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Determinant Solutions to a (3+1)-Dimensional Generalized KP Equation with Variable Coefficients*

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Abstract A system of linear conditions is presented for Wronskian and Grammian solutions to a (3+1)-dimensional generalized vcKP equation. The formulations of these solutions require a constraint on variable coefficients.

Keywords Hirota bilinear form, Wronskian solution, Grammian solution
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1 Introduction

Although partial differential equations that govern the motion of solitons are nonlinear, many of them can be put into the bilinear form. Hirota, in 1971, developed an ingenious method to obtain exact solutions to nonlinear partial differential equations in the soliton theory, such as the KdV equation, the Boussinesq equation and the KP equation (see [1–2]). The multiple exp-function method, oriented toward the ease of use and the capability of computer algebra systems, provides a direct and efficient way to search for generic multi-exponential wave solutions to nonlinear equations including bilinear equations (see [3]). Interestingly, some nonlinear equations even possess linear subspaces of their solution spaces (see [4]). Moreover, a necessary and sufficient condition was given for Hirota bilinear equations to check whether they possess linear combination solutions to exponential waves (see [5]).

Solitons and positons (a kind of periodic solutions) can be expressed as Wronskian determinants (see [6–7]). Particular solutions combining exponential functions and trigonometrical functions are presented and called complexiton (or briefly complexitons) (see [8]). Lattice soliton equations have a similar situation (see [9]). Complexitons are also shown to exist for source solution equations (see [10]) and soliton equations with sources (see [11]). For higher-

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dimensional soliton equations, there exist Grammian solutions and Pfaffian solutions (see [1]). Grammian solutions to the KP equation were constructed by Nakamura [12]. Pfaffian solutions to the BKP equation were presented by Hirota [13].

Recently, Wronskian and Grammian solutions, nonsingular and singular soliton solutions and the Bäcklund transformation in the bilinear form to a (3+1)-dimentional generalized KP equation

$$u_{xxxy}+3(u_xu_y)_x+u_{tx}+u_{ty}-u_{zz}=0$$

have been presented in [14], [15] and [16], respectively. This equation can be written in the Hirota bilinear form and reduced to the KP equation by taking $y=x$, but does not belong to a class of generalized KP and Boussinesq equations (see [17])

$$(u_{x_1x_1x_1}-6uu_{x_1})_{x_1}+\sum_{i,j=1}^{M}a_{ij}u_{x_ix_j}=0,\quad a_{ij}=\text{constant},\quad M\in\mathbb{N}.$$

In this paper, we consider the following generalized KP equation with variable coefficients:

$$(u_t+\alpha_1(t)u_{xxy}+3\alpha_2(t)u_xu_y)_x+\alpha_3(t)u_{ty}-\alpha_4(t)u_{zz}+\alpha_5(t)(u_x+\alpha_3(t)u_y)=0,$$

where α_i $(i=1,2,3,4,5)$ are nonzero arbitrary analytic functions with respect to t. Under a certain constraint, we show that this generalized vcKP equation has a class of Wronskian solutions and a class of Grammian solutions, with all generating functions for matrix entries satisfying a linear system of partial differential equations. The Plücker relation and the Jacobi identity for determinants are the tools of constructing the corresponding Wronskian and Grammian formulations. Two particular cases are discussed in Section 4.

2 Wronskian Formulation

Let us introduce the following helpful notation:

$$\begin{aligned}&|\widehat{N-j-1},i_1,\cdots,i_j|\\&=|\Phi^{(0)},\Phi^{(1)},\cdots,\Phi^{(N-j-1)},\ \Phi^{(i_1)},\cdots,\Phi^{(i_j)}|\\&=\det(\Phi^{(0)},\Phi^{(1)},\cdots,\Phi^{(N-j-1)},\Phi^{(i_1)},\cdots,\Phi^{(i_j)}),\quad 1\le j\le N-1,\end{aligned}\tag{2.1}$$

where $i_1,\cdots,i_j$ are non-negative integers, and the vectors of functions $\Phi^{(j)}$ are defined by

$$\Phi^{(j)}=(\phi_1^{(j)},\phi_2^{(j)},\cdots,\phi_N^{(j)})^{\mathrm{T}},\quad \phi_i^{(j)}=\frac{\partial^j}{\partial x^j}\phi_i.\tag{2.2}$$

A Wronskian determinant is given by

$$W(\phi_1,\phi_2,\cdots,\phi_N)=|\widehat{N-1}|.\tag{2.3}$$

We also use the assumption for convenience that if $i<0$, the column vector $\Phi^{(i)}$ does not appear in the determinant $\det(\cdots,\Phi^{(i)},\cdots)$. We consider the following (3+1)-dimensional nonlinear equation:

$$(u_t+\alpha_1(t)u_{xxy}+3\alpha_2(t)u_xu_y)_x+\alpha_3(t)u_{ty}-\alpha_4(t)u_{zz}+\alpha_5(t)(u_x+\alpha_3(t)u_y)=0,\tag{2.4}$$

where $\alpha_i, i = 1,2,3,4,5$ are nonzero arbitrary analytic functions with respect to t. When $\alpha_i \equiv 1$ for $i = 1,2,3,4,\ \alpha_5 \equiv 0$ and $x = y$, the equation (2.4) is reduced to the KP equation. So we call it a generalized vcKP. The KP equation was also generalized by constructing decomposition of (2+1)-dimensional equations into (1+1)-dimensional equations (see [18]).

Through the dependent variable transformation

$$u = 2\frac{\alpha_1(t)}{\alpha_2(t)}(\ln f)_x, \tag{2.5}$$

the above (3+1)-dimensional generalized vcKP equation is mapped into the Hirota bilinear equation

$$(\alpha_1(t)D_x^3 D_y + D_t D_x + \alpha_3(t)D_t D_y - \alpha_4(t)D_z^2) f \cdot f = 0, \tag{2.6}$$

under the constraint

$$\alpha_1(t) = C_0 \alpha_2(t) \mathrm{e}^{-\int \alpha_5(t)\mathrm{d}t}, \tag{2.7}$$

where $C_0 \neq 0$ is an arbitrary constant, and D_x, D_y, D_z and D_t are Hirota bilinear differential operators (see [1, 19]), which are defined by

$$D_x^n D_y^m g(x,y) \cdot f(x,y) = \Big(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\Big)^n \Big(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\Big)^m g(x,y) f(x',y')\Big|_{x=x', y=y'},$$

where $n, m \geq 0$.

Indeed, the vcKP equation (2.4) is written in the form

$$\begin{aligned}&\alpha_1(t)(\ln f)_{xxxxy} + 6C_0\alpha_2(t)\mathrm{e}^{-\int \alpha_5(t)\mathrm{d}t}[(\ln f)_{xx}(\ln f)_{xy}]_x + (\ln f)_{txx}\\ &+ \alpha_3(t)(\ln f)_{txy} - \alpha_4(t)(\ln f)_{xzz} = 0.\end{aligned} \tag{2.8}$$

By integrating with respect to x and taking the integration constant to be zero, we get

$$\begin{aligned}&\alpha_1(t)\frac{D_x^3 D_y f \cdot f}{2f^2} - \frac{3\alpha_1(t)}{2}\Big(\frac{D_x^2 f \cdot f}{f^2}\Big)\Big(\frac{D_x D_y f \cdot f}{f^2}\Big) + 6\alpha_1(t)\Big(\frac{D_x^2 f \cdot f}{2f^2}\Big)\Big(\frac{D_x D_y f \cdot f}{2f^2}\Big)\\ &+ \frac{D_x D_t f \cdot f}{2f^2} + \alpha_3(t)\frac{D_y D_t f \cdot f}{2f^2} - \alpha_4(t)\frac{D_z^2 f \cdot f}{2f^2} = 0,\end{aligned} \tag{2.9}$$

from which the equation (2.4) can be written in the bilinear form (2.6).

Equivalently, we have

$$\begin{aligned}&(\alpha_1(t)f_{xxxy} + f_{tx} + \alpha_3(t)f_{ty} - \alpha_4(t)f_{zz})f - 3\alpha_1(t)f_{xxy}f_x + 3\alpha_1(t)f_{xy}f_{xx}\\ &- \alpha_1(t)f_y f_{xxx} - f_t f_x - \alpha_3(t)f_t f_y + \alpha_4(t)(f_z)^2 = 0.\end{aligned} \tag{2.10}$$

In the next theorem, we would like to present a system of three linear partial differential equations for which the Nth order Wronskian determinant solves the generalized Hirota bilinear vcKP equation (2.6).

Theorem 2.1 *Let the set of functions* $\phi_i = \phi_i(x, y, z, t)$ *satisfy the following linear partial differential equations:*

$$\begin{aligned}\phi_{i,y} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}\phi_{i,x},\\ \phi_{i,z} &= a\phi_{i,xx},\\ \phi_{i,t} &= \beta(t)\phi_{i,xxx},\end{aligned}\tag{2.11}$$

where

$$\beta(t) = \frac{4a^2\alpha_1(t)\alpha_4(t)}{3\alpha_1(t) - a^2\alpha_3(t)\alpha_4(t)},$$

$1 \le i \le N$, *a is an arbitrary nonzero constant,* $\frac{\alpha_4}{\alpha_1}$ *is an arbitrary constant, and* $\frac{3}{a^2\alpha_3(t)}$ *is not equal to that constant* $\frac{\alpha_4}{\alpha_1}$ *for all values of t. Then the Wronskian determinant* $f_N = |\widehat{N-1}|$ *defined by* (2.3) *solves the* $(3+1)$*-dimensional generalized bilinear vcKP equation* (2.6).

Proof Using (2.11) and the following equality:

$$\sum_{k=1}^{N}|A|_{lk} = \sum_{i,j=1}^{N} A_{ij}\frac{\partial^l a_{ij}}{\partial x^l},$$

where $A = (a_{ij})_{N\times N}$, and $|A|_{lk}$ denotes the determinant resulting from $|A|$ with its kth column differentiated l times with respect to x, whereas A_{ij} denotes the co-factor of a_{ij}, we can compute various derivatives of the Wronskian determinant $f_N = |\widehat{N-1}|$ with respect to the variables x, y, z, t,

$$\begin{aligned}
f_{N,x} &= |\widehat{N-2}, N|,\\
f_{N,xx} &= |\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|,\\
f_{N,xxx} &= |\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|,\\
f_{N,y} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}|\widehat{N-2}, N|,\\
f_{N,xy} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}(|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|),\\
f_{N,xxy} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}(|\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|),\\
f_{N,xxxy} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}(|\widehat{N-5}, N-3, N-2, N-1, N| + 3|\widehat{N-4}, N-2, N-1, N+1|\\
&\quad + 2|\widehat{N-3}, N, N+1| + 3|\widehat{N-3}, N-1, N+2| + |\widehat{N-2}, N+3|),\\
f_{N,z} &= a(|\widehat{N-2}, N+1| - |\widehat{N-3}, N-1, N|),\\
f_{N,zz} &= a^2(-|\widehat{N-4}, N-2, N-1, N+1| + 2|\widehat{N-3}, N, N+1| + |\widehat{N-2}, N+3|\\
&\quad + |\widehat{N-5}, N-3, N-2, N-1, N| - |\widehat{N-3}, N-1, N+2|),\\
f_{N,t} &= \beta(t)(|\widehat{N-4}, N-2, N-1, N| - |\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|),\\
f_{N,tx} &= \beta(t)(|\widehat{N-5}, N-3, N-2, N-1, N| - |\widehat{N-3}, N, N+1| + |\widehat{N-2}, N+3|),
\end{aligned}$$

$$\begin{aligned}f_{N,ty} = &-\frac{a^2\alpha_4(t)\beta(t)}{3\alpha_1(t)}(|\widehat{N-5}, N-3, N-2, N-1, N|\\ &- |\widehat{N-3}, N, N+1| + |\widehat{N-2}, N+3|).\end{aligned}$$

In these derivatives, we use the condition that $\frac{\alpha_4}{\alpha_1}$ is an arbitrary constant.

In the above expressions, the column $\Phi^{(N-5)}$ does not appear if $N < 5$, as we assumed before. Therefore, we can now compute that

$$\begin{aligned}&\alpha_1(t)f_{N,xxxy} + f_{N,tx} + \alpha_3(t)f_{N,ty} - \alpha_4(t)f_{N,zz}\\ =& -4a^2\alpha_4(t)|\widehat{N-3}, N, N+1|,\\ &-3\alpha_1(t)f_{N,xxy}f_{N,x} - \alpha_1(t)f_{N,y}f_{N,xxx} - f_{N,t}f_{N,x} - \alpha_3(t)f_{N,t}f_{N,y}\\ =& 4a^2\alpha_4(t)|\widehat{N-2}, N||\widehat{N-3}, N-1, N+1|,\\ &3\alpha_1(t)f_{N,xy}f_{N,xx} + \alpha_4(t)(f_{N,z})^2\\ =& -4a^2\alpha_4(t)|\widehat{N-3}, N-1, N||\widehat{N-2}, N+1|.\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}&(\alpha_1(t)D_x^3D_y + D_tD_x + \alpha_3(t)D_tD_y - \alpha_4(t)D_z^2)f_N\cdot f_N\\ =& 2(\alpha_1(t)f_{N,xxxy} + f_{N,tx} + \alpha_3(t)f_{N,ty} - \alpha_4(t)f_{N,zz})f_N - \alpha_1(t)(6f_{N,xxy}f_{N,x}\\ &- 6f_{N,xy}f_{N,xx} + 2f_{N,y}f_{N,xxx}) - 2f_{N,t}f_{N,x} - 2\alpha_3(t)f_{N,t}f_{N,y} + 2\alpha_4(t)(f_{N,z})^2\\ =& -8a^2\alpha_4(t)(|\widehat{N-1}||\widehat{N-3}, N, N+1| - |\widehat{N-2}, N||\widehat{N-3}, N-1, N+1|\\ &+ |\widehat{N-3}, N-1, N||\widehat{N-2}, N+1|)\\ =& 0.\end{aligned}$$

This last equality is nothing but the Plücker relation for determinants

$$|B, A_1, A_2||B, A_3, A_4| - |B, A_1, A_3||B, A_2, A_4| + |B, A_1, A_4||B, A_2, A_3| = 0,$$

where B denotes an $N\times(N-2)$ matrix, and A_i $(1\le i\le 4)$ are four N-dimensional column vectors. Therefore, we have shown that $f = f_N$ solves the (3+1)-dimensional generalized Hirota bilinear vcKP equation (2.6), under the condition (2.11).

The condition (2.11) is a linear system of partial differential equations. It has an exponential-type function solution

$$\phi_i = \sum_{j=1}^{p} d_{ij}\mathrm{e}^{\eta_{ij}}, \quad \eta_{ij} = k_{ij}x - \frac{a^2\alpha_4(t)}{3\alpha_1(t)}k_{ij}y + ak_{ij}^2z + k_{ij}^3h(t), \tag{2.12}$$

where

$$h(t) = \int\beta(t)\mathrm{d}t, \tag{2.13}$$

d_{ij}, k_{ij} are free parameters, and p is an arbitrary natural number.

3 Grammian Formulation

Let us now introduce the following Grammian determinant:

$$f_N = \det(a_{ij})_{1\le i,j\le N}, \quad a_{ij} = c_{ij} + \int^x \phi_i\psi_j \mathrm{d}x, \quad c_{ij} = \text{constant} \tag{3.1}$$

with ϕ_i and ψ_j satisfying

$$\phi_{i,y} = -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}\phi_{i,x}, \quad \phi_{i,z} = a\phi_{i,xx}, \qquad \phi_{i,t} = \beta(t)\phi_{i,xxx}, \quad 1\le i\le N, \tag{3.2}$$

$$\psi_{i,y} = -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}\psi_{i,x}, \quad \psi_{i,z} = -a\psi_{i,xx}, \quad \psi_{i,t} = \beta(t)\psi_{i,xxx}, \quad 1\le i\le N, \tag{3.3}$$

where $\beta, \alpha_1, \alpha_3, \alpha_4$ and a are as in Theorem 2.1.

Theorem 3.1 *Let ϕ_i and ψ_j satisfy* (3.2) *and* (3.3)*, respectively. Then the Grammian determinant $f_N = \det(a_{ij})_{1\le i,j\le N}$ defined by* (3.1) *solves the* $(3+1)$*-dimensional generalized bilinear vcKP equation* (2.6)*.*

Proof Let us express the Grammian determinant f_N by means of a Pfaffian as

$$f_N = (1, 2, \cdots, N, N^*, \cdots, 2^*, 1^*), \tag{3.4}$$

where $(i, j^*) = a_{ij}$ and $(i, j) = (i^*, j^*) = 0$.

To compute derivatives of the entries a_{ij} and the Grammian f_N, we introduce the new Pfaffian entries

$$\begin{aligned} (d_n, j^*) &= \frac{\partial^n}{\partial x^n}\psi_j, \\ (d_n^*, i) &= \frac{\partial^n}{\partial x^n}\phi_i, \\ (d_m, d_n^*) &= (d_n, i) = (d_m^*, j^*) = 0, \quad m, n\ge 0 \end{aligned} \tag{3.5}$$

as usual. In terms of these new entries, by using (3.2)–(3.3), derivatives of the entries $a_{ij} = (i, j^*)$ are obtained

$$\begin{aligned} \frac{\partial}{\partial x}a_{ij} &= \phi_i\psi_j = (d_0, d_0^*, i, j^*), \\ \frac{\partial}{\partial y}a_{ij} &= \int^x (\phi_{i,y}\psi_j + \phi_i\psi_{j,y})\mathrm{d}x = -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}\int^x (\phi_{i,x}\psi_j + \phi_i\psi_{j,x})\mathrm{d}x \\ &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}\phi_i\psi_j = -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}(d_0, d_0^*, i, j^*), \\ \frac{\partial}{\partial z}a_{ij} &= \int^x (\phi_{i,z}\psi_j + \phi_i\psi_{j,z})\mathrm{d}x = a\int^x (\phi_{i,xx}\psi_j - \phi_i\psi_{j,xx})\mathrm{d}x \\ &= a(\phi_{i,x}\psi_j - \phi_i\psi_{j,x}) = a[-(d_1, d_0^*, i, j^*) + (d_0, d_1^*, i, j^*)], \\ \frac{\partial}{\partial t}a_{ij} &= \int^x (\phi_{i,t}\psi_j + \phi_i\psi_{j,t})\mathrm{d}x = \beta(t)\int^x (\phi_{i,xxx}\psi_j + \phi_i\psi_{j,xxx})\mathrm{d}x \\ &= \beta(t)(\phi_{i,xx}\psi_j - \phi_{i,x}\psi_{j,x} + \phi_i\psi_{j,xx}) \\ &= \beta(t)[(d_2, d_0^*, i, j^*) - (d_1, d_1^*, i, j^*) + (d_0, d_2^*, i, j^*)]. \end{aligned}$$

Then, we can develop differential rules for Pfaffians as in [1], and compute various derivatives of the Grammian determinant $f_N = \det(a_{ij})$ with respect to the variables x, y, z, t as follows:

$$\begin{aligned}
f_{N,x} &= (d_0, d_0^*, \bullet),\\
f_{N,xx} &= (d_1, d_0^*, \bullet) + (d_0, d_1^*, \bullet),\\
f_{N,xxx} &= (d_2, d_0^*, \bullet) + 2(d_1, d_1^*, \bullet)) + (d_0, d_2^*, \bullet),\\
f_{N,y} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}(d_0, d_0^*, \bullet),\\
f_{N,xy} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}[(d_1, d_0^*, \bullet) + (d_0, d_1^*, \bullet)],\\
f_{N,xxy} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}[(d_2, d_0^*, \bullet) + 2(d_1, d_1^*, \bullet)) + (d_0, d_2^*, \bullet)],\\
f_{N,xxxy} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}[(d_3, d_0^*, \bullet) + 3(d_2, d_1^*, \bullet) + 2(d_0, d_0^*, d_1, d_1^*, \bullet) + 3(d_1, d_2^*, \bullet) + (d_0, d_3^*, \bullet)],\\
f_{N,z} &= a[-(d_1, d_0^*, \bullet) + (d_0, d_1^*, \bullet)],\\
f_{N,zz} &= a^2[(d_3, d_0^*, \bullet) - (d_2, d_1^*, \bullet) + 2(d_0, d_0^*, d_1, d_1^*, \bullet) - (d_1, d_2^*, \bullet) + (d_0, d_3^*, \bullet)],\\
f_{N,t} &= \beta(t)[(d_2, d_0^*, \bullet) - (d_1, d_1^*, \bullet) + (d_0, d_2^*, \bullet)],\\
f_{N,tx} &= \beta(t)[(d_3, d_0^*, \bullet) - (d_0, d_0^*, d_1, d_1^*, \bullet) + (d_0, d_3^*, \bullet)],\\
f_{N,ty} &= -\frac{a^2\alpha_4(t)\beta(t)}{3\alpha_1(t)}[(d_3, d_0^*, \bullet) - (d_0, d_0^*, d_1, d_1^*, \bullet) + (d_0, d_3^*, \bullet)],
\end{aligned}$$

where the abbreviated notation $\bullet$ denotes the list of indices $1, 2, \cdots, N, N^*, \cdots, 2^*, 1^*$ common to each Pfaffian.

Under the conditions on α_1, α_3, α_4 and a, we can now compute that

$$\begin{aligned}
&\alpha_1(t) f_{N,xxxy} + f_{N,tx} + \alpha_3(t) f_{N,ty} - \alpha_4(t) f_{N,zz}\\
={}& -4a^2\alpha_4(t)(d_0, d_0^*, d_1, d_1^*, \bullet),\\
&- 3\alpha_1(t) f_{N,xxy} f_{N,x} - \alpha_1(t) f_{N,y} f_{N,xxx} - f_{N,t} f_{N,x} - \alpha_3(t) f_{N,t} f_{N,y}\\
={}& 4a^2\alpha_4(t)(d_0, d_0^*, \bullet)(d_1, d_1^*, \bullet),\\
&3\alpha_1(t) f_{N,xy} f_{N,xx} + \alpha_4(t)(f_{N,z})^2\\
={}& -4a^2\alpha_4(t)(d_1, d_0^*, \bullet)(d_0, d_1^*, \bullet),
\end{aligned}$$

and further obtain that

$$\begin{aligned}
&(\alpha_1(t) D_x^3 D_y + D_t D_x + \alpha_3(t) D_t D_y - \alpha_4(t) D_z^2) f_N \cdot f_N\\
={}& 2(\alpha_1(t) f_{N,xxxy} + f_{N,tx} + \alpha_3(t) f_{N,ty} - \alpha_4(t) f_{N,zz}) f_N - 2\alpha_1(t)(3 f_{N,xxy} f_{N,x}\\
&- 3 f_{N,xy} f_{N,xx} + f_{N,y} f_{N,xxx}) - 2 f_{N,t} f_{N,x} - 2\alpha_3(t) f_{N,t} f_{N,y} + 2\alpha_4(t)(f_{N,z})^2\\
={}& -8a^2\alpha_4(t)[(\bullet)(d_0, d_0^*, d_1, d_1^*, \bullet) - (d_0, d_0^*, \bullet)(d_1, d_1^*, \bullet) + (d_1, d_0^*, \bullet)(d_0, d_1^*, \bullet)]\\
={}& 0.
\end{aligned}$$

The last equality is nothing but the Jacobi identity for determinants. Therefore, we have shown that $f_N = \det(a_{ij})_{1\le i,j\le N}$ defined by (3.1) solves the (3+1)-dimensional generalized Hirota bilinear vcKP equation (2.6) under the conditions of (3.2)–(3.3).

The systems (3.2)–(3.3) have solutions

$$\phi_i = \sum_{j=1}^{p} d_{ij}\mathrm{e}^{\eta_{ij}}, \quad \eta_{ij} = k_{ij}x - \frac{a^2\alpha_4(t)}{3\alpha_1(t)}k_{ij}y + ak_{ij}^2 z + k_{ij}^3 h(t), \tag{3.6}$$

$$\psi_j = \sum_{i=1}^{q} e_{ji}\mathrm{e}^{\zeta_{ji}}, \quad \zeta_{ji} = l_{ji}x - \frac{a^2\alpha_4(t)}{3\alpha_1(t)}l_{ji}y - al_{ji}^2 z + l_{ji}^3 h(t), \tag{3.7}$$

where

$$h(t) = \int \beta(t)\mathrm{d}t, \tag{3.8}$$

d_{ij}, e_{ji}, k_{ij}, l_{ji} are free parameters, and p, q are two arbitrary natural numbers.

4 Conclusions and Remarks

Under certain constraint on the variable coefficients, we have verified that the (3+1)-dimensional generalized vcKP equation

$$(u_t + \alpha_1(t)u_{xxy} + 3\alpha_2(t)u_xu_y)_x + \alpha_3(t)u_{ty} - \alpha_4(t)u_{zz} + \alpha_5(t)(u_x + \alpha_3(t)u_y) = 0$$

has two classes of exact determinant solutions. One is formulated in the Wronskian determinant and the other in the Grammian determinant. Indeed, we have shown that the above vcKP equation was reduced to the Plücker relation for determinants and the Jacobi identity for determinants in the cases of the obtained determinant solutions. In our solutions, there is a free parameter a which satisfies

$$3\alpha_1(t) - a^2\alpha_3(t)\alpha_4(t) \neq 0 \quad \text{for all values of } t.$$

Theorems 2.1 and 3.1 present the main results on these solutions.

We remark that in order to get more solutions to the above vcKP equation, we have tried to replace this arbitrary constant with an arbitrary function with respect to t. But we faced a problem with a compatibility condition of the system of the linear differential equations (2.11). It is unavoidable that $\frac{\alpha_4}{\alpha_1}$ must be a constant. Actually, if we computed the derivative $f_{N,ty}$ without this condition, the term

$$\frac{a^2}{3}\frac{d}{dt}\Big(\frac{\alpha_4}{\alpha_1}\Big)|\widehat{N-2}, N|$$

would appear and the vcKP equation could not be reduced to the Plücker relation for determinants or the Jacobi identity for determinants, in addition $f_{N,ty} \neq f_{N,yt}$.

In particular, if we put $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 \equiv 1$ and $\alpha_5 \equiv 0$, then we will get an equivalent solution to the one given in [14, Theorem 2.1] with a condition on the parameter a, which accepts any real number except $\pm\sqrt{3}$ for a.

On the other hand, if we choose $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 \equiv -1$ and $\alpha_5 \equiv 0$, then we will have the equation

$$u_{xxxy} + 3(u_x u_y)_x - u_{tx} + u_{ty} - u_{zz} = 0.$$

Note here that the coefficient of the term u_{tx} is -1. By using Theorem 2.1, one can get the following Wronskian solution:

$$u = 2(\ln f_N)_x, \quad f_N = W(\phi_1, \phi_2, \cdots, \phi_N),$$

where

$$\phi_i = \sum_{j=1}^{p} d_{ij} \mathrm{e}^{\eta_{ij}}, \quad \eta_{ij} = k_{ij} x - \frac{1}{3} a^2 k_{ij} y + a k_{ij}^2 z - \frac{4a^2}{a^2+3} k_{ij}^3 t,$$

d_{ij} and k_{ij} are free parameters, and p is an arbitrary natural number. There are not any restrictions on our parameter a here.

However, it should be mentioned that this generalization is non-trivial. For example, the KdV equation with the variable constraint

$$u_t + \alpha_1(t) u_{xxx} + \alpha_2(t) u u_x = 0$$

is a trivial generalization of

$$u_t + u_{xxx} + 6 u u_x = 0$$

by a simple change of variables (x, t, u). In this paper, one can always set $\alpha_1(t) = 1$, $\alpha_2 = 3$, $\alpha_5 = 0$ under the constraint (2.7) by a similar change of variables. But introducing $\alpha_3(t)$, $\alpha_4(t)$ makes our generalization non-trivial.

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