The Teodorescu Operator in Clifford Analysis

F. BRACKX[1]   H. De SCHEPPER[1]
M. E. LUNA-ELIZARRARÁS[2]   M. SHAPIRO[2]

Abstract Euclidean Clifford analysis is a higher dimensional function theory centred around monogenic functions, i.e., null solutions to a first order vector valued rotation invariant differential operator $\underline{\partial}$ called the Dirac operator. More recently, Hermitian Clifford analysis has emerged as a new branch, offering yet a refinement of the Euclidean case; it focuses on the simultaneous null solutions, called Hermitian monogenic functions, to two Hermitian Dirac operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}^\dagger}$ which are invariant under the action of the unitary group. In Euclidean Clifford analysis, the Teodorescu operator is the right inverse of the Dirac operator $\underline{\partial}$. In this paper, Teodorescu operators for the Hermitian Dirac operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}^\dagger}$ are constructed. Moreover, the structure of the Euclidean and Hermitian Teodorescu operators is revealed by analyzing the more subtle behaviour of their components. Finally, the obtained inversion relations are still refined for the differential operators issuing from the Euclidean and Hermitian Dirac operators by splitting the Clifford algebra product into its dot and wedge parts. Their relationship with several complex variables theory is discussed.

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1 Introduction

In the complex plane, the Teodorescu integral operator T is well-known. In a bounded domain Ω with smooth boundary $\partial\Omega$, its action on a function $u : \mathbb{R}^2 \to \mathbb{C}$ is given by

$$T[u](x,y) = -\frac{1}{\pi} \int_\Omega \frac{1}{\zeta - z}\, u(\zeta) \mathrm{d}V(\xi,\eta), \quad \zeta = \xi + \mathrm{i}\eta, \quad z = x + \mathrm{i}y.$$

It is the right inverse in Ω to the Cauchy-Riemann operator $\overline{\partial}$, i.e., $\overline{\partial}\, T[u] = u$. The Teodorescu integral kernel thus coincides with the Cauchy kernel, which itself is a convolution kernel obtained by translating the fundamental solution $\frac{1}{\pi z}$ to the Cauchy-Riemann operator $\overline{\partial}$.

In order to obtain an analogue of the Teodorescu operator in higher dimension, one first looks for a generalization of the Cauchy kernel. Now the Cauchy integral formula for holomorphic functions in the complex plane may be generalized to the case of several complex variables in two ways: either one takes a holomorphic kernel and an integral over the distinguished boundary

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[1]Clifford Research Group, Faculty of Engineering and Architecture, Ghent University, Belgium.
   E-mail: fb@cage.ugent.be   hds@cage.ugent.be
[2]Instituto Politécnico Nacional, Mexico City, Mexico. E-mail: eluna@esfm.ipn.mx   shapiro@esfm.ipn.mx

$\partial_0 \widetilde{D} = \prod\limits_{j=1}^{n} \partial \widetilde{D}_j$ of a polydisk $\widetilde{D} = \prod\limits_{j=1}^{n} \widetilde{D}_j$ in $\mathbb{C}^n$, leading to

$$f(z_1, \cdots, z_n) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \widetilde{D}} \frac{f(\xi_1, \cdots, \xi_n)}{(\xi_1 - z_1) \cdots (\xi_n - z_n)} d\xi_1 \wedge \cdots \wedge d\xi_n, \quad z_j \in \overset{\circ}{\widetilde{D}}_j, \tag{1.1}$$

or one takes an integral over the (piecewise) smooth boundary ∂D of a bounded domain D in $\mathbb{C}^n$, yielding

$$f(z) = \int_{\partial D} f(\xi) \, U(\xi, z) d\xi, \quad z \in \overset{\circ}{D}, \tag{1.2}$$

where, with $\cdot^c$ denoting the complex conjugate, the kernel

$$U(\xi, z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} (-1)^{j-1} \frac{\xi_j^c - z_j^c}{|\xi - z|^{2n}} d\xi_1^c \wedge \cdots \wedge d\xi_{j-1}^c \wedge d\xi_{j+1}^c \wedge \cdots \wedge d\xi_n^c \wedge d\xi_1 \wedge \cdots \wedge d\xi_n$$

is the so-called Martinelli-Bochner kernel (see, e.g., [27]), which is not holomorphic but still harmonic. For the history of formula (1.2), obtained independently by Martinelli and Bochner, we refer to [26]. So we may expect that a Teodorescu operator in several complex variables will involve a harmonic integral kernel which will formally mimic the Martinelli-Bochner kernel. It is explained in Section 5 that this indeed is the case.

There is however an alternative for generalizing the two-dimensional Cauchy integral formula, offered by Clifford analysis, where functions defined in Euclidean space $\mathbb{R}^m$ and taking values in a Clifford algebra are studied. The theory focuses on so-called monogenic functions, i.e., null solutions to the elliptic Dirac operator $\partial_{\underline{X}}$ factorizing the Laplace operator: $\partial_{\underline{X}}^2 = -\Delta_m$. The Dirac operator being rotation invariant, the name Euclidean Clifford analysis is used nowadays to refer to this setting (see, e.g., [12, 22–23, 25]). Here the kernel appearing in the Clifford-Cauchy formula is monogenic, up to a pointwise singularity, while the integral is taken over the complete boundary:

$$f(\underline{X}) = \int_{\partial D} E(\underline{\Xi} - \underline{X}) d\sigma_{\underline{\Xi}} f(\underline{\Xi}), \quad \underline{X} \in \overset{\circ}{D}$$

with

$$E(\underline{\Xi} - \underline{X}) = \frac{1}{a_m} \frac{\overline{\underline{\Xi}} - \overline{\underline{X}}}{|\underline{\Xi} - \underline{X}|^m},$$

a_m denoting the area of the unit sphere $S^m \subset \mathbb{R}^m$, $\bar{\cdot}$ denoting the main conjugation in the Clifford algebra, and $d\sigma_{\underline{\Xi}}$ being a Clifford algebra valued differential form of order $(m-1)$. This Clifford-Cauchy integral formula has been a corner stone in the development to the function theory. The function $E(\underline{X}) = \frac{1}{a_m} \frac{\overline{\underline{X}}}{|\underline{X}|^m}$ is the fundamental solution to the Euclidean Dirac operator $\partial_{\underline{X}}$, i.e., it holds in the distributional sense that $\partial_{\underline{X}} E(\underline{X}) = \delta(\underline{X})$, where $\delta(\underline{X})$ is the Dirac distribution in $\mathbb{R}^m$. As can be expected, the Teodorescu inversion in Clifford analysis indeed involves a convolution with the Clifford-Cauchy kernel. This is recalled in Section 3, where the underlying mechanisms leading to this result are also revealed.

In the even dimensional case, Hermitian Clifford analysis emerged as a refinement of the Euclidean setting; it focuses on the simultaneous null solutions to the complex Hermitian Dirac operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}^\dagger}$, which decompose the Laplacian in the sense that $4(\partial_{\underline{z}} + \partial_{\underline{z}^\dagger})^2 = 4(\partial_{\underline{z}} \partial_{\underline{z}^\dagger} +$

$\partial_{\underline{z}^\dagger}\partial_{\underline{z}}) = \Delta_{2n}$ and which are invariant under the action of the special unitary group. The study of complex Dirac operators was initiated in [28–30]. A systematic development of the function theory in the Hermitian Clifford analysis context, including the invariance properties with respect to the underlying Lie groups and Lie algebras, is still in full progress (see, e.g., [20, 1–4, 16, 19, 8]). In this framework, a Cauchy integral formula for Hermitian monogenic functions taking values in the complex Clifford algebra $\mathbb{C}_{2n}$ or in complex spinor space $\mathbb{S}$ was established in [11], and further integral representation formulae were developed in [17]. However, from the start, it was clear that the desired formula could not have the traditional form of (1.1) or (1.2). Indeed, it is known (see [4]) that in the special case where the functions considered take their values in a specific part of spinor space, Hermitian monogenicity is equivalent to holomorphy in the underlying complex variables. It turned out that a matrix approach was the key to obtaining the desired result. Moreover and as could be expected, the obtained Hermitian Cauchy integral formula reduces to the traditional Martinelli-Bochner formula (1.2) in the particular case mentioned. This also means that the theory of Hermitian monogenic functions not only refines Euclidean Clifford analysis (and thus harmonic analysis as well), but also has strong connections with the theory of functions of several complex variables, in some sense even encompassing its results.

In Section 4, we construct Hermitian Teodorescu inversion formulae not only for the Hermitian Dirac operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}^\dagger}$, but also for the associated differential operators $\partial_{\underline{z}}\bullet$, $\partial_{\underline{z}}\wedge$, $\partial_{\underline{z}^\dagger}\bullet$, $\partial_{\underline{z}^\dagger}\wedge$, obtained by splitting the Clifford algebra or geometric product into its "dot" and "wedge" parts. These associated differential operators are the counterparts in the multi-vector language of Clifford analysis of well-known differential operators for real and complex differential forms in Euclidean space.

As was already announced above, the results of this section are then interpreted in Section 5 for scalar valued functions of several complex variables, leading to the expected connections with the Martinelli-Bochner approach.

2 Preliminaries of Clifford Analysis

The real Clifford algebra $\mathbb{R}_{0,m}$ is constructed over the vector space $\mathbb{R}^{0,m}$ endowed with a non-degenerate quadratic form of signature $(0, m)$ and generated by the orthonormal basis $(e_1, \cdots, e_m)$. The non-commutative Clifford or geometric multiplication in $\mathbb{R}_{0,m}$ is governed by the rules

$$e_\alpha e_\beta + e_\beta e_\alpha = -2\delta_{\alpha\beta}, \quad \alpha, \beta = 1, \cdots, m. \tag{2.1}$$

As a basis for $\mathbb{R}_{0,m}$, one takes for any set $A = \{j_1, \cdots, j_h\} \subset \{1, \cdots, m\}$, the element $e_A = e_{j_1} \cdots e_{j_h}$ with $1 \le j_1 < j_2 < \cdots < j_h \le m$, together with $e_\varnothing = 1$, the identity element. Any Clifford number a in $\mathbb{R}_{0,m}$ may thus be written as $a = \sum_A e_A a_A$, $a_A \in \mathbb{R}$, or still as $a = \sum_{k=0}^m [a]_k$, where $[a]_k = \sum_{|A|=k} e_A a_A$ is the so-called k-vector part of a. Euclidean space $\mathbb{R}^{0,m}$ is embedded in $\mathbb{R}_{0,m}$ by identifying $(X_1, \cdots, X_m)$ with the Clifford vector $\underline{X} = \sum_{\alpha=1}^m e_\alpha X_\alpha$, for which it holds that $\underline{X}^2 = -\langle \underline{X}, \underline{X}\rangle = -|\underline{X}|^2$. The Clifford or geometric product of two vectors splits into the

sum of a scalar part, their commuting "dot"-product, and a 2-vector or bivector part, their anti-commuting "wedge"-product: $\underline{X}\,\underline{Y} = \underline{X} \bullet \underline{Y} + \underline{X} \wedge \underline{Y}$ with

$$\underline{X} \bullet \underline{Y} = \sum_{\alpha=1}^{m} X_\alpha Y_\alpha = \frac{1}{2}(\underline{X}\,\underline{Y} + \underline{Y}\,\underline{X}),$$

$$\underline{X} \wedge \underline{Y} = \sum_{\alpha<\beta} e_\alpha e_\beta (X_\alpha Y_\beta - Y_\alpha X_\beta) = \frac{1}{2}(\underline{X}\,\underline{Y} - \underline{Y}\,\underline{X}).$$

The Fischer dual of $\underline{X}$ is the vector valued Dirac operator $\partial_{\underline{X}} = \sum_{\alpha=1}^{m} e_\alpha\, \partial_{X_\alpha}$, underlying the notion of monogenicity of a function, a notion which is the higher dimensional counterpart of holomorphy in the complex plane. More explicitly, a function f defined and continuously differentiable in an open region Ω of $\mathbb{R}^m$ and taking values in $\mathbb{R}_{0,m}$ is called (left) monogenic in Ω iff $\partial_{\underline{X}}[f] = 0$ in Ω. As the Dirac operator factorizes the Laplacian, monogenicity can be regarded as a refinement of harmonicity. We refer to this setting as the Euclidean case, since the fundamental group leaving the Dirac operator $\partial_{\underline{X}}$ invariant is the special orthogonal group $SO(m;\mathbb{R})$, which is doubly covered by the $\mathrm{Spin}(m)$ group of the Clifford algebra $\mathbb{R}_{0,m}$. For this reason, the Dirac operator $\partial_{\underline{X}}$ is also called rotation invariant.

When allowing for complex constants and taking the dimension to be even: $m = 2n$, the generators $(e_1, \cdots, e_{2n})$, still satisfying the multiplication rules (2.1), produce the complex Clifford algebra $\mathbb{C}_{2n} = \mathbb{R}_{0,2n} \oplus i\mathbb{R}_{0,2n}$. Any complex Clifford number $\lambda \in \mathbb{C}_{2n}$ may thus be written as $\lambda = a + ib$, $a, b \in \mathbb{R}_{0,2n}$, an observation leading to the definition of the Hermitian conjugation $\lambda^\dagger = (a + ib)^\dagger = \bar{a} - i\bar{b}$, where the bar notation stands for the usual Clifford conjugation in $\mathbb{R}_{0,2n}$, i.e., the main anti-involution for which $\bar{e}_\alpha = -e_\alpha$, $\alpha = 1, \cdots, 2n$. This Hermitian conjugation also leads to a Hermitian inner product and its associated norm on $\mathbb{C}_{2n}$ given by $(\lambda, \mu) = [\lambda^\dagger \mu]_0$ and $|\lambda| = \sqrt{[\lambda^\dagger \lambda]_0} = \left(\sum_A |\lambda_A|^2\right)^{\frac{1}{2}}$. This is the framework of the so-called Hermitian Clifford analysis, a refinement of Euclidean Clifford analysis.

An elegant way of introducing it consists in considering a complex structure, i.e., a specific $SO(2n;\mathbb{R})$-element J for which $J^2 = -\mathbf{1}$ (see [3–4]). Here, J is chosen to act upon the generators $e_1, \cdots, e_{2n}$ of the Clifford algebra as $J[e_j] = -e_{n+j}$ and $J[e_{n+j}] = e_j$, $j = 1, \cdots, n$. With J, one associates two projection operators $\frac{1}{2}(\mathbf{1} \pm iJ)$ which produce the main objects of the Hermitian setting by acting upon the corresponding ones in the Euclidean framework. First, the so-called Witt basis elements $(\mathfrak{f}_j, \mathfrak{f}_j^\dagger)_{j=1}^n$ for $\mathbb{C}_{2n}$ are obtained from the orthogonal basis elements e_α,

$$\mathfrak{f}_j = \frac{1}{2}(\mathbf{1} + iJ)[e_j] = \frac{1}{2}(e_j - i\,e_{n+j}), \quad j = 1, \cdots, n,$$

$$\mathfrak{f}_j^\dagger = -\frac{1}{2}(\mathbf{1} - iJ)[e_j] = -\frac{1}{2}(e_j + i\,e_{n+j}), \quad j = 1, \cdots, n.$$

The Witt basis elements satisfy the respective Grassmann and duality identities

$$\mathfrak{f}_j \mathfrak{f}_k + \mathfrak{f}_k \mathfrak{f}_j = \mathfrak{f}_j^\dagger \mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger \mathfrak{f}_j^\dagger = 0, \quad j, k = 1, \cdots, n,$$

$$\mathfrak{f}_j \mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger \mathfrak{f}_j = \delta_{jk}, \quad j, k = 1, \cdots, n,$$

including their isotropy. A vector in $\mathbb{R}^{0,2n}$ is now denoted by $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ and

identified with the Clifford vector

$$\underline{X} = \sum_{j=1}^{n} (e_j\, x_j + e_{n+j}\, y_j).$$

The action of the complex structure J on $\underline{X}$ yields

$$\underline{X}| = J[\underline{X}] = \sum_{j=1}^{n} (e_j\, y_j - e_{n+j}\, x_j),$$

the so-called twisted vector. The Clifford vectors $\underline{X}$ and $\underline{X}|$ anti-commute, since they are orthogonal w.r.t. the standard Euclidean scalar product. More precisely, the following lemma holds (see [18]).

Lemma 2.1 *One has*
(i) $\underline{X} \bullet \underline{X}| = 0$,
(ii) $\underline{X} \wedge \underline{X}| = \sum\limits_{j \neq k} x_j y_k (e_j e_k - e_{n+k} e_{n+j}) - \sum\limits_{j,k} e_j e_{n+k}(x_j x_k + y_j y_k)$,
(iii) $\underline{X}| \wedge \underline{X} = \sum\limits_{j \neq k} x_j y_k (e_k e_j - e_{n+j} e_{n+k}) - \sum\limits_{j,k} e_{n+k} e_j(x_j x_k + y_j y_k)$,
(iv) $\underline{X}\,\underline{X}| + \underline{X}|\,\underline{X} = \underline{X} \wedge \underline{X}| + \underline{X}| \wedge \underline{X} = 0$.

The actions of the projection operators on the Clifford vector $\underline{X}$ then produce the conjugate Hermitian Clifford vectors $\underline{z}$ and $\underline{z}^{\dagger}$,

$$\underline{z} \;=\; \frac{1}{2}(\mathbf{1} + \mathrm{i}J)[\underline{X}] = \frac{1}{2}(\underline{X} + \mathrm{i}\,\underline{X}|) = \sum_{j=1}^{n} \mathfrak{f}_j\, z_j,$$

$$\underline{z}^{\dagger} = -\frac{1}{2}(\mathbf{1} - \mathrm{i}J)[\underline{X}] = -\frac{1}{2}(\underline{X} - \mathrm{i}\,\underline{X}|) = \sum_{j=1}^{n} \mathfrak{f}_j^{\dagger}\, z_j^{c},$$

where n complex variables $z_j = x_j + \mathrm{i}y_j$ have been introduced with complex conjugates $z_j^{c} = x_j - \mathrm{i}y_j$, $j = 1, \cdots, n$. Finally, the Hermitian Dirac operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}^{\dagger}}$ are obtained from the Euclidean Dirac operator $\partial_{\underline{X}}$,

$$\partial_{\underline{z}^{\dagger}} = \frac{1}{4}(\mathbf{1} + \mathrm{i}J)[\partial_{\underline{X}}] = \frac{1}{4}(\partial_{\underline{X}} + \mathrm{i}\,\partial_{\underline{X}|}) = \sum_{j=1}^{n} \mathfrak{f}_j^{\dagger}\, \partial_{z_j},$$

$$\partial_{\underline{z}} \;=\; -\frac{1}{4}(\mathbf{1} - \mathrm{i}J)[\partial_{\underline{X}}] \;=\; -\frac{1}{4}(\partial_{\underline{X}} - \mathrm{i}\,\partial_{\underline{X}|}) = \sum_{j=1}^{n} \mathfrak{f}_j\, \partial_{z_j^{c}},$$

involving the classical Cauchy-Riemann operators $\partial_{z_j^{c}} = \frac{1}{2}(\partial_{x_j} + \mathrm{i}\partial_{y_j})$ and their complex conjugates $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - \mathrm{i}\partial_{y_j})$ in the complex z_j-planes, $j = 1, \cdots, n$. Here also the so-called twisted Dirac operator

$$\partial_{\underline{X}|} = J[\partial_{\underline{X}}] = \sum_{j=1}^{n} (e_j\, \partial_{y_j} - e_{n+j}\, \partial_{x_j})$$

arises. As the case for $\partial_{\underline{X}}$, a notion of (twisted) monogenicity may be associated in a natural way to $\partial_{\underline{X}|}$ as well.

The Hermitian vector variables and Dirac operators are isotropic, i.e.,

$$(\underline{z})^2 = (\underline{z}^\dagger)^2 = 0 \quad \text{and} \quad (\partial_{\underline{z}})^2 = (\partial_{\underline{z}^\dagger})^2 = 0,$$

whence the Laplacian $\Delta_{2n} = -\partial_{\underline{X}}^2 = -\partial_{\underline{X}|}^2$ allows for the decomposition and factorization

$$\Delta_{2n} = 4(\partial_{\underline{z}}\partial_{\underline{z}^\dagger} + \partial_{\underline{z}^\dagger}\partial_{\underline{z}}) = 4(\partial_{\underline{z}} + \partial_{\underline{z}^\dagger})^2,$$

while also

$$(\underline{z} + \underline{z}^\dagger)^2 = \underline{z}\,\underline{z}^\dagger + \underline{z}^\dagger\underline{z} = |\underline{z}|^2 = |\underline{z}^\dagger|^2 = |\underline{X}|^2 = |\underline{X}||^2.$$

A continuously differentiable function g on an open region Ω of $\mathbb{R}^{2n}$ with values in the complex Clifford algebra $\mathbb{C}_{2n}$ is called (left) Hermitian monogenic (or h-monogenic) in Ω iff it simultaneously is $\partial_{\underline{X}}$- and $\partial_{\underline{X}|}$-monogenic in Ω, i.e., it satisfies in Ω the system $\partial_{\underline{X}}\, g = 0 = \partial_{\underline{X}|}\, g$ or the equivalent system $\partial_{\underline{z}}\, g = 0 = \partial_{\underline{z}^\dagger}\, g$. It remains to recall the group invariance underlying this system. To that end we consider the group $\widetilde{\mathrm{U}}(n) \subset \mathrm{Spin}(2n)$, given by

$$\widetilde{\mathrm{U}}(n) = \{s \in \mathrm{Spin}(2n) \mid \exists\, \theta \geq 0 : \overline{s}I = \exp(-\mathrm{i}\theta)I\},$$

its definition involving the self-adjoint primitive idempotent $I = I_1 \cdots I_n$ with $I_j = \mathfrak{f}_j\mathfrak{f}_j^\dagger = \frac{1}{2}(1 - \mathrm{i}e_j e_{n+j})$, $j = 1, \cdots, n$. It was proved that this group constitutes a realisation in the Clifford algebra of the unitary group $\mathrm{U}(n)$, and that moreover, its action leaves the Hermitian Dirac operators invariant (see [20]). Less precisely, one thus says that the operators as well as the notion of h-monogenicity are invariant under the action of the unitary group.

3 The Euclidean Teodorescu Operators

Let Ω be a fixed open bounded domain in $\mathbb{R}^{2n} \cong \mathbb{C}^n$ with smooth boundary $\partial\Omega$. It is well-known that the right inverses of the Euclidean Dirac operators $\partial_{\underline{X}}$ and $\partial_{\underline{X}|}$ are the Teodorescu operators T and $T| : C_1(\Omega) \to C_1(\Omega)$, given by

$$T[u](\underline{X}) = -\int_\Omega E(\underline{Y} - \underline{X})\, u(\underline{Y})\mathrm{d}V(\underline{Y}),$$

$$T|[u](\underline{X}) = -\int_\Omega E|(\underline{Y} - \underline{X})\, u(\underline{Y})\mathrm{d}V(\underline{Y}),$$

where

$$E(\underline{X}) = \frac{1}{a_{2n}}\frac{\overline{\underline{X}}}{|\underline{X}|^{2n}} = -\frac{1}{a_{2n}}\frac{\underline{X}}{r^{2n}} = \sum_{\alpha=1}^{2n}\left(-\frac{1}{a_{2n}}\frac{X_\alpha}{r^{2n}}\right)e_\alpha = \sum_{\alpha=1}^{2n}E_\alpha e_\alpha$$

$$= \sum_{j=1}^{n}\left(-\frac{1}{a_{2n}}\frac{x_j}{r^{2n}}\right)e_j + \left(-\frac{1}{a_{2n}}\frac{y_j}{r^{2n}}\right)e_{n+j} = \sum_{j=1}^{n}E_j e_j + E_{n+j}e_{n+j}$$

and

$$E|(\underline{X}) = \frac{1}{a_{2n}}\frac{\overline{\underline{X}|}}{|\underline{X}|^{2n}} = -\frac{1}{a_{2n}}\frac{\underline{X}|}{r^{2n}} = \sum_{\alpha=1}^{2n}\left(-\frac{1}{a_{2n}}\frac{X|_\alpha}{r^{2n}}\right)e_\alpha = \sum_{\alpha=1}^{2n}E|_\alpha e_\alpha$$

$$= \sum_{j=1}^{n}\left(-\frac{1}{a_{2n}}\frac{y_j}{r^{2n}}\right)e_j - \left(-\frac{1}{a_{2n}}\frac{x_j}{r^{2n}}\right)e_{n+j} = \sum_{j=1}^{n}E_{n+j}e_j - E_j e_{n+j}$$

are the fundamental solutions to the respective Euclidean Dirac operators considered, which means that in the distributional sense $\partial_{\underline{X}} E(\underline{X}) = \delta(\underline{X})$ and $\partial_{\underline{X}|} E|(\underline{X}) = \delta(\underline{X}|) = \delta(\underline{X})$, where δ stands for the Dirac distribution in $\mathbb{R}^{2n}$. It thus holds that for each function $u \in C_1(\Omega)$ and each $\underline{X} \in \Omega$,

$$\partial_{\underline{X}} T[u](\underline{X}) = u(\underline{X}), \quad \partial_{\underline{X}|} T|[u](\underline{X}) = u(\underline{X}),$$

which also can be written as

$$\partial_{\underline{X}} T = \mathbf{1}_\Omega, \quad \partial_{\underline{X}|} T| = \mathbf{1}_\Omega. \tag{3.1}$$

As a matter of fact, this result is quite remarkable: the a priori actions of $\partial_{\underline{X}}$ on T and of $\partial_{\underline{X}|}$ on $T|$ are expected to be para-bivector valued, i.e., resulting in a sum of a scalar or "dot" and a bivector or "wedge" part, since the Euclidean Dirac operators and the corresponding Teodorescu operators are both Clifford vector valued. As the function u may in particular be chosen to be scalar valued, (3.1) thus implies that

$$\partial_{\underline{X}} \bullet T = \mathbf{1}_\Omega, \quad \partial_{\underline{X}|} \bullet T| = \mathbf{1}_\Omega, \tag{3.2}$$

while

$$\partial_{\underline{X}} \wedge T = 0, \quad \partial_{\underline{X}|} \wedge T| = 0. \tag{3.3}$$

In other words, the Teodorescu operators T and $T|$ are right inverses to the differential operators $\partial_{\underline{X}} \bullet$ and $\partial_{\underline{X}|} \bullet$, which are the Clifford analysis counterparts to the well-known operators of exterior co-derivative d^* and its twisted counterpart $d|^*$ for smooth differential forms in Euclidean space. The counterparts of the exterior derivative d and its twisted version $d|$ are the respective operators $\partial_{\underline{X}} \wedge$ and $\partial_{\underline{X}|} \wedge$, which apparently are annihilating the respective Teodorescu operators. For an overview of the similarities between real and complex differential forms in open subsets of $\mathbb{R}^m$ and $\mathbb{C}^n$ on one hand, and multivector functions in Euclidean and Hermitian Clifford analysis on the other hand, we refer the reader to [15, 21].

For a better understanding of the meaning of (3.1)–(3.3), we now consider the derivatives of the components of the Teodorescu operators. The expressions obtained involve the multi-dimensional Fp-distributions "finite part", which are defined by using the Fp-distribution on the real line; for a thorough study of these and related families of distributions in Clifford analysis, we refer the reader to [13–14, 5–7, 9–10]. Restricting our considerations to the operator T, we put $T = \sum_{\alpha=1}^{m} e_\alpha T_\alpha$, where

$$T_\alpha[u](\underline{X}) = -\int_{\partial\Omega} E_\alpha(\underline{Y} - \underline{X})\, u(\underline{Y}) \mathrm{d}V(\underline{Y}), \quad \alpha = 1, \cdots, m.$$

As an auxiliary result, we need the following formulae concerning the distributional derivatives of the components E_α ($\alpha = 1, \cdots, m$) of the fundamental solution E (see [18]). Note that in the present context, we have $m = 2n$. However, when we only consider the operator T, the results below remain valid for the general dimension m.

Lemma 3.1 *In the distributional sense, one has*

(i) $\partial_{X_\beta} E_\alpha = \dfrac{m}{a_m} \operatorname{Fp} \dfrac{X_\alpha X_\beta}{r^{m+2}}$ *for all* $\alpha \neq \beta$, $\alpha, \beta = 1, \cdots, m$;

(ii) $\partial_{X_\alpha} E_\alpha = \dfrac{m}{a_m} \operatorname{Fp} \dfrac{X_\alpha^2}{r^{m+2}} - \dfrac{1}{m} \operatorname{Fp} \dfrac{1}{r^m} - \dfrac{1}{m} \delta$ *for all* $\alpha = 1, \cdots, m$.

Proposition 3.1 *For* $u \in C_1(\overline{\Omega})$ *and* $\underline{X} \in \Omega$, *one has*

$$
\begin{aligned}
\partial_{X_\alpha} T_\alpha[u](\underline{X}) = {}& -\frac{1}{m} u(\underline{X}) - \frac{1}{a_m} \operatorname{Fp} \int_\Omega \frac{u(\underline{Y})}{|\underline{X} - \underline{Y}|^m} dV(\underline{Y}) \\
& + \frac{m}{a_m} \operatorname{Fp} \int_\Omega \frac{(X_\alpha - Y_\alpha)^2}{|\underline{X} - \underline{Y}|^{m+2}} u(\underline{Y}) dV(\underline{Y}),
\end{aligned}
\tag{3.4}
$$

$$
\partial_{X_\beta} T_\alpha[u](\underline{X}) = \frac{m}{a_m} \operatorname{Fp} \int_\Omega \frac{(X_\alpha - Y_\alpha)(X_\beta - Y_\beta)}{|\underline{X} - \underline{Y}|^{m+2}} u(\underline{Y}) dV(\underline{Y}).
\tag{3.5}
$$

Proof Let the function $u \in C_1(\overline{\Omega})$ be extended by zero to $\mathbb{R}^m \backslash \overline{\Omega}$, this extension $\widetilde{u}$ being interpreted as a compactly supported distribution. Then one has

$$
T_\alpha[\widetilde{u}](\underline{X}) = \int_\Omega E_\alpha(\underline{X} - \underline{Y}) \widetilde{u}(\underline{Y}) dV(\underline{Y}) = E_\alpha * \widetilde{u}(\underline{X}),
$$

and hence, using the formulae of Lemma 3.1, one gets

$$
\partial_{X_\alpha} T_\alpha[\widetilde{u}] = \partial_{X_\alpha} E_\alpha * \widetilde{u} = \left(\frac{m}{a_m} \operatorname{Fp} \frac{X_\alpha^2}{r^{m+2}} - \frac{1}{m} \operatorname{Fp} \frac{1}{r^m} - \frac{1}{m} \delta \right) * \widetilde{u}.
$$

Restricting to $\underline{X} \in \Omega$, formula (3.4) is readily obtained. One also has that

$$
\partial_{X_\beta} T_\alpha[\widetilde{u}] = \partial_{X_\beta} E_\alpha * \widetilde{u} = \left(\frac{m}{a_m} \operatorname{Fp} \frac{X_\alpha X_\beta}{r^{m+2}} \right) * \widetilde{u}
$$

yields formula (3.5).

Remark 3.1 Formulae (3.4)–(3.5) may also be written as

$$
\partial_{X_\alpha} \int_\Omega E_\alpha(\underline{Y} - \underline{X}) u(\underline{Y}) dV(\underline{Y}) = \frac{1}{m} u(\underline{X}) + \operatorname{pv} \int_\Omega \partial_{X_\alpha} E_\alpha(\underline{Y} - \underline{X}) u(\underline{Y}) dV(\underline{Y}),
\tag{3.6}
$$

$$
\partial_{X_\beta} \int_\Omega E_\alpha(\underline{Y} - \underline{X}) u(\underline{Y}) dV(\underline{Y}) = \operatorname{pv} \int_\Omega \partial_{X_\beta} E_\alpha(\underline{Y} - \underline{X}) u(\underline{Y}) dV(\underline{Y}),
\tag{3.7}
$$

where pv stands for the "Cauchy principal value" of the considered integral.

Remark 3.2 For the technique used in the above proof, the condition $u \in C_1(\overline{\Omega})$ is not really necessary. It suffices to have, e.g., $u \in C_1(\Omega) \cap L_1^{\mathrm{loc}}(\Omega)$. In fact, the results may be obtained for functions $u \in C_1(\Omega)$, but then an approach in e.g. [24] is needed.

Corollary 3.1 *For* $u \in C_1(\overline{\Omega})$ *and* $\underline{X} \in \Omega$, *one has*

$$
\partial_{X_\beta} T[u](\underline{X}) = -\frac{1}{m} e_\beta \, u(\underline{X}) - \operatorname{pv} \int_\Omega \partial_{X_\beta} E(\underline{Y} - \underline{X}) u(\underline{Y}) dV(\underline{Y}).
$$

Proof Using the results of Proposition 3.1, one obtains

$$\partial_{X_\beta} T[u](\underline{X}) = \sum_{\alpha=1}^{m} e_\alpha \, \partial_{X_\beta} T_\alpha[u](\underline{X}) = e_\beta \partial_{X_\beta} T_\beta[u](\underline{X}) + \sum_{\alpha\neq\beta} e_\alpha \, \partial_{X_\beta} T_\alpha[u](\underline{X})$$

$$= -e_\beta \Big(\frac{1}{m} u(\underline{X}) + \text{pv} \int_\Omega \partial_{X_\beta} E_\beta(\underline{Y} - \underline{X}) \, u(\underline{Y}) \mathrm{d}V(\underline{Y}) \Big)$$

$$- \sum_{\alpha\neq\beta} e_\alpha \Big(\text{pv} \int_\Omega \partial_{X_\beta} E_\alpha(\underline{Y} - \underline{X}) \, u(\underline{Y}) \mathrm{d}V(\underline{Y}) \Big)$$

$$= -\frac{1}{m} e_\beta \, u(\underline{X}) - \text{pv} \int_\Omega \partial_{X_\beta} \sum_{\alpha=1}^{m} e_\alpha \, E_\alpha(\underline{Y} - \underline{X}) \, u(\underline{Y}) \mathrm{d}V(\underline{Y}).$$

The above results now enable us to prove directly the finer structure of the action of the Euclidean Teodorescu operator, already observed above.

Proposition 3.2 *The Teodorescu formulae take the form*

(i) $\partial_{\underline{X}} \bullet T = \mathbf{1}_\Omega$,

(ii) $\partial_{\underline{X}} \wedge T = 0$.

Proof Take a function $u \in C_1(\overline{\Omega})$. Then

$$\partial_{\underline{X}} \bullet T[u](\underline{X}) = -\sum_{\alpha=1}^{m} \partial_{X_\alpha} T_\alpha[u](\underline{X})$$

$$= -\sum_{\alpha=1}^{m} \Big(-\frac{1}{m} u(\underline{X}) - \frac{1}{a_m} \text{Fp} \int_\Omega \frac{u(\underline{Y})}{|\underline{X} - \underline{Y}|^m} \mathrm{d}V(\underline{Y})$$

$$+ \frac{m}{a_m} \text{Fp} \int_\Omega \frac{(X_\alpha - Y_\alpha)^2}{|\underline{X} - \underline{Y}|^{m+2}} \, u(\underline{Y}) \mathrm{d}V(\underline{Y}) \Big)$$

$$= u(\underline{X}) + \frac{m}{a_m} \text{Fp} \int_\Omega \frac{u(\underline{Y})}{|\underline{X} - \underline{Y}|^m} \mathrm{d}V(\underline{Y})$$

$$- \frac{m}{a_m} \text{Fp} \int_\Omega \sum_{\alpha=1}^{m} \frac{(X_\alpha - Y_\alpha)^2}{|\underline{X} - \underline{Y}|^{m+2}} \, u(\underline{Y}) \mathrm{d}V(\underline{Y}),$$

from which (i) follows since the last two terms are cancelled. Furthermore,

$$\partial_{\underline{X}} \wedge T[u](\underline{X}) = \sum_{\alpha<\beta} e_\alpha e_\beta (\partial_{X_\alpha} T_\beta[u](\underline{X}) - \partial_{X_\beta} T_\alpha[u](\underline{X})),$$

which is zero since, in view of (3.5), each term vanishes. This yields (ii).

4 The Hermitian Teodorescu Operators

Since the kernel of the Teodorescu integral operator is the fundamental solution to the Dirac operator, subject to translation, we start by constructing the Hermitian counterparts of the pair of fundamental solutions $(E, E|)$ to the Dirac operators $(\partial_{\underline{X}}, \partial_{\underline{X}|})$. They are given by

$$\mathcal{E} = -(E + \mathrm{i}E|), \quad \mathcal{E}^\dagger = E - \mathrm{i}E|,$$

or, more explicitly,

$$\mathcal{E}(\underline{z}) = \frac{2}{a_{2n}} \frac{\underline{z}}{|\underline{z}|^{2n}} = \sum_{j=1}^{n} \mathfrak{f}_j \left(\frac{2}{a_{2n}} \frac{z_j}{r^{2n}} \right) = \sum_{j=1}^{n} \mathfrak{f}_j \, \mathcal{E}_j,$$

$$\mathcal{E}^\dagger(\underline{z}) = \frac{2}{a_{2n}} \frac{\underline{z}^\dagger}{|\underline{z}|^{2n}} = \sum_{j=1}^{n} \mathfrak{f}_j^\dagger \left(\frac{2}{a_{2n}} \frac{z_j^c}{r^{2n}} \right) = \sum_{j=1}^{n} \mathfrak{f}_j^\dagger \, \mathcal{E}_j^c.$$

However, they are not the fundamental solutions to the respective Hermitian Dirac operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}^\dagger}$. To show this, we first rephrase the results of Lemma 3.1 to the Hermitian case.

Lemma 4.1 *For all $j, k = 1, \cdots, n$, $j \neq k$, one has in the distributional sense,*

(i) $\partial_{z_j} \mathcal{E}_j = \frac{1}{n} \delta + \frac{2}{a_{2n}} \mathrm{Fp} \frac{1}{r^{2n}} - \frac{2n}{a_{2n}} \mathrm{Fp} \frac{|z_j|^2}{r^{2n+2}},$

(ii) $\partial_{z_k} \mathcal{E}_j = -\frac{2n}{a_{2n}} \mathrm{Fp} \frac{z_j z_k^c}{r^{2n+2}},$

(iii) $\partial_{z_j^c} \mathcal{E}_j^c = \frac{1}{n} \delta + \frac{2}{a_{2n}} \mathrm{Fp} \frac{1}{r^{2n}} - \frac{2n}{a_{2n}} \mathrm{Fp} \frac{|z_j|^2}{r^{2n+2}},$

(iv) $\partial_{z_k^c} \mathcal{E}_j^c = -\frac{2n}{a_{2n}} \mathrm{Fp} \frac{z_k z_j^c}{r^{2n+2}},$

(v) $\partial_{z_k} \mathcal{E}_j^c = \partial_{z_j} \mathcal{E}_k^c = -\frac{2n}{a_{2n}} \mathrm{Fp} \frac{z_j^c z_k^c}{r^{2n+2}},$

(vi) $\partial_{z_k^c} \mathcal{E}_j = \partial_{z_j^c} \mathcal{E}_k = -\frac{2n}{a_{2n}} \mathrm{Fp} \frac{z_j z_k}{r^{2n+2}}.$

A rather straightforward computation then leads to the following results. They involve the so-called Spin-Euler operator β, which is para-bivector valued and given by

$$\beta = \sum_{j=1}^{n} \mathfrak{f}_j^\dagger \mathfrak{f}_j = \frac{n}{2} + \frac{\mathrm{i}}{2} \sum_{j=1}^{n} e_j e_{n+j}.$$

Proposition 4.1 *One has in the distributional sense,*

(i) $\partial_{\underline{z}} \mathcal{E}(\underline{z}) = \frac{1}{n} \beta \, \delta(\underline{z}, \underline{z}^\dagger) + \frac{2}{a_{2n}} \beta \, \mathrm{Fp} \frac{1}{r^{2n}} - \frac{2}{a_{2n}} n \mathrm{Fp} \frac{\underline{z}^\dagger \underline{z}}{r^{2n+2}},$

(ii) $\partial_{\underline{z}^\dagger} \mathcal{E}(\underline{z}) = 0,$

(iii) $\partial_{\underline{z}} \mathcal{E}^\dagger(\underline{z}) = 0,$

(iv) $\partial_{\underline{z}^\dagger} \mathcal{E}^\dagger(\underline{z}) = \frac{1}{n} (n - \beta) \delta(\underline{z}, \underline{z}^\dagger) + \frac{2}{a_{2n}} (n - \beta) \mathrm{Fp} \frac{1}{r^{2n}} - \frac{2}{a_{2n}} n \mathrm{Fp} \frac{\underline{z} \, \underline{z}^\dagger}{r^{2n+2}},$

(v) $\partial_{\underline{z}} \mathcal{E}(\underline{z}) + \partial_{\underline{z}^\dagger} \mathcal{E}^\dagger(\underline{z}) = \delta(\underline{z}, \underline{z}^\dagger).$

For a more thorough study of the fundamental solutions to the Euclidean and Hermitian Dirac operators and in particular their behaviour under the action of the operators $\partial_{\underline{z}} \bullet$, $\partial_{\underline{z}^\dagger} \bullet$, $\partial_{\underline{z}} \wedge$ and $\partial_{\underline{z}^\dagger} \wedge$, we refer the reader to [18]. These operators originate by splitting the Clifford algebra product — which is always tacitly involved when a Clifford vector operator acts on a Clifford vector function — into its scalar or "dot" part and its bivector or "wedge" part. Moreover, these operators are the counterparts in Hermitian Clifford analysis to the well-known operators $\overline{\partial}$, $\overline{\partial}^*$, ∂ and ∂^*, the bar here denoting complex conjugation, for complex differential forms (see [21]). Here, let us just mention that

$$\partial_{\underline{z}} \bullet \mathcal{E}(\underline{z}) = \frac{1}{2} \delta(\underline{z}, \underline{z}^\dagger) = \partial_{\underline{z}^\dagger} \bullet \mathcal{E}^\dagger(\underline{z}), \tag{4.1}$$

whence $2\mathcal{E}(\underline{z})$ may be interpreted as a fundamental solution to the operator $(\partial_{\underline{z}}\bullet)$ and $2\mathcal{E}^\dagger(\underline{z})$ as a fundamental solution to $(\partial_{\underline{z}^\dagger}\bullet)$.

Moreover, on the basis of the results of Proposition 4.1, an important result was obtained in [11], by considering the particular circulant (2×2) matrices,

$$\boldsymbol{\mathcal{D}}_{(\underline{z},\underline{z}^\dagger)} = \begin{pmatrix} \partial_{\underline{z}} & \partial_{\underline{z}^\dagger} \\ \partial_{\underline{z}^\dagger} & \partial_{\underline{z}} \end{pmatrix}, \quad \boldsymbol{\mathcal{E}} = \begin{pmatrix} \mathcal{E} & \mathcal{E}^\dagger \\ \mathcal{E}^\dagger & \mathcal{E} \end{pmatrix}, \quad \boldsymbol{\delta} = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix}.$$

Indeed, it then holds that

$$\boldsymbol{\mathcal{D}}_{(\underline{z},\underline{z}^\dagger)}\boldsymbol{\mathcal{E}}(\underline{z}) = \boldsymbol{\delta}(\underline{z})$$

meaning that $\boldsymbol{\mathcal{E}}$ may be considered as a fundamental solution to $\boldsymbol{\mathcal{D}}_{(\underline{z},\underline{z}^\dagger)}$. The analogue of this result, in the setting of differential forms, was already obtained in [28, Chapter 7]. It is precisely this simple observation that has inspired the idea of a matrix approach to establish a Cauchy integral formula in the Hermitian setting. Also note that the Dirac matrix $\boldsymbol{\mathcal{D}}_{(\underline{z},\underline{z}^\dagger)}$ in some sense factorizes the Laplacian, since

$$4\,\boldsymbol{\mathcal{D}}_{(\underline{z},\underline{z}^\dagger)}(\boldsymbol{\mathcal{D}}_{(\underline{z},\underline{z}^\dagger)})^\dagger = \begin{pmatrix} \Delta_{2n} & 0 \\ 0 & \Delta_{2n} \end{pmatrix} = \boldsymbol{\Delta}_{2n}$$

(see also [28]). Now, in order to establish a Hermitian version of the Teodorescu inversion formula, we repeat the above procedure for the Euclidean Teodorescu operators, making suitable complex linear combinations:

$$\mathcal{T}^{(1)}[(\,\cdot\,)](\underline{z}) = (-1)^{\frac{n(n+1)}{2}}(2\mathrm{i})^n(-T - \mathrm{i}T|) = -\int_\Omega \mathcal{E}(\underline{w} - \underline{z})(\,\cdot\,)(\underline{w})\mathrm{d}W,$$

$$\mathcal{T}^{(2)}[(\,\cdot\,)](\underline{z}) = (-1)^{\frac{n(n+1)}{2}}(2\mathrm{i})^n(T - \mathrm{i}T|) = -\int_\Omega \mathcal{E}^\dagger(\underline{w} - \underline{z})(\,\cdot\,)(\underline{w})\mathrm{d}W,$$

where we have used the notations $\underline{w}$ and $\underline{w}^\dagger$ for the Hermitian variables associated to the variables $\underline{Y}$ and $\underline{Y}|$ and have put

$$\mathrm{d}W = (\mathrm{d}w_1 \wedge \mathrm{d}w_1^c) \wedge (\mathrm{d}w_2 \wedge \mathrm{d}w_2^c) \wedge \cdots \wedge (\mathrm{d}w_n \wedge \mathrm{d}w_n^c) = (-1)^{\frac{n(n+1)}{2}}(2\mathrm{i})^n\mathrm{d}V(\underline{Y}).$$

Using these definitions, we arrive at the desired result.

Theorem 4.1 (Hermitian Teodorescu Formulae) *Let Ω be a bounded domain in $\mathbb{R}^{2n} \cong \mathbb{C}^n$ with smooth boundary $\partial\Omega$, and let $u \in C_1(\overline{\Omega})$. Then $\mathcal{T}^{(1)}[u]$ and $\mathcal{T}^{(2)}[u]$ are in $C_1(\Omega)$ and they satisfy in Ω*

(i) $\partial_{\underline{z}}\mathcal{T}^{(1)}[u] + \partial_{\underline{z}^\dagger}\mathcal{T}^{(2)}[u] = (-1)^{\frac{n(n+1)}{2}}(2\mathrm{i})^n\, u$,

(ii) $\partial_{\underline{z}^\dagger}\mathcal{T}^{(1)}[u] + \partial_{\underline{z}}\mathcal{T}^{(2)}[u] = 0$.

Proof We only prove (i), with the proof of (ii) being similar. Temporarily denoting $C = (-1)^{\frac{n(n+1)}{2}}(2\mathrm{i})^n$, we have

$$\partial_{\underline{z}}\mathcal{T}^{(1)}[u] + \partial_{\underline{z}^\dagger}\mathcal{T}^{(2)}[u] = -\frac{C}{4}(\partial_{\underline{X}} - \mathrm{i}\partial_{\underline{X}|})(-T[u] - \mathrm{i}T|[u]) + \frac{C}{4}(\partial_{\underline{X}} + \mathrm{i}\partial_{\underline{X}|})(T[u] - \mathrm{i}T|[u])$$

$$= \frac{C}{4}(2\partial_{\underline{X}}T[u] + 2\partial_{\underline{X}|}T|[u]) = \frac{C}{2}(u + u) = Cu.$$

The results of the foregoing theorem may be written in the matrix form as

$$\begin{pmatrix} \partial_{\underline{z}} & \partial_{\underline{z}^\dagger} \\ \partial_{\underline{z}^\dagger} & \partial_{\underline{z}} \end{pmatrix} \begin{pmatrix} \mathcal{T}^{(1)}[u] & \mathcal{T}^{(2)}[u] \\ \mathcal{T}^{(2)}[u] & \mathcal{T}^{(1)}[u] \end{pmatrix} = (-1)^{\frac{n(n+1)}{2}} (2\mathrm{i})^n \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix},$$

or still more explicitly

$$-\boldsymbol{\mathcal{D}}_{(\underline{z},\underline{z}^\dagger)} \int_\Omega \boldsymbol{\mathcal{E}}(\underline{w}-\underline{z}) \begin{pmatrix} u(\underline{w}) & 0 \\ 0 & u(\underline{w}) \end{pmatrix} \mathrm{d}W = (-1)^{\frac{n(n+1)}{2}} (2\mathrm{i})^n \begin{pmatrix} u(\underline{w}) & 0 \\ 0 & u(\underline{w}) \end{pmatrix},$$

which is an exact analogue to the traditional Teodorescu inversion formula (3.1).

Our aim now is to refine these results, first with respect to the action of the associated operators $\partial_{\underline{z}}\bullet$, $\partial_{\underline{z}^\dagger}\bullet$, $\partial_{\underline{z}}\wedge$ and $\partial_{\underline{z}^\dagger}\wedge$, next with respect to the individual action of the Hermitian Dirac operators on both Hermitian Teodorescu operators. To that end, we first calculate the Teodorescu formulae at the component level, putting for each function $u \in C_1(\Omega)$,

$$\mathcal{T}_j^{(1)}[u] = \int_\Omega \mathcal{E}_j(\underline{z}-\underline{w}) \, u(\underline{w}) \mathrm{d}W,$$

$$\mathcal{T}_j^{(2)}[u] = \int_\Omega \mathcal{E}_j^c(\underline{z}-\underline{w}) \, u(\underline{w}) \mathrm{d}W.$$

Proposition 4.2 *For a function $u \in C_1(\overline{\Omega})$ and for $j, k = 1, \cdots, n$, $j \neq k$, one has*

(i) $\partial_{z_j} \int_\Omega \mathcal{E}_j(\underline{z}-\underline{w}) \, u(\underline{w}) \mathrm{d}W = (-1)^{\frac{n(n+1)}{2}} (2\mathrm{i})^n \frac{1}{n} u(\underline{z}) + \mathrm{pv} \int_\Omega \partial_{z_j} \mathcal{E}_j(\underline{z}-\underline{w}) \, u(\underline{w}) \mathrm{d}W,$

(ii) $\partial_{z_k} \int_\Omega \mathcal{E}_j(\underline{z}-\underline{w}) \, u(\underline{w}) \mathrm{d}W = \mathrm{pv} \int_\Omega \partial_{z_k} \mathcal{E}_j(\underline{z}-\underline{w}) \, u(\underline{w}) \mathrm{d}W,$

(iii) $\partial_{z_j^c} \int_\Omega \mathcal{E}_j^c(\underline{z}-\underline{w}) \, u(\underline{w}) \mathrm{d}W = (-1)^{\frac{n(n+1)}{2}} (2\mathrm{i})^n \frac{1}{n} u(\underline{z}) + \mathrm{pv} \int_\Omega \partial_{z_j^c} \mathcal{E}_j^c(\underline{z}-\underline{w}) \, u(\underline{w}) \mathrm{d}W,$

(iv) $\partial_{z_k^c} \int_\Omega \mathcal{E}_j^c(\underline{z}-\underline{w}) \, u(\underline{w}) \mathrm{d}W = \mathrm{pv} \int_\Omega \partial_{z_k^c} \mathcal{E}_j^c(\underline{z}-\underline{w}) \, u(\underline{w}) \mathrm{d}W,$

(v) $\partial_{z_k} \int_\Omega \mathcal{E}_j^c(\underline{z}-\underline{w}) \, u(\underline{w}) \mathrm{d}W = \mathrm{pv} \int_\Omega \partial_{z_k} \mathcal{E}_j^c(\underline{z}-\underline{w}) \, u(\underline{w}) \mathrm{d}W,$

(vi) $\partial_{z_k^c} \int_\Omega \mathcal{E}_j(\underline{z}-\underline{w}) \, u(\underline{w}) \mathrm{d}W = \mathrm{pv} \int_\Omega \partial_{z_k^c} \mathcal{E}_j(\underline{z}-\underline{w}) \, u(\underline{w}) \mathrm{d}W.$

Proof The proofs of all these relations being similar, we only prove the first one. To this end, we extend the function $u \in C_1(\overline{\Omega})$ by zero in $\mathbb{R}^{2n} \setminus \overline{\Omega}$ to $\widetilde{u}$, and consider this extension $\widetilde{u}$ as a compactly supported distribution. Denoting temporarily $C = (-1)^{\frac{n(n+1)}{2}} (2\mathrm{i})^n$, we have

$$\partial_{z_j} \mathcal{T}_j^{(1)}[\widetilde{u}] = C \partial_{z_j}(\mathcal{E}_j * \widetilde{u}) = C(\partial_{z_j}\mathcal{E}_j) * \widetilde{u} = C\Big(\frac{1}{n}\delta + \frac{2}{a_{2n}}\mathrm{Fp}\frac{1}{r^{2n}} - \frac{2n}{a_{2n}}\mathrm{Fp}\frac{|z_j|^2}{r^{2n+2}}\Big) * \widetilde{u},$$

which restricting to Ω also may be written as

$$\partial_{z_j} \int_\Omega \mathcal{E}_j(\underline{z}-\underline{w}) \, u(\underline{w}) \mathrm{d}W = \frac{C}{n} u(\underline{z}) + \mathrm{pv} \int_\Omega \partial_{z_j} \mathcal{E}_j(\underline{z}-\underline{w}) \, u(\underline{w}) \mathrm{d}W,$$

since for $\underline{w} \neq \underline{z}$,

$$\partial_{z_j} \mathcal{E}_j(\underline{z}-\underline{w}) = \frac{2}{a_{2n}} \frac{1}{\rho^{2n}} - \frac{2n}{a_{2n}} \frac{|z_j-w_j|^2}{\rho^{2n+2}}$$

with $\rho = |\underline{z}-\underline{w}|$.

By making the appropriate combinations of the formulae obtained in the foregoing proposition, we obtain the Hermitian Teodorescu formulae for the associated Hermitian Dirac operators $\partial_{\underline{z}}\bullet$, $\partial_{\underline{z}^\dagger}\bullet$, $\partial_{\underline{z}}\wedge$ and $\partial_{\underline{z}^\dagger}\wedge$.

Theorem 4.2 *For a function $u \in C_1(\overline{\Omega})$, one has*

$$\partial_{\underline{z}} \bullet \mathcal{T}^{(1)}[u](\underline{z}) = \frac{1}{2}(-1)^{\frac{n(n+1)}{2}}(2\mathrm{i})^n u(\underline{z}),$$

$$\partial_{\underline{z}} \wedge \mathcal{T}^{(1)}[u](\underline{z}) = \frac{1}{n}(-1)^{\frac{n(n+1)}{2}}(2\mathrm{i})^n \Big(\sum_{j=1}^{n} \mathfrak{f}_j^\dagger \wedge \mathfrak{f}_j \Big) u(\underline{z}) + \mathrm{pv}\int_\Omega \partial_{\underline{z}}\mathcal{E}(\underline{z}-\underline{w})u(\underline{w})\mathrm{d}W,$$

$$\partial_{\underline{z}^\dagger} \bullet \mathcal{T}^{(1)}[u](\underline{z}) = 0,$$

$$\partial_{\underline{z}^\dagger} \wedge \mathcal{T}^{(1)}[u](\underline{z}) = 0,$$

$$\partial_{\underline{z}} \bullet \mathcal{T}^{(2)}[u](\underline{z}) = 0,$$

$$\partial_{\underline{z}} \wedge \mathcal{T}^{(2)}[u](\underline{z}) = 0,$$

$$\partial_{\underline{z}^\dagger} \bullet \mathcal{T}^{(2)}[u](\underline{z}) = \frac{1}{2}(-1)^{\frac{n(n+1)}{2}}(2\mathrm{i})^n u(\underline{z}),$$

$$\partial_{\underline{z}^\dagger} \wedge \mathcal{T}^{(2)}[u](\underline{z}) = \frac{1}{n}(-1)^{\frac{n(n+1)}{2}}(2\mathrm{i})^n \Big(\sum_{j=1}^{n} \mathfrak{f}_j \wedge \mathfrak{f}_j^\dagger \Big) u(\underline{z}) + \mathrm{pv}\int_\Omega \partial_{\underline{z}^\dagger}\mathcal{E}^\dagger(\underline{z}-\underline{w})u(\underline{w})\mathrm{d}W.$$

Proof The proofs of all these relations being similar, we only prove the second one. Keeping the same notations as above, we have

$$\partial_{\underline{z}} \wedge \mathcal{T}^{(1)}[u](\underline{z}) = \sum_{k=1}^{n}\sum_{j=1}^{n} \mathfrak{f}_k^\dagger \wedge \mathfrak{f}_j\, \partial_{z_k} \int_\Omega \mathcal{E}_j(\underline{z}-\underline{w})\, u(\underline{w})\mathrm{d}W$$

$$= \sum_{j=1}^{n} \mathfrak{f}_j^\dagger \wedge \mathfrak{f}_j \Big(\frac{C}{n}u(\underline{z}) + \mathrm{pv}\int_\Omega \partial_{z_j}\mathcal{E}_j(\underline{z}-\underline{w})u(\underline{w})\mathrm{d}W \Big)$$

$$\quad + \sum_{j\neq k} \mathfrak{f}_k^\dagger \wedge \mathfrak{f}_j\, \mathrm{pv}\int_\Omega \partial_{z_k}\mathcal{E}_j(\underline{z}-\underline{w})u(\underline{w})\mathrm{d}W$$

$$= \frac{C}{n}\Big(\sum_{j=1}^{n} \mathfrak{f}_j^\dagger \wedge \mathfrak{f}_j \Big) u(\underline{z}) + \mathrm{pv}\int_\Omega \sum_{j=1}^{n}\sum_{k=1}^{n} \mathfrak{f}_k^\dagger \wedge \mathfrak{f}_j\, \partial_{z_k}\mathcal{E}_j(\underline{z}-\underline{w})\, u(\underline{w})\mathrm{d}W$$

$$= \frac{C}{n}\Big(\sum_{j=1}^{n} \mathfrak{f}_j^\dagger \wedge \mathfrak{f}_j \Big) u(\underline{z}) + \mathrm{pv}\int_\Omega \partial_{\underline{z}} \wedge \mathcal{E}(\underline{z}-\underline{w})\, u(\underline{w})\mathrm{d}W,$$

from which the desired result follows since, by (4.1), $\partial_{\underline{z}}\mathcal{E} = \partial_{\underline{z}}\wedge\mathcal{E}$ as long as $\underline{z} \neq \underline{w}$.

Corollary 4.1 *One has*
(i) $\partial_{\underline{z}} \bullet \mathcal{T}^{(1)}[u](\underline{z}) + \partial_{\underline{z}^\dagger} \bullet \mathcal{T}^{(2)}[u](\underline{z}) = \frac{1}{2}(-1)^{\frac{n(n+1)}{2}}(2\mathrm{i})^n u(\underline{z}),$
(ii) $\partial_{\underline{z}} \wedge \mathcal{T}^{(1)}[u](\underline{z}) + \partial_{\underline{z}^\dagger} \wedge \mathcal{T}^{(2)}[u](\underline{z}) = 0.$

Finally, making the appropriate combinations of the formulae of the foregoing theorem, we obtain expressions for the action of the Hermitian Dirac operators on the Hermitian Teodorescu operators.

Proposition 4.3 *For a function $u \in C_1(\overline{\Omega})$, one has*

(i) $\partial_{\underline{z}} \mathcal{T}^{(1)}[u] = (-1)^{\frac{n(n+1)}{2}} (2\mathrm{i})^n \frac{1}{n} \beta\, u + \mathrm{pv} \int_\Omega \partial_{\underline{z}} \mathcal{E}(\underline{z} - \underline{w})\, u(\underline{w}) \mathrm{d}W,$

(ii) $\partial_{\underline{z}^\dagger} \mathcal{T}^{(1)}[u] = 0,$

(iii) $\partial_{\underline{z}} \mathcal{T}^{(2)}[u] = 0,$

(iv) $\partial_{\underline{z}^\dagger} \mathcal{T}^{(2)}[u] = (-1)^{\frac{n(n+1)}{2}} (2\mathrm{i})^n \left(1 - \frac{1}{n}\beta\right) u + \mathrm{pv} \int_\Omega \partial_{\underline{z}^\dagger} \mathcal{E}^\dagger(\underline{z} - \underline{w})\, u(\underline{w}) \mathrm{d}W.$

Remark 4.1 Note that formula (i) of Theorem 4.1 follows from formulae (i) and (iv) of Proposition 4.3 due to formula (v) of Proposition 4.1. Note also that formulae (ii) and (iii) of Proposition 4.3 are stronger than formula (ii) of Theorem 4.1.

5 The Case of Several Complex Variables

In this section, we will restrict ourselves to complex valued functions $f : \mathbb{R}^m \cong \mathbb{C}^{2n} \to \mathbb{C}$. Such functions which a priori are functions of the real variables $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ become a functions of n complex variables and their complex conjugates: $f(z_1, \cdots, z_n, z_1^c, \cdots, z_n^c)$ or in shorthand $f(\underline{z}, \underline{z}^\dagger)$. It is only when a function is holomorphic that it becomes a function of the variables $(z_1, \cdots, z_n)$ only. Naturally, all the results obtained in the foregoing section apply to such scalar valued functions.

When concentrating on Proposition 4.2, we see that quite naturally, only formulae (iii) and (iv) are to be found in the literature on several complex variables (see, e.g., [27, (1.11)]), since there the focus is on holomorphic functions and the related Cauchy-Riemann operators $\partial_{z_j^c}, j = 1, \cdots, n$, while there is not one unique differential operator defining multidimensional holomorphy. The mirror formulae (i) and (ii) involving the conjugate Cauchy-Riemann operators $\partial_{z_j^c}, j = 1, \cdots, n$ are then tacitly assumed. However, the formulae from Proposition 4.2 do not really deserve the qualification of Teodorescu inversion. It becomes interesting when summing them up, leading to the results already contained in Theorem 4.2,

$$\partial_{\underline{z}} \bullet \mathcal{T}^{(1)}[f] = \frac{1}{2} \sum_{j=1}^n \partial_{z_j} \int_\Omega \mathcal{E}_j(\underline{z} - \underline{w})\, f(\underline{w}) \mathrm{d}W = \frac{1}{2}(-1)^{\frac{n(n+1)}{2}} (2\mathrm{i})^n f(\underline{z})$$

or

$$\sum_{j=1}^n \partial_{z_j} \int_\Omega \frac{2}{a_{2n}} \frac{z_j - w_j}{\rho^{2n}}\, f(\underline{w}) \mathrm{d}W = (-1)^{\frac{n(n+1)}{2}} (2\mathrm{i})^n f(\underline{z}),$$

and

$$\partial_{\underline{z}^\dagger} \bullet \mathcal{T}^{(2)}[f] = \frac{1}{2} \sum_{j=1}^n \partial_{z_j^c} \int_\Omega \mathcal{E}_j^c(\underline{z} - \underline{w})\, f(\underline{w}) \mathrm{d}W = \frac{1}{2}(-1)^{\frac{n(n+1)}{2}} (2\mathrm{i})^n f(\underline{z})$$

or

$$\sum_{j=1}^n \partial_{z_j^c} \int_\Omega \frac{2}{a_{2n}} \frac{z_j^c - w_j^c}{\rho^{2n}}\, f(\underline{w}) \mathrm{d}W = (-1)^{\frac{n(n+1)}{2}} (2\mathrm{i})^n f(\underline{z}),$$

expressing the fact that $\mathcal{T}^{(1)}$ and $\mathcal{T}^{(2)}$ indeed are the right inverses of the associated Dirac operators $\partial_{\underline{z}}\bullet$ and $\partial_{\underline{z}^\dagger}\bullet$, respectively. Similar formulae, in the context of differential forms, can be found in [28, Chapter 7].

As was expected, the involved integral kernels are not (anti-)holomorphic, but still harmonic, since they are the components of the fundamental solutions to both Hermitian Dirac operators.

Moreover, they do coincide with the ones appearing in the Martinelli-Bochner representation formula for holomorphic functions (1.2) and its anti-holomorphic counterpart.

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