f_{k+1}| \\
&= 2\sum_{i=0}^{\infty}\mu_i\sum_{j=i}^{\infty}|f_j - f_{j+1}|\sum_{k=j}^{\infty}|f_k - f_{k+1}| \\
&= 2\sum_{j=0}^{\infty}|f_j - f_{j+1}|\sum_{k=j}^{\infty}|f_k - f_{k+1}|\cdot\frac{\sum_{i=0}^{j}\mu_i}{\mu_j}\cdot\mu_j \\
&\le 2\Big\{\sum_{j=0}^{\infty}|f_j - f_{j+1}|^2\sigma_j^2\mu_j\cdot\sum_{j=0}^{\infty}\Big(\sum_{k=j}^{\infty}|f_k - f_{k+1}|\Big)^2\mu_j\Big\}^{\frac{1}{2}}.
\end{aligned}$$

Thus we have $\sum_{i=0}^{\infty}\mu_i\Big\{\sum_{j=i}^{\infty}(f_j - f_{j+1})\Big\}^2 \le 4\sum_{j=0}^{\infty}|f_j - f_{j+1}|^2\sigma_j^2\mu_j$. Using the Schwarz inequality, we have

$$\begin{aligned}
\|f\|^2 &\le 4\sum_{j=0}^{\infty}|f_j - f_{j+1}|^2\sigma_j^2\mu_j \\
&\le 4\Big\{\sum_{j=0}^{\infty}\mu_j b_j|f_j - f_{j+1}|^2\Big\}^{\frac{1}{p}}\cdot\Big\{\sum_{j=0}^{\infty}|f_j - f_{j+1}|^2\mu_j b_j\cdot\frac{\sigma_j^{2q}}{b_j^q}\Big\}^{\frac{1}{q}} \\
&\le C\cdot D(f)^{\frac{1}{p}}\widetilde{V}_1(f)^{\frac{1}{q}},
\end{aligned}$$

where

$$C = 4\sum_{j=0}^{\infty}\mu_j b_j^{1-q}u_j^2\sigma_j^{2q} = 4\sum_{j=0}^{\infty}\mu_j b_j\Big(\frac{\sigma_j^2}{b_j}\Big)^q u_j^2 < \infty.$$

Finally, we need to test the contraction property of $\widetilde{V}_1$. We select the classical coupling $\widetilde{\Omega}$ (see [1, Lemma 4.1] or [5, Theorem 3.3] for details), such that for all $j > i \ge 0$,

$$\widetilde{\Omega}\phi_{ij} = b_j u_j - a_j u_{j-1} - b_i u_i + a_i u_{i-1} \le 0, \quad u_{-1} := 1.$$

Thus, $\widetilde{V}_1(P_t f) \le \widetilde{V}_1(f)$. By Theorem 1.1, we get the conclusion.

4 Examples

In this section, we examine some examples. We note that the V_δ in the following examples is used as the first type given by (2.1).

Example 4.1 Let $b_i = 1$, $a_i = 1 - \frac{1}{i^\alpha}$ $(i \ge 2)$, $a_1 = 1$, where $0 < \alpha < 1$. It is known that the process does not have exponential convergence rate due to [6]. By using Theorem 2.1, we see that the process has algebraic decay with respect to V_1 in terms of the sequence $\phi_i = i^s$ $(s \ge \alpha)$.

Proof It is easy to get $\mu_k \sim \mathrm{e}^{\frac{1}{(1-\alpha)}k^{1-\alpha}}$. So when $0 < \alpha < 1$, we have $\sum\limits_k \mu_k = \infty$ and $\sum\limits_k \frac{1}{\mu_k b_k} < \infty$, that is, the process is transient. Take $\phi_i = i^s \vee 1$ for $s \geq \alpha$. Now, we test the conditions (a), (b) and (c) of Theorem 2.1(1).

(a) For simplicity of notation, we write $g(i) = \sqrt{b_{i-1}}\phi_{i-1} - \frac{a_i\phi_i}{\sqrt{b_i}} = (i-1)^s - \frac{(1-i^{-\alpha})\cdot i^s}{1}$. It is easy to test that $g(i)$ is an increasing sequence when i is large enough because of $\lim\limits_{i\to+\infty} \frac{g(i)}{i^{s-\alpha}} = 1$.

(b) Next, consider $\sigma(i) = \frac{\phi_i}{\mu_i\sqrt{b_i}}\sum\limits_{k=0}^{i}\frac{\mu_k}{\phi_k^2}$. Because $\frac{\mu_i}{\phi_{i+1}} \sim \frac{\mathrm{e}^{\frac{1}{1-\alpha}i^{1-\alpha}}}{(i+1)^s}$ is increasing with i large enough, by using the Stokes theorem, we get

$$\lim_{i\to\infty}\sigma(i) = \lim_{i\to\infty}\frac{\mu_i\cdot\phi_i^{-2}}{\mu_i\phi_i^{-1} - \mu_{i-1}\phi_{i-1}^{-1}} = \lim_{i\to\infty}\frac{1}{\frac{i^{2s}}{(i+1)^s} - \frac{i^s\cdot\mu_{i-1}}{\mu_i}} = \lim_{i\to\infty}\frac{1}{\frac{i^{2s}}{(i+1)^s} - i^s(1-i^{-\alpha})} < \infty.$$

Thus, we obtain $\sup\limits_i \sigma(i) < \infty$.

(c) Because $\frac{1}{\mu_i b_i}\phi_i^{2q} \sim \frac{1}{\mathrm{e}^{(1-\alpha)i^{1-\alpha}}}i^{2qs}$, we can get that $\sum \frac{1}{\mu_i b_i}\phi_i^{2q}$ is finite for all $q > 1$.

Thus due to Theorem 2.1(1), the process has algebraic decay.

Example 4.2 Let $b_i = i+\alpha,\ a_i = i$, where $\alpha > 1$. The process has algebraic decay rate iff $\alpha > 2$ with respect to V_1 in terms of the sequence $\phi_i = i^s$ $(\frac{1}{2} \leq s \leq \frac{2\alpha-1}{2})$.

Proof (1) Firstly, we can obtain that $\mu_k \sim k^{\alpha-1}$. It is easy to know that $\sum\mu_k = \infty$ because of $\alpha\in(1,+\infty)$, and $\sum\frac{1}{\mu_k b_k} < \infty$ because of $\frac{1}{\mu_k b_k} \sim \frac{1}{k^\alpha}$. So the process is transient. Take $\phi_i = i^s \vee 1$ $(\frac{1}{2} \leq s < \frac{2\alpha-1}{2})$. Now, we test the conditions (a), (b) and (c) of Theorem 2.2(1).

(a) It is easy to check the monotony of $\sqrt{b_{i-1}}\phi_{i-1} - \frac{a_i}{\sqrt{b_i}}\phi_i$. So V_1 satisfies (1.4).

(b) Now, consider the condition (b) of Theorem 2.1. We have

$$\sigma(i) = \frac{i^s}{\mu_i\sqrt{i+\alpha}}\sum_{k=0}^{i}\frac{\mu_k}{\phi_k^2} = \frac{\sum\limits_{k=0}^{i}\frac{\mu_k}{\phi_k^2}}{\mu_i(i+\alpha)^{\frac{1}{2}}\cdot i^{-s}}.$$

When $s \leq \frac{2\alpha-1}{2}$, $\mu_i(i+\alpha)^{\frac{1}{2}}i^{-s}$ is increasing for i large enough. Then, we have

$$\lim_{i\to\infty}\sigma(i) = \lim_{i\to\infty}\frac{i^{-2s}}{(i+\alpha)^{\frac{1}{2}}\cdot i^{-s} - \frac{\mu_{i-1}}{\mu_i}(i+\alpha-1)^{\frac{1}{2}}\cdot(i-1)^{-s}} < \infty.$$

Hence, we get $\sup\limits_i \sigma(i) < \infty$.

(c) Finally, consider the condition (c) of Theorem 2.1. Because $\frac{1}{\mu_i b_i}\phi_i^{2q} \sim i^{2qs-\alpha}$, we can obtain $\sum\frac{1}{\mu_i b_i}\phi_i^{2q} < \infty$ for $\alpha > 2$ and $1 < q < \frac{\alpha-1}{2s}$.

Thus, due to Theorem 2.1(1), the process has algebraic decay. This finishes the proof of sufficiency.

(2) We examine the necessary conditions. Take $\phi_i = i^s$. Then ϕ satisfies condition (i). Let $\beta = \frac{1}{2s}$. We have

$$\lim_{i\to\infty}\frac{(a_{i+1}\phi_{i+1}^m - \sqrt{b_i b_{i+1}}\phi_i^m)^2}{b_{i+1}\phi_i^{2m-2\beta}} = \lim_{i\to\infty}\frac{((i+1)^{sm+1} - \sqrt{(i+\alpha)(i+\alpha+1)}\cdot i^{sm})^2}{(i+1+\alpha)\cdot i^{2s(m-\beta)}} = (m+1)^2.$$

Thus we get $c(m) < \infty$. According to Theorem 2.1(2), if the process has algebraic decay with respect to V_1, we have

$$\sum \frac{1}{i^\alpha} \cdot i^{sk} < \infty \quad \text{for all} \quad k < 2(q-1)\beta + 2\delta = \frac{q-1}{s} + 2.$$

But we know this is impossible when $1 < \alpha \leq 2$, that is, the process does not have algebraic decay when $1 < \alpha \leq 2$.

Example 4.3 Let $b_i = i$ $(i \geq 1)$, $a_i = i - \sqrt{i}$ $(i \geq 1)$, $a_1 = 1$. Then the process has algebraic decay with respect to V_0 in term of the sequences $\phi_i = \mu_i$ with decay rate and $q \in (1, \frac{3}{2})$. And the discussed process does not have exponential decay rate.

Proof We can get $\mu_k \sim \mathrm{e}^{2\sqrt{k}}$. It is easy to see $\sum \mu_k = \infty$ and $\sum \frac{1}{\mu_k b_k} < \infty$. So the process is transient.

Firstly, let

$$g(i) := \sqrt{b_{i-1}} - \frac{a_i}{\sqrt{b_i}} = \sqrt{i-1} - \frac{i-\sqrt{i}}{\sqrt{i}} = \frac{-1}{\sqrt{i-1}+\sqrt{i}} + 1.$$

Obviously, $g(i)$ is increasing for $i > 1$.

Secondly, take $\phi_k = \mu_k$ $(k \geq 1)$. Then we have $\widetilde{\sigma}(i) = \frac{1}{\sqrt{i}} \sum\limits_{j=i}^{\infty} \frac{1}{\mu_j}$. It is easy to see $\lim\limits_{i\to+\infty} \widetilde{\sigma}(i) = 0$. So we get $\sup\limits_i \widetilde{\sigma}(i) < \infty$.

Finally, we have $\sum\limits_{i=1}^{\infty} \frac{1}{\mu_i b_i} \phi_i^{2(q-1)} = \sum\limits_{i=1}^{\infty} \frac{1}{\mu_i^{1-2(q-1)} b_i} < \infty$, when $1 - 2(q-1) > 0$.

Thus due to Theorem 2.1(1), the process has algebraic decay when $q \in (1, \frac{3}{2})$.

Example 4.4 Let $b_i = i$ $(i \geq 1)$, $a_1 = 1$, $a_i = i(1 - \frac{1}{i^\alpha})$ $(\frac{1}{2} \leq \alpha < 1,\ i \geq 2)$. Then the process has algebraic decay with respect to V_1 in terms of the sequence of $\phi_i = i^s$ $(s \geq \alpha - \frac{1}{2})$, and the processes discussed above do not have exponential decay rate.

Proof As in Example 4.1, we can check that the process is transient. Let $\phi_i = i^s \vee 1$ $(s > \alpha - \frac{1}{2})$. It is easy to check that V_0 does not satisfy the contraction condition, so we only need to consider V_1.

(a) Firstly, consider the monotonicity of $\sqrt{b_{i-1}}\phi_{i-1} - \frac{a_i\phi_i}{\sqrt{b_i}}$. Let

$$g(i) = \sqrt{b_{i-1}}\phi_{i-1} - \frac{a_i\phi_i}{\sqrt{b_i}} = (i-1)^{s+\frac{1}{2}} - i^{s+\frac{1}{2}} + i^{s-\alpha+\frac{1}{2}}.$$

It is easy to see that $g(i)$ is nondecreasing when i is large enough.

(b) Then, we have

$$\mu_1 = 1, \quad \mu_k = \frac{b_1 \cdots b_k}{a_2 \cdots a_k} = \frac{(k-1)!}{k!(1-\frac{1}{2^\alpha})\cdots(1-\frac{1}{k^\alpha})} \sim \frac{1}{k} \exp\Big\{\frac{1}{1-\alpha} k^{1-\alpha}\Big\}.$$

Obviously, $\frac{\mu_i\sqrt{b_i}}{\phi_i}$ is increasing when i is large enough. Hence,

$$\lim_{i\to\infty} \sigma(i) = \lim_{i\to\infty} \frac{\sum\limits_{k=1}^{i} \frac{\mu_k}{\phi_k^2}}{\frac{\mu_i\sqrt{b_i}}{\phi_i}} = \lim_{i\to\infty} \frac{\frac{\mu_i}{\phi_i^2}}{\frac{\mu_i\sqrt{b_i}}{\phi_i} - \frac{\mu_{i-1}\sqrt{b_{i-1}}}{\phi_{i-1}}} = \lim_{i\to\infty} \frac{1}{i^{\frac{1}{2}+s} - \frac{\mu_{i-1}}{\mu_i} \cdot (i-1)^{\frac{1}{2}-s} \cdot i^{2s}}.$$

As $\frac{\mu_{i-1}}{\mu_i} = (1 - i^{-\alpha})\frac{i}{i-1}$, we have

$$\lim_{i\to\infty} \frac{1}{i^{\frac{1}{2}+s} - \frac{\mu_{i-1}}{\mu_i}\cdot(i-1)^{\frac{1}{2}-s}\cdot i^{2s}} = \lim_{i\to\infty} \frac{1}{i^{\frac{1}{2}+s} - (1-i^{-\alpha})\cdot(i-1)^{-\frac{1}{2}-s}\cdot i^{2s+1}} < \infty.$$

It is easy to see that $\sup\limits_i \sigma(i) < \infty$.

(c) Obviously, we have $\sum\limits_{i=1}^{\infty} \frac{1}{\mu_i b_i}\cdot\phi_i^{2q} < \infty$.

Due to Theorem 2.1(1), the process has algebraic decay with respect to V_1.

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