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# On the Regularity Theories for Harmonic Maps from Finsler Manifolds*

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**Abstract** The author studies the regularity of energy minimizing maps from Finsler manifolds to Riemannian manifolds. It is also shown that the energy minimizing maps are smooth, when the target manifolds have no focal points.

**Keywords** Finsler manifolds, Harmonic map, Regularity
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## 1 Introduction

Harmonic maps between Riemannian manifolds are very important in both differential geometry and mathematical physics. One method to establish their existence is the direct method of the calculus of variations (see [1–3]) and the regularity was proved by Morrey [1], Schoen and Uhlenbeck [4–5].

Harmonic maps in Finsler manifolds were first considered by Mo [6], which were also defined as the critical points of their energy functionals, and were extensively studied later (see [7–9]). As the Riemannian case, it is also important to study their existence and regularity. Mo and Yang proved the fundamental existence theorem for harmonic maps from Finsler manifolds to Riemannian manifolds in 2005 (see [11]). von der Mosel and Winklmann showed that weakly harmonic maps with the image contained in a regular ball are locally Hölder continuous (see [12]). Recently, Mo and Zhao proved that a weakly harmonic map from a boundless Finsler surface to a sphere is smooth actually (see [13]). The key point of the treatment for the maps from Finsler manifolds to Riemannian manifolds is to construct a Riemannian metric from the fundamental tensor of the Finsler metric, so that the problem of these maps was transferred to that from Riemannian manifolds with induced Riemannian metric and induced volume measure to Riemannian manifolds.

In this paper, we use the above idea and follow Schoen and Uhlenbeck's method to get the interior and boundary regularity theories:

**Theorem 1.1** *Let $(M,F)$ be a compact $n$-dimensional Finsler manifold $(n \geq 3)$, and $N$ be a compact Riemannian manifold. Let $\phi : M \to N$ be an E-minimizing map in $W^{1,2}(M,N)$. Then $\dim(\mathfrak{S} \cap \mathrm{int}M) \leq n-3$, where $\dim A$ is the Hausdorff dimension of a set $A$, and $\mathfrak{S}$ is the singular set of $\phi$. If $n = 3$, then $\mathfrak{S}$ is a discrete set of points. Moreover, if there is an integer $l \geq 3$, such that every* MTM *from $\mathbb{R}^j \to N$ is trivial, $3 \leq j \leq l$, then $\dim(\mathfrak{S} \cap \mathrm{int}M) \leq n-l-1$. If $n = l+1$, then $\mathfrak{S}$ is a discrete set of points, and if $n < l+1$, $\mathfrak{S} = \varnothing$.*

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**Theorem 1.2** *Let $(M,F)$ be a compact $n$-dimensional Finsler manifold ($n \geq 3$), and $N$ be a compact Riemannian manifold. Let $\phi : M \to N$ be an E-minimizing map in $W^{1,2}(M,N)$. Suppose $\psi \in C^{2,\alpha}(\partial M, N)$ and $\phi|_{\partial M} = \psi$. Then the singular set $\mathfrak{S}$ of $\phi$ is a compact subset of the interior of $M$, in particular, $\phi$ is $C^{2,\alpha}$ in a neighborhood of $\partial M$.*

**Theorem 1.3** *Any E-minimizing map $\phi \in W^{1,2}(M,N)$ between Finsler surface $(M,F)$ and compact Riemannian manifold $N$ is smooth.*

**Remark 1.1** When $(M,F)$ is a boundless Finsler surface and $N$ is a sphere, Theorem 1.3 is a corollary of the theorem stated in [13].

The contents of this paper are arranged as follows. In Section 2, some definitions and fundamental formulas which are necessary for the present paper are given. In Section 3, we give the proof of the main theorems and get two corollaries as in [4, 14].

## 2 Preliminary

Let $M$ be an $n$-dimensional ($n \geq 2$) smooth manifold, and $\pi : \mathrm{TM} \to M$ be the natural projection from the tangent bundle TM. Let $(x,y)$ be a point of TM with $x \in M$, $y \in \mathrm{T}_x\mathrm{M}$, and let $(x^i, y^i)$ be the local coordinates on TM with $y = y^i \frac{\partial}{\partial x^i}$. A Finsler metric on $M$ is a function $F : \mathrm{TM} \to [0, +\infty)$ satisfying the following properties:
(i) Regularity: $F(x,y)$ is smooth in $\mathrm{TM}\backslash 0$;
(ii) Positive homogeneity: $F(x, \lambda y) = \lambda F(x,y)$ for $\lambda > 0$;
(iii) Strong convexity: The fundamental quadratic form

$$g_y := g_{ij}(x,y)\mathrm{d}x^i \otimes \mathrm{d}x^j, \quad g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}$$

is positively definite. Here and from now on, we use the following convention of index ranges unless otherwise stated:

$$1 \leq i, j, \cdots \leq n, \quad 1 \leq \alpha, \beta, \gamma, \cdots \leq m.$$

Let $\phi : (M,F) \to (N,h)$ be a smooth map from an $n$-dimensional Finsler manifold $(M,F)$ to an $m$-dimensional Riemannian manifold $(N,h)$. The energy functional of $\phi$ is defined as

$$\begin{aligned} E(\phi) &= \frac{1}{2c_{n-1}} \int_{\mathrm{SM}} |\mathrm{d}\phi|^2 \mathrm{d}V_{\mathrm{SM}} \\ &= \frac{1}{2c_{n-1}} \int_{\mathrm{SM}} g^{ij}(x,y) \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j} h_{\alpha\beta}(\phi(x)) \mathrm{d}V_{\mathrm{SM}}, \end{aligned} \tag{2.1}$$

where $(g^{ij}) = (g_{ij})^{-1}$, $\mathrm{d}V_{\mathrm{SM}} = \Omega \mathrm{d}\tau \wedge \mathrm{d}x$, $\Omega = \det\left(\frac{g_{ij}}{F}\right)$, $\mathrm{d}\tau = \sum\limits_{i=1}^{n} (-1)^{i-1} y^i \mathrm{d}y^1 \wedge \cdots \wedge \widehat{\mathrm{d}y^i} \wedge \cdots \wedge \mathrm{d}y^n$, $\mathrm{d}x = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, and $c_{n-1}$ denotes the volume of the unit Euclidean sphere $S^{n-1}$. Let

$$\overline{g}^{ij}(x) := \frac{\int_{\mathrm{S}_x\mathrm{M}} g^{ij}(x,y) \det(g_{kl}(x,y)) \mathrm{d}\tau}{\int_{\mathrm{S}_x\mathrm{M}} \det(g_{kl}(x,y)) \mathrm{d}\tau}, \tag{2.2}$$

$$\sigma(x) := \frac{1}{c_{n-1}} \int_{\mathrm{S}_x\mathrm{M}} \det(g_{kl}(x,y)) \mathrm{d}\tau, \tag{2.3}$$

$$(\overline{g}_{ij}) := (\overline{g}^{ij})^{-1}. \tag{2.4}$$

Then $\overline{g} := \overline{g}_{ij}(x)\mathrm{d}x^i \otimes \mathrm{d}x^j$ is a Riemannian metric on $M$ (see [11]). Hence,

$$E(\phi) = \frac{1}{2}\int_M |\mathrm{d}\phi|^2_{\overline{g}}\sigma(x)\mathrm{d}x, \tag{2.5}$$

where $|\mathrm{d}\phi|^2_{\overline{g}}$ denote the norm by Riemannian metric $\overline{g}$, i.e., $|\mathrm{d}\phi|^2_{\overline{g}} = \overline{g}^{ij}(x)\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}h_{\alpha\beta}(\phi(x))$.

By the Nash embedding theorem, we embed $N$ isometrically in some Euclidean space $\mathbb{R}^k$. Let $W^{1,2}(M,\mathbb{R}^k)$ be the Sobolev space of the maps from $M$ to $\mathbb{R}^k$, whose squares of weak derivation is integral with Hilbert norm

$$\|\phi\| = \Big[\int_M (|\phi|^2_{\overline{g}} + |\mathrm{d}\phi|^2_{\overline{g}})\sigma(x)\mathrm{d}x\Big]^{\frac{1}{2}}. \tag{2.6}$$

Define

$$W^{1,2}(M,N) = \{\phi \in W^{1,2}(M,\mathbb{R}^k) : \phi(x) \in N \text{ a.e. } x \in M\}. \tag{2.7}$$

As is known, $W^{1,2}(M,N)$ is weakly closed in $W^{1,2}(M,\mathbb{R}^k)$, while $\{\phi \in W^{1,2}(M,N) : \|\phi\| \le C\}$ is weakly compact. Moreover, the energy functional is lower semi-continuous with respect to the weak convergence in $W^{1,2}(M,\mathbb{R}^k)$.

We call $\phi \in W^{1,2}(M,N)$ a weakly harmonic map, if it is a critical point of energy functional, that is, $\phi$ satisfies the Euler-Lagrange equation for $E$ in the sense of distributions, i.e.,

$$\triangle_\sigma\phi + \mathrm{Tr}_{\overline{g}}A(\phi)(\mathrm{d}\phi,\mathrm{d}\phi) = 0, \tag{2.8}$$

where $\triangle_\sigma = \frac{1}{\sigma(x)}\frac{\partial}{\partial x^i}\big(\overline{g}^{ij}(x)\sigma(x)\frac{\partial}{\partial x^j}\big)$, and $A(\phi)$ is the second fundamental form of $N$ in $\mathbb{R}^k$ (see [11]). An $E$-minimizing map means a map $\phi \in W^{1,2}(M,N)$, such that $E(\phi) \le E(\psi)$ for any map $\psi \in W^{1,2}(M,N)$ with $\phi = \psi$ on $\partial M$. It is easy to check that an $E$-minimizing map $\phi \in W^{1,2}(M,N)$ is a weakly harmonic map.

A point $x \in M$ is a regular point if $\phi$ is continuous in a neighborhood of $x$. Let $\mathfrak{R} = \mathfrak{R}(\phi)$ be the set of all regular points and $\mathfrak{S} = \mathfrak{S}(\phi)$ be the complement of $\mathfrak{R}$. A homogeneous harmonic map with an isolated singularity at 0 will be referred to as a tangent map (TM). A tangent map which is energy minimizing on compact subsets of $\mathbb{R}^n$ is a minimizing tangent map (MTM).

## 3 Proof of Main Results

In this section, we assume that $(M,F)$ and $(N,h)$ are compact. The existence of $E$-minimizing maps can be obtained by the direct method in calculus of variations. We are going to prove their regularity.

Let $B_\lambda$ be the unit ball in $\mathbb{R}^n$ with radius of $\lambda$. For $\Lambda > 0$, let $\mathcal{F}_\Lambda$ denote the class of functionals $E$ on $B_1$ with metric $\overline{g}$ and volume measure $\sigma(x)\mathrm{d}x$, such that $\overline{g}_{ij}(0) = \delta_{ij}$, and the lower order terms satisfy, for $x \in B_1$,

$$\sum_{i,j,k}\Big|\frac{\partial}{\partial x^k}\overline{g}_{ij}(x)\Big| + \sum_i\Big|\frac{\partial\omega(x)}{\partial x^i}\Big| \le \Lambda, \tag{3.1}$$

where $\omega(x) := \frac{\sigma(x)}{\sqrt{\det\overline{g}}}$ which is a smooth function on $M$. Let $\mathcal{H}_\Lambda := \{\phi \in W^{1,2}(M,N) : \phi$ is $E$-minimizing for some $E \in \mathcal{F}_\Lambda\}$. Let $B_\lambda(p)$ be a geodesic ball in $M$ of radius $\lambda$ centered at $p \in \mathrm{int}M$. Define a functional $E^{p,\lambda}$ on $B_1$, by setting

$$\begin{aligned} E^{p,\lambda}(\psi) &:= \frac{1}{2}\int_{B_1}|\mathrm{d}\psi|^2_{\overline{g}^\lambda}(y)\omega^\lambda(y)\sqrt{\det\overline{g}^\lambda(y)}\mathrm{d}y \\ &= \lambda^{2-n}E(\psi)|_{B_\lambda(p)}, \end{aligned} \tag{3.2}$$

where $\psi(y) := \phi(\lambda y)$, $\overline{g}^{\lambda}_{ij}(y) = \overline{g}_{ij}(\lambda y)$ and $\omega^{\lambda}(y) = \omega(\lambda y)$. Then we can state the following lemma.

**Lemma 3.1** *Given $\Lambda > 0$, there is $\lambda_0 = \lambda_0(n, F, \Lambda) > 0$, such that for $0 < \lambda \leq \lambda_0$ and $p \in M$, if $\phi$ is $E$-minimizing, then $\psi(y) := \phi(\lambda y)$ is $E^{p,\lambda}$-minimizing where $E^{p,\lambda} \in \mathcal{F}_\Lambda$.*

**Proof** Choose a normal coordinate around $p$. Since

$$\Big|\frac{\partial \overline{g}^{\lambda}_{ij}(x)}{\partial x^k}\Big| = \Big|\frac{\partial \overline{g}_{ij}(x)}{\partial x^k}\lambda\Big| \leq C_1\lambda, \quad \Big|\frac{\partial \omega^{\lambda}(x)}{\partial x^k}\Big| \leq C_2\lambda,$$

where $C_1 = \sup\limits_{\substack{x\in M\\ i,j,k=1,\cdots,n}} \big|\frac{\partial \overline{g}_{ij}(x)}{\partial x^k}\big|$, $C_2 = \sup\limits_{\substack{x\in M\\ k=1,\cdots,n}} \big|\frac{\partial \omega(x)}{\partial x^k}\big|$, letting $\lambda_0 \leq \frac{\Lambda}{3nC_1+nC_2}$, for all $0 < \lambda \leq \lambda_0$, we have

$$\sum_{i,j,k}\Big|\frac{\partial \overline{g}^{\lambda}_{ij}(x)}{\partial x^k}\Big| + \sum_i\Big|\frac{\partial \omega^{\lambda}(x)}{\partial x^i}\Big| \leq \lambda(3nC_1 + nC_2) \leq \Lambda.$$

Since $\overline{g}^{\lambda}_{ij}(0) = \overline{g}_{ij}(0) = \delta_{ij}$, $E^{p,\lambda} \in \mathcal{F}_\Lambda$, by $E^{p,\lambda}(\psi) = \lambda^{2-n}E(\phi)|_{B_\lambda(p)}$, we get that $\psi(z)$ is $E^{p,\lambda}$-minimizing.

Thus from now on, we consider $E \in \mathcal{F}_\lambda$, where $\Lambda$ is sufficiently small.

For $0 < \lambda \leq 1$, let

$$^{\delta}E_\lambda(\phi) := \frac{1}{2}\int_{B_\lambda} |\mathrm{d}\phi|^2_\delta \mathrm{d}x, \tag{3.3}$$

where $\delta$ denotes the Euclidean metric, and let $E_\lambda$ denote energy $E$ taken over $B_\lambda$. By the Taylor expansion, the following inequality holds:

$$|E_\lambda(\psi) - \omega(0)^{\delta}E_\lambda(\psi)| \leq C_1\Lambda\lambda^{\delta}E_\lambda(\phi) \tag{3.4}$$

for any $\psi \in W^{1,2}(M, N)$, where $C_1$ is a positive constant depending on $n$ and $F$. In fact, given the Taylor expansion of $\overline{g}^{ij}(x)\omega(x)\sqrt{\det(\overline{g}_{ij})(x)}$ around $x = 0$, using (3.1), we have

$$\begin{aligned}
&|E_\lambda(\psi) - \omega(0)E^{\delta}_\lambda(\psi)| \\
&\leq \frac{1}{2}\int_{B_\lambda}\sum_{\alpha,i,j}|\psi^\alpha_i\psi^\beta_j(\overline{g}^{ij}(x)\omega(x)\sqrt{\det(\overline{g}_{ij})(x)} - \delta^{ij}\omega(0))|\mathrm{d}x \\
&\leq \frac{1}{2}\int_{B_\lambda}\sum_{\alpha,i,j}\lambda\Lambda C_4|\psi^\alpha_i\psi^\beta_j|\mathrm{d}x \\
&\leq C_1\lambda\Lambda E^{\delta}_\lambda(\psi).
\end{aligned}$$

By the same arguments in [4] and using (3.4), we have the results as follows.

**Lemma 3.2** *Let $\phi \in \mathcal{H}_\Lambda$ for $\Lambda$ sufficiently small. Then, we have*

$$\lambda^{2-n}\int_{B_\lambda(x)} |\mathrm{d}\phi|^2_\delta \mathrm{d}y \leq C_2\rho^{2-n}\int_{B_\rho(x)} |\mathrm{d}\phi|^2_\delta \mathrm{d}y \tag{3.5}$$

*for any $x \in B_1$ and $0 < \lambda \leq \rho \leq \mathrm{dist}(x, \partial B_1)$, where $C_2$ is a positive constant depending on $n$ and $F$.*

**Theorem 3.1** *There exists a constant $\epsilon = \epsilon(n, F, N) > 0$, such that if $\phi \in \mathcal{H}_\Lambda$, $\Lambda \leq \epsilon$ and $^{\delta}E_1(\phi) \leq \epsilon$, then $\phi$ is Hölder continuous on $B_{\frac{1}{2}}$.*

The proof of the above $\epsilon$-regularity theorem is almost the same as the proof of [4, Theorem 3.1]. We need to notice that in this case the Euler-Lagrange equation and the monotonicity formula for $E$-minimizing map are in the form of (2.8) and (3.5). The following corollary is proved as in [4, Corollary 2.7].

**Corollary 3.1** *If $\phi \in W^{1,2}(B_1, N)$ is in $\mathcal{H}_\Lambda$, then $\mathcal{H}^{n-2}(\mathfrak{S} \cap B_{\frac{1}{2}}) = 0$. More generally, if $\phi \in W^{1,2}(M, N)$ is E-minimizing, then $\mathcal{H}^{n-2}(\mathfrak{S} \cap \mathrm{int}M) = 0$.*

**Proof of Theorem 1.1** The direct consequence of Corollary 3.1 is that $\dim(\mathfrak{S} \cap B_{\frac{1}{2}}) \leq n-2$. To refine the Hausdorff dimension, we need to show the compactness theorem firstly, which says that weak convergent sequence $\{\phi_i\} \subset \mathcal{H}_\Lambda$ (in $W^{1,2}(B_1, N)$) with limit $\phi_0$ and uniform energy bound is strongly convergent in $W^{1,2}(B_{\frac{1}{2}}, N)$ with limit $\phi_0$ actually. This proof is similar to that of [4, Proposition 4.6] with the help of Lemma 3.2 and Theorem 3.1.

For $\phi \in \mathcal{H}_\Lambda$, set $\phi_\lambda(x) = \phi(\lambda x)$ for $\lambda \in (0, 1]$. As a consequence of Lemma 3.2, we have

$$ {}^{\delta}E_1(\phi_\lambda) = \lambda^{2-n}\,{}^{\delta}E_\lambda(\phi) \leq C_2\,{}^{\delta}E_1(\phi) \leq M \tag{3.6} $$

for all $\lambda \in (0, 1]$ and some positive constant $M$. Therefore, we get a weakly convergent sequence $\phi_{\lambda_i}$ with limit $\phi_0$ in $W^{1,2}(B_1, N)$. By the compactness theorem, $\{\phi_{\lambda_i}\}$ is strongly convergent. Since $\phi_\lambda \in \mathcal{H}_{\lambda\Lambda}$, it satisfies the Euler-Lagrange equation (2.8). Hence, $\phi_0$ is harmonic with $\frac{\partial \phi_0}{\partial r} = 0$ a.e. in $B_1$, that is, $\phi_0$ is the tangent map if $p \in \mathfrak{S} \cap \mathrm{int}M$. At last, applying Federer's dimension reduction argument to this setting as the one in [4], we complete the proof.

**Proof of Theorem 1.2** Choose coordinates $(x^i)$ centered at a point $p \in \partial M$, such that locally $M$ is the upper $\frac{1}{2}$-space $\mathbb{R}^n_+ := \{(x^i) : x^n > 0\}$. After scaling, we will deal with maps $\phi \in W^{1,2}(B^+_\lambda, N)$, which are $E$-minimizing and $\phi|_{T_\lambda} = \psi \in C^{2,\alpha}(T_\lambda, N)$, where $T_\lambda = \{x \in B_\lambda : x_n = 0\}$. For $\Lambda > 0$, let $\mathcal{F}_\Lambda$ denote the class of functionals $E$ on $B_1$ with metric $\overline{g}$ and volume measure $\sigma(x)\mathrm{d}x$ such that $\overline{g}_{ij}(0) = \delta_{ij}$, and the lower order terms satisfy for $x \in B^+_1$,

$$ \sum_{i,j,k} \Big| \frac{\partial}{\partial x^k} \overline{g}_{ij}(x) \Big| + \sum_i \Big| \frac{\partial \omega(x)}{\partial x^i} \Big| \leq \Lambda. \tag{3.7} $$

Let $\mathcal{H}_\Lambda$ denote the space of maps $\phi \in W^{1,2}(B^+_1, N)$, such that $\phi$ is $E$-minimizing for some $E \in \mathcal{F}_\Lambda$ and $\phi = \psi$ on $T_1$. Proceeding as the proof of Lemma 3.2, we have the following inequality for a given $\phi \in \mathcal{H}_\Lambda$ with a sufficiently small $\Lambda$:

$$ \lambda^{2-n} \int_{B_\lambda(x)} |\mathrm{d}\widehat{\phi}|^2_\delta \mathrm{d}y \leq C_3 \Big[\rho^{2-n} \int_{B_\rho(x)} |\mathrm{d}\widehat{\phi}|^2_\delta \mathrm{d}y + \Lambda\rho\Big], \tag{3.8} $$

where $x \in B^+_{\frac{1}{2}}$, $0 < \lambda \leq \rho \leq \frac{1}{2}$, $\widehat{\phi}(x', x_n) := -(\phi(x', -x_n) - \psi(x', -x_n))$ for $x_n \leq 0$, and $\widehat{\phi}(x', x_n) := \phi(x', x_n) - \psi(x', x_n)$ for $x_n > 0$, $x' = (x_1, \cdots, x_{n-1})$ denotes the first $(n-1)$-coordinates. Then the $\epsilon$-regularity estimate on the boundary can be given: There exists a constant $\epsilon = \epsilon(n, F, N) > 0$, such that if $\phi \in \mathcal{H}_\Lambda$, $\Lambda \leq \epsilon$ and ${}^{\delta}E^+_1(\phi) \leq \epsilon$, then $\phi$ is Hölder continuous on $B^+_{\frac{1}{2}}$. Hence, $\mathcal{H}^{n-2}(\mathfrak{S} \cap (B^+_1 \cup T_1)) = 0$. Given the compactness theorem, the tangent map $\phi_0$ is established, which is nontrivial if $p \in \mathfrak{S}$. The above proceedings are similar to the interior case. However, $\phi_0$ is constant according the Geometric Lemma, which is proved in [5, Proposition 2.6] with the help of (3.8) and the compactness theorem. It is a contradiction. At last, since $\phi$ is continuous in a neighborhood of $\partial M$ and $\phi = \psi$ on $\partial M$, $\psi \in C^{2,\alpha}(\partial M, N)$, we have that $\phi$ is $C^{2,\alpha}$ in the neighborhood of $\partial M$.

**Proof of Theorem 1.3** For any $p \in \mathrm{int}M$, consider $\psi(y) = \phi(\lambda y)$, where $\lambda$ is sufficinetly small and $y \in B_1(p)$. By Corollary 3.1 and $\mathcal{H}^{n-2}(\mathfrak{S} \cap B_{\frac{1}{2}}) = 0$, we conclude that $\mathfrak{S} \cap B_{\frac{1}{2}} = \varnothing$, when $n = 2$, i.e., $p$ is a regular point of $\phi$. For $p \in \partial M$, the proof is the same.

As mentioned in [14], $c(t_1)$ is called a focal point of $p = c(0)$ along a geodesic $c(t)$ in a Riemannian manifold if there exists a nontrivial Jacobi field $J(t)$ satisfying the following conditions:

$$g(\nabla_{c'}J(0), J(0)) = 0, \quad J(t_1) = 0. \tag{3.9}$$

It is evident that Riemannian manifolds with nonpositive sectional curvature should not have focal points.

**Corollary 3.2** *Let $M$ be a compact Finsler manifold, and $N$ is a Riemannian manifold without focal points. Then any E-minimizing map $\phi \in W^{1,2}(M,N)$ is smooth.*

**Proof** By Theorem 1.2, it suffices to show that any tangent map $\phi : R^j \to N$, $j \geq 3$ is trivial, that is, any harmonic map $\psi : S^{j-1} \to N$, $\psi(\frac{x}{|x|}) = \phi(x)$ is trivial, which was proved in [14].

**Corollary 3.3** *Let $(M,F)$ be a compact Finsler manifold with boundary and $N$ a compact Riemannian manifold without focal points. Then for a given map $\psi : \partial M \to N$, there is a map $\phi \in W^{1,2}(M,N)$ with $\phi(\partial M) = \psi$, and $\phi$ is a smooth harmonic map in the interior of $M$.*

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