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Higher-Order Schwarz-Pick Estimates for Holomorphic Self-mappings on Classical Domains*

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Abstract In this paper, Schwarz-Pick estimates for high order Fréchet derivatives of holomorphic self-mappings on classical domains are presented. Moreover, the obtained result can deduce the early work on Schwarz-Pick estimates of higher-order partial derivatives for bounded holomorphic functions on classical domains.

Keywords Schwarz-Pick estimate, Holomorphic self-mapping, Classical domain, Holomorphic expansion
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1 Introduction

For notation, let $\mathbf{D}$ be the unit disk in $\mathbb{C}$, and $\mathbf{B}_n$ be the unit ball in $\mathbb{C}^n$. A multi-index $v = (v_{11}, \cdots, v_{mn})$ consists of $m \times n$ nonnegative integers v_{ij}, $1 \le i \le m$, $1 \le j \le n$. The degree of a multi-index v is the sum

$$|v| = \sum_{\substack{1\le i\le m\\ 1\le j\le n}} v_{ij}.$$

Denote $v! = v_{11}! \cdots v_{mn}!$. Given another multi-index $\alpha = (\alpha_{11}, \cdots, \alpha_{mn})$, let

$$v^\alpha = (v_{11}^{\alpha_{11}}, \cdots, v_{mn}^{\alpha_{mn}}).$$

For vectors $Z = (z_{ij})_{1\le i\le m, 1\le j\le n} \in \mathbb{C}^{m\times n}$, and the multi-index v, let

$$z^v = \prod_{\substack{1\le i\le m\\ 1\le j\le n}} z_{ij}^{v_{ij}}.$$

Now we recall the definition of the four classical domains in the sense of [1]. The first classical domain $\mathcal{R}_{\mathrm{I}}(m, n) \subset \mathbb{C}^{m\times n}$ consists of matrices $Z \in \mathbb{C}^{m\times n}$, such that $I_m - Z\overline{Z}^{\mathrm{T}} > 0$, where I_m is the identity matrix of rank m, $\overline{Z}^{\mathrm{T}}$ means the transpose and complex conjugate of

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Z, and the inequality sign means $I_m - Z\overline{Z}^{\mathrm{T}}$ is positive definite. The second classical domain $\mathcal{R}_{\mathrm{II}}(n) \subset \mathbb{C}^{n\times n}$ consists of matrices $Z \in \mathbb{C}^{n\times n}$, such that $Z = Z^{\mathrm{T}}$ and $I_n - Z\overline{Z}^{\mathrm{T}} > 0$. The third classical domain $\mathcal{R}_{\mathrm{III}}(n) \subset \mathbb{C}^{n\times n}$ consists of matrices $Z \in \mathbb{C}^{n\times n}$, such that $Z = -Z^{\mathrm{T}}$ and $I_n - Z\overline{Z}^{\mathrm{T}} > 0$. The fourth classical domain $\mathcal{R}_{\mathrm{IV}}(n) \subset \mathbb{C}^n$ consists of vectors or $1\times n$ matrices $Z \in \mathbb{C}^n$, such that $|ZZ^{\mathrm{T}}|^2 + 1 - 2|Z|^2 > 0,\ |ZZ^{\mathrm{T}}| < 1$.

Let $\mathcal{R}$ denote one of the four classical domains, and $\partial_0\mathcal{R}$ be the distinguished boundary of $\mathcal{R}$. For $\mathcal{R}$, we consider the Minkowski functional $\|\cdot\|_{\mathcal{R}}$ (see [2]) instead of the Euclidean norm in several complex variables

$$\|Z\|_{\mathcal{R}} = \sup\{|\alpha Z\beta^{\mathrm{T}}| : Z \in \mathbb{C}^{m\times n},\ \alpha \in \partial\mathbf{B}_m,\ \beta \in \partial\mathbf{B}_n\},$$

where $\mathcal{R} = \mathcal{R}_{\mathrm{I}}(m,n),\ \mathcal{R}_{\mathrm{II}}(n),\ \mathcal{R}_{\mathrm{III}}(n)$ ($m = n$ for $\mathcal{R}_{\mathrm{II}}(n),\ \mathcal{R}_{\mathrm{III}}(n)$). When $\mathcal{R}_{\mathrm{I}}(1,n) = \mathbf{B}_n$, the Minkowski functional equals the Euclidean norm.

Denote by $\Omega_{X,Y}$ the space of holomorphic mappings from bounded domains X to Y, and by $\mathrm{Aut}(\mathcal{R})$ the group of holomorphic automorphisms of $\mathcal{R}$. If T is a linear operator between normed linear spaces, we denote by $\|T\|$ its norm. $Df(Z)$ denotes the Fréchet derivative of $f(Z)$ with $Z \in \mathbb{C}^{m\times n}$, and $Df(Z)\cdot\beta$ denotes the Fréchet derivative of $f(Z)$ in the direction $\beta \in \mathbb{C}^{m\times n}$. For a non-negative integer k, the kth order Fréchet derivative of $f(Z)$ and its evaluation on $(\beta_1,\cdots,\beta_k)$, $\beta_i \in \mathbb{C}^{m\times n}$, $i = 1,\cdots,k$ are denoted by $D^kf(Z)$ and $D^kf(Z)\cdot(\beta_1,\cdots,\beta_k)$ respectively. The norm of $D^kf(Z)$ is defined as

$$\|D^kf(Z)\| = \sup\{|D^kf(Z)\cdot(\beta_1,\cdots,\beta_k)| : \|\beta_1\|_{\mathcal{R}} = \cdots = \|\beta_k\|_{\mathcal{R}} = 1\}. \tag{1.1}$$

In addition, denote $D^kf(Z)\cdot(\beta,\cdots,\beta)$ by $D^kf(Z)\cdot\beta^k$. Then we have

$$D^kf(Z)\cdot\beta^k = \sum_{|v|=k}\frac{k!}{v!}\frac{\partial^k f(Z)}{\partial z_{11}^{v_{11}}\cdots\partial z_{mn}^{v_{mn}}}\beta^v, \qquad Z \in \mathcal{R},\ \beta \in \mathbb{C}^{m\times n}. \tag{1.2}$$

In particular, when $m = n = 1$, we have

$$D^kf(z)\cdot\beta^k = f^{(k)}(z)\beta^k, \quad z \in \mathbf{D},\ \beta \in \mathbb{C}.$$

The estimate for the derivatives of mappings is an interesting topic in the geometry function theory. There are many results on it (see [3–5] and references therein). The classical Schwarz-Pick estimate for bounded holomorphic functions in the unit disk was given as follows:

$$|\varphi'(z)| \le \frac{1-|\varphi(z)|^2}{1-|z|^2}, \qquad |z| < 1,$$

where $\varphi(z)$ is a holomorphic function on $\mathbf{D}$ and $|\varphi(z)| < 1$. Furthermore, Anderson and Rovnyak [6] proved the inequalities for operator valued functions, and discussed in detail the optimality of the Schwarz-Pick inequality. Pan and Liao [7] first gave the Schwarz-Pick inequality of order $2,3,4$ for bounded holomorphic functions in the unit disk. Later, the above inequality was generalized to the derivatives of arbitrary order in [8–14]. The following estimate is the best one so far.

Theorem 1.1 (see [8, 14]) *If $\varphi(z) \in \Omega_{\mathbf{D},\mathbf{D}}$, then*

$$|\varphi^{(k)}(z)| \le \frac{k!(1-|\varphi(z)|^2)}{(1-|z|^2)^k}(1+|z|)^{k-1}, \quad |z|<1. \tag{1.3}$$

In the case of several complex variables, the authors [15] generalized the result of [14] to bounded holomorphic functions on $\mathbf{B}_n$. Using operator theory, Anderson, Dritschel and Rovnyak [16] estimated derivatives of arbitrary order of functions in the Schur-Agler class function on the unit ball and polydisc of $\mathbb{C}^n$, where the Schur-Agler class only contains the bounded holomorphic functions satisfying a strict assumption. Later, Dai, Chen and Pan [17] got a more precise Schwarz-Pick estimate than that in [15] for bounded holomorphic functions in the unit ball.

Theorem 1.2 (see [17]) *If $\varphi(z) \in \Omega_{\mathbf{B}_n,\mathbf{D}}$, then for any multi-index $v=(v_1,\cdots,v_n)\neq 0$,*

$$\Big|\frac{\partial^{|v|}\varphi(z)}{\partial z_1^{v_1}\cdots\partial z_n^{v_n}}\Big| \le \sqrt{\frac{|v|^{|v|}}{v^v}}\, v!\frac{1-|\varphi(z)|^2}{(1-|z|^2)^{|v|}}(1+|z|)^{|v|-1}. \tag{1.4}$$

Most recently, the authors have obtained the Schwarz-Pick estimates for the bounded holomorphic functions in classical domains (see [18]), which deduces the corresponding results in the unit ball and disk (see [14–15, 17]).

Theorem 1.3 (see [18]) *Let $\varphi(Z) \in \Omega_{\mathcal{R},\mathbf{D}}$, where $\mathcal{R}=\mathcal{R}_{\mathrm{I}}(m,n)$, $\mathcal{R}_{\mathrm{II}}(n)$, $\mathcal{R}_{\mathrm{III}}(n)$. Then*

$$|D^k\varphi(Z)\cdot W^k| \le k!(1-|\varphi(Z)|^2)\frac{(1+\|Z\|_{\mathcal{R}})^{k-1}}{(1-\|Z\|_{\mathcal{R}}^2)^k}, \tag{1.5}$$

where $Z\in\mathcal{R}$ and $W\in\partial_0\mathcal{R}$.

The holomorphic self-mapping is an interesting and important topic for bounded domains with several complex variables. Using homogeneous expansions of holomorphic mappings, Liu and Wang got a Bohr's theorem in classical domains (see [19]). In this paper, based on the Minkowski norm of classical domains, we give a new generalization of Schwarz-Pick estimate to holomorphic self-mapping of classical domains.

2 The Schwarz-Pick Estimate for $\Omega_{\mathbf{D},\mathcal{R}}$

We first give some important lemmas.

Lemma 2.1 (see [18]) *Let $\mathcal{R}=\mathcal{R}_{\mathrm{I}}(m,n)$, or $\mathcal{R}_{\mathrm{II}}(n)$, or $\mathcal{R}_{\mathrm{III}}(n)$. For two given points $p,q\in\mathcal{R}$ with $q-p\in\mathcal{R}$, let $L(z)=p+z(q-p)$ for $z\in\mathbb{C}$. Then*

$$L(\mathbf{D}^*)\subset\mathcal{R},$$

where

$$\mathbf{D}^* = \Big\{ z \in \mathbb{C} : \Big| z + \frac{\overline{a}b}{|a|^2} \Big|^2 < \frac{1}{|a|^2} \Big\}, \quad a = \alpha(q-p)\beta^{\mathrm{T}} \neq 0,\ b = \alpha p \beta^{\mathrm{T}}$$

for any given $\alpha \in \partial \mathbf{B}_m,\ \beta \in \partial \mathbf{B}_n$.

Lemma 2.2 *Let* $f \in \Omega_{\mathbf{D},\mathcal{R}}$*, where* $\mathcal{R}$ *is one of the calssical domains and* $f(0) = P \in \mathcal{R}$. $\phi_P \in \mathrm{Aut}(\mathcal{R})$ *which maps* P *to* 0. *Then*

$$\frac{\|D_{\phi_P}(P)[D^k f(0) \cdot z^k]\|_{\mathcal{R}}}{k!} < 1, \quad z \in \mathbf{D}.$$

Proof Without loss of generality, we assume $\mathcal{R} = \mathcal{R}_{\mathrm{I}}(m,n)$. Since $\mathcal{R}$ is convex, for a fixed integer k, we can define

$$h(z) = \frac{1}{k} \sum_{j=1}^{k} f(\mathrm{e}^{\frac{\mathrm{i}2\pi j}{k}} z).$$

Then, $h(z) \in \Omega_{\mathbf{D},\mathcal{R}}$, $h(0) = P$. From the holomorphic expansion of the holomorphic mapping f, we get

$$h(z) = \frac{1}{k} \Big(\sum_{j=1}^{k} \Big(f(0) + \sum_{l=1}^{\infty} \mathrm{e}^{\frac{\mathrm{i}2\pi l}{k}} \frac{D^l f(0) \cdot z^l}{l!} \Big) \Big).$$

This implies that

$$\begin{aligned} \phi_P(h(z)) &= \phi_P \Big(P + \sum_{l=1}^{\infty} \frac{D^{lk} f(0) \cdot z^{lk}}{(lk)!} \Big) \\ &= \phi_P(P) + D_{\phi_P}(P) \Big(\sum_{l=1}^{\infty} \frac{D^{lk} f(0) \cdot z^{lk}}{(lk)!} \Big) + \cdots \\ &= \frac{D_{\phi_P}(P)[D^k f(0) \cdot z^k]}{k!} + \frac{D_{\phi_P}(P)[D^{2k} f(0) \cdot z^{2k}]}{(2k)!} + \cdots, \end{aligned} \tag{2.1}$$

and hence

$$\frac{D_{\phi_P}(P)[D^k f(0) \cdot z^k]}{k!} = \frac{1}{2\pi} \int_0^{2\pi} \phi_P(h(z\mathrm{e}^{\mathrm{i}\theta})) \mathrm{e}^{-\mathrm{i}k\theta} \mathrm{d}\theta.$$

Since $\phi_P(h(z)) \in \Omega_{\mathbf{D},\mathcal{R}}$, we can obtain

$$\frac{\|D_{\phi_P}(P)[D^k f(0) \cdot z^k]\|_{\mathcal{R}}}{k!} < 1, \quad z \in \mathbf{D}.$$

Theorem 2.1 *Let* $f \in \Omega_{\mathbf{D},\mathcal{R}}$*, where* $\mathcal{R} = \mathcal{R}_{\mathrm{I}}(m,n),\ \mathcal{R}_{\mathrm{II}}(n),\ \mathcal{R}_{\mathrm{III}}(n)$ *and* $f(z) = P \in \mathcal{R}$. $\phi_P \in \mathrm{Aut}(\mathcal{R})$ *which maps* P *to* 0. *Then*

$$\|D_{\phi_P}(P)[D^k f(z) \cdot 1^k]\|_{\mathcal{R}} \leq \frac{k!(1+|z|^{k-1})}{(1-|z|^2)^k} \tag{2.2}$$

holds for $k \geq 1$ *and* $z \in \mathbf{D}$.

Proof Let $\zeta \in \mathbf{D}$ and a positive integer k be fixed. We consider $g = f(\tau_\zeta) \in \Omega_{\mathbf{D},\mathcal{R}}$, where

$$\tau_\zeta(z) = \frac{\zeta - z}{1 - \overline{\zeta} z}.$$

For g, $g(0) = f(\zeta) = P$ and

$$g(z) = \sum_{n=0}^{\infty} \frac{D^n g(0) \cdot z^n}{n!} = \sum_{n=0}^{\infty} \frac{D^n g(0) \cdot 1^n}{n!} z^n.$$

It is easy to verify that

$$\frac{\mathrm{d}^n (\tau_\zeta(z)^j)}{\mathrm{d}z^n}\Big|_{z=\zeta} = \begin{cases} 0, & n < j, \\ \dfrac{(-1)^j \overline{\zeta}^{n-j}}{(1-|\zeta|^2)^n} \dfrac{n!(n-1)!}{(n-j)!(j-1)!}, & n \geq j. \end{cases}$$

Let

$$A_j = \frac{(-1)^j \overline{\zeta}^{k-j}}{(1-|\zeta|^2)^k} \frac{k!(k-1)!}{(k-j)!(j-1)!}.$$

Since $f = g(\tau_\zeta)$, we have

$$D^k f(\zeta) \cdot 1^k = \sum_{j=1}^{k} \frac{D^j g(0) \cdot 1^j}{j!} A_j.$$

Using Lemma 2.2, we can obtain

$$\begin{aligned}
&\|D_{\phi_P}(P)[D^k f(\zeta) \cdot 1^k]\|_{\mathcal{R}} \\
&= \sup\{|\alpha[D_{\phi_P}(P)(D^k f(\zeta) \cdot 1^k)]\beta'| : \alpha \in \partial \mathbf{B}_m, \beta \in \partial \mathbf{B}_n\} \\
&= \sup\Big\{\Big|\alpha\Big[D_{\phi_P}(P)\Big(\sum_{j=1}^{k} \frac{D^j g(0) \cdot 1^j}{j!} A_j\Big)\Big]\beta'\Big| : \alpha \in \partial \mathbf{B}_m, \beta \in \partial \mathbf{B}_n\Big\} \\
&\leq \sup\Big\{\max_{j=1,\cdots,k}\Big\{\Big|\alpha\Big[D_{\phi_P}(P)\Big(\frac{D^j g(0) \cdot 1^j}{j!}\Big)\Big]\beta'\Big|\Big\} \sum_{j=1}^{k} |A_j| : \alpha \in \partial \mathbf{B}_m, \beta \in \partial \mathbf{B}_n\Big\} \\
&= \max_{j=1,\cdots,k}\Big\{\Big\|D_{\phi_P}(P)\Big(\frac{D^j g(0) \cdot 1^j}{j!}\Big)\Big\|_{\mathcal{R}}\Big\} \sum_{j=1}^{k} |A_j| \\
&\leq \sum_{j=1}^{k} |A_j|,
\end{aligned} \tag{2.3}$$

where the last inequality comes from Lemma 2.2 with $z \to \partial \mathbf{D}$.

On the other hand,

$$\sum_{j=1}^{k} |A_j| = \frac{k!}{(1-|\zeta|^2)^k} \sum_{j=1}^{k} \frac{(k-1)!|\zeta|^{k-j}}{(k-j)!(j-1)!} = \frac{k!(1+|\zeta|^{k-1})}{(1-|\zeta|^2)^k}.$$

At last, replacing ζ with z, we complete the proof of Theorem 2.1.

3 The Schwarz-Pick Estimate for $\Omega_{\mathcal{R},\mathcal{R}}$

Lemma 3.1 *Let $g(z) \in \Omega_{\mathbf{D}_{z_0,r},\mathcal{R}}$, where $\mathbf{D}_{z_0,r} = \{z \in \mathbb{C} : |z - z_0| < r\}$. Then*

$$\|D_{\phi_P}(P)[D^k g(z) \cdot 1^k]\|_{\mathcal{R}} \le \frac{k!r(r + |z - z_0|)^{k-1}}{(r^2 - |z - z_0|^2)^k}, \tag{3.1}$$

where $z \in \mathbf{D}_{z_0,r}$ and $g(z) = P$.

Proof Let $\varphi(z) = g(rz + z_0)$ for $z \in \mathbf{D}$. By Theorem 2.1, we have

$$\|D_{\phi_P}(P)[D^k \varphi(z) \cdot 1^k]\|_{\mathcal{R}} \le \frac{k!(1 + |z|)^{k-1}}{(1 - |z|^2)^k}.$$

For $\xi \in \mathbf{D}_{z_0,r}$, $r^k D^k g(\xi) \cdot 1^k = D^k g(\xi) \cdot r^k = D^k \varphi(\frac{\xi - z_0}{r}) \cdot 1^k$. Let $z = \frac{\xi - z_0}{r}$. Then we have

$$\|D_{\phi_P}(P)[D^k g(\xi) \cdot 1^k]\|_{\mathcal{R}} \le \frac{k!r(r + |\xi - z_0|)^{k-1}}{(r^2 - |\xi - z_0|^2)^k}, \quad \xi \in \mathbf{D}_{z_0,r}.$$

The proof is completed.

Now we give our main results.

Theorem 3.1 *Let $\varphi(Z) \in \Omega_{\mathcal{R},\mathcal{R}}$, where $\mathcal{R} = \mathcal{R}_{\mathrm{I}}(m,n)$, $\mathcal{R}_{\mathrm{II}}(n)$, $\mathcal{R}_{\mathrm{III}}(n)$. Then*

$$\frac{\|D_{\phi_P}(P)[D^k \varphi(Z) \cdot W^k]\|_{\mathcal{R}}}{\|D_{\phi_P}(P)\|} \le k!(1 - \|\varphi(Z)\|_{\mathcal{R}}^2) \frac{(1 + \|Z\|_{\mathcal{R}})^{k-1}}{(1 - \|Z\|_{\mathcal{R}}^2)^k}, \tag{3.2}$$

where $Z \in \mathcal{R}$, $\varphi(Z) = P$ and $W \in \partial_0 \mathcal{R}$.

Proof For a given $p \in \mathcal{R}$, $q \in \mathcal{R}$ such that $q - p \in \mathcal{R}$. Without loss of generality, assume $p, q \in \mathcal{R}_{\mathrm{I}}(m,n)$. Let $g(z) = \varphi(L(z))$ for $z \in \mathbf{D}^*$, where $L(z)$, $\mathbf{D}^*$ are defined in Lemma 2.1. Apply Lemma 3.1 to $z = 0$. Then

$$\|D_{\phi_P}(P)[D^k g(0) \cdot 1^k]\|_{\mathcal{R}} \le \frac{k!r(r + |z_0|)^{k-1}}{(r^2 - |z_0|^2)^k},$$

where $r = \frac{1}{|a|}$, $z_0 = -\frac{\overline{a}b}{|a|^2}$, and $a = \alpha(q-p)\beta^{\mathrm{T}} \ne 0$, $b = \alpha p \beta^{\mathrm{T}}$ for any given $\alpha \in \partial \mathbf{B}_m$, $\beta \in \partial \mathbf{B}_n$ by Lemma 2.1.

Note $g(0) = \varphi(p) = P$, $D^k g(0) \cdot 1^k = D^k \varphi(p) \cdot (q - p)^k$, and furthermore,

$$\frac{r(r + |z_0|)^{k-1}}{(r^2 - |z_0|^2)^k} = \frac{|a|^k (1 + |b|)^{k-1}}{(1 - |b|^2)^k}.$$

So, we have

$$\|D_{\phi_P}(P)[D^k \varphi(p) \cdot (q - p)^k]\|_{\mathcal{R}} \le \frac{k!|a|^k (1 + |b|)^{k-1}}{(1 - |b|^2)^k}.$$

Since $q - p \in \mathcal{R}$, multiplying both sides of above inequality by $\frac{1}{\|q-p\|_{\mathcal{R}}^k}$, we have

$$\Big\|D_{\phi_P}(P)\Big[D^k \varphi(p) \cdot \Big(\frac{q - p}{\|q - p\|_{\mathcal{R}}}\Big)^k\Big]\Big\|_{\mathcal{R}} \le \frac{k!|c|^k (1 + |b|)^{k-1}}{(1 - |b|^2)^k}, \tag{3.3}$$

where $c = \frac{a}{\|q-p\|_{\mathcal{R}}}$. Now, from the notations of $a,\ b,\ c$ for any given $\alpha \in \partial \mathbf{B}_m,\ \beta \in \partial \mathbf{B}_n$, we have

$$\frac{|c|^k(1+|b|)^{k-1}}{(1-|b|^2)^k} \leq \frac{(1+\|p\|_{\mathcal{R}})^{k-1}}{(1-\|p\|_{\mathcal{R}}^2)^k}.$$

Hence, replacing p with Z, we have the following result from (3.3):

$$\|D_{\phi_P}(P)[D^k\varphi(Z)\cdot W^k]\|_{\mathcal{R}} \leq \frac{k!(1+\|Z\|_{\mathcal{R}})^{k-1}}{(1-\|Z\|_{\mathcal{R}}^2)^k}, \tag{3.4}$$

where $W \in \partial_0 \mathcal{R}$.

On the other hand, it comes from [19] that

$$\|D_{\phi_P}(P)\| = \frac{1}{1-\|\varphi(Z)\|_{\mathcal{R}}^2}.$$

Combining the above equality with (3.4), we have

$$\frac{\|D_{\phi_P}(P)[D^k\varphi(Z)\cdot W^k]\|_{\mathcal{R}}}{\|D_{\phi_P}(P)\|} \leq k!(1-\|\varphi(Z)\|_{\mathcal{R}}^2)\frac{(1+\|Z\|_{\mathcal{R}})^{k-1}}{(1-\|Z\|_{\mathcal{R}}^2)^k}.$$

The theorem is proved.

Remark 3.1 Especially, when $\varphi(Z) \in \Omega_{\mathcal{R},\mathbf{D}}$, Theorem 1.3 can be easily deduced from Theorem 3.1. Hence, we can also obtain the early work (see [18]) on Schwarz-Pick estimates of higher-order derivatives for bounded holomorphic functions in $\mathbf{D}$ (see [14]) and in $\mathbf{B}_n$ (see [15, 17]).

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