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Note on Co-split Lie Algebras*

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Abstract It is proved that, any finite dimensional complex Lie algebra $\mathcal{L} = [\mathcal{L}, \mathcal{L}]$, hence, over a field of characteristic zero, any finite dimensional Lie algebra $\mathcal{L} = [\mathcal{L}, \mathcal{L}]$ possessing a basis with complex structure constants, should be a weak co-split Lie algebra. A class of non-semi-simple co-split Lie algebras is given.

Keywords Co-split, Casimir operator, Killing form
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1 Introduction

In the recent paper [1], the authors introduced a new "Lie bialgebra" structure called a co-split Lie algebra, which is a Lie algebra $(\mathcal{L}, \mu)$ endowed with a Lie coalgebra structure $(\mathcal{L}, \delta)$, such that the composition $\mu\circ\delta$ coincides with the identity. In the case that $\mu\circ\delta$ is non-degenerate diagonal corresponding to some basis, $(\mathcal{L}, \mu, \delta)$ is called a weak co-split Lie algebra.

It is an important result in [1] that any finite dimensional complex simple Lie algebra has a co-split Lie algebra structure. In [2], the construction of δ is interpreted as finding some map of $\mathcal{L}$-modules, and the result of [1] is extended to a wider context. Moreover, the cobracket in [2] is proved to be a scalar (multiple) of

$$\sum_{i=1}^{\dim \mathcal{L}} [y_i, -] \otimes x_i, \tag{1.1}$$

where $\sum\limits_{i=1}^{\dim \mathcal{L}} y_i \otimes x_i$ is the Casimir operator of adjoint representation.

It is clear that any (weak) co-split Lie algebra $\mathcal{L}$ must satisfy the condition $[\mathcal{L}, \mathcal{L}] = \mathcal{L}$. The last result listed in [2] can be roughly stated as follows: a semi-simple Lie algebra over field k with suitable characteristic p possesses a non-singular Casimir operator, and has a co-split Lie algebra structure.

Naturally, we want to know the answers to the following problems: (1) Can any Lie algebra $\mathcal{L} = [\mathcal{L}, \mathcal{L}]$ be endowed with a weak co-split Lie algebra structure? (2) Is any co-split Lie algebra semi-simple?

In this note, we obtain the following results: (1) Over a field of characteristic zero, any finite dimensional Lie algebra $\mathcal{L} = [\mathcal{L}, \mathcal{L}]$ possessing a basis with real structure constants is a weak co-split Lie algebra; (2) A class of non-semi-simple co-split Lie algebras is given.

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Let k be a field. For a given finite-dimensional k-vector space V, we consider the k-linear maps

$$\tau_V : V \otimes_k V \to V \otimes_k V; \quad u \otimes v \mapsto v \otimes u \tag{1.2}$$

and

$$\xi_V = (\tau_V \otimes_k \mathrm{id}_V) \circ (\mathrm{id}_V \otimes_k \tau_V) : V \otimes_k V \otimes_k V \to V \otimes_k V \otimes_k V. \tag{1.3}$$

Identifying $(V \otimes_k \cdots \otimes_k V)^*$ with $V^* \otimes_k \cdots \otimes_k V^*$, the transpose maps satisfy

$$\tau_V^* = \tau_{V^*}^{-1} \quad \text{and} \quad \xi_V^* = \xi_{V^*}^{-1}.$$

A Lie coalgebra is a k-vector space $\mathcal{L}$ endowed with a k-linear map $\delta : \mathcal{L} \to \mathcal{L} \otimes_k \mathcal{L}$, which satisfies:

(i) $(\mathrm{id}_{\mathcal{L}\otimes_k\mathcal{L}} + \tau_{\mathcal{L}}) \circ \delta = 0$,

(ii) $(\mathrm{id}_{\mathcal{L}\otimes_k\mathcal{L}\otimes_k\mathcal{L}} + \xi_{\mathcal{L}} + \xi_{\mathcal{L}}^2) \circ (\mathrm{id}_{\mathcal{L}} \otimes \delta) \circ \delta = 0$.

We know that a module of Lie algebra $(\mathcal{L}, \mu)$ is a k-vector space M with an $\mathcal{L}$-linear map $\rho_M : \mathcal{L} \otimes M \to M$, such that

$$\rho_M \circ (\mathrm{id}_{\mathcal{L}} \otimes \rho_M) \circ ((\mathrm{id}_{\mathcal{L}\otimes_k\mathcal{L}} - \tau_{\mathcal{L}}) \otimes_k \mathrm{id}_M) = \rho_M \circ (\mu \otimes \mathrm{id}_M). \tag{1.4}$$

Then we can consider its dual concept.

Definition 1.1 *A comodule of Lie coalgebra $(\mathcal{L}, \delta)$ is a k-vector space M with an $\mathcal{L}$-linear map $\delta_M : M \to \mathcal{L} \otimes M$, such that*

$$((\mathrm{id}_{\mathcal{L}\otimes_k\mathcal{L}} - \tau_{\mathcal{L}}) \otimes_k \mathrm{id}_M) \circ (\mathrm{id}_{\mathcal{L}} \otimes \delta_M) \circ \delta_M = (\delta \otimes \mathrm{id}_M) \circ \delta_M. \tag{1.5}$$

2 Weak Co-split Theorem

Theorem 2.1 *Any finite dimensional complex Lie algebra $\mathcal{L} = [\mathcal{L}, \mathcal{L}]$ is a weak co-split Lie algebra.*

Proof Suppose that $c_{i,j}^k$'s are structure constants associated with a basis $\{x_1, \cdots, x_n\}$. Define a map $\delta : \mathcal{L} \to \mathcal{L} \otimes \mathcal{L}$ as

$$x_k \mapsto \sum_{i,j} \overline{c}_{i,j}^k x_i \otimes x_j. \tag{2.1}$$

Then $(\mathcal{L}, \delta)$ is a Lie co-algebra, and when $c_{i,j}^k$'s are real numbers, it is isomorphic to $(\mathcal{L}^*, \mu^*)$, where μ is the bracket.

Denote by A the $n \times n^2$ matrix of μ. Then $\delta = \overline{A^{\mathrm{T}}}$, which is the convolution transpose of A. The composition of μ and δ, i.e. $M_n = A\overline{A^{\mathrm{T}}}$, is a Hermitian symmetric square matrix. Since $[\mathcal{L}, \mathcal{L}] = \mathcal{L}$, A is of rank n, by the fundamental knowledge of linear algebras, M_n is non-degenerate and there exists a Hermitian orthogonal matrix $U = (u_{i,j})$, such that $D = \overline{U^{\mathrm{T}}} M_n U = (d_1, \cdots, d_n)$ is diagonal and non-degenerate. Choosing a new basis

$$\{y_1, \cdots, y_n\} = U\{x_1, \cdots, x_n\}, \quad y_j = \sum_{i=1}^{n} u_{i,j} x_i,$$

it is easy to check that

$$\mu|_{\{y_1,\cdots,y_n\}} = \overline{U^{\mathrm{T}}}A(U\otimes U), \tag{2.2}$$

$$\delta|_{\{y_1,\cdots,y_n\}} = (\overline{U^{\mathrm{T}}}\otimes\overline{U^{\mathrm{T}}})\overline{A^{\mathrm{T}}}U. \tag{2.3}$$

Then

$$\begin{aligned}(\mu\circ\delta)|_{\{y_1,\cdots,y_n\}} &= \overline{U^{\mathrm{T}}}A(U\otimes U)(\overline{U^{\mathrm{T}}}\otimes\overline{U^{\mathrm{T}}})\overline{A^{\mathrm{T}}}U\\ &= \overline{U^{\mathrm{T}}}A\overline{A^{\mathrm{T}}}U\\ &= \overline{U^{\mathrm{T}}}M_nU\\ &= D.\end{aligned}$$

Thus this theorem is proved.

Remark 2.1 However, the method used in the proof of Theorem 2.1 does not work for the positive characteristic field k. For example, $\mathcal{L}=\mathrm{span}\{x_i \mid i=0,1,\cdots,p-1\}$, $\mathrm{char}(k)=p>3$, $[x_i,x_j]=(j-i)x_{i+j}$. Using the symbols above, we have

$$(\mu\circ\delta)(x_0)=\sum_{i=1}^{\frac{p-1}{2}}2(2i)^2x_0=8\sum_{i=1}^{\frac{p-1}{2}}i^2x_0. \tag{2.4}$$

So when $p>3$, we have

$$\sum_{i=1}^{\frac{p-1}{2}}i^2=\frac{1}{6}\frac{p-1}{2}\Big(\frac{p-1}{2}+1\Big)\Big(2\times\frac{p-1}{2}+1\Big)=0, \tag{2.5}$$

that is, $\mu\circ\delta$ is degenerate.

3 Simple Co-split Lie Algebras and Their Modules

Suppose that $(\mathcal{L},\mu)$ is a simple Lie algebra with non-singular Killing form $(\ ,\ )$ over an algebraically closed field k (when $p=\mathrm{char}(k)>0$, p is suitable).

Let $\theta:\mathcal{L}\to\mathcal{L}^*$ be the isomorphism given by $x\mapsto(x,-)$, and μ^* be the transpose of μ. Define an $\mathcal{L}$-linear map $\delta:\mathcal{L}\to\mathcal{L}\otimes_k\mathcal{L}$ via

$$\delta=(\theta^{-1}\otimes\theta^{-1})\circ\mu^*\circ\theta. \tag{3.1}$$

Fix a basis $\{x_1,\cdots,x_n\}$ of $\mathcal{L}$ along with its dual basis $\{y_1,\cdots,y_n\}$, that is, $(x_i,y_j)=\delta_{i,j}$. Then for any $x\in\mathcal{L}$,

$$\delta(x)=\sum_{i=1}^{n}[y_j,x]\otimes x_j=-\sum_{i=1}^{n}y_j\otimes[x_j,x], \tag{3.2}$$

and $\mu\circ\delta$ is a non-zero scalar (see [2]).

For an $\mathcal{L}$-module M, define an $\mathcal{L}$-linear map $\delta_M:M\to L\otimes M$ via

$$m\mapsto\sum_{i=1}^{n}y_i\otimes(x_i\cdot m). \tag{3.3}$$

Lemma 3.1 *(M,δ_M) is a comodule of $(\mathcal{L},\delta)$.*

Proof For any element $m \in M$,

$$\begin{aligned}
&((\mathrm{id}_{\mathcal{L}\otimes_k\mathcal{L}} - \tau_{\mathcal{L}}) \otimes_k \mathrm{id}_M) \circ (\mathrm{id}_{\mathcal{L}} \otimes \delta_M) \circ \delta_M(m)\\
&= \sum_{i,j=1}^{n} (y_i \otimes y_j \otimes (x_j x_i m) - y_j \otimes y_i \otimes (x_j x_i m))\\
&= \sum_{i,j=1}^{n} y_i \otimes y_j \otimes ([x_j, x_i] m)\\
&= \sum_{i,j=1}^{n} y_i \otimes [x_i, y_j] \otimes (x_j m)\\
&= (\delta \otimes \mathrm{id}_M) \circ \delta_M(m).
\end{aligned}$$

Hence Lemma 3.1 holds.

Define a k-linear map as

$$\Delta_M = (\mathrm{id}_{\mathcal{L}\otimes M} - \tau) \circ \delta_M : M \to (\mathcal{L} \oplus M) \otimes (\mathcal{L} \oplus M),$$

and $\Delta = \delta + \Delta_M$, where $\tau : \mathcal{L} \otimes M \to M \otimes \mathcal{L}$ maps $l \otimes m$ to $m \otimes l$.

Lemma 3.2 *$(\mathcal{L} \oplus M, \Delta)$ is a Lie coalgebra.*

Proof It is sufficient to check that

$$(\mathrm{id}_{(\mathcal{L}\oplus M)\otimes_k(\mathcal{L}\oplus M)\otimes_k(\mathcal{L}\oplus M)} + \xi_{(\mathcal{L}\oplus M)} + \xi^2_{(\mathcal{L}\oplus M)}) \circ (\mathrm{id}_{(\mathcal{L}\oplus M)} \otimes \Delta) \circ \Delta_M = 0.$$

By a direct computation, we have

$$\begin{aligned}
&(\mathrm{id}_{(\mathcal{L}\oplus M)} \otimes \Delta) \circ \Delta_M(m)\\
&= -\sum_{i=1}^{n} (\mathrm{id}_{(\mathcal{L}\oplus M)} \otimes \Delta)(y_i \otimes (x_i m) - (x_i m) \otimes y_i)\\
&= \sum_{i=1}^{n} (x_i m) \otimes \delta(y_i) + \sum_{i,j=1}^{n} y_i \otimes y_j \otimes (x_j x_i m) - y_i \otimes (x_j x_i m) \otimes y_j.
\end{aligned}$$

Since (M, δ_M) is a comodule, we get

$$\sum_{i=1}^{n} (x_i m) \otimes \delta(y_i) = \sum_{i,j=1}^{n} (x_j x_i m) \otimes y_i \otimes y_j - (x_j x_i m) \otimes y_j \otimes y_i.$$

Then the proof of this lemma is completed.

4 Examples of Non-semi-simple Co-split Lie Algebras

In this section, we give some non-semi-simple co-split Lie algebras.

(a) Suppose that

$$\mathcal{L} = sl_2 + M,$$

where M is an $(r+1)$-dimensional irreducible sl_2-module, $[M,M]=0$, and $sl_2=\mathrm{span}\{x,y,h\}$ with brackets

$$[x,y]=h, \quad [h,x]=2x, \quad [h,y]=-2y.$$

Then the Killing form is given by

$$(x,y)=4, \quad (h,h)=8.$$

Hence, the Casimir operator is

$$c=\frac{1}{8}(2x\otimes y+2y\otimes x+h\otimes h),$$

and $\mu\circ\Delta_M=\frac{1}{4}(r^2+2r)\mathrm{id}_M$.

$\mathcal{L}$ is a co-split Lie algebra with cobracket Δ defined in Section 4, if r and the characteristic p of field k satisfy

$$(r+1)^2\equiv 5(\mathrm{mod}\ p) \quad \text{and} \quad r<p. \tag{4.1}$$

The inequality above comes from the irreducibility of M. Some (r,p) pairs are listed as follows:

Table 1 Some (r,p) pairs

r	p	$(r+1)^2-5$
3	11	11
4	5	20
5	31	31
6	11	44
7	59	59
8	19	76
9	19	100
10	29	116
11	139	139
12	41	164
13	191	191
15	251	251
16	71	284

In this way, we can find much more non-semi-simple co-split Lie algebras.

(b) Let $\mathbb{F}$ be a field with $\mathrm{char}\mathbb{F}=p=2k+1$, and

$$W=\mathrm{span}_{\mathbb{F}}\{x_i\mid i\in\mathbb{Z}_p\}=\mathrm{span}_{\mathbb{F}}\{x_{-k},x_{-k+1},\cdots,x_k\}, \tag{4.2}$$

$$\mu_W(x_i,x_j)=[x_i,x_j]=(j-i)x_{i+j}. \tag{4.3}$$

Let

$$E=-2x_k, \quad F=2x_{-k}, \quad H=-4x_0, \tag{4.4}$$

$$u_i=x_i, \quad v_i=-x_{i-k}, \quad i=1,\cdots,k-1. \tag{4.5}$$

Define $\Delta: W\to W\otimes W$,

$$\Delta_W(H)=2E\otimes F-2F\otimes E, \tag{4.6}$$

$$\Delta_W(E)=H\otimes E-E\otimes H, \tag{4.7}$$

$$\Delta_W(F)=F\otimes H-H\otimes F, \tag{4.8}$$

and for $i = 1, \cdots, k-1$,

$$\Delta_W(u_i) = E \otimes v_i - v_i \otimes E + \frac{1}{2}(H \otimes u_i - u_i \otimes H), \tag{4.9}$$

$$\Delta_W(v_i) = F \otimes u_i - u_i \otimes F + \frac{1}{2}(v_i \otimes H - H \otimes v_i), \tag{4.10}$$

and $\delta_W = \frac{1}{4}\Delta_W = k^2\Delta_W$. Then (W, μ_W, δ_W) is a co-split Lie algebra. Particularly, (W^*, δ_W^*) is not semi-simple while (W, μ_W) is simple.

References

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