Chin. Ann. Math.
33B(5), 2012, 673–680
DOI: 10.1007/s11401-012-0735-y

Chinese Annals of Mathematics, Series B

On Regularity of Incompressible Fluid with Shear Dependent Viscosity*

Hongjun YUAN[1] Qiu MENG[1]

Abstract The authors consider a non-Newtonian fluid governed by equations with p-structure in a cubic domain. A fluid is said to be shear thinning (or pseudo-plastic) if $1 < p < 2$, and shear thickening (or dilatant) if $p > 2$. The case $p > 2$ is considered in this paper. To improve the regularity results obtained by Crispo, it is shown that the second-order derivatives of the velocity and the first-order derivative of the pressure belong to suitable spaces, by appealing to anisotropic Sobolev embeddings.

Keywords Non-Newtonian fluid, Regularity, Shear dependent viscosity
2000 MR Subject Classification 35Q35, 76D03

1 Introduction

We are interested in studying regularity results for the weak solution to the following equations for flows with shear dependent viscosity:

$$\begin{cases}(u\cdot\nabla)u+\nabla\pi-\nabla\cdot[(\nu_0+\nu_1|\mathcal{D}u|^{p-2})\mathcal{D}u]=f,\\ \nabla\cdot u=0,\end{cases}\tag{1.1}$$

where u and π denote the velocity and the pressure of viscous incompressible fluid, respectively, $\frac{1}{2}\mathcal{D}u$ denotes the symmetric gradient of u, i.e.,

$$\mathcal{D}u=\nabla u+\nabla u^{\mathrm{T}}.$$

Here ν_0 and ν_1 are strictly positive constants. The system (1.1) is a classical model, considered by von Neumann and Richtmeyer in the 50's, and by Smagorinsky at the beginning of the 60's (for $p = 3$). The model was extended to other physical situations, and deeply studied from a mathematical point of view by Ladyzhenskaya in the second half of the 60's.

An extensive discussion of incompressible fluids with shear dependent viscosity can be found, for instance, in [1, 8, 11, 14–15]. In order to avoid additional calculation, we assume that $p \le 4$. However, this restriction is not at all necessary. The case $2 \le p \le 4$ was applied in the last forty years to model turbulence phenomena in fluid flows, for instance, in [5–6, 9–10, 12, 16–18].

We are interested in the regularity results up to the boundary for all the second derivatives of the velocity and the first derivatives of the pressure. In [2], particularly sharp regularity

Manuscript received November 8, 2011. Revised April 11, 2012.
[1]Institute of Mathematics, Jilin University, Changchun 130012, China.
E-mail: hjy@jlu.edu.cn mengqiu15@163.com
*Project supported by the National Natural Science Foundation of China (No. 10971080).

results in the half-space $\mathbb{R}^n_+$ $(n \geq 3)$ were obtained for the system (1.1) without the convective term

$$\begin{cases} -\nabla \cdot [(\nu_0 + \nu_1 |\mathcal{D}u|^{p-2})\mathcal{D}u] + \nabla \pi = f, \\ \nabla \cdot u = 0, \end{cases} \tag{1.2}$$

under slip and non-slip boundary conditions. More recently, in [3], the presentation of the above method has been simplified by considering a three dimensional cubic domain $\Omega = [0,1]^3$ instead of the half-space $\mathbb{R}^n_+$. To impose the non-slip boundary condition on two opposite sides, on the remaining sides Beirão da Veiga assumed periodicity conditions (in this way to avoid singularities due to the corner points). In [4], the method was applied to the shear thinning case. On the other hand, in [7], by introducing a new device in the proof given in [4], the author improved the results stated in [3]. More precisely, the author succeeds in taking advantage of the better regularity of solutions in the tangential directions, with respect to the normal one. This allows the use of anisotropic Sobolev embedding theorems (see [19]), instead of the classical ones. In this paper, by applying this device to shear thickening fluids, we improve the results stated in [7].

2 Notations and Main Theorems

Throughout this paper, Ω denotes a three dimensional cube $\Omega = [0,1]^3$, and we set

$$\Gamma = \{x : |x_1| < 1, |x_2| < 1, x_3 = 0\} \cup \{x : |x_1| < 1, |x_2| < 1, x_3 = 1\}.$$

We will impose Dirichlet boundary conditions on the two opposite faces defining Γ and periodicity in the other two directions. We set $x' = (x_1, x_2)$ and say that a function is x'-periodic if it is periodic in both two directions x_1 and x_2. Therefore, we can write the boundary conditions as follows:

$$\begin{cases} u|_\Gamma = 0, \\ u \text{ is } x'\text{-periodic.} \end{cases} \tag{2.1}$$

If X is a Banach space, we denote by X' its strong dual space. We use the same notation for functional spaces and norms for both scalar and vector fields. $L^p(\Omega)$, $p \in [1, \infty)$, denotes the usual Lebesgue space with norm $\|\cdot\|_p$, and we set $\|\cdot\| = \|\cdot\|_2$. By $W^{m,p}(\Omega)$, where m is a nonnegative integer and $p \in (1, \infty)$, we denote the usual Sobolev space.

We set

$$V_p = \{v \in W^{1,p}(\Omega) : (\nabla \cdot v)|_\Omega = 0;\ v|_\Gamma = 0;\ v \text{ is } x'\text{-periodic}\}. \tag{2.2}$$

Lemma 2.1 (see [16]) *There is a positive constant c, such that the estimate*

$$\|\nabla v\|_p + \|v\|_p = c\|\mathcal{D}v\|_p \tag{2.3}$$

holds for each $v \in V_p$. Hence the two above quantities are equivalent norms in V_p.

Definition 2.1 *Assume $f \in (V_p)'$. We say that u is a weak solution to problem* (1.2) *and* (2.1)*, if $u \in V_p$ satisfies*

$$\frac{1}{2}\int_\Omega (\nu_0 + \nu_1 |\mathcal{D}u|^{p-2})\mathcal{D}u \cdot \mathcal{D}v \mathrm{d}x = \int_\Omega f \cdot v \mathrm{d}x \tag{2.4}$$

for all $v \in V_p$, and π is a distribution determined up to a constant, by de Rham's theorem.

Existence and uniqueness of the weak solution can be shown (see [13]).

Replacing v by u in (2.4), one gets

$$\nu_0\|\mathcal{D}u\|^2+\nu_1\|\mathcal{D}u\|^p=\langle f,u\rangle. \tag{2.5}$$

From (2.5), there readily follow the basic estimates

$$\begin{cases}\nu_0\|\nabla u\|\le c\|f\|,\\ \nu_1\|\nabla u\|_p\le c\|f\|_{p'}^{\frac{1}{p-1}}.\end{cases} \tag{2.6}$$

We denote by D^2u the set of all the second derivatives of u. The symbol D^2_*u denotes any second order derivatives $\frac{\partial^2 u_j}{\partial x_i\partial x_k}$ except the derivatives $\frac{\partial^2 u_j}{\partial x_3^2}$, if $j=1$ or $j=2$. Moreover,

$$|D^2_*u|^2\doteq\Big|\frac{\partial^2 u_3}{\partial x_3^2}\Big|^2+\sum_{\substack{(i,j,k)=1\\(i,k)\neq(3,3)}}^{3}\Big|\frac{\partial^2 u_j}{\partial x_i\partial x_k}\Big|^2.$$

So, similarly, ∇_* may denote any first-order partial derivative, other than $\frac{\partial}{\partial x_3}$.

As far as the regularity problem is concerned, the best regularity results for the Dirichlet boundary value problem are obtained in [7]. Here we recall the main theorems of [7] as follows.

Theorem 2.1 *Assume that $p\in[2,4]$, $f\in L^2(\Omega)$. Let (u,π) be the weak solution to problem* (1.2) *and* (2.1). *Then D^2_*u and $|\mathcal{D}u|^{\frac{p-2}{2}}\nabla_*\mathcal{D}u$ belong to $L^2(\Omega)$ with*

$$\nu_0\|D^2_*u\|+(\nu_0\nu_1)^{\frac12}\||\mathcal{D}u|^{\frac{p-2}{2}}\nabla_*\mathcal{D}u\|\le c\|f\|. \tag{2.7}$$

Moreover, D^2u, $\nabla_*\pi$ and $|\mathcal{D}u|^{p-2}\nabla_*\mathcal{D}u$ belong to $L^{\frac{8-p}{3}}(\Omega)$, and

$$\partial_3\pi\in L^{\overline{s}}(\Omega),$$

where

$$\frac{1}{s(q)}=\frac{p-2}{q}+\frac12,\quad \overline{s}=\min\Big\{s(8-p),\frac{8-p}{3}\Big\}=s(8-p).$$

Moreover, for $p\in[2,4)$, there hold

$$\|\nabla_*\pi\|_{\frac{8-p}{3}}+\|D^2u\|_{\frac{8-p}{3}}+\||\mathcal{D}u|^{p-2}\nabla_*\mathcal{D}u\|_{\frac{8-p}{3}}\le c\|f\|+c\|f\|^{\frac{2}{4-p}} \tag{2.8}$$

and

$$\|\nabla\pi\|_{\overline{s}}\le c\|f\|+c\|f\|^{\frac{2p-2}{4-p}}. \tag{2.9}$$

Theorem 2.2 *Assume that $p\in[2,4]$, $f\in L^2(\Omega)$. Let (u,π) be the weak solution to problem* (1.1) *and* (2.1). *Then all the regularity results of Theorem* 2.1 *still hold. Moreover, all the estimates still hold, provided that one replaces $\|f\|$ by $\|f\|+\|\mathcal{D}u\|_p^2$.*

Our aim is to improve the previous regularity results for both systems (1.1) and (2.1). The main step is to prove Theorem 2.3 below for the solution to system (2.1), by treating the convective term as a “right-hand side”.

Set

$$\frac{1}{T(q)} = \frac{p-2}{\beta(q)} + \frac{1}{2}, \tag{2.10}$$

where

$$\frac{1}{\beta(q)} = \frac{5(p-2)}{6(p-1)q} + \frac{1}{6(p-1)}. \tag{2.11}$$

Theorem 2.3 *Assume that $p \in [2,4]$, $f \in L^2(\Omega)$. Let (u,π) be the weak solution to problem* (1.2) *and* (2.1)*. Then in addition to* (2.7)*, one gets*

$$D^2u,\ \nabla_*\pi,\ |\mathcal{D}u|^{p-2}\nabla_*\mathcal{D}u \in L^{\frac{p+4}{p+1}}(\Omega), \quad \partial_3\pi \in L^{\overline{T}}(\Omega),$$

where

$$\overline{T} = \min\Big\{T(p+4), \frac{p+4}{p+1}\Big\} = T(p+4).$$

Moreover, for $p \in [2,4)$, there hold

$$\|\nabla_*\pi\|_{\frac{p+4}{p+1}} + \|D^2u\|_{\frac{p+4}{p+1}} + \||\mathcal{D}u|^{p-2}\nabla_*\mathcal{D}u\|_{\frac{p+4}{p+1}} \le c\|f\| + c\|f\|^{\frac{4p-2}{4+p}} \tag{2.12}$$

and

$$\|\nabla\pi\|_{\overline{T}} \le c\|f\|(1+\mathcal{A}_{p+4}^{p-2}) + c\mathcal{K}_{p+4}. \tag{2.13}$$

Theorem 2.4 *Assume that $p \in [2,4]$, $f \in L^2(\Omega)$. Let (u,π) be the weak solution to problem* (1.1) *and* (2.1)*. Then all the regularity results of Theorem* 2.3 *still hold.*

Remark 2.1 We note that $\beta(q) > q$, since $q < p+4$. We set $\overline{T} = \min\{T(p+4), r(q)\} = T(p+4)$. We need to note that $r(q) > T(q)$ is equivalent to $q \ge 7-2p$.

3 Proof of Theorem 2.3

We recall some preliminary results, which play a key role in the proof of the theorem, and introduce some further notations.

In order to make our way of reasoning more clear, it is convenient to recall the main steps and results of [3], without any claim of completeness. The first step is the following basic result (see [3, Theorem 3.1]).

Theorem 3.1 *Assume that $p \ge 2$, $f \in L^2(\Omega)$. Let (u,π) be the weak solution to problem* (1.2) *and* (2.1)*. Then, in addition to* (2.7)*, one has D^2u, $|\mathcal{D}u|^{p-2}\nabla_*\mathcal{D}u$ and $\nabla_*\pi$, which belong to $L^{p'}(\Omega)$, satisfy the following estimate:*

$$\|\nabla_*\pi\|_{p'} + \|D^2u\|_{p'} + \||\mathcal{D}u|^{p-2}\nabla_*\mathcal{D}u\|_{p'} \le \mathcal{K}_p \tag{3.1}$$

with

$$\mathcal{K}_p \le c\|f\|(1+\|\mathcal{D}u\|_p^{\frac{p-2}{2}}).$$

This theorem is the first step for a bootstrap argument, which leads to estimates of Theorem 2.1. The second tool is an "intermediate" regularity result, which gives higher regularity results in the extra hypothesis of higher integrability of $\mathcal{D}u$ (see [3, Theorem 3.2]).

Theorem 3.2 *Assume that $p \in [2,4]$. Let the assumptions of Theorem 3.1 be satisfied. In addition,*

$$\mathcal{D}u \in L^q(\Omega) \quad \text{for some } q \in [p,6]. \tag{3.2}$$

Then, besides estimate (2.7), one has that D^2u, $|\mathcal{D}u|^{p-2}\nabla_*\mathcal{D}u$ and $\nabla_*\pi$ belong to $L^r(\Omega)$ with

$$\frac{1}{r(q)} = \frac{p-2}{2q} + \frac{1}{2}, \tag{3.3}$$

and more precisely

$$\|\nabla_*\pi\|_{r(q)} + \|D^2u\|_{r(q)} + \||\mathcal{D}u|^{p-2}\nabla_*\mathcal{D}u\|_{r(q)} \le \mathcal{K}_q \tag{3.4}$$

with

$$\mathcal{K}_q \le c\|f\|(1 + \|\mathcal{D}u\|_q^{\frac{p-2}{2}}). \tag{3.5}$$

Remark 3.1 The previous theorem is proved in [3] on slightly different assumptions on p and q, i.e., $p \in [2,3]$ and $q \in [3,6]$. However, as the author's remarks, the result still holds on our assumptions on the ranges of p and q, without any change in the proof. Further, the assumption $p > 2$ implies that r is a nondecreasing function of q, if $2 \le p \le 4$ and $p \le q \le 6$, and then $\frac{4}{3} \le r \le 2$. The lack of dependence of the constant c on p, q, r follows from this fact, since in our hypotheses for these parameters, they are bounded from above.

In this paper, we also appeal to a bootstrap argument and the anisotropic Sobolev embeddings theorem (see [19]), furnishing a regularity result corresponding to a stronger integrability hypothesis on $\mathcal{D}u$.

Proposition 3.1 (see [19]) *Let Ω be as above, $1 \le p_i < \infty$ be real numbers with a harmonic mean*

$$\frac{1}{\overline{p}} = \frac{1}{3}\sum_{i=1}^{3}\frac{1}{p_i}.$$

Assume that $\partial_i v \in L^{p_i}$ for $i = 1,2,3$. Let $v \in W^{1,1}(\Omega)$. Then $v \in L^{\overline{p}^*}$, and

$$\|v\|_{\overline{p}^*} \le c\prod_{i=1}^{3}\|\partial_i v\|_{p_i}^{\frac{1}{3}} + c\|v\|_p, \tag{3.6}$$

where $\overline{p}^*$ is the Sobolev conjugate of $\overline{p}$, i.e., $\frac{1}{\overline{p}^*} = \frac{1}{\overline{p}} - \frac{1}{3}$.

Obviously, we may replace $\|v\|_p$ by any other L^s norm, $s \ge 1$.

Lemma 3.1 *Assume that the hypotheses in Theorem 3.2 hold. Then*

$$\nabla u \in L^{\beta(q)}.$$

Moreover

$$\|\nabla u\|_{\beta(q)} \le \mathcal{A}_q \doteq c\|f\|^{\frac{1}{p-1}}(1 + \|\mathcal{D}u\|_q^{\frac{5(p-2)}{6(p-1)}}) + c\|\nabla u\|_p. \tag{3.7}$$

Proof We note that

$$|\partial_{x_i}|\mathcal{D}u|^{p-1}| \le (p-1)|\mathcal{D}u|^{p-2}|\partial_{x_i}\mathcal{D}u|. \tag{3.8}$$

For $i = 1, 2$, by using (3.4) and (3.8), we have

$$\|\nabla^*|\mathcal{D}u|^{p-1}\|_r \le \mathcal{K}_q. \tag{3.9}$$

On the other hand, by applying Hölder's inequality to (3.8), one gets

$$\|\partial_{x_3}|\mathcal{D}u|^{p-1}\|_l \le c\|\mathcal{D}u\|_q^{p-2}\|\partial_{x_3}\mathcal{D}u\|_r,$$

where $\frac{1}{l(q)} = \frac{p-2}{q} + \frac{1}{r(q)}$.

By appealing to (3.4), we have

$$\|\partial_{x_3}|\mathcal{D}u|^{p-1}\|_l \le c\|\mathcal{D}u\|_q^{p-2}\mathcal{K}_q. \tag{3.10}$$

Define $\alpha = \alpha(q)$ by

$$\frac{1}{\alpha(q)} = \frac{1}{3}\Big(\frac{2}{r(q)} + \frac{1}{l(q)}\Big) - \frac{1}{3}. \tag{3.11}$$

Note that $\frac{1}{\alpha(q)} = \frac{1}{r(q)} + \frac{p-2}{3q} - \frac{1}{3}$.

In addition, recalling (2.11), one gets

$$\beta(q) = (p-1)\alpha(q).$$

From (3.9), (3.11) and Proposition 3.1, it follows that

$$\||\nabla u|^{p-1}\|_\alpha \le c\mathcal{K}_q\|\mathcal{D}u\|_q^{\frac{p-2}{3}} + c\|\nabla u\|_p^{p-1}. \tag{3.12}$$

Our result follows from (3.5) and (3.12).

Proof of Theorem 2.3 For the exponent $p = 2$, the result follows directly from Theorem 3.1. Hence, we assume $p \in (2, 4]$. From Lemma 3.1, we see that $u \in W^{1,q}(\Omega)$ implies $u \in W^{1,\beta(q)}(\Omega)$, where $\beta(q)$ is given above. Moreover, by (3.7), we have

$$\|u\|_{1,\beta(q)} \le c\|f\|^{\frac{1}{p-1}} + c\|\nabla u\|_p + c\|f\|^{\frac{1}{p-1}}\|u\|_{1,q}^{\frac{5(p-2)}{6(p-1)}}. \tag{3.13}$$

Define the recurse sequence $\{q_m\}$ as

$$\begin{cases} q_1 = p, \\ q_{m+1} = \beta(q_m) = \frac{6(p-1)q_m}{5(p-2)+q_m}. \end{cases}$$

Clearly, the sequence $\{q_m\}$ monotonically converges to the limit $p+4$, which shows that $\mathcal{D}u \in L^{\overline{q}}(\Omega)$ for any $\overline{q} < p+4$. Moreover, from (3.13),

$$\|u\|_{1,q_{m+1}} \le c\|f\|^{\frac{1}{p-1}} + c\|f\|^{\frac{1}{p-1}}\|u\|_{1,q_m}^{\frac{5(p-2)}{6(p-1)}}, \tag{3.14}$$

where we have used estimate (2.6). Taking into account that if $p \in (2, 4]$, then $0 < \xi = \frac{5(p-2)}{6(p-1)} < 1$, and following the arguments used in [3], we can easily verify that $\|u\|_{1,q_m}$ is uniformly bounded, at least for large values of m, and one gets

$$\|u\|_{1,q_m} \le c\|f\|^{\frac{1}{p-1}} + c\|f\|^{\frac{6}{p+4}}. \tag{3.15}$$

Remark 3.2 For readers' convenience, we reproduce the proof here. Set $b = \|f\|^{\frac{1}{p-1}}$ and $\xi = \frac{5(p-2)}{6(p-1)}$. Moreover, set $a_m = \|u\|_{1,q_m}$, $b_1 = a_1$ and $b_{m+1} = cb + cbb_m^{\xi}$. From (3.14), $a_m \le b_m$ for each m. Denote by λ the unique solution to the equation $\lambda = cb + cb\lambda^{\xi}$. If $b_1 < \lambda$, then b_m is an increasing sequence and converges to the fixed point λ, and hence $a_m < \lambda$ for any m. If $b_1 > \lambda$, then the sequence b_m decreases and converges to the value λ, and hence $a_m < 2\lambda$ for large values of m. On the other hand, if $\lambda \le 1$, then $\lambda = cb + cb\lambda^{\xi} \le 2cb$. If $\lambda > 1$, then $\lambda = cb + cb\lambda^{\xi} \le 2cb\lambda^{\xi}$, which gives $\lambda < (2cb)^{\frac{1}{1-\xi}}$. This shows that $\lambda \le 2cb + (2cb)^{\frac{1}{1-\xi}}$. Therefore, one obtains (3.15) at least for large values of m.

Hence, the limit exponent $\overline{q} = p + 4$ can be actually reached. Recalling (2.6), we get

$$\|u\|_{1,p+4} \le c\|f\|^{\frac{1}{p-1}} + c\|f\|^{\frac{6}{p+4}}. \tag{3.16}$$

Applying once again Theorem 3.2 with $\mathcal{D}u \in L^{p+4}$, we also get $\nabla_* \pi, |\mathcal{D}u|^{p-2}\nabla_* \mathcal{D}u$, $D^2 u$ which belong to $L^{r(\overline{q})}$ with $r(\overline{q}) = \frac{p+4}{p+1}$. Using estimate (3.16), we arrive at estimate (2.12). Finally, $\partial_3 \pi \in L^{\overline{T}}$ with $\overline{T} = \min\{T(p+4), r(p+4)\} = T(p+4)$, follows from Lemma 3.3 in [3]. Using (3.5) and (3.16), we can easily get (2.13). By (3.16), $\mathcal{K}_{p+4}$ and $\mathcal{A}_{p+4}$ are bounded in terms of $\|f\|$. The proof is then completed.

Remark 3.3 The proof of Theorem 2.4 follows step by step the proof of the corresponding Theorem in [7], so here we abbreviate it.

Acknowledgement The authors would like to thank the anonymous referees for their helpful comments and advice.

References

[1] Astarita, G. and Marucci, G., Principles of Non-Newtonian Fluid Mechanics, McGraw-Hill, London, 1974.

[2] Beirão da Veiga, H., On the regularity of flows with Ladyzhenskaya shear dependent viscosity and slip and non-slip boundary conditions, *Comm. Pure Appl. Math.,* **58**, 2005, 552–577.

[3] Beirão da Veiga, H., Navier-stokes equations with shear thickening viscosity. Regularity up to the boundary, *J. Math. Fluid Mech.,* **11**, 2009, 233–257. DOI: 10.1007/s00021-008-0257-2

[4] Beirão da Veiga, H., Navier-stokes equations with shear thinning viscosity. Regularity up to the boundary, *J. Math. Fluid. Mech.,* **11**, 2009, 258–273. DOI: 10.1007/s00021-008-0258-1

[5] Berselli, L. C., Iliescu, T. and Layton, W. J., Mathematics of Large Eddy Simulation of Turbulent Flows, Scientific Computation, Springer-Verlag, Berlin, 2006.

[6] Cottet, G. H., Jiroveanu, D. and Michaux, B., Vorticity dynamics and turbulence models of large eddy simulations, *M2AN Math. Model Numer. Anal.,* **37**, 2003, 187–207.

[7] Crispo, F., On the regularity of shear thickening viscous fluids, *Chin. Ann. Math.,* **30B**(3), 2009, 273–280.

[8] Galdi, G. P., Mathematical problems in classical and non-Newtonian fluid mechanics, Hemodynamical Flows: Modeling, Analysis and Simulation (Oberwolfach Seminars), G. P. Galdi, A. M. Robertson, R. Rannacher, et al. (eds.), Birkhaeuser-Verlag, Basel, 2007.

[9] Germano, M., Piomelli, U., Moin, P., et al., A dynamic subgrid-scale eddy viscosity model, *Phys. Fluids,* **A**(3), 1991, 1760–1765.

[10] Hughes, T. J. R., Mazzei, L. and Oberai, A. A., The multiscale formulation of large eddy simulation: decay of homogeneous isotropic turbulence, *Phys. Fluids,* **13**, 2001, 505–512.

[11] Hughes, T. J. R., Wells, G. N. and Wray, A. A., Energy transfers and spectral eddy viscosity in large-eddy simulations of homogeneous isotropic turbulence: comparison of dynamic Smagorinsky and multiscale models over a range of discretizations, *Phys. Fluids,* **16**(11), 2004, 4044–4052.

[12] Lesieur, M., Metais, O. and Comte, P., Large-eddy Simulations of Turbulence, Cambridge University Press, New York, 2005.

[13] Lions, J. L., Quelques Méthodes de Résolution des Problèmes aux Limites Non-linéaires, Dunod, Paris, 1969.

[14] Málek, J., Neč, J., Rokyta, M, et al., Weak and measure-valued solutions to evolutionary PDEs, *Applied Mathematics and Mathematical Computation,* **13**, Chapman and Hall, London, 1996.

[15] Málek, J., Nečas, J. and Růžička, M., On the non-Newtonian incompressible fluids, *Math. Models Methods Appl. Sci.*, 1993, 35–63.

[16] Parés, C., Existence, uniqueness and regularity of solutions of the equations of a turbulence model for incompressible fluids, *Appl. Analysis,* **43**, 1992, 245–296.

[17] Rajagopal, K. R., Mechanics of non-Newtonian fluids, Recent Developments in Theoretical Fluid Mechanics, Galdi, G. P. and Nečas, J. (eds.), Pitman Research Notes in Mathematics Series, **291**, Longman, Harlow, 1993, 129–162.

[18] Smagorinsky, J. S., General circulation experiments with the primitive equations. I. The basic experiment, *Mon. Weather Rev.,* **91**, 1963, 99–164.

[19] Troisi, M., Teoremi di inclusione per spazi di Sobolev non isotropi, *Ricerche di Matematica,* **18**, 1969, 3–24.