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Slowly Increasing Cohomology for Discrete Metric Spaces with Polynomial Growth*

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Abstract The authors introduce a kind of slowly increasing cohomology $\mathrm{HS}^*(X)$ for a discrete metric space X with polynomial growth, and construct a character map from the slowly increasing cohomology $\mathrm{HS}^*(X)$ into $\mathrm{HC}^*_{\mathrm{cont}}(S(X))$, the continuous cyclic cohomology of the smooth subalgebra $S(X)$ of the uniform Roe algebra $B^*(X)$. As an application, it is shown that the fundamental cocycle, associated with a uniformly contractible complete Riemannian manifold M with polynomial volume growth and polynomial contractibility radius growth, is slowly increasing.

Keywords Slowly increasing cohomology, Polynomial growth, Uniform Roe algebra
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1 Introduction

Let M be a noncompact complete Riemannian manifold. The indices $\mathrm{Ind}(D)$ of a generalized Dirac operator D on M which lies in the K-theory of the uniform Roe algebra $B^*(M)$, i.e., the completed algebra of locally traceable operators with finite propagation, carry the topological and geometrical information of the manifold M. One of the approaches to detect the indices of D is Connes' pairing theory between cyclic cohomology and the K-theory for the appropriate smooth subalgebra of the uniform Roe algebra $B^*(M)$. This approach was proved to be successful in studying various problems in topology and geometry such as the Novikov conjecture, the positive scalar curvature problem and the zero spectrum conjecture (see [1–2]).The notion of coarse cohomology was introduced by Roe in [3] for the purpose of the index theory on the noncompact manifolds. An important feature of the coarse cohomology is that it can be considered as the cyclic cohomology of the precomplete uniform Roe algebra $B(M)$, i.e., the algebra of locally traceable operators with finite propagation. Extending cyclic cocycles of the precomplete uniform Roe algebra to the appropriate smooth subalgebra of the uniform Roe algebra is crucial for the Connes approach and its applications. This paper focuses on studying which cyclic cocycles from the coarse cohomology can be extended to the appropriate smooth algebra of the uniform Roe algebra. The Schwartz-type space $S(X)$ is an explicit smooth subalgebra of the uniform Roe algebra $B^*(X)$ for a discrete metric space X with polynomial growth

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(see [2, 4]). We will introduce a kind of the slowly increasing cohomology $\mathrm{HS}^*(X)$, and show that this cohomology can be considered as the continuous cyclic cohomology of the smooth subalgebra $S(X)$. An explicit character map from the slowly increasing cohomology $\mathrm{HS}^*(X)$ into $\mathrm{HC}^*_{\mathrm{cont}}(S(X))$, the continuous cyclic cohomology of the smooth subalgebra $S(X)$ of the uniform Roe algebra $B^*(X)$, will be constructed.

Recall that a discrete metric space X is said to have polynomial growth if there exist positive constants m and p, such that the number of points in the ball $B(x,r)$ is less than $m(1+r)^p$ for all $x \in X$. Throughout this paper, we always assume that X is a discrete metric space with polynomial growth.

This paper is organized as follows. In Section 2, we introduce the definition of slowly increasing cohomology. In Section 3, we show that this kind of cohomology is a coarse isometric invariant. In Section 4, a character map from the slowly increasing cohomology $\mathrm{HS}^*(X)$ into $\mathrm{HC}^*_{\mathrm{cont}}(S(X))$ is established. In Section 5, we consider the relationship between the slowly increasing cohomology and the coarse cohomology. In particular, we get a sufficient condition for a coarse cocycle to be slowly increasing. This paper ends with Section 6, where an application is given. We recapture that $\mathrm{Ind}(D)$ is not zero in the K-theory of the uniform Roe algebra $B^*(M)$ for a uniformly contractible complete Riemannian manifold M with polynomial volume growth and polynomial contractibility radius growth, by proving that the fundamental cocycle associated with M is slowly increasing.

2 Slowly Increasing Cohomology

In this section, we first give the definition of slowly increasing cohomology for a discrete metric space X with polynomial growth. X^{n+1} denotes the Cartesian product of $n+1$ copies of X, and $\Delta_{n+1} \subseteq X^{n+1}$ denotes the multi-diagonal $\{(x,x,\cdots,x) : x \in X\}$. For $\overline{x} = (x_0, x_1, \cdots, x_n)$ and $\overline{y} = (y_0.y_1, \cdots, y_n)$ in X^{n+1}, we define the distance

$$d(\overline{x}, \overline{y}) = \max_i d(x_i, y_i).$$

$P(\Delta_{n+1}, k)$ denotes the set $\{\overline{x} \in X^{n+1} : d(\overline{x}, \Delta_{n+1}) \leq k\}$ for $k \geq 0$.

We shall define a kind of slowly increasing cohomology with coefficients in $\mathbb{R}$. Given a function $\phi : X^{n+1} \to \mathbb{R}$, we say that ϕ is slowly increasing, if $\{\phi_k\}$ is a slowly increasing sequence, which means that there exist positive constants c and l, such that $\phi_k \leq c(1+k)^l$ for all positive integer k, where

$$\phi_k = \sum_{\overline{x} \in P(\Delta_{n+1}, k)} |\phi(\overline{x})|.$$

For $n = 0, 1, 2, \cdots$, we define

$$\mathrm{CS}^n(X) = \{\phi : \Gamma^{n+1} \to \mathbb{R} \mid \phi \text{ is slowly increasing}\}$$

and

$$(\partial\phi)(x_0, x_1, \cdots, x_{n+1}) = \sum_{i=0}^{n+1} (-1)^i \phi(x_0, x_1, \cdots, \widehat{x}_i, \cdots, x_{n+1})$$

for any $\phi \in \mathrm{CS}^n(X)$. One can easily check that $\mathrm{CS}^n(X)$ is a linear space. Since

$$\begin{aligned}
&\sum_{(x_0,\cdots,x_{n+1})\in P(\Delta_{n+2},k)} |\partial\phi(x_0,x_1,\cdots,x_{n+1})| \\
&\le \sum_{i=0}^{n+1} \sum_{(x_0,\cdots,\widehat{x}_i,\cdots,x_{n+1})\in P(\Delta_{n+1},k)} \sum_{\{x_i|d(x_i,x_j)\le 2k,\forall j\neq i\}} |\phi(x_0,x_1,\cdots,\widehat{x}_i,\cdots,x_{n+1})| \\
&\le \sum_{i=0}^{n+1} \phi_k m(1+2k)^p \\
&\le \sum_{i=0}^{n+1} c(1+k)^l m(1+2k)^p \\
&\le (n+2)cm(1+k)^{l+2p},
\end{aligned}$$

we have that $\{(\partial\phi)_k\}$ is a slowly increasing sequence. So, ∂ maps $\mathrm{CS}^n(X)$ into $\mathrm{CS}^{n+1}(X)$. Note that ∂ is the usual coboundary of Alexander-Spanier cohomology, $\partial^2 = 0$. Thus, $(\mathrm{CS}^*(X), \partial)$ is a subcomplex of the acyclic Alexander-Spanier complex. Therefore, we have a cochain complex $(\mathrm{CS}^*(X), \partial)$.

Definition 2.1 *The cohomology of the cochain complex* $(\mathrm{CS}^*(X), \partial)$ *is called the slowly increasing cohomology, and is denoted by* $\mathrm{HS}^*(X)$.

In the following, we give an equivalent description of a slowly increasing cochain. We also show that any slowly increasing cohomology class can be represented by a totally antisymmetric cocycle. These two results will be frequently used later.

Theorem 2.1 *Let* ϕ *be a function from* X^{n+1} *into* $\mathbb{R}$. *Then* $\phi \in \mathrm{CS}^n(X)$ *if and only if there exists an* $s > 0$ *such that*

$$\sum_{(x_0,\cdots,x_n)\in X^{n+1}} |\phi(x_0,\cdots,x_n)|(1+d(x_0,x_1))^{-s}\cdots(1+d(x_n,x_0))^{-s} < \infty.$$

Proof If $\phi \in \mathrm{CS}^n(X)$, then, by the definition of $\mathrm{CS}^n(X)$, there exist positive constants c and l, such that $\phi_k \le c(1+k)^l$. Choosing $s > 2+l$, one has

$$\begin{aligned}
&\sum_{(x_0,\cdots,x_n)\in X^{n+1}} |\phi(x_0,\cdots,x_n)|(1+d(x_0,x_1))^{-s}\cdots(1+d(x_n,x_0))^{-s} \\
&= \sum_{k=0}^{\infty} \sum_{(x_0,\cdots,x_n)\in P(\Delta_{n+1},k)\setminus P(\Delta_{n+1},k-1)} |\phi(x_0,\cdots,x_n)|(1+d(x_0,x_1))^{-s}\cdots(1+d(x_n,x_0))^{-s} \\
&\le \sum_{k=0}^{\infty} \sum_{(x_0,\cdots,x_n)\in P(\Delta_{n+1},k)\setminus P(\Delta_{n+1},k-1)} |\phi(x_0,\cdots,x_n)|k^{-s} \\
&\le \sum_{k=0}^{\infty} \sum_{(x_0,\cdots,x_n)\in P(\Delta_{n+1},k)} |\phi(x_0,\cdots,x_n)|k^{-s} \\
&= \sum_{k=0}^{\infty} \phi_k k^{-s} \le \sum_{k=0}^{\infty} c(1+k)^l k^{-s} \le 2^s c \sum_{k=0}^{\infty} (1+k)^{l-s} \\
&< \infty.
\end{aligned}$$

For the converse, if there exists an $s > 0$ such that

$$\sum_{(x_0,\cdots,x_n)\in X^{n+1}} |\phi(x_0,\cdots,x_n)|(1+d(x_0,x_1))^{-s}\cdots(1+d(x_n,x_0))^{-s} = c < \infty,$$

then

$$\begin{aligned}
\phi_k &= \sum_{(x_0,\cdots,x_n)\in P(\Delta_{n+1},k)} |\phi(x_0,x_1,\cdots,x_n)| \\
&\le \sum_{(x_0,\cdots,x_n)\in P(\Delta_{n+1},k)} |\phi(x_0,x_1,\cdots,x_n)|(1+d(x_0,x_1))^{-s}\cdots(1+d(x_n,x_0))^{-s} \\
&\quad \cdot \underbrace{(1+2k)^s\cdots(1+2k)^s}_{n+1} \\
&\le c(1+k)^{2(n+1)s}.
\end{aligned}$$

It follows that $\{\phi_k\}$ is a slowly increasing sequence, i.e., $\phi \in \mathrm{CS}^n(X)$.

Define

$$\mathrm{CS}^n_\alpha(X) = \{\phi \in \mathrm{CS}^n(X) \mid \phi(x_{\sigma(0)},\cdots,x_{\sigma(n)}) = (-1)^\sigma\phi(x_0,\cdots,x_n),\ \forall\sigma\in S_{n+1}\},$$

where S_{n+1} denotes the $n+1$ order permutation group. It is easy to see that $(\mathrm{CS}^*_\alpha(X), \partial)$ is a subcomplex of $(\mathrm{CS}^*(X), \partial)$. Moreover, we have the following theorem.

Theorem 2.2 *The inclusion*

$$\mathrm{i}:\ \mathrm{CS}^*_\alpha(X) \longrightarrow \mathrm{CS}^*(X)$$

induces an isomorphism on cohomology. The operation of the complete antisymmetrization

$$\mathrm{A}:\ \mathrm{CS}^*(X) \longrightarrow \mathrm{CS}^*_\alpha(X)$$

defined by

$$(\mathrm{A}\phi)(x_0,\cdots,x_n) = \frac{1}{(n+1)!}\sum_{\alpha\in S_{n+1}}(-1)^\sigma\phi(x_0,\cdots,x_n)$$

is a cochain map and induces the inverse isomorphism on cohomology.

Proof It is easy to see that A is a cochain map and $\mathrm{A}\circ\mathrm{i} = \mathrm{Id}_X$. In the following, we will construct homotopy operators D_n by induction and show that $\mathrm{i}\circ\mathrm{A}$ and Id_X are homotopic, i.e., $\mathrm{i}\circ\mathrm{A}\sim\mathrm{Id}_X$. Let $D_0 = 0$, and for $n \ge 1$

$$(D_n\phi)(x_0,\cdots,x_n) = (\mathrm{Id}_X\phi_{x_0})(x_0,\cdots,x_n) - (\mathrm{A}\phi_{x_0})(x_0,\cdots,x_n) - (\partial_{n-1}D_{n-1}\phi_{x_0})(x_0,\cdots,x_n),$$

where $\phi_{x_0}(x_1,\cdots,x_n) = \phi(x_0,x_1,\cdots,x_n)$. It is straightforward to check that D_n is a linear map from $\mathrm{CS}^{n+1}(X)$ into $\mathrm{CS}^n(X)$, and

$$D_n\partial_n + \partial_{n-1}D_{n-1} = \mathrm{Id}_X - \mathrm{i}\circ\mathrm{A}.$$

Thus $\mathrm{i}\circ\mathrm{A}\sim\mathrm{Id}_X$.

A direct computation shows that $\mathrm{HS}^{2n+1}(X) = 0$, $\mathrm{HS}^{2n}(X) = \mathbb{R}$, $\forall n \ge 0$ for the space $X = \{pt\}$ with a single point, and hence for any compact discrete space X by its coarse isometric invariant (see Section 3). On the contrary, one has that $\mathrm{HS}^0(X) = 0$ for any noncompact discrete metric space X. In the end of this section, we give an example to see that $\mathrm{HS}^1(X) \neq 0$.

Example 2.1 Let $X = \{\pm n^2 : n \in \mathbb{Z}\}$ with the metric which inherits as a subspace of $\mathbb{Z}$. Then $\mathrm{HS}^1(X) \neq 0$.

Proof Let

$$\phi_g(x_0, x_1) = g(x_1) - g(x_0), \quad \forall g \in l^\infty(X).$$

Then ϕ_g is a slowly increasing cocycle, since g is bounded and X is of polynomial growth. Thus

$$\begin{aligned}
[\phi_g] = 0 &\Leftrightarrow \exists f \in l^1(X), \text{ such that } \partial f = \phi_g \\
&\Leftrightarrow \exists f \in l^1(X), \text{ such that } f(x_1) + g(x_0) = g(x_1) + f(x_0)), \quad \forall x_0, x_1 \in X \\
&\Rightarrow \exists f \in l^1(X), \text{ such that } f(x) + g(0) = g(x) + f(0), \quad \forall x \in X.
\end{aligned}$$

Now let $g \in l^\infty(X) \backslash l^1(X)$, such that $g(0) = 0$, $g(-n^2) = 0$. We claim that $[\phi_g] \neq 0$. Otherwise, $\exists f \in l^1(X)$ such that $f(n^2) = g(n^2) + f(0)$, $f(-n^2) = f(0)$. Then $f \in l^1(X)$ implies $f(0) = 0$. Hence $f(n^2) = g(n^2)$. Let

$$f_1(x) = \begin{cases} f(x), & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that $f_1(x) \in l^1(X)$ while $f_1(x) = g(x) \notin l^1(X)$, a contradiction.

3 Coarse Isometric Invariant

We first recall some standard definitions coming from coarse geometry (see [5]).

Definition 3.1 *Let X and Y be discrete metric spaces.*

A map $f : X \to Y$ is said to be a coarse isometry, if there exist positive constants a, b and c, such that

$$a^{-1}(x, x') - b \leq d(f(x), f(x')) \leq ad(x, x') + b, \quad \forall x, x' \in X$$

and

$$d(y, f(X)) \leq c, \quad \forall y \in Y.$$

Two coarse maps $f, f' : X \to Y$ are said to be close, if there exists a $c > 0$, such that

$$d(f(x), f'(x)) \leq c, \quad \forall x \in X.$$

Two metric spaces X and Y are said to be coarse isometric, if there exists a coarse isometry between X and Y.

Let X and Y be discrete metric spaces with polynomial growth, and let f be a coarse isometry from X into Y. For any $\phi \in \mathrm{CS}^n(Y)$, define $f^*(\phi)$ by

$$f^*(\phi)(x_0, x_1, \cdots, x_n) = \phi(f(x_0), f(x_1), \cdots, f(x_n)).$$

Then $f^*(\phi) \in \mathrm{CS}^n(X)$. In fact, since f is a coarse isometry from X into Y, there exist constants $a > 0$ and $b > 0$, such that

$$d(f(x), f(x')) \leq ad(x, x') + b$$

for any $x, x' \in X$. Thus, for any $s > 0$, there exists a $c > 0$, such that

$$(1 + d(x, x'))^{-s} \le c(1 + d(f(x), f(x')))^{-s}.$$

Hence,

$$\begin{aligned}
&\sum_{(x_0,\cdots,x_n)\in X^{n+1}} |f^*(\phi)(x_0,\cdots,x_n)|(1+d(x_0,x_1))^{-s}\cdots(1+d(x_n,x_0))^{-s} \\
&= \sum_{(x_0,,\cdots,x_n)\in X^{n+1}} |\phi(f(x_0),\cdots,f(x_n))|(1+d(x_0,x_1))^{-s}\cdots(1+d(x_n,x_0))^{-s} \\
&\le c^{n+1} \sum_{(x_0,\cdots,x_n)\in X^{n+1}} |\phi(f(x_0),\cdots,f(x_n))|(1+d(f(x_0),f(x_1)))^{-s}\cdots(1+d(f(x_n),f(x_0)))^{-s} \\
&\le c^{n+1} \sum_{(y_0,\cdots,y_n)\in Y^{n+1}} |\phi(y_0,\cdots,y_n)|(1+d(y_0,y_1))^{-s}\cdots(1+d(y_n,y_0))^{-s} \\
&< \infty.
\end{aligned}$$

It follows that $f^*(\phi) \in \mathrm{CS}^n(X)$.

It is straightforward to check that the diagram

$$\begin{array}{ccc}
\mathrm{CS}^n(Y) & \xrightarrow{f^*} & \mathrm{CS}^n(X) \\
\partial\downarrow & & \downarrow\partial \\
\mathrm{CS}^{n+1}(Y) & \xrightarrow{f^*} & \mathrm{CS}^{n+1}(X)
\end{array}$$

is commutative. Hence, f^* is a cochain map from $\mathrm{CS}^*(Y)$ into $\mathrm{CS}^*(X)$. Moreover, f^* induces a homomorphism from $\mathrm{HS}^*(Y)$ into $\mathrm{HS}^*(X)$.

Theorem 3.1 *Let X and Y be discrete metric spaces with polynomial growth, and let f and g be coarse isometries from X into Y. If f and g are close, then $f^* = g^*$.*

Proof To prove $f^* = g^*$, it suffices to construct cochain homotopy operators $D^n : \mathrm{CS}^{n+1}(Y) \longrightarrow \mathrm{CS}^n(X)$ such that

$$D^n\partial + \partial D^{n-1} = f^* - g^*.$$

Put

$$D_i^n\phi(x_0, x_1, \cdots, x_n) = \phi(g(x_0), g(x_1), \cdots, g(x_i), f(x_i), f(x_{i+1}), \cdots, f(x_n))$$

for any $\phi \in \mathrm{CS}^{n+1}(Y)$ and any $n \ge 0$. We claim that $D_i^n\phi \in \mathrm{CS}^n(X)$. Indeed, there exists a $c > 0$, such that

$$d(f(x), g(x)) \le c, \quad \forall x \in X,$$

since f and g are close; and for any $s > 0$, there exists a $c > 0$, such that

$$(1 + d(x, x'))^{-s} \le c(1 + d(f(x), f(x')))^{-s}$$

and

$$(1 + d(x, x'))^{-s} \le c(1 + d(g(x), g(x')))^{-s}$$

for all $x, x' \in X$, since f and g are coarse isometries. Therefore, one has

$$\begin{aligned}
&\sum_{(x_0,\cdots,x_n)\in X^{n+1}} |D_i^n\phi(x_0,\cdots,x_n)|(1+d(x_0,x_1))^{-s}\cdots(1+d(x_n,x_0))^{-s}\\
=&\sum_{(x_0,\cdots,x_n)\in X^{n+1}} |\phi(g(x_0),\cdots,g(x_i),f(x_i),f(x_{i+1}),\cdots,f(x_n))|\\
&\cdot(1+d(x_0,x_1))^{-s}\cdots(1+d(x_n,x_0))^{-s}\\
\le& c^n \sum_{(x_0,\cdots,x_n)\in X^{n+1}} |\phi(g(x_0),\cdots,g(x_i),f(x_i),f(x_{i+1}),\cdots,f(x_n))|\\
&\cdot(1+d(g(x_0),g(x_1)))^{-s}\cdots(1+d(g(x_{i-1}),g(x_i)))^{-s}(1+d(g(x_i),f(x_i)))^{-s}\\
&\cdot(1+d(f(x_i),f(x_{i+1})))^{-s}\cdots(1+d(f(x_{n-1}),f(x_n)))^{-s}(1+d(f(x_n),g(x_0)))^{-s}\\
&\cdot(1+d(g(x_i),f(x_i)))^{s}(1+d(f(x_n),g(x_0)))^{s}(1+d(x_n,x_0))^{-s}\\
\le& c^{n+1}(1+c)^{2s}\sum_{(y_0,\cdots,y_{n+1})\in Y^{n+2}} |\phi(y_0,y_1,\cdots,y_{n+1})|\\
&\cdot(1+d(y_0,y_1))^{-s}(1+d(y_1,y_2))^{-s}\cdots(1+d(y_{n+1},y_0))^{-s}\\
<&\infty.
\end{aligned}$$

It follows that $D_i^n\phi \in \mathrm{CS}^n(X)$ by Theorem 2.1. Let

$$D^n = \sum_{i=0}^{n}(-1)^i D_i^n.$$

It is straightforward to check that D^n are the desired cochain homotopy operators.

Theorem 3.2 *Let X and Y be discrete metric spaces with polynomial growth. If X and Y are coarse isometric, then* $\mathrm{HS}^*(X) \cong \mathrm{HS}^*(Y)$.

Proof Let f be a coarse isometry from X into Y. Then there exists another coarse isometry g from Y into X as a coarse inverse of f, that is, $g \circ f$ and $f \circ g$ are close to Id_X and Id_Y, respectively. It follows that $g^* \circ f^* = \mathrm{Id}_Y^*$ and $f^* \circ g^* = \mathrm{Id}_X^*$ by Theorem 3.1. Therefore, $\mathrm{HS}^*(X) \cong \mathrm{HS}^*(Y)$.

4 Character Map

This section mainly concerns constructing continuous cyclic cocycles on the smooth subalgebra $S(X)$ of the uniform Roe algebra $B^*(X)$ from the slowly increasing cohomology. We first review some basic definitions, results concerning uniform Roe algebras and their smooth subalgebras.

Let (X, d) be a discrete metric space, and $l^2(X)$ be the natural l^2-space of X. Given a function $k : X \times X \to \mathbb{C}$, k is said to be finitely propagated, if there is a constant $c_k > 0$, such that $k(x, y) = 0$ whenever $d(x, y) > c_k$. k is said to be bounded, if it defines a bounded operator on $l^2(X)$ by convolution, that is, $k : l^2(X) \to l^2(X)$ defined by

$$k * \xi(x) = \sum_{y\in X} k(x, y)\xi(y)$$

is a bounded operator.

Definition 4.1 (see [6]) *Let (X,d) be a discrete metric space. The precompleted uniform Roe algebra of X is defined to be*

$$B(X)=\{k:X\times X\to\mathbb{C}\mid k \text{ is bounded and finitely propagated}\}.$$

The norm closure of $B(X)$ in $l^2(X)$ is called the uniform Roe algebra, and is denoted by $B^*(X)$.

Definition 4.2 *A dense Fréchet subalgebra A^∞ of a C^* algebra A is said to be smooth, if A^∞ is closed in A under holomorphic functional calculus.*

Let $S(X)$ be the Fréchet space of functions k on $X\times X$ satisfying

$$\sup_{y\in X}\sum_{x\in X}|k(x,y)|^2(1+d(x,y))^{2s}<\infty$$

for all $s\geq 0$, where the seminorms are defined by

$$\|k\|_s=\Big(\sup_{y\in X}\sum_{x\in X}|k(x,y)|^2(1+d(x,y))^{2s}\Big)^{\frac{1}{2}},\quad s=0,1,2,\cdots.$$

It is clear that $B(X)$ is contained in $S(X)$. However, $S(X)$ is not always contained in $B^*(X)$. In [4], we showed that $S(X)\subseteq B^*(X)$ if and only if X has polynomial growth and in the case that $S(X)$ is a smooth subalgebra of $B^*(X)$.

It is the suitable position to recall some notations on the cyclic cohomology (see [7]). Let A be a locally convex algebra. $A^{\widehat{\otimes} n}$ denotes the n-fold topological projective tensor product of A. Let

$$C^n_{\text{cont}}(A)=\mathrm{Hom}_{\text{cont}}(A^{\widehat{\otimes}(n+1)},\mathbb{C})$$

be the space of continuous $(n+1)$-linear functionals on A. $\phi\in C^n_{\text{cont}}(A)$ is called an n-cochain. A cochain $\phi\in C^n_{\text{cont}}(A)$ is said to be cyclic if

$$\phi(a_n,a_0,\cdots,a_{n-1})=(-1)^n\phi(a_0,a_1,\cdots,a_n).$$

$C^n_{\text{cont},\lambda}(A)$ denotes the space of continuous cyclic cochains on A. Define $b:C^n_{\text{cont},\lambda}(A)\to C^{n+1}_{\text{cont},\lambda}(A)$ by

$$(b\phi)(a_0,\cdots,a_{n+1})=\sum_{i=0}^{n}(-1)^n\phi(a_0,\cdots,a_ia_{i+1},\cdots,a_{n+1})+(-1)^{n+1}\phi(a_{n+1}a_0,a_1,\cdots,a_n).$$

A direct computation shows that b is well-defined and $b^2=0$. The complex

$$C^0_{\text{cont},\lambda}(A)\xrightarrow{b}C^1_{\text{cont},\lambda}(A)\xrightarrow{b}C^2_{\text{cont},\lambda}(A)\xrightarrow{b}\cdots$$

is called the Connes (continuous) complex of A. The cohomology of this complex is called the continuous cyclic cohomology of A and will be denoted by $\mathrm{HC}^n_{\text{cont}}(A)$, $n=0,1,2,\cdots$.

Lemma 4.1 *For any positive integer n, define a linear map*

$$l:S(X)^{\otimes n}\to S(X^n)$$

by

$$l(k_1 \otimes k_2 \otimes \cdots \otimes k_n)((x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots, y_n)) = k_1(x_1, y_1)k_2(x_2, y_2) \cdots k_n(x_n, y_n).$$

Then l is continuous with respect to the projective topology, and furthermore l can be continuously extended to the n-fold projective tensor product $S(X)^{\widehat{\otimes} n}$ of $S(X)$.

Proof Without loss of generality, we assume that $n = 2$. Note that for any $k_1, k_2 \in S(X)$ and $s \geq 0$, one has

$$\begin{aligned}
\|l(k_1 \otimes k_2)\|_s^2 &= \sup_{(x_1,x_2)\in X^2} \sum_{(y_1,y_2)\in X^2} |l(k_1 \otimes k_2)((x_1, x_2), (y_1, y_2))|^2 (1 + d((x_1, x_2), (y_1, y_2))^{2s} \\
&= \sup_{(x_1,x_2)\in X^2} \sum_{(y_1,y_2)\in X^2} |k_1(x_1, y_1)k_2(x_2, y_2)|^2 (1 + \max\{d(x_1, y_1), d(x_2, y_2)\})^{2s} \\
&\leq \sup_{x_1\in X} \sum_{y_1\in X} |k_1(x_1, y_1)|^2 (1 + d(x_1, y_1))^{2s} \sup_{x_2\in X} \sum_{y_2\in X} |k_2(x_2, y_2)|^2 (1 + d(x_2, y_2))^{2s} \\
&= \|k_1\|_s^2 \|k_2\|_s^2.
\end{aligned}$$

Therefore, for any $k \in S(X) \otimes S(X)$ and any of its representations $k = \sum\limits_{i=0}^{m} k_{1i} \otimes k_{2i}$, we have

$$\|l(k)\|_s \leq \sum_{i=0}^{m} \|k_{1i} \otimes k_{2i}\|_s \leq \sum_{i=0}^{m} \|k_{1i}\|_s \|k_{2i}\|_s.$$

It follows that $\|l(k)\|_s \leq \|k\|_{s,s}$ by the definition of the projective seminorms, which implies that l is continuous.

Now we construct a character map from the slowly increasing cohomology of X to the continuous cyclic cohomology of $S(X)$.

Define

$$\chi(\phi)(k) = \sum_{(x_0,\cdots,x_n)\in X^{n+1}} l(k)((x_0, \cdots, x_{n-1}, x_n), (x_1, \cdots, x_n, x_0))\phi(x_0, x_1, \cdots, x_n)$$

for any $\phi \in \mathrm{CS}^n(X)$ and any $k \in S(X)^{\widehat{\otimes}(n+1)}$. Then $\chi(\phi)(k)$ is well-defined and $\chi(\phi)$ is a continuous linear functional on $S(X)^{\widehat{\otimes}(n+1)}$, i.e., $\chi(\phi) \in C^n_{\mathrm{cont}}(S(X))$. Indeed, there exists an $s > 0$, such that

$$c_\phi = |\phi(x_0, \cdots, x_n)|(1 + d(x_0, x_1))^{-s} \cdots (1 + d(x_n, x_0))^{-s} < \infty,$$

since $\phi \in \mathrm{CS}^n(X)$ is slowly increasing. Therefore

$$\begin{aligned}
|\chi(\phi)(k)| &\leq \sum_{(x_0,\cdots,x_n)\in X^{n+1}} |l(k)((x_0, \cdots, x_{n-1}, x_n), (x_1, \cdots, x_n, x_0))||\phi(x_0, x_1, \cdots, x_n)| \\
&= \sum_{(x_0,\cdots,x_n)\in X^{n+1}} |l(k)((x_0, \cdots, x_{n-1}, x_n), (x_1, \cdots, x_n, x_0))|(1 + d(x_0, x_1))^s \cdots \\
&\quad \cdot (1 + d(x_n, x_0))^s |\phi(x_0, \cdots, x_n)|(1 + d(x_0, x_1))^{-s} \cdots (1 + d(x_n, x_0))^{-s} \\
&\leq \|l(k)\|_{(n+1)s} \sum_{(x_0,\cdots,x_n)\in X^{n+1}} |\phi(x_0, \cdots, x_n)|(1 + d(x_0, x_1))^{-s} \cdots (1 + d(x_n, x_0))^{-s} \\
&\leq c_\phi \|k\|_{(n+1)s,(n+1)s},
\end{aligned}$$

which implies that $\chi(\phi)(k)$ is well-defined and $\chi(\phi)$ is continuous on $S(X)^{\widehat{\otimes}(n+1)}$. Furthermore, one can regard $\chi(\phi)$ as a continuous $(n+1)$-linear functional on $S(X)$, that is,

$$\chi(\phi)(k_0, k_1, \cdots, k_n) \sum_{(x_0,\cdots,x_n)\in X^{n+1}} k_0(x_0, x_1)k_1(x_1, x_2)\cdots k_n(x_n, x_0)\phi(x_0, x_1, \cdots, x_n).$$

Lemma 4.2 *χ defined as above maps $\mathrm{CS}^n_\alpha(X)$ into $C^n_{\mathrm{cont},\lambda}(S(X))$.*

Proof By the assumption $\phi \in \mathrm{CS}^n_\alpha(X)$, one has

$$\phi(x_1, x_2, \cdots, x_n, x_0) = (-1)^n\phi(x_0, x_1, \cdots, x_n).$$

Hence

$$\begin{aligned}
&\chi(\phi)(k_n, k_0, \cdots, k_{n-1})\\
&= \sum_{(x_0,\cdots,x_n)\in X^{n+1}} k_n(x_0, x_1)k_0(x_1, x_2)\cdots k_{n-1}(x_n, x_0)\phi(x_0, x_1, \cdots, x_n)\\
&= \sum_{(x_0,\cdots,x_n)\in X^{n+1}} k_0(x_1, x_2)k_1(x_2, x_3)\cdots k_{n-1}(x_n, x_0)k_n(x_0, x_1)\cdot(-1)^n\phi(x_1, x_2, \cdots, x_n, x_0)\\
&= (-1)^n\chi(\phi)(k_0, k_1, \cdots, k_n),
\end{aligned}$$

which implies that $\chi(\phi) \in C^n_{\mathrm{cont},\lambda}(S(X))$.

Theorem 4.1 *χ defined as above induces a homomorphism from the slowly increasing cohomology $\mathrm{HS}^*(X)$ into the continuous cyclic cohomology $\mathrm{HC}^*_{\mathrm{cont}}(S(X))$ of the smooth subalgebra $S(X)$.*

Proof By Lemma 4.2, it suffices to show that for any $\phi \in \mathrm{CS}^n_\alpha(X)$,

$$(\chi \circ \partial)(\phi) = (-b \circ \chi)(\phi),$$

where ∂ and b are coboundary operators over the cochain complexes $(\mathrm{CS}^*_\alpha X, \partial)$ and $(C^n_{\mathrm{cont},\lambda}(S(X)), b)$, respectively. In fact, by the assumption $\phi \in \mathrm{CS}^n_\alpha(X)$, one has

$$(-1)^{n+1}\phi(x_{n+1}, x_1, \cdots, x_n) = -\phi(x_1, x_2, \cdots, x_{n+1})$$

and

$$\begin{aligned}
&b(\chi(\phi))(k_0, k_1, \cdots, k_{n+1})\\
&= \sum_{i=0}^{n}(-1)^n\chi(\phi)(k_0, \cdots, k_ik_{i+1}, \cdots, k_{n+1}) + (-1)^{n+1}\chi(\phi)(k_{n+1}k_0, k_1, \cdots, k_n)\\
&= \sum_{(x_0,\cdots,x_n)\in X^{n+1}} \phi(x_0, x_1, \cdots, x_n)\cdot\Big[\sum_{y\in X} k_0(x_0, y)k_1(y, x_1)k_2(x_1, x_2)\cdots k_{n+1}(x_n, x_0)\\
&\quad - \sum_{y\in X} k_0(x_0, x_1)k_1(x_1, y)k_2(y, x_2)\cdots k_{n+1}(x_n, x_0) + \cdots\\
&\quad + (-1)^i\sum_{y\in X} k_0(x_0, x_1)k_1(x_1, x_2)\cdots k_i(x_i, y)k_{i+1}(y, x_{i+1})\cdots k_{n+1}(x_n, x_0) + \cdots\\
&\quad + (-1)^{n+1}\sum_{y\in X} k_{n+1}(x_0, y)k_0(y, x_1)k_1(, x_1, x_2)\cdots k_n(x_n, x_0)\Big]
\end{aligned}$$

$$= \sum_{(x_0,\cdots,x_{n+1})\in X^{n+2}} k_0(x_0,x_1)k_1(x_1,x_2)\cdots k_{n+1}(x_{n+1},x_0)\cdot[\phi(x_0,x_2,\cdots,x_{n+1})+\cdots$$
$$+(-1)^i\phi(x_0,x_1,\cdots,x_i,x_{i+1},\cdots,x_{n+1})+\cdots+(-1)^{n+1}\phi(x_{n+1},x_1,x_2,\cdots,x_n)]$$
$$= \sum_{(x_0,\cdots,x_{n+1})\in X^{n+2}} k_0(x_0,x_1)k_1(x_1,x_2)\cdots k_{n+1}(x_{n+1},x_0)\sum_{i=0}^{n+1}(-1)^i\phi(x_0,\cdots,\widehat{x}_i,\cdots,x_{n+1})$$
$$= \chi(\partial\phi)(k_0,k_1,\cdots,k_{n+1}).$$

Therefore, $\chi\partial=-b\chi$, which implies that χ is a homomorphism from $\mathrm{HS}^n(X)$ into $\mathrm{HC}^n_{\mathrm{cont}}(S(X))$.

5 Relationship with Coarse Cohomology

Recall that the coarse cohomology, which is introduced by John Roe in [3] as one of the tools to study the index theory on the noncompact manifolds, is defined as the homology $\mathrm{HX}^*(X)$ of the cocomplex $(\mathrm{CX}^*(X),\ \partial)$, where $\mathrm{CX}^n(X)$ is the space of functions $\phi: X^{n+1}\to\mathbb{R}$ which satisfy the following support condition: For each $k>0$, the set

$$\mathrm{supp}(\phi)\cap P(\Delta_{n+1};k)$$

is finite in X^{n+1}, and

$$\partial\phi(x_0,x_1,\cdots,x_{n+1})=\sum_{i=0}^{n+1}(-1)^i\phi(x_0,\cdots,x_{i-1},\widehat{x}_i,x_{i+1},\cdots,x_{n+1}).$$

Roe proved that any totally antisymmetric coarse cochain $\phi\in\mathrm{CX}^n_\alpha(X)$ can induce a cyclic cochain $\chi(\phi)$ on the precomplete uniform Roe algebra as follows:

$$\chi(\phi)(k_0,k_1,\cdots,k_n)=\sum_{(x_0,\cdots,x_n)\in X^{n+1}} k_0(x_0,x_1)k_1(x_1,x_2)\cdots k_n(x_n,x_0)\phi(x_0,x_1,\cdots,x_n),$$

where $k_0,k_1,\cdots,k_n\in B(X)$. Furthermore, the map χ defined as above is a homomorphism from $\mathrm{HX}^*(X)$ into $\mathrm{HC}^*(B(X))$.

Let $\mathrm{CS}^n_c(X)=\mathrm{CX}^n(X)\cap\mathrm{CS}^n(X)$ and $\mathrm{CS}^n_{c,\alpha}(X)=\mathrm{CX}^n_\alpha(X)\cap\mathrm{CS}^n_\alpha(X)$. Then $(\mathrm{CS}^*_c(X),\,\partial)$ and $(\mathrm{CS}^*_{c,\alpha}(X),\,\partial)$ are subcomplexes of the coarse complex $(\mathrm{CX}^*(X),\ \partial)$ and its antisymmetrization $(\mathrm{CX}^*_\alpha(X),\ \partial)$, respectively. $\phi\in\mathrm{CS}^*_c(X)$ is called a slowly increasing coarse cochain. Denote $\mathrm{HS}^*_c(X)$ by the homology of the complex $(\mathrm{CS}^*_c(X),\,\partial)$, and call it the slowly increasing cohomology with compact support. It is obvious that the inclusion

$$l:\mathrm{CS}^n_c(X)\hookrightarrow\mathrm{CX}^n(X)$$

induces a natural homomorphism

$$l^*:\mathrm{HS}^n_c(X)\longrightarrow\mathrm{HX}^n(X),$$

and the map χ defined as above gives a homomorphism from $\mathrm{HS}^*_c(X)$ into $\mathrm{HC}^*_{\mathrm{cont}}(S(X))$ by Theorem 4.1. Moreover, the following diagram:

$$\begin{array}{ccc}\mathrm{HS}^*_c(X) & \xrightarrow{\chi} & \mathrm{HC}^*(S(X))\\ l^*\downarrow & & \downarrow l^*\\ \mathrm{HX}^*(X) & \xrightarrow{\chi} & \mathrm{HC}^*(B(X))\end{array}$$

is commutative.

In the end of this section, we will give a condition for a coarse cochain to be slowly increasing in terms of the growth of the support of coarse cochains.

Theorem 5.1 *Let (X, d) be a discrete metric space with polynomial growth, and ϕ be a uniformly bounded coarse antisymmetric n-cocycle. If ϕ satisfies*

$$\#\{\mathrm{supp}\phi \cap P(\Delta_{n+1}, k)\} \leq c(1+k)^l, \quad \forall k \geq 0$$

for some $c > 0$ and $l > 0$, then ϕ is slowly increasing.

Proof Since ϕ is uniformly bounded, there exists a $c' > 0$, such that $|\phi(x_0, \cdots, x_n)| \leq c'$ for any $(x_0, \cdots, x_n) \in X^{n+1}$. Since the support of ϕ satisfies

$$\#\{\mathrm{supp}\phi \cap P(\Delta_{n+1}, k)\} \leq c(1+k)^l,$$

we have

$$\phi_k = \sum_{(x_0,\cdots,x_n)\in P(\Delta_{n+1},k)} |\phi(x_0, x_1, \cdots, x_n)| \leq c'c(1+k)^l,$$

which implies that $\{\phi_k\}$ is slowly increasing.

Theorem 5.2 *Let X be a discrete metric space with polynomial growth. If ϕ satisfies the conditions in Theorem 5.1, then $\chi(\phi)$ is a continuous cyclic cocycle on the smooth subalgebra $S(X)$ of the uniform Roe algebra $B^*(X)$.*

Proof It immediately follows from Theorems 4.1 and 5.1.

6 Application

Let M be a uniformly contractible manifold of dimension m, and X be a separate net in M. It is well-known that the uniform Roe algebra $B^*(M)$ is isomorphic to $B^*(X) \otimes K(H)$, where $K(H)$ is the algebra of compact operators on an infinitely dimensional separable Hilbert space H. If M has polynomial growth, then $S(X)\widehat{\otimes}L^1(H)$ is a smooth subalgebra of $B^*(X)$, where $L^1(H)$ is the algebra of trace class operators on H.

Theorem 6.1 *Let M be a uniformly contractible complete Riemannian manifold of dimension m with polynomial volume growth and polynomial contractibility radius growth, and X be a separate net in M. Let ϕ be an antisymmetric slowly increasing cocycle on X. Define*

$$\begin{aligned}\widetilde{\phi}(k_0 \otimes c_0, k_1 \otimes c_1, \cdots, k_n \otimes c_n) = &\sum_{(x_0,\cdots,x_n)\in X^{n+1}} k_0(x_0, x_1)k_1(x_1, x_2)\cdots k_n(x_n, x_0)\\ &\cdot \mathrm{tr}(\varphi_{x_0}c_0\varphi_{x_1}c_1\cdots\varphi_{x_n}c_n)\phi(x_0, x_1, \cdots, x_n),\end{aligned}$$

where $\{\varphi_x\}_{x\in X}$ is a partition of unity subordinate to a uniformly bounded open cover $\{O_x\}_{x\in X}$. Then $\widetilde{\phi}$ defined as above is a continuous cyclic cocycle on the smooth subalgebra $S(X)\widehat{\otimes}L^1(H)$ of $B^*(X)$.

Proof Since ϕ is slowly increasing, there exist $s > 0$ and $c > 0$, such that

$$|\phi(x_0, \cdots, x_n)|(1 + d(x_0, x_1))^{-s} \cdots (1 + d(x_n, x_0))^{-s} < c.$$

Hence

$$|\widetilde{\phi}(k_0 \otimes c_0, k_1 \otimes c_1, \cdots, k_n \otimes c_n)| \leq c\|k_0\|_s \cdots \|k_n\|_s \|c_0\|_1 \cdots \|c_n\|_s,$$

which implies that $\widetilde{\phi}$ can be extended as a continuous $n+1$ linear functional on $S(X)\widehat{\otimes}L^1(H)$. It is straightforward to check that $\widetilde{\phi}$ is a cyclic cocycle.

Next, we will consider the fundamental cocycle on M. Recall that one can define a continuous map from the Rip complex $P(X)$ to M, such that

(1) $f(x) = x$ for all $x \in X$;

(2) f is smooth when it is restricted to a simplex in $P(X)$;

(3) for any $d > 0$, $k > 0$, there exists a $d_k > d$, such that $f([x_0, \cdots, x_n])$ is contained in $B(x_0, \cdots, x_n, d_k)$ for any k-simplex $[x_0, \cdots, x_k]$ in $P_d(X)$.

Let ω be a compactly supported differential form representing the generator in $H_c^m(M)$, such that $\int_M \omega = 1$. Define

$$\tau(x_0, \cdots, x_m) = \int_{[x_0, \cdots, x_m]} f^*\omega.$$

It is easy to see that τ is an antisymmetric m-coarse cocycle, which is called the fundamental cocycle on M (see [12]). Moreover, one has the following theorem.

Theorem 6.2 *Let M be a uniformly contractible complete Riemannian manifold of dimension m with polynomial volume growth and polynomial contractibility radius growth, and X be a net in M. Then τ defined as above is a slowly increasing m-cocycle with compact support.*

Proof It suffices to show that τ satisfies the condition of Theorem 5.1. Denote $R(r)$ by the contractibility radius of M. Let $y_0 \in \operatorname{supp}\omega$. There exists an $r > 0$, such that $d(x, y_0) \leq r$ for any $x \in \operatorname{supp}\omega$, since ω is compactly supported. Now, for any $(x_0, \cdots, x_n) \in X^{n+1}$, we may assume that $\exists 0 \leq l < p \leq m$, such that $d(x_l, x_p) \geq d(x_i, x_j)$ for any $0 \leq i, j \leq m$. We claim that $f([x_0, \cdots, x_m]) \cap \operatorname{supp}\omega \neq \emptyset$ implies $d(x_i, y_0) \leq 3R(d(x_l, x_p)) + r$ for any $0 \leq i \leq m$. Otherwise, $d(y_0, x_j) > 3R(d(x_l, x_p)) + r$ for some j. Since $\{x_0, \cdots, x_m\} \subset B(x_i, d(x_l, x_p))$ for $i = 0, 1, \cdots, m$, we have that $f([x_0, \cdots, x_m]) \subset B(x_i, R(d(x_l, x_p)))$ for $i = 0, 1, \cdots, m$. It follows that

$$\begin{aligned} d(y_0, x_i) &\geq d(y_0, x_j) - d(x_j, x_i) \\ &> 3R(d(x_l, x_p)) + r - d(x_j, x_i) \\ &> R(d(x_l, x_p)) + r, \end{aligned}$$

$i = 0, 1, \cdots, m$. So $f([x_0, \cdots, x_m]) \cap \operatorname{supp}\omega = \emptyset$. It is a contradiction. Therefore,

$$\begin{aligned} \operatorname{supp}\tau \cap P(\Delta_{n+1}, k) &\subset \{(x_0, x_1, \cdots, x_n) \,|\, d(x_i, y_0) \leq 3R(k) + r\} \\ &\subset \{(x_0, x_1, \cdots, x_n) \,|\, d(x_i, y_0) \leq c(1+k)^s\} \end{aligned}$$

for some $c > 0$ and $s > 0$, since the contractibility radius of M has polynomial growth. It follows that $\#\{\operatorname{supp}\tau \cap P(\Delta_{n+1}, k)\} \leq c'(1+k)^{s'}$, since the volume of M has polynomial growth.

Theorem 6.3 (see [2]) *Let M be a uniformly contractible complete Riemannian manifold with polynomial volume growth and polynomial contractibility radius growth, and D be the generalized Dirac operator on the Clifford bundle over M. Then* $\operatorname{Ind}D$ *is nonzero in $K_0(B^*(M))$.*

Proof It follows from Theorem 4.1 in [2], together with Theorems 6.1 and 6.2.

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