Chin. Ann. Math.
33B(5), 2012, 695–714
DOI: 10.1007/s11401-012-0733-0

Chinese Annals of Mathematics, Series B

Regular Submanifolds in Conformal Space $\mathbb{Q}_p^{n*}$

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Abstract The authors study the regular submanifolds in the conformal space $\mathbb{Q}_p^n$ and introduce the submanifold theory in the conformal space $\mathbb{Q}_p^n$. The first variation formula of the Willmore volume functional of pseudo-Riemannian submanifolds in the conformal space $\mathbb{Q}_p^n$ is given. Finally, the conformal isotropic submanifolds in the conformal space $\mathbb{Q}_p^n$ are classified.

Keywords Conformal space, Conformal invariants, Willmore submanifolds, Conformal isotropic
2000 MR Subject Classification 53A30, 53C50

1 Introduction

A pseudo-Riemannian manifold is a manifold with an indefinite metric of index p ($p \geq 1$). Such structures arise naturally in the theory of relativity, and more recently, string theory (for more details, see [11]). In this paper, we study the conformal submanifold geometry in pseudo-Riemannian space forms.

Let $\mathbb{R}_s^N$ denote a pseudo-Euclidean space, which is the real vector space $\mathbb{R}^N$ with the non-degenerate inner product $\langle\, ,\,\rangle$ given by

$$\langle \xi, \eta \rangle = \sum_{i=1}^{N-s} x_i y_i - \sum_{i=N-s+1}^{N} x_i y_i, \tag{1.1}$$

where $\xi = (x_1, \cdots x_N)$, $\eta = (y_1, \cdots, y_N) \in \mathbb{R}^N$. Let

$$C^{n+1} := \{\xi \in \mathbb{R}_{p+1}^{n+2} \mid \langle \xi, \xi \rangle = 0, \xi \neq 0\}, \tag{1.2}$$

$$\mathbb{Q}_p^n := \{[\xi] \in \mathbb{R}P^{n+1} \mid \langle \xi, \xi \rangle = 0\} = C^{n+1}/(\mathbb{R}\backslash\{0\}). \tag{1.3}$$

We call C^{n+1} the light cone in $\mathbb{R}_{p+1}^{n+2}$ and $\mathbb{Q}_p^n$ the conformal space (or the projective light cone) in $\mathbb{R}P^{n+1}$. We know that the light cone has a degenerate symmetric 2-form (a semi-Riemannian metric) $\widetilde{h}$. In fact, there is an orthonormal decomposition of the tangent space of the light cone

Manuscript received August 16, 2011. Revised April 30, 2012.
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*Project supported by the National Natural Science Foundation of China (No. 10971055) and the Natural Science Foundation of the Educational Commission of Hubei Province (Key Program) (No. D1120111007).

at a point $u \in C^{n+1}$:

$$\mathrm{T}_u C^{n+1} = \{\xi \in \mathbb{R}^{n+2}_{p+1} \mid \langle u, \xi \rangle = 0\} = \mathbb{R}u \oplus E_u^{\perp}.$$

Along the line $\mathbb{R}u$, the two subspaces of $\mathbb{R}^{n+2}_{p+1}$, $E^{\perp}_{\lambda u}$ and $E^{\perp}_u$ are equal, where $\lambda \in \mathbb{R}\backslash\{0\}$. For any vectors $\alpha, \beta \in E_u^{\perp}$, we may define

$$\widetilde{h}(\alpha, \beta) = \frac{2}{|u|^2}\langle \alpha, \beta \rangle,$$

where $|\cdot|^2$ is the standard square norm of $\mathbb{R}^{n+2}$.

The standard metric h of the conformal space $\mathbb{Q}^n_p$ can be obtained through the pseudo-Riemannian immersion

$$\pi : C^{n+1} \to \mathbb{Q}^n_p.$$

Define the metric h as follows: for any $X, Y \in \mathrm{T}_{[u]}\mathbb{Q}^n_p$, there exist horizontal lifts α, β of X, Y, such that

$$h(X, Y) = \widetilde{h}(\alpha, \beta),$$

where $\alpha, \beta \in E_u^{\perp}$. This definition is well-defined. We can check that $(\mathbb{Q}^n_p, h)$ is a pseudo-Riemannian manifold. Topologically, $\mathbb{Q}^n_p$ is $\mathbb{S}^{n-p} \times \mathbb{S}^p/\mathbb{Z}_2$, which is endowed by the standard pseudo-Riemannian metric $h = g_{_{\mathbb{S}^{n-p}}} \oplus (-g_{_{\mathbb{S}^p}})$ and the corresponding conformal structure $[h] := \{\mathrm{e}^{2\tau} h \mid \tau \in C^{\infty}(\mathbb{Q}^n_p)\}$.

We define the pseudo-Riemannian sphere space $\mathbb{S}^n_p$ and the pseudo-Riemannian hyperbolic space $\mathbb{H}^n_p$ by

$$\mathbb{S}^n_p = \{u \in \mathbb{R}^{n+1}_p \mid \langle u, u \rangle = 1\}, \quad \mathbb{H}^n_p = \{u \in \mathbb{R}^{n+1}_{p+1} \mid \langle u, u \rangle = -1\}.$$

We call $\mathbb{R}^n_p, \mathbb{S}^n_p$ and $\mathbb{H}^n_p$ pseudo-Riemannian space forms with an index p ($p \geq 1$). When $p = 1$, we call them de Sitter space $\mathbb{S}^n_1$ and anti-de Sitter space $\mathbb{H}^n_1$.

Denote $\pi = \{[x] \in \mathbb{Q}^n_p \mid x_1 = x_{n+2}\}$, $\pi_+ = \{[x] \in \mathbb{Q}^n_p \mid x_{n+2} = 0\}$, $\pi_- = \{[x] \in \mathbb{Q}^n_p \mid x_1 = 0\}$. There exist three conformal diffeomorphisms

$$\begin{aligned} \sigma : \mathbb{R}^n_p \to \mathbb{Q}^n_p \backslash \pi, &\quad u \mapsto \Big[\Big(\frac{\langle u, u \rangle - 1}{2}, u, \frac{\langle u, u \rangle + 1}{2}\Big)\Big], \\ \sigma_+ : \mathbb{S}^n_p \to \mathbb{Q}^n_p \backslash \pi_+, &\quad u \mapsto [(u, 1)], \\ \sigma_- : \mathbb{H}^n_p \to \mathbb{Q}^n_p \backslash \pi_-, &\quad u \mapsto [(1, u)]. \end{aligned}$$

We may assume that $\mathbb{Q}^n_p$ is the common compactification of $\mathbb{R}^n_p$, $\mathbb{S}^n_p$ and $\mathbb{H}^n_p$, where $\mathbb{R}^n_p$, $\mathbb{S}^n_p$ and $\mathbb{H}^n_p$ are the subsets of $\mathbb{Q}^n_p$ under the conformal geometry. Therefore, we only need to study the conformal geometry in the conformal space $\mathbb{Q}^n_p$ with the index p.

When $p = 0$, our analysis in this text can be reduced to the Moebius submanifold geometry in the sphere space (see [13]). For more details of Moebius submanifold geometry, see [2, 4–6, 12–14], etc. For some other results about the Lorentzian conformal geometry (when $p = 1$), see [7–10], etc.

This paper is organized as follows. In Section 2, we prove that the conformal group of the conformal space $\mathbb{Q}^n_p$ is $O(n - p + 1, p + 1)/\{\pm E\}$. In Section 3, we construct the general submanifold theory in the conformal space $\mathbb{Q}_p^n$, and give the relationship between conformal invariants and isometric ones for hypersurfaces in pseudo-Riemannian space forms. In Section 4, we give the first variation formula of the Willmore volume functional of regular pseudo-Riemannian submanifolds in the conformal space $\mathbb{Q}_p^n$. In Section 5, we classify the conformal isotropic submanifolds in the conformal space $\mathbb{Q}_p^n$.

2 The Conformal Group of the Conformal Space $\mathbb{Q}_p^n$

In this section, we will prove that the conformal group of the conformal space $\mathbb{Q}_p^n$ is $O(n-p+1,p+1)/\{\pm E\}$.

First we introduce the following lemma.

Lemma 2.1 *Let $\varphi:\mathbf{M}\to\mathbf{M}$ be a conformal transformation on m $(m>2)$ dimensional pseudo-Riemannian submanifold $(\mathbf{M},g)$, i.e., φ is a diffeomorphism, such that $\varphi^*g=\mathrm{e}^{2\tau}g$, $\tau\in C^\infty(\mathbf{M})$. If $\mathbf{M}$ is connected, then φ is determined by the tangent map φ_{*p} and the value of 1-form $\mathrm{d}\tau_p$ at one fixed point $p\in\mathbf{M}$.*

Proof For any point $p\in\mathbf{M}$, suppose that (x^i) is a local coordinate around p, and (y^i) is a local coordinate around $\varphi(p)$.

For the pseudo-Riemannian metric $\widetilde{g}=\mathrm{e}^{2\tau}g=\varphi^*g$ on $\mathbf{M}$, we denote $\widetilde{D}$ the connection of $\widetilde{g}$, $\widetilde{R}$ the curvature tensor, and $\widetilde{\mathrm{Ric}}$ the Ricci curvature tensor. With respect to g, the corresponding operators are D,R and Ric, respectively. The relations between these operators are as follows:

$$\widetilde{D}_XY=D_XY+X(\tau)Y+Y(\tau)X-g(X,Y)\nabla\tau, \tag{2.1}$$

$$\begin{aligned}\widetilde{R}(X,Y)Z=&R(X,Y)Z+g(X,Z)D_Y\nabla\tau-g(Y,Z)D_X\nabla\tau\\&+[g(X,\nabla\tau)g(Y,Z)-g(Y,\nabla\tau)g(X,Z)]\nabla\tau\\&+[D_YZ(\tau)+Y(\tau)Z(\tau)-YZ(\tau)-g(Y,Z)g(\nabla\tau,\nabla\tau)]X\\&-[D_XZ(\tau)+X(\tau)Z(\tau)-XZ(\tau)-g(X,Z)g(\nabla\tau,\nabla\tau)]Y,\end{aligned} \tag{2.2}$$

$$\begin{aligned}\widetilde{R}(X,Y,W,Z)=&\mathrm{e}^{2\tau}\{R(X,Y,W,Z)+g(X,Z)g(W,D_Y\nabla\tau)-g(Y,Z)g(W,D_X\nabla\tau)\\&+[g(X,\nabla\tau)g(Y,Z)-g(Y,\nabla\tau)g(X,Z)]g(W,\nabla\tau)\\&+[D_YZ(\tau)+Y(\tau)Z(\tau)-YZ(\tau)-g(Y,Z)g(\nabla\tau,\nabla\tau)]g(W,X)\\&-[D_XZ(\tau)+X(\tau)Z(\tau)-XZ(\tau)-g(X,Z)g(\nabla\tau,\nabla\tau)]g(W,Y)\},\end{aligned} \tag{2.3}$$

where X,Y,Z,W are smooth vector fields on $\mathbf{M}$, and $\nabla\tau$ is the gradient of τ with respect to g.

Locally, let

$$\begin{aligned}&D_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i}=\sum_k\Gamma_{ij}^k\frac{\partial}{\partial x^k},\quad D_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i}=\sum_k\Gamma_{ij}^{\prime k}\frac{\partial}{\partial y^k},\\&g_{ij}=g\Big(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\Big),\quad(g^{ij})=(g_{ij})^{-1},\quad\varphi_*\frac{\partial}{\partial x^i}=\sum_jA_i^j\frac{\partial}{\partial y^j},\quad\mathrm{d}\tau=\sum_iB_i\mathrm{d}x^i.\end{aligned}$$

First we have

$$g\Big(\varphi_*\frac{\partial}{\partial x^i},\varphi_*\frac{\partial}{\partial x^j}\Big)\circ\varphi=\mathrm{e}^{2\tau}g\Big(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\Big). \tag{2.4}$$

Acting the both sides of (2.4) with $\frac{\partial}{\partial x^k}$, we get

$$2B_k g\Big(\varphi_*\frac{\partial}{\partial x^i},\varphi_*\frac{\partial}{\partial x^j}\Big)=g\Big(D_{\varphi_*\frac{\partial}{\partial x^k}}\varphi_*\frac{\partial}{\partial x^i}-\varphi_* D_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^i},\varphi_*\frac{\partial}{\partial x^j}\Big)$$
$$+g\Big(D_{\varphi_*\frac{\partial}{\partial x^k}}\varphi_*\frac{\partial}{\partial x^j}-\varphi_* D_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^j},\varphi_*\frac{\partial}{\partial x^i}\Big).$$

Alternating the positions of i,j,k, and using

$$D_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}=D_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i},\quad D_{\varphi_*\frac{\partial}{\partial x^i}}\varphi_*\frac{\partial}{\partial x^j}=D_{\varphi_*\frac{\partial}{\partial x^j}}\varphi_*\frac{\partial}{\partial x^i},$$

one obtains

$$B_i g\Big(\varphi_*\frac{\partial}{\partial x^j},\varphi_*\frac{\partial}{\partial x^k}\Big)+B_j g\Big(\varphi_*\frac{\partial}{\partial x^i},\varphi_*\frac{\partial}{\partial x^k}\Big)-B_k g\Big(\varphi_*\frac{\partial}{\partial x^i},\varphi_*\frac{\partial}{\partial x^j}\Big)$$
$$=g\Big(D_{\varphi_*\frac{\partial}{\partial x^i}}\varphi_*\frac{\partial}{\partial x^j}-\varphi_* D_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j},\varphi_*\frac{\partial}{\partial x^k}\Big)$$

and

$$B_k g\Big(\varphi_*\frac{\partial}{\partial x^i},\varphi_*\frac{\partial}{\partial x^j}\Big)=g\Big(\nabla\tau,\frac{\partial}{\partial x^k}\Big)\mathrm{e}^{2\tau}g_{ij}=g_{ij}g\Big(\varphi_*\nabla\tau,\varphi_*\frac{\partial}{\partial x^k}\Big),$$

where

$$\nabla\tau=\sum_{ij}g^{ij}B_i\frac{\partial\tau}{\partial x^j}.$$

Therefore,

$$D_{\varphi_*\frac{\partial}{\partial x^i}}\varphi_*\frac{\partial}{\partial x^j}-\varphi_* D_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}=B_i\varphi_*\frac{\partial}{\partial x^j}+B_j\varphi_*\frac{\partial}{\partial x^k}-g_{ij}\varphi_*\nabla\tau.$$

We collect the terms of $\frac{\partial}{\partial y^k}$ and get

$$\frac{\partial A_j^k}{\partial x^i}=B_iA_j^k+B_jA_i^k+\Gamma_{ij}^tA_t^k-g_{ij}\sum_{st}g^{st}B_sA_t^k-\sum_{st}A_i^sA_j^t\Gamma'^k_{st}. \tag{2.5}$$

Denote

$$r_{ij}=\mathrm{Ric}\Big(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\Big),\quad \widetilde{r}_{ij}=\widetilde{\mathrm{Ric}}\Big(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\Big).$$

On the one hand, from (2.3), we have

$$\widetilde{r}_{ij}=r_{ij}-g_{ij}\triangle\tau+(m-2)\Big[B_iB_j-\frac{\partial B_i}{\partial x^j}+\sum_t\Gamma_{ij}^tB_t-g_{ij}g(\nabla\tau,\nabla\tau)\Big], \tag{2.6}$$

where $\triangle$ is the Laplacian with respect to g. On the other hand, we have

$$\widetilde{\mathrm{Ric}}(X,Y)=\mathrm{Ric}(\varphi_*X,\varphi_*Y)\circ\varphi. \tag{2.7}$$

Therefore,

$$\widetilde{r}_{ij}=\widetilde{\mathrm{Ric}}\Big(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\Big)=\sum_{st}A_i^sA_j^tr'_{st},\quad r'_{st}=\mathrm{Ric}\Big(\frac{\partial}{\partial y^s},\frac{\partial}{\partial y^t}\Big). \tag{2.8}$$

Combining (2.6) and (2.8), we have

$$\begin{aligned}\frac{\partial B_j}{\partial x^i} &= B_iB_j + \sum_t \Gamma^t_{ij}B_t - g_{ij}\sum_{st} g^{st}B_sB_t \\ &\quad + \frac{1}{m-2}\Big(r_{ij} - g_{ij}\triangle\tau - \sum_{st} A^s_iA^t_j r'_{st}\Big).\end{aligned} \tag{2.9}$$

Combining the first order PDEs (2.5) and (2.9), we get the first order PDEs

$$\begin{cases}\dfrac{\partial\varphi_k}{\partial x^j} = A^k_j,\\ \dfrac{\partial\tau}{\partial x^j} = B_j,\\ \dfrac{\partial A^k_j}{\partial x^i} = B_iA^k_j + B_jA^k_i + \Gamma^t_{ij}A^k_t - g_{ij}\displaystyle\sum_{st} g^{st}B_sA^k_t - \sum_{st} A^s_iA^t_j\Gamma'^k_{st},\\ \dfrac{\partial B_j}{\partial x^i} = B_iB_j + \displaystyle\sum_t \Gamma^t_{ij}B_t - g_{ij}\sum_{st} g^{st}B_sB_t + \frac{1}{m-2}\Big(r_{ij} - g_{ij}\triangle\tau - \sum_{st} A^s_iA^t_j r'_{st}\Big).\end{cases}$$

If $\mathbf{M}$ is connected, by the existence and uniqueness theorem of initial values of PDEs, then φ is determined by the tangent map φ_* and 1-form $\mathrm{d}\tau$ at one fixed point.

Theorem 2.1 *Suppose that φ is a conformal transformation on $\mathbb{Q}^n_p$, $\varphi^*h = \mathrm{e}^{2\tau}h$, and x_0 is a fixed point of φ. Then there is an $A \in O(n-p+1, p+1)$, such that $\varphi = \Phi_A$ and $\Phi_A([X]) = [XA]$.*

Proof Let $(\mathbf{U}, x^i)$ be a coordinate chart around x_0. At the point x_0, denote

$$\frac{\partial\varphi_i}{\partial x^j}\Big|_{x_0} = A^i_j, \quad \frac{\partial\tau}{\partial x^j}\Big|_{x_0} = B_j, \quad h_{ij} = h\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big)\Big|_{x_0}, \quad (h^{ij}) = (h_{ij})^{-1}.$$

Suppose that

$$x_0 = [u_0], \quad u_0 = (u_1, u_2) \in \mathbb{S}^{n-p}\times\mathbb{S}^p \subset \mathbb{R}^{n+p}_{p+1}, \quad Ju_0 = (u_1, -u_2).$$

For every $\frac{\partial}{\partial x^i}|_{x_0}$, there exists a unique $e_i \in E^{\perp}_{u_0}$, such that

$$\pi_* e_i = \frac{\partial}{\partial x^i}\Big|_{x_0}.$$

$\{u_0, Ju_0, e_1, \cdots, e_n\}$ provides a basis of $\mathbb{R}^{n+p}_{p+1}$, and then there is an orthonormal decomposition of $\mathbb{R}^{n+p}_{p+1}$,

$$\mathbb{R}^{n+p}_{p+1} = \mathrm{span}\{u_0, Ju_0\} \oplus \mathrm{span}\{e_1, \cdots, e_n\}.$$

Define a linear transformation $A : \mathbb{R}^{n+p}_{p+1} \to \mathbb{R}^{n+p}_{p+1}$ on the basis of

$$A(u_0) = \mathrm{e}^{-\tau(x_0)}u_0, \quad A(e_i) = \mathrm{e}^{-\tau(x_0)}\Big(\sum_j A^j_i e_j - B_iu_0\Big), \tag{2.10}$$

$$A(Ju_0) = \mathrm{e}^{\tau(x_0)}Ju_0 + 2\mathrm{e}^{-\tau(x_0)}\Big(\sum_{ijk} h^{jk}B_jA^i_k e_i - \sum_{ij} h^{ij}B_iB_ju_0\Big). \tag{2.11}$$

First, it is easy to check that $A \in O(n-p+1, p+1)$. In fact, it is guaranteed by $\sum\limits_{st} A_i^s A_j^t h_{st} = h_{ij} \mathrm{e}^{2\tau(x_0)}$ (it suffices to check it on the basis).

Furthermore, we have

$$\Phi_A(x_0) = \varphi(x_0) = x_0, \tag{2.12}$$

$$\Phi_{A*}\Big|_{x_0}\Big(\frac{\partial}{\partial x^i}\Big) = \pi_*|_{\mathrm{T}_{x_0}\mathbb{Q}_p^n} \circ A \circ (\pi_*|_{\mathrm{T}_{x_0}\mathbb{Q}_p^n})^{-1}\Big(\frac{\partial}{\partial x^i}\Big) = \sum_j A_i^j e_j = \varphi_*\Big|_{x_0}\Big(\frac{\partial}{\partial x^i}\Big). \tag{2.13}$$

Suppose that $[u] \in \mathbb{Q}_p^n$, for any $X, Y \in \mathrm{T}_{[u]}\mathbb{Q}_p^n$, and there are $\alpha, \beta \in E_u^{\perp} \subset \mathrm{T}_u C^{n+1}$, such that

$$\pi_*\alpha = X, \quad \pi_*\beta = Y, \quad h(X,Y) = \frac{2}{|u|^2}\langle \alpha, \beta \rangle.$$

Therefore, from the above, we have

$$\begin{aligned}
(\Phi_A^* h)_{[u]}(X,Y) &= (\Phi_A^* h)_{[u]}(\pi_*\alpha, \pi_*\beta) = (\pi^* \circ \Phi_A^* h)_u(\alpha, \beta) \\
&= (A \circ \pi^* h)_u(\alpha,\beta) = (\pi^* h)_{A(u)}(\alpha A, \beta A) = \frac{2}{|A(u)|^2}\langle \alpha A, \beta A\rangle \\
&= \frac{|u|^2}{|A(u)|^2} \cdot \frac{2}{|u|^2}\langle \alpha, \beta\rangle = \frac{|u|^2}{|A(u)|^2} h_{[u]}(X,Y).
\end{aligned} \tag{2.14}$$

Therefore, $\Phi_A^* h = \frac{|u|^2}{|uA|^2} h$. Next we prove that

$$\frac{\partial}{\partial x^i}\Big|_{x_0}\Big(\frac{|u|^2}{|uA|^2}\Big) = \mathrm{e}^{2\tau(x_0)} B_i. \tag{2.15}$$

Suppose that there is a local lift of $\mathbb{Q}_p^n$ around $x_0 \in \mathbb{Q}_p^n$, such that $u : \mathbf{U} \subset \mathbb{Q}_p^n \to C^{n+1}$. Then $\pi \circ u = \mathrm{id}$, and

$$\frac{\partial u}{\partial x^i}\Big|_{x_0} = u_*\Big(\frac{\partial}{\partial x^i}\Big|_{x_0}\Big) = u_* \circ \pi_*(e_i) = (\pi \circ u)_*(e_i) = e_i. \tag{2.16}$$

Suppose that

$$u = a u_0 + b J u_0 + \sum_i c^i e_i,$$

where a, b, c^i are local smooth functions. We may assume that

$$a(x_0) = 1, \quad b(x_0) = 0, \quad c^i(x_0) = 0. \tag{2.17}$$

Using (2.10) and (2.11), we have

$$\begin{aligned}
A(u) &= \Big(a - 2h(\nabla\tau(x_0), \nabla\tau(x_0))b - \sum_i B_i c^i\Big) u_0 + b\mathrm{e}^{\tau(x_0)} J u_0 \\
&\quad + \mathrm{e}^{-\tau(x_0)} \sum_{ik}\Big(2b\sum_j B_j h^{jk} + c^k\Big) A_k^i e_i \\
&:= a' u_0 + b' J u_0 + \sum_i c'^i e_i.
\end{aligned} \tag{2.18}$$

It is easy to check that

$$\frac{\partial a}{\partial x^i}\Big|_{x_0} = 0, \quad \frac{\partial b}{\partial x^i}\Big|_{x_0} = 0, \quad \frac{\partial c^j}{\partial x^i}\Big|_{x_0} = \delta_i^j. \tag{2.19}$$

Consequently,

$$\frac{\partial}{\partial x^i}\Big|_{x_0}(|u|^2) = \frac{\partial}{\partial x^i}\Big|_{x_0}\Big(2a^2 + 2b^2 + \sum_{jk} c^j c^k \langle e_j, e_k\rangle\Big) = 0. \tag{2.20}$$

$$\frac{\partial}{\partial x^i}\Big|_{x_0}(|A(u)|^2) = 4\Big\langle \frac{\partial a'}{\partial x^i}\Big|_{x_0}, a'(x_0)\Big\rangle = -2\mathrm{e}^{-2\tau(x_0)}B_i. \tag{2.21}$$

Therefore

$$\frac{\partial}{\partial x^i}\Big|_{x_0}\Big(\frac{|u|^2}{|A(u)|^2}\Big) = -\frac{|u_0|^2 \frac{\partial}{\partial x^i}\Big|_{x_0}(|A(u)|^2)}{|A(u_0)|^4} = \mathrm{e}^{2\tau(x_0)}B_i. \tag{2.22}$$

From Lemma 2.1, we have $\Phi_A = \varphi$.

Remark 2.1 Theorem 2.1 is a generalization of the Liouville theorem on S^n (see [15, Chapter 6, Theorem 1.1]).

Suppose that for some fixed point $x_0 = [(a,b)] \in \mathbb{Q}_p^n$, a conformal transformation $\varphi : \mathbb{Q}_p^n \to \mathbb{Q}_p^n$ has

$$\varphi([a,b)] = [(c,d)],$$

where

$$(a,b),(c,d) \in \mathbb{S}^{n-p} \times \mathbb{S}^p.$$

We can certainly find $C \in O(n-p+1),\ D \in O(p+1)$, such that $a = cC,\ b = dD$. That is, $A_p = \mathrm{diag}(C,D) \in O(n-p+1,p+1)$, such that $\Phi_{A_p}[(c,d)] = [(a,b)]$. Clearly, the conformal transformation $\Phi_{A_p} \circ \varphi$ of $\mathbb{Q}_p^n$ has a fixed point x_0. From the above theorem, there is an $A \in O(n-p+1,p+1)$, such that $\Phi_{A_p} \circ \varphi = \Phi_A$. Thus $\varphi = \Phi_{AA_p^{-1}}$. Because

$$\Phi : O(n-p+1,p+1) \to \text{the conformal group of } \mathbb{Q}_p^n, \quad A \mapsto \Phi_A$$

is an epimorphism and $\ker(\Phi) = \{\pm E\}$, we obtain the following theorem.

Theorem 2.2 *The conformal group of the conformal space $\mathbb{Q}_p^n$ is $O(n-p+1,p+1)/\{\pm E\}$.*

Remark 2.2 Theorem 2.2 was proved by Cahen and Kerbrat in 1983 (see [3]).

3 Fundamental Equations of Submanifolds

Suppose that $x : \mathbf{M} \to \mathbb{Q}_p^n$ $(p \geq 1)$ is an m-dimensional Riemannian or pseudo-Riemannian submanifold with an index s $(0 \leq s \leq p)$, that is, $x_*(\mathrm{T}\mathbf{M})$ is a non-degenerate subbundle of $(\mathrm{T}\mathbb{Q}_p^n, h)$ with the index s $(0 \leq s \leq p)$. When $s = 0$, we call an $\mathbf{M}$ space-like submanifold. When $s > 0$, we call $\mathbf{M}$ a pseudo-Riemannian submanifold. Especially, when $s = 1$, $\mathbf{M}$ is called a Lorentzian submanifold or a time-like submanifold. From now on, we always assume that the submanifold x has an index s $(0 \leq s \leq p)$.

Let $y : U \to C^{n+1}$ be a lift of $x : \mathbf{M} \to \mathbb{Q}^n_p$ defined in an open subset U of $\mathbf{M}$. We denote by Δ and κ the Laplacian operator and the normalized scalar curvature of the local non-degenerate metric $\langle \mathrm{d}y, \mathrm{d}y\rangle$. Then we have the following theorem.

Theorem 3.1 *Suppose that $x : \mathbf{M} \to \mathbb{Q}^n_p$ is an m-dimensional Riemannian or pseudo-Riemannian submanifold with the index s $(0 \le s \le p)$. On $\mathbf{M}$, the 2-form $g := \pm(\langle \Delta y, \Delta y\rangle - m^2\kappa)\langle \mathrm{d}y, \mathrm{d}y\rangle$ is a globally defined invariant of $x : \mathbf{M} \to \mathbb{Q}^n_p$ under the Lorentzian group transformations of $\mathbb{Q}^n_p$, where $y : U \to C^{n+1}$ is a lift of $x : \mathbf{M} \to \mathbb{Q}^n_p$ defined in an open subset U of $\mathbf{M}$.*

Proof We should prove that for any $T \in O(n+1, p+1)$, $\widetilde{x} = \Phi_T \circ x$ has the same 2-form g.

First we check that the expression of g is invariant under different local lifts. Suppose that $y : U \to C^{n+1}$, and $\widetilde{y} : \widetilde{U} \to C^{n+1}$ are different lifts of $x : \mathbf{M} \to \mathbb{Q}^n_p$ defined in open subsets U and $\widetilde{U}$ of $\mathbf{M}$. For the local non-degenerate metrics $\langle\, ,\, \rangle_y = \langle \mathrm{d}y, \mathrm{d}y\rangle$, we denote by Δ the Laplacian, by ∇f the gradient of a function f, and by κ the normalized scalar curvatures. For $\langle \mathrm{d}\widetilde{y}, \mathrm{d}\widetilde{y}\rangle$, we denote by $\widetilde{\Delta}$ the Laplacian, and by $\widetilde{\kappa}$ the normalized scalar curvatures. On $U \cap \widetilde{U}$, we find that $\widetilde{y} = \mathrm{e}^{\tau} y$, where τ is a local smooth function on $U \cap \widetilde{U}$. Therefore, $\langle \mathrm{d}\widetilde{y}, \mathrm{d}\widetilde{y}\rangle = \mathrm{e}^{2\tau}\langle \mathrm{d}y, \mathrm{d}y\rangle$, and they are conformal on $U \cap \widetilde{U}$. We have

$$\widetilde{\omega^j_i} = \omega^j_i + \tau_i\omega^j - \tau^j\omega_i + \delta^j_i \mathrm{d}\tau, \tag{3.1}$$

$$\mathrm{e}^{2\tau}\widetilde{\Delta} f = \Delta f + (m-2)\langle \nabla\tau, \nabla f\rangle_y, \tag{3.2}$$

$$\mathrm{e}^{2\tau}\widetilde{\kappa} = \kappa - \frac{2}{m}\Delta\tau - \frac{m-2}{m}\langle \nabla\tau, \nabla\tau\rangle_y, \tag{3.3}$$

where $\{\omega^j_i\}$ and $\{\widetilde{\omega^j_i}\}$ are connection forms with respect to the local non-degenerate metrics $\langle \mathrm{d}y, \mathrm{d}y\rangle$ and $\langle \mathrm{d}\widetilde{y}, \mathrm{d}\widetilde{y}\rangle$.

It follows that

$$(\langle \Delta y, \Delta y\rangle - m^2\kappa)\langle \mathrm{d}y, \mathrm{d}y\rangle = (\langle \widetilde{\Delta}\widetilde{y}, \widetilde{\Delta}\widetilde{y}\rangle - m^2\widetilde{\kappa})\langle \mathrm{d}\widetilde{y}, \mathrm{d}\widetilde{y}\rangle. \tag{3.4}$$

If there is an isometric transformation $T \in O(n-p+1, p+1)$ of $\mathbb{R}^{n-p+1}_{p+1}$ acting on $\mathbb{Q}^n_p$ and $y : U \to C^{n+1}$ is a lift of $x : \mathbf{M} \to \mathbb{Q}^n_p$ defined in open subsets U, then the submanifold $\widetilde{x} = x \circ T$ must have a local lift like $\widetilde{y} = \mathrm{e}^{\tau} yT$. Since T perserves the pseudo-Riemannian inner product and the dilatation of the local lift y will not impact the term $(\langle \Delta y, \Delta y\rangle - m^2\kappa)\langle \mathrm{d}y, \mathrm{d}y\rangle$, the 2-form g is conformally invariant.

Definition 3.1 *We call an m-dimensional submanifold $x : \mathbf{M} \to \mathbb{Q}^n_p$ a regular submanifold if the 2-form $g := \pm(\langle \Delta y, \Delta y\rangle - m^2\kappa)\langle \mathrm{d}y, \mathrm{d}y\rangle$ is non-degenerate. g is called the conformal metric of the regular submanifold $x : \mathbf{M} \to \mathbb{Q}^n_p$.*

Remark 3.1 If the regular submanifold x is a space-like hypersurface $(n = m+1)$ and $p = 1$, then the conformal metric must be $g := -(\langle \Delta y, \Delta y\rangle - m^2\kappa)\langle \mathrm{d}y, \mathrm{d}y\rangle$. If the regular submanifold x is a Lorentzian (or time-like) hypersurface $(n = m+1)$ and $p = 1$, then the conformal metric must be $g := (\langle \Delta y, \Delta y\rangle - m^2\kappa)\langle \mathrm{d}y, \mathrm{d}y\rangle$. When the co-dimension $n - m > 1$, the above two forms of the conformal metric g are both possible.

In this paper, we assume that $x : \mathbf{M} \to \mathbb{Q}_p^n$ is a regular submanifold. Since the metric g is non-degenerate (we call it the conformal metric), there exists a unique lift $Y : \mathbf{M} \to C^{n+1}$, such that $g = \langle \mathrm{d}Y, \mathrm{d}Y \rangle$ up to sign. We call Y the canonical lift of x. By taking $y := Y$ in (3.1), we get

$$\langle \Delta Y, \Delta Y \rangle = m^2 \kappa \pm 1. \tag{3.5}$$

Definition 3.2 *The two submanifolds $x, \widetilde{x}$ are conformal equivalent, if there exists a conformal transformation $\sigma : \mathbb{Q}_p^n \to \mathbb{Q}_p^n$, such that $\widetilde{x} = \sigma \circ x$.*

It is easy to check that the following theorem holds.

Theorem 3.2 *Two submanifolds $x, \widetilde{x} : \mathbf{M} \to \mathbb{Q}_p^n$ are conformal equivalent, if and only if there exists a $T \in O(n-p+1, p+1)$, such that $\widetilde{Y} = YT$, where $Y, \widetilde{Y}$ are canonical lifts of $x, \widetilde{x}$, respectively.*

Let $\{e_1, \cdots, e_m\}$ be a local basis of $\mathbf{M}$ with a dual basis $\{\omega^1, \cdots, \omega^m\}$. Let $g_{ij} = g(e_i, e_j)$. If $(g_{ij}) = (-I_s) \oplus (I_{m-s})$, we call $\{e_1, \cdots, e_m\}$ an orthonormal basis with respect to g. If $(g_{ij}) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus (-I_{s-1}) \oplus (I_{m-s-1})$, we call $\{e_1, \cdots, e_m\}$ a pseudo-orthonormal basis with respect to g. But in this section, we need not choose an orthonormal or a pseudo-orthonormal basis.

Denote $Y_i = e_i(Y)$. We define

$$N := -\frac{1}{m}\Delta Y - \frac{1}{2m^2}\langle \Delta Y, \Delta Y \rangle Y. \tag{3.6}$$

In a similar way to the corresponding calculation of [12], we have

$$\langle N, Y \rangle = 1, \quad \langle N, N \rangle = 0, \quad \langle N, Y_k \rangle = 0, \quad 1 \le k \le m. \tag{3.7}$$

We may decompose $\mathbb{R}_{p+1}^{n+2}$, such that

$$\mathbb{R}_{p+1}^{n+2} = \mathrm{span}\{Y, N\} \oplus \mathrm{span}\{Y_1, \cdots, Y_m\} \oplus \mathbb{V}, \tag{3.8}$$

where $\mathbb{V} \perp \mathrm{span}\{Y, N, Y_1, \cdots, Y_m\}$. We call $\mathbb{V}$ the conformal normal bundle for $x : \mathbf{M} \to \mathbb{Q}_p^n$. Let $\{\xi_{m+1}, \cdots, \xi_n\}$ be a local basis for the bundle $\mathbb{V}$ over $\mathbf{M}$. Then $\{Y, N, Y_1, \cdots, Y_m, \xi_{m+1}, \cdots, \xi_n\}$ forms a moving frame in $\mathbb{R}_{p+1}^{n+2}$ along $\mathbf{M}$. We adopt the conventions on the ranges of indices in this paper,

$$1 \le i, j, k, l, r, q \le m, \quad m+1 \le \alpha, \beta, \gamma, \nu \le n. \tag{3.9}$$

Let $g_{\alpha\beta} = \langle \xi_\alpha, \xi_\beta \rangle$. If $(g_{\alpha\beta}) = (-I_{p-s}) \oplus (I_{n-m-p+s})$, we call $\{\xi_{m+1}, \cdots, \xi_n\}$ an orthonormal normal basis of x. If $(g_{\alpha\beta}) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus (-I_{p-s-1}) \oplus (I_{n-m-p+s-1})$, we call $\{\xi_{m+1}, \cdots, \xi_n\}$ a pseudo-orthonormal normal basis of x.

We may write the structure equations as follows:

$$\mathrm{d}Y = \sum_i \omega^i Y_i, \quad \mathrm{d}N = \sum_i \psi^i Y_i + \sum_\alpha \phi^\alpha \xi_\alpha, \tag{3.10}$$

$$\mathrm{d}Y_i = -\psi_i Y - \omega_i N + \sum_j \omega_i^j Y_j + \sum_\alpha \omega_i^\alpha \xi_\alpha, \tag{3.11}$$

$$\mathrm{d}\xi_\alpha = -\phi_\alpha Y + \sum_i \omega_\alpha^i Y_i + \sum_\beta \omega_\alpha^\beta \xi_\alpha, \tag{3.12}$$

where the coefficients of $\{Y, N, Y_i, \xi_\alpha\}$ are 1-forms on $\mathbf{M}$.

Remark 3.2 If $\{e_1, \cdots, e_m\}$ and $\{\xi_{m+1}, \cdots, \xi_n\}$ are both orthonormal basis, then

$$\begin{aligned} &\omega_i^j + \omega_j^i = 0, \quad 1 \le i, j \le s, && \omega_\alpha^\beta + \omega_\beta^\alpha = 0, \quad m+1 \le \alpha, \beta \le m+p-s, \\ &\omega_i^j - \omega_j^i = 0, \quad s+1 \le i, j \le m-s, && \omega_\alpha^\beta - \omega_\beta^\alpha = 0, \quad m+p-s+1 \le \alpha, \beta \le n. \end{aligned}$$

It is clear that $\mathbb{A} := \sum_i \psi_i \otimes \omega^i$, $\mathbb{B} := \sum_{i,\alpha} \omega_i^\alpha \otimes \omega^i e_\alpha$, $\Phi := \sum_\alpha \phi^\alpha \xi_\alpha$ are globally defined conformal invariants. Let

$$\psi_i = \sum_j A_{ij} \omega^j, \quad \omega_i^\alpha = \sum_j B_{ij}^\alpha \omega^j, \quad \phi^\alpha = \sum_i C_i^\alpha \omega^i. \tag{3.13}$$

Denote the covariant derivatives of these tensors with respect to conformal metric g as follows:

$$\sum_j C_{i,j}^\alpha \omega^j = \mathrm{d}C_i^\alpha - \sum_j C_j^\alpha \omega_i^j + \sum_\beta C_i^\beta \omega_\beta^\alpha, \tag{3.14}$$

$$\sum_k A_{ij,k} \omega^k = \mathrm{d}A_{ij} - \sum_k A_{ik} \omega_j^k - \sum_k A_{kj} \omega_i^k, \tag{3.15}$$

$$\sum_k B_{ij,k}^\alpha \omega^k = \mathrm{d}B_{ij}^\alpha - \sum_k B_{ik}^\alpha \omega_j^k - \sum_k B_{kj}^\alpha \omega_i^k + \sum_\beta B_{ij}^\beta \omega_\beta^\alpha. \tag{3.16}$$

The curvature forms $\{\Omega_j^i\}$ and the normal curvature forms $\{\Omega_\beta^\alpha\}$ of submanifold $x : \mathbf{M} \to \mathbb{Q}_p^n$ can be written as

$$\Omega_j^i = \frac{1}{2} \sum_{kl} R^i{}_{jkl} \omega^k \wedge \omega^l = \omega^i \wedge \psi_j + \psi^i \wedge \omega_j - \sum_\alpha \omega_\alpha^i \wedge \omega_j^\alpha, \tag{3.17}$$

$$\Omega_\beta^\alpha = \frac{1}{2} \sum_{kl} R^\alpha{}_{\beta kl} \omega^k \wedge \omega^l = -\sum_i \omega_i^\alpha \wedge \omega_\beta^i. \tag{3.18}$$

Denote

$$\begin{aligned} &g_{ij} = \langle Y_i, Y_j \rangle, \quad g_{\beta\gamma} = \langle \xi_\beta, \xi_\gamma \rangle, \quad (g^{ij}) = (g_{ij})^{-1}, \quad (g^{\beta\gamma}) = (g_{\beta\gamma})^{-1}, \\ &R_{ijkl} = \sum_p g_{it} R^p{}_{jkl}, \quad R_{\alpha\beta kl} = \sum_\nu g_{\alpha\nu} R^\nu{}_{\beta kl}. \end{aligned}$$

Then the integrable conditions of the structure equations contain

$$A_{ij,k} - A_{ik,j} = -\sum_{\alpha\beta} g_{\alpha\beta}(B^{\alpha}_{ij}C^{\beta}_{k} - B^{\alpha}_{ik}C^{\beta}_{j}), \quad B^{\alpha}_{ij,k} - B^{\alpha}_{ik,j} = g_{ij}C^{\alpha}_{k} - g_{ik}C^{\alpha}_{j}, \tag{3.19}$$

$$C^{\alpha}_{i,j} - C^{\alpha}_{j,i} = \sum_{kl} g^{kl}(B^{\alpha}_{ik}A_{lj} - B^{\alpha}_{jk}A_{li}), \quad R_{\alpha\beta ij} = \sum_{kl\gamma\nu} g_{\alpha\gamma}g_{\beta\nu}g^{kl}(B^{\gamma}_{ik}B^{\nu}_{lj} - B^{\nu}_{ik}B^{\gamma}_{lj}), \tag{3.20}$$

$$R_{ijkl} = \sum_{\alpha\beta} g_{\alpha\beta}(B^{\alpha}_{ik}B^{\beta}_{jl} - B^{\alpha}_{il}B^{\beta}_{jk}) + (g_{ik}A_{jl} - g_{il}A_{jk}) + (A_{ik}g_{jl} - A_{il}g_{jk}). \tag{3.21}$$

Furthermore, we have

$$\mathrm{tr}(\mathbb{A}) = \frac{1}{2m}(m^2\kappa \pm 1), \quad R_{ij} = \mathrm{tr}(\mathbb{A})g_{ij} + (m-2)A_{ij} - \sum_{kl\alpha\beta} g^{kl}g_{\alpha\beta}B^{\alpha}_{ik}B^{\beta}_{lj}, \tag{3.22}$$

$$(1-m)C^{\alpha}_{i} = \sum_{jk} g^{jk}B^{\alpha}_{ij,k}, \quad \sum_{ijkl\alpha\beta} g^{ij}g^{kl}g_{\alpha\beta}B^{\alpha}_{ik}B^{\beta}_{jl} = \frac{m-1}{m}, \quad \sum_{ij} g^{ij}B^{\alpha}_{ij} = 0, \quad \forall\alpha. \tag{3.23}$$

From the above, we know that in the case $m \geq 3$ all coefficients in the PDE system (3.10)–(3.12) are determined by the conformal metric g, the conformal second fundamental form $\mathbb{B}$ and the normal connection $\{\omega^{\beta}_{\alpha}\}$ in the conformal normal bundle $\mathbb{V}$. Then we have the following theorem.

Theorem 3.3 *Two hypersurfaces $x : \mathbf{M}^m \to \mathbb{Q}^{m+1}_p$ and $\widetilde{x} : \widetilde{\mathbf{M}}^m \to \mathbb{Q}^{m+1}_p$ $(m \geq 3)$ are conformal equivalent if and only if there exists a diffeomorphism $f : \mathbf{M} \to \widetilde{\mathbf{M}}$ which preserves the conformal metric and the conformal second fundamental form. In other word, $\{g, \mathbb{B}\}$ is a complete invariants system of the hypersurface $x : \mathbf{M}^m \to \mathbb{Q}^{m+1}_p$ $(m \geq 3)$.*

Next we give the relations between the conformal invariants induced above and isometric invariants of $u : \mathbf{M} \to \mathbb{R}^n_p$. We also give a conformal fundamental theorem for hypersurfaces in $\mathbb{R}^n_p$. The pseudo-Euclidean space $\mathbb{R}^n_p$ has a non-degenerate inner product $\langle\, ,\, \rangle$, whose signature is $(\underbrace{+,\cdots,+}_{(n-p)\text{-tiple}}, \underbrace{-,\cdots,-}_{p\text{-tiple}})$. From the conformal map

$$\sigma : \mathbb{R}^n_p \to \mathbb{Q}^n_p, \quad u \mapsto \Big[\Big(\frac{\langle u,u\rangle - 1}{2}, u, \frac{\langle u,u\rangle + 1}{2}\Big)\Big], \tag{3.24}$$

we may recognize that $\mathbb{R}^n_p \subset \mathbb{Q}^n_p$. Let $u : \mathbf{M} \to \mathbb{R}^n_p$ be a submanifold, $\{e_1, \cdots, e_m\}$ a local basis for u with a dual basis $\{\omega^1, \cdots, \omega^m\}$, and $\{e_{m+1}, \cdots, e_n\}$ a local basis of the normal bundle of u in $\mathbb{R}^n_p$. Then we have the first and second fundamental forms I, II and the mean curvature vector $\mathbf{H}$. We may write

$$\mathrm{I} = \sum_{ij} I_{ij}\omega^i \otimes \omega^j, \quad \mathrm{II} = \sum_{ij\alpha} h^{\alpha}_{ij}\omega^i \otimes \omega^j e_{\alpha},$$

$$(I^{ij}) = (I_{ij})^{-1}, \quad \mathbf{H} = \frac{1}{m}\sum_{ij\alpha} I^{ij}h^{\alpha}_{ij}e_{\alpha} := \sum_{\alpha} H^{\alpha}e_{\alpha}.$$

From the structure equations

$$\mathrm{d}u = \sum_i \omega^i u_i, \quad \mathrm{d}u_i = \sum_j \theta^j_i u_j + \sum_{\alpha} \theta^{\alpha}_i e_{\alpha}, \quad \mathrm{d}e_{\alpha} = \sum_j \theta^j_{\alpha} u_j + \sum_{\beta} \theta^{\beta}_{\alpha} e_{\beta}, \tag{3.25}$$

we have

$$\sum_j u_{i,j}\omega^j = \mathrm{d}u_i - \sum_j \theta_i^j u_j = \sum_\alpha \theta_i^\alpha e_\alpha, \quad u_{i,j} = \sum_\alpha h_{ij}^\alpha e_\alpha, \tag{3.26}$$

where $\{\theta_i^j\}$ are the connection forms, $\{\theta_i^\alpha\}$ are the second fundamental forms, and $\{\theta_\alpha^\beta\}$ are the normal connection forms of the submanifold u. Denote by $\Delta_{\mathbf{M}}$ the Laplacian, and by $\kappa_{\mathbf{M}}$ the normalized scalar curvature for I. It is easy to see that

$$\Delta_{\mathbf{M}} u = m\mathbf{H}, \quad \kappa_{\mathbf{M}} = \frac{1}{m(m-1)}(m^2|\mathbf{H}|^2 - |\mathrm{II}|^2), \tag{3.27}$$

where

$$|\mathbf{H}|^2 = \sum_{\alpha\beta} I_{\alpha\beta} H^\alpha H^\beta, \quad I_{\alpha\beta} = (e_\alpha, e_\beta), \quad |\mathrm{II}|^2 = \sum_{ijkl\alpha\beta} I_{\alpha\beta} I^{ik} I^{jl} h_{ij}^\alpha h_{kl}^\beta.$$

For $x = \sigma \circ u : M \to \mathbb{R}_p^n$, there is a global lift

$$y : \mathbf{M} \to C^{n+1}, \quad y = \Big(\frac{\langle u, u\rangle - 1}{2}, u, \frac{\langle u, u\rangle + 1}{2}\Big).$$

So we get

$$\langle \mathrm{d}y, \mathrm{d}y\rangle = \langle \mathrm{d}u, \mathrm{d}u\rangle = \mathrm{I}, \quad \Delta = \Delta_{\mathbf{M}}, \quad \kappa = \kappa_{\mathbf{M}}. \tag{3.28}$$

It follows from (3.25) that

$$\langle \Delta Y, \Delta Y\rangle - m^2\kappa = \frac{m}{m-1}(|\mathrm{II}|^2 - m|\mathbf{H}|^2). \tag{3.29}$$

Therefore, we get the conformal metric of x

$$g = \pm\frac{m}{m-1}(|\mathrm{II}|^2 - m|\mathbf{H}|^2)\langle \mathrm{d}u, \mathrm{d}u\rangle := \mathrm{e}^{2\tau}\mathrm{I}. \tag{3.30}$$

Let

$$y_i = e_i(y) = (0, u_i, 0) + \langle u, u_i\rangle(1, \mathbf{0}, 1), \quad \zeta_\alpha = (0, e_\alpha, 0) + \langle u, e_\alpha\rangle(1, \mathbf{0}, 1).$$

By a direct calculation, we get

$$Y = \mathrm{e}^\tau y, \quad Y_i = e_i(Y) = \mathrm{e}^\tau(\tau_i y + y_i), \quad \xi_\alpha = H_\alpha y + \zeta_\alpha, \tag{3.31}$$

$$-\mathrm{e}^\tau N = \frac{1}{2}(|\nabla\tau|^2 + |\mathbf{H}|^2)y + \sum_i \tau^i y_i + \sum_\alpha H^\alpha \zeta_\alpha + (1, \mathbf{0}, 1), \tag{3.32}$$

where $\tau^i = \sum_j I^{ij}\tau_j$, $(I^{ij}) = (I_{ij})^{-1}$, $|\nabla\tau|^2 = \sum_i \tau_i\tau^i$, $H_\alpha = \sum_\beta I_{\alpha\beta}H^\beta$.

By a direct calculation, we get the following expression of the conformal invariants $\mathbb{A}$, $\mathbb{B}$ and Φ:

$$A_{ij} = \tau_i\tau_j - \sum_\alpha h_{ij}^\alpha H_\alpha - \tau_{i,j} - \frac{1}{2}(|\nabla\tau|^2 + |\mathbf{H}|^2)I_{ij}, \tag{3.33}$$

$$B_{ij}^\alpha = \mathrm{e}^\tau(h_{ij}^\alpha - H^\alpha I_{ij}), \quad \mathrm{e}^\tau C_i^\alpha = H^\alpha\tau_i - \sum_j h_{ij}^\alpha\tau^j - H_{,i}^\alpha, \tag{3.34}$$

where $\tau_{i,j}$ is the Hessian of τ respect to I, and $H^{\alpha}_{,i}$ is the covariant derivative of the mean curvature vector field of u in the normal bundle $N(\mathbf{M})$ with respect to I.

Now we consider the case when $u : \mathbf{M} \to \mathbb{R}^n_p$ is a hypersurface. Observing the PDE system (3.10)–(3.12), from Theorem 3.3, we have the following theorem.

Theorem 3.4 *Two hypersurfaces* $u, \widetilde{u} : \mathbf{M} \to \mathbb{R}^n_p$ $(n \geq 4)$ *are conformally equivalent if and only if there exists a diffeomorphism* $f : \mathbf{M} \to \mathbf{M}$*, which preserves the conformal metric and the conformal second fundamental form* $\{g, \mathbb{B}\}$.

Remark 3.3 For the pseudo-Riemannian sphere $\mathbb{S}^n_p$ and the pseudo-Riemannian hyperbolic space $\mathbb{H}^n_p$, we know that their sectional curvatures are ± 1. We have the similar equations about the conformal invariants of the submanifolds u of a pseudo-Riemannian sphere space or a pseudo-Riemannian hyperbolic space with an index p,

$$\Delta_{\mathbf{M}} u = m(\mathbf{H} - \epsilon u), \quad \kappa_{\mathbf{M}} = \frac{1}{m(m-1)}(m^2|\mathbf{H}|^2 - |\mathrm{II}|^2), \tag{3.35}$$

$$A_{ij} = \tau_i \tau_j - \sum_{\alpha} h^{\alpha}_{ij} H_{\alpha} - \tau_{i,j} - \frac{1}{2}(|\nabla \tau|^2 + |\mathbf{H}|^2 - \epsilon) I_{ij}, \tag{3.36}$$

$$B^{\alpha}_{ij} = \mathrm{e}^{\tau}(h^{\alpha}_{ij} - H^{\alpha} I_{ij}), \quad \mathrm{e}^{\tau} C^{\alpha}_i = H^{\alpha}\tau_i - \sum_j h^{\alpha}_{ij}\tau^j - H^{\alpha}_{,i}, \tag{3.37}$$

where ϵ corresponds to the sectional curvature of the pseudo-Riemannian sphere space or the pseudo-Riemannian hyperbolic space with the index p. When $\epsilon = 1$, the above equations are due to the pseudo-Riemannian sphere space $\mathbb{S}^n_p$; when $\epsilon = -1$, the above ones are due to the pseudo-Riemannian hyperbolic space $\mathbb{H}^n_p$.

4 The First Variation of the Conformal Volume Functional

Let $x_0 : \mathbf{M} \to \mathbb{Q}^n_p$ be a compact oriented regular submanifold with the index s $(0 \leq s \leq p)$ and a boundary $\partial\mathbf{M}$. Suppose that the local basis $\{e_1, \cdots, e_m\}$ on $\mathbf{M}$ satisfies the orientation. Denote $g_{ij} = g(e_i, e_j)$. We recall that if $(g_{ij}) = (-I_s) \oplus (I_{m-s})$, we call $\{e_1, \cdots, e_m\}$ a local orthonormal basis with respect to g. In what follows, let $\{e_1, \cdots, e_m\}$ be a local orthonormal basis for g with a dual basis $\{\omega^1, \cdots, \omega^m\}$.

We define the generalized Willmore functional $\mathbb{W}(\mathbf{M})$ as the volume functional of the conformal metric g:

$$\mathbb{W}(\mathbf{M}) = \mathrm{Vol}_g(\mathbf{M}) = \int_{\mathbf{M}} \mathrm{d}\mathbf{M}_g.$$

The conformal volume element $\mathrm{d}\mathbf{M}_g$ is defined by

$$\mathrm{d}\mathbf{M}_g = \omega^1 \wedge \cdots \wedge \omega^m,$$

which is well-defined.

Let $x : \mathbf{M} \times \mathbb{R} \to \mathbb{Q}^n_p$ be an admissible variation of x_0, such that $x(\,\cdot\,, t) = x_t$ and $x_{t*}(\mathrm{T}_p\mathbf{M}) = x_{0*}(\mathrm{T}_p\mathbf{M})$ on $\partial\mathbf{M}$ for each small t. For each t, x_t has the conformal metric g_t. As the similar procedure in Section 3, for each small t, we have a moving frame $\{Y, N, Y_i, \xi_\alpha\}$ in $\mathbb{R}^{n+2}_{p+1}$ along

$\mathbf{M}\times\mathbb{R}$ and the conformal volume $W(t)=\mathbb{W}(x_t)$. Let $\{\xi_\alpha\}$ be a local orthonormal basis for the conformal normal bundle $\mathbb{V}_t$ of x_t. Denote by $\widetilde{\mathrm{d}}$ and d the differential operators on $\mathbf{M}\times\mathbb{R}$ and $\mathbf{M}$, respectively. Then we have

$$\widetilde{\mathrm{d}}=\mathrm{d}+\mathrm{d}t\wedge\frac{\partial}{\partial t}, \tag{4.1}$$

on $\mathrm{T}^*(\mathbf{M}\times\mathbb{R})=\mathrm{T}^*\mathbf{M}\oplus\mathrm{T}^*\mathbb{R}$. We also have

$$\mathrm{d}\circ\frac{\partial}{\partial t}=\frac{\partial}{\partial t}\circ\mathrm{d}. \tag{4.2}$$

Denote $P=(Y,N,Y_1,\cdots,Y_m,\xi_{m+1},\cdots,\xi_n)^{\mathrm{T}}$. Suppose that

$$\mathrm{d}P=\Omega P,\quad \frac{\partial}{\partial t}P=LP,$$

where

$$\Omega=\begin{pmatrix} 0 & 0 & \omega^1 & \cdots & \omega^m & 0 & \cdots & 0\\ 0 & 0 & \psi^1 & \cdots & \psi^m & \phi^{m+1} & \cdots & \phi^n\\ -\psi_1 & -\omega_1 & \omega_1^1 & \cdots & \omega_1^m & \omega_1^{m+1} & \cdots & \omega_1^n\\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots\\ -\psi_m & -\omega_m & \omega_m^1 & \cdots & \omega_m^m & \omega_m^{m+1} & \cdots & \omega_m^n\\ -\phi_{m+1} & 0 & \omega_{m+1}^1 & \cdots & \omega_{m+1}^m & \omega_{m+1}^{m+1} & \cdots & \omega_{m+1}^n\\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots\\ -\phi_n & 0 & \omega_n^1 & \cdots & \omega_n^m & \omega_n^{m+1} & \cdots & \omega_n^n \end{pmatrix},$$

$$L=\begin{pmatrix} w & 0 & v^1 & \cdots & v^m & v^{m+1} & \cdots & v^n\\ 0 & -w & u^1 & \cdots & u^m & u^{m+1} & \cdots & u^n\\ -u_1 & -v_1 & L_1^1 & \cdots & L_1^m & L_1^{m+1} & \cdots & L_1^n\\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots\\ -u_m & -v_m & L_m^1 & \cdots & L_m^m & L_m^{m+1} & \cdots & L_m^n\\ -u_{m+1} & -v_{m+1} & L_{m+1}^1 & \cdots & L_{m+1}^m & L_{m+1}^{m+1} & \cdots & L_{m+1}^n\\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots\\ -u_n & -v_n & L_n^1 & \cdots & L_n^m & L_n^{m+1} & \cdots & L_n^n \end{pmatrix}.$$

From (4.2), it is easy to get

$$\frac{\partial}{\partial t}\Omega=\mathrm{d}L+L\Omega-\Omega L. \tag{4.3}$$

Therefore, we have

$$\frac{\partial\omega^i}{\partial t}=\sum_j\Big(v^i_{,j}+L^i_j-\sum_{k\alpha\beta}g_{\alpha\beta}v^\alpha B^\beta_{kj}g^{ik}\Big)\omega^j+\sum_\alpha v^\alpha\omega^i_\alpha+w\omega^i,\quad L^\alpha_i=v^\alpha_{,i}+\sum_j B^\alpha_{ij}v^j, \tag{4.4}$$

where $\{v^i_{,j}\}$ is the covariant derivative of $\sum v^ie_i$ with respect to g_t and $\{v^\alpha_{,i}\}$ is the covariant derivative of $\sum v^\alpha\xi_\alpha$. Here we use the notations of conformal invariants $\{A_{ij},B^\alpha_{ij},C^\alpha_i\}$ for x_t defined in Section 3. Furthermore, we have

$$\frac{\partial\omega^\alpha_i}{\partial t}=\sum_j\Big(L^\alpha_{i,j}+\sum_k L^k_iB^\alpha_{kj}-\sum_\beta B^\beta_{ij}L^\alpha_\beta+A_{ij}v^\alpha-v_iC^\alpha_j\Big)\omega^j+u^\alpha\omega_i, \tag{4.5}$$

where $\{L^\alpha_{i,j}\}$ is the covariant derivative of $\sum\limits_{i\alpha} L^\alpha_i \omega^i \xi_\alpha$. Using (4.4) and (4.5), we get

$$\begin{aligned}\frac{\partial B^\alpha_{ij}}{\partial t} + wB^\alpha_{ij} = v^\alpha_{,ij} + A_{ij}v^\alpha + \sum_{kl\gamma} g^{kl}B^\alpha_{ik}B^\gamma_{lj}v_\gamma + u^\alpha g_{ij} + \sum_k L^k_i B^\alpha_{kj} \\ - \sum_\gamma B^\gamma_{ij}L^\alpha_\gamma + \sum_k v^k B^\alpha_{ik,j} - v_i C^\alpha_j. \end{aligned} \tag{4.6}$$

It follows from (3.19) and (3.23) that

$$\frac{m-1}{m}w = \sum_{ijkl\alpha\beta} g_{\alpha\beta}g^{ik}g^{jl}B^\beta_{kl}\Big(v^\alpha_{,ij} + A_{ij}v^\alpha + \sum_{kl\gamma} g^{kl}B^\alpha_{ik}B^\gamma_{lj}v_\gamma\Big). \tag{4.7}$$

Definition 4.1 *For a regular submanifold $x_0 : \mathbf{M} \to \mathbb{Q}^n_p$, the conformal volume functional of an admissible variation $x : \mathbf{M} \times \mathbb{R} \to \mathbb{Q}^n_p$ is denoted by*

$$W(t) = \mathrm{vol}(g_t) = \int_{\mathbf{M}} \mathrm{d}\mathbf{M}_g,$$

where $\mathrm{d}\mathbf{M}_g$ is the volume for g_t. When $W'(0) = 0$ for any admissible variation x, we call x_0 a Willmore submanifold of the conformal space $\mathbb{Q}^n_p$.

Now we calculate the first variation of the conformal volume functional

$$W(t) = \int_{\mathbf{M}} \omega^1 \wedge \cdots \wedge \omega^m = \int_{\mathbf{M}} \mathrm{d}\mathbf{M}_g.$$

From (4.4), we get

$$\begin{aligned} W'(t) &= \sum_i \int_{\mathbf{M}} \omega^1 \wedge \cdots \wedge \frac{\partial \omega^i}{\partial t} \wedge \cdots \wedge \omega^m \\ &= \sum_i \int_{\mathbf{M}} \omega^1 \wedge \cdots \wedge \Big[\sum_j \Big(v^i_{,j} + L^i_j - \sum_{k\alpha\beta} g_{\alpha\beta}v^\alpha B^\beta_{kj}g^{ik}\Big)\omega^j + \sum_\alpha v^\alpha \omega^i_\alpha + w\omega^i\Big] \wedge \cdots \wedge \omega^m \\ &= \int_{\mathbf{M}} \sum_i \Big(v^i_{,i} + L^i_i - \sum_{k\alpha\beta} g_{\alpha\beta}v^\alpha B^\beta_{ki}g^{ik}\Big)\mathrm{d}\mathbf{M}_g \\ &\quad - \int_{\mathbf{M}} \sum_{i,j}\sum_{\alpha,\beta} g_{\alpha\beta}g^{ij}v^\alpha B^\beta_{ij}\mathrm{d}\mathbf{M}_g + m\int_{\mathbf{M}} w\mathrm{d}\mathbf{M}_g \\ &= \int_{\mathbf{M}} \sum_i v^i_{,i}\mathrm{d}\mathbf{M}_g + m\int_{\mathbf{M}} w\mathrm{d}\mathbf{M}_g. \end{aligned} \tag{4.8}$$

From the assumption that the variation is admissible, we know $v^i = 0, v^\alpha = 0$ and $v^\alpha_{,i} = 0$ on $\partial\mathbf{M}$. It follows from (4.7) and Green's formula that

$$\begin{aligned} W'(t) = \frac{m^2}{m-1}\int_{\mathbf{M}} \sum_\alpha v^\alpha \Big[\sum_{ijkl\beta} g_{\alpha\beta}g^{ik}g^{jl}\Big(B^\beta_{ij,kl} + A_{ij}B^\beta_{kl} \\ + \sum_{rq\gamma\nu} g_{\gamma\nu}g^{rq}B^\beta_{ir}B^\gamma_{qj}B^\nu_{kl}\Big)\Big]\mathrm{d}\mathbf{M}_g. \end{aligned} \tag{4.9}$$

It follows from (4.9) that the following theorem holds.

Theorem 4.1 *The variation of the conformal volume functional depends only on the normal component of the variation field* $\frac{\partial Y}{\partial t}$. *A submanifold* $x : \mathbf{M} \to \mathbb{Q}^n_p$ *is a Willmore submanifold* (*i.e., a critical submanifold to the conformal volume functional*) *if and only if*

$$\sum_{ijkl\beta} g_{\alpha\beta} g^{ik} g^{jl} \Big(B^\beta_{ij,kl} + A_{ij} B^\beta_{kl} + \sum_{rq\gamma\nu} g_{\gamma\nu} g^{rq} B^\beta_{ir} B^\gamma_{qj} B^\nu_{kl} \Big) = 0, \quad \forall \alpha. \tag{4.10}$$

We call equation (4.10) the Euler-Lagrange equations or the Willmore equations. Using (3.22) and (3.23), we can write the Willmore equations (4.10) as

$$\sum_\beta g_{\alpha\beta} \Big[\sum_{ij} g^{ij} C^\beta_{i,j} + \sum_{ijkl} g^{ik} g^{jl} \Big(\frac{1}{m-1} R_{ij} - A_{ij} \Big) B^\beta_{kl} \Big] = 0, \quad \forall \alpha. \tag{4.11}$$

Definition 4.2 (see [1]) *A submanifold in a pseudo-Riemannian manifold is called stationary when its mean curvature vector is vanishing.*

Theorem 4.2 *Any stationary regular surface in the pseudo-Euclidean space* $\mathbb{R}^n_p$, *the pseudo-Riemannian sphere space* $\mathbb{S}^n_p$ *and the pseudo-Riemannian hyperbolic space* $\mathbb{H}^n_p$ *is Willmore.*

Proof Let $u : \mathbf{M} \to \mathbb{R}^n_p$ be a regular surface, whether space-like or time-like. Let $\{e_1, e_2\}$ be a local basis of $\langle \mathrm{d}u, \mathrm{d}u \rangle$ and $\{e_\alpha\}^n_{\alpha=3}$ a local basis for the normal bundle. If x is a stationary regular surface, we have $H^\alpha \equiv 0$, $\forall \alpha$. From (3.33) and (3.34), we get

$$\sum_{ijkl} g^{ik} g^{jl} A_{ij} B^\beta_{kl} = \sum_{ijkl} g^{ik} g^{jl} B^\beta_{kl} (\tau_i \tau_j - \tau_{i,j}) = \mathrm{e}^{-3\tau} \sum_{ijkl} I^{ik} I^{jl} h^\beta_{kl} (\tau_i \tau_j - \tau_{i,j}). \tag{4.12}$$

Now we know from (3.34) that

$$-\mathrm{e}^\tau C^\beta_i = \sum_{kl} I^{kl} h^\beta_{ik} \tau_l := W^\beta_i. \tag{4.13}$$

From (3.14), we have

$$\begin{aligned} \sum_j \mathrm{e}^\tau C^\beta_{i,j} \omega^j &= \mathrm{d}(\mathrm{e}^\tau C^\beta_i) - \mathrm{e}^\tau C^\beta_i \mathrm{d}\tau + \sum_\gamma \mathrm{e}^\tau C^\gamma_i \theta^\beta_\gamma - \sum_k \mathrm{e}^\tau C^\beta_k \omega^k_i \\ &= -\mathrm{d}W^\beta_i + W^\beta_i \mathrm{d}\tau - \sum_\gamma W^\gamma_i \theta^\beta_\gamma + \sum_k W^\beta_k \omega^k_i. \end{aligned} \tag{4.14}$$

Combining

$$\omega^k_i = \theta^k_i + \tau^k \sum_j I_{ij} \omega^j - \tau_i \omega^k + \delta^k_i \mathrm{d}\tau$$

and (4.14), we get

$$\mathrm{e}^\tau C^\beta_{i,j} = 2W^\beta_i \tau_j + W^\beta_j \tau_i - \sum_k W^\beta_k \tau^k I_{ij} - W^\beta_{i,j}, \tag{4.15}$$

where $W^\beta_{i,j}$ is the covariant differential of W^β_i with respect to the first fundamental form I of u. Therefore

$$\sum_{ij} g^{ij} C^\beta_{i,j} = \mathrm{e}^{-3\tau} \sum_{ijkl} I^{ik} I^{jl} h^\beta_{kl} (\tau_i \tau_j - \tau_{i,j}). \tag{4.16}$$

Whether the regular surface u is space-like or time-like, if we choose orthonormal $\{e_1, e_2\}$, then a direct calculation leads to

$$\sum_{ijkl} g^{ik} g^{jl} R_{ij} B^{\beta}_{kl} = 0. \tag{4.17}$$

Thus we have (4.11) from (4.12), (4.16) and (4.17), which implies that u is Willmore.

One can verify that stationary regular surfaces in $\mathbb{S}^n_p$ and $\mathbb{H}^n_p$ are also Willmore.

5 Conformal Isotropic Submanifolds in $\mathbb{Q}^n_p$

In this section let $x : \mathbf{M} \to \mathbb{Q}^n_p$ be an m-dimensional submanifold with the index s $(0 \le s \le p)$.

Definition 5.1 *We call an m-dimensional submanifold $x : \mathbf{M} \to \mathbb{Q}^n_p$ conformal isotropic, if there exists a smooth function λ on $\mathbf{M}$, such that*

$$\mathbb{A} \equiv \lambda g, \quad \Phi \equiv 0. \tag{5.1}$$

From previous discussions in Section 3, one can easily verify the following proposition.

Proposition 5.1 *If $u : \mathbf{M} \to \mathbb{R}^n_p$ is a stationary regular submanifold with a constant scalar curvature, then $x = \sigma \circ u$ is a conformal isotropic submanifold in $\mathbb{Q}^n_p$.*

Remark 5.1 The same conclusion holds for such submanifolds in $\mathbb{S}^n_p$ or $\mathbb{H}^n_p$.

Suppose that $x : \mathbf{M} \to \mathbb{Q}^n_p$ is a conformal isotropic submanifold. Then from (5.1) and (3.10), we get

$$\mathrm{d}N = \lambda \mathrm{d}Y, \quad \mathrm{d}\lambda \wedge \mathrm{d}Y = \sum_{i=1}^{m} (\mathrm{d}\lambda \wedge \omega^i) Y_i = 0. \tag{5.2}$$

Since $\{Y_1, \cdots, Y_m\}$ are linearly independent,

$$\mathrm{d}\lambda \wedge \omega^i = \sum_{j=1}^{m} E_j(\lambda) \omega^j \wedge \omega^i = 0.$$

If $\mathbf{M}$ is connected, we get

$$\lambda = \text{constant}, \tag{5.3}$$

which implies by (3.22) that

$$\kappa = \text{constant}.$$

In fact, if we take the trace of the first equation of (5.1), we will find that

$$\lambda = \frac{1}{m} \mathrm{tr}(\mathbb{A}) = \frac{1}{2m^2} (m^2 \kappa \pm 1). \tag{5.4}$$

Combining (3.5) and (3.6), we get

$$N = -\frac{1}{m} \Delta Y - \lambda Y. \tag{5.5}$$

Therefore, by (5.2), we can find a constant vector $\mathbf{c} \in \mathbb{R}^{n+2}_{p+1}$, such that

$$N = \lambda Y + \mathbf{c}. \tag{5.6}$$

It follows that

$$\langle Y, \mathbf{c}\rangle = 1, \quad \langle \mathbf{c}, \mathbf{c}\rangle = -2\lambda = \text{constant}, \quad \Delta Y = -m(2\lambda Y + \mathbf{c}). \tag{5.7}$$

Then we distinguish three cases.

Case 1 $\boldsymbol{\langle \mathbf{c}, \mathbf{c}\rangle = -2\lambda = 0}$

By use of an isometric transformation of $\mathbb{R}^{n+2}_{p+1}$, if it is necessary, we assume

$$\mathbf{c} = (-1, \mathbf{0}, -1). \tag{5.8}$$

Letting

$$Y = (x_1, u, x_{n+2}), \tag{5.9}$$

by (5.7) and $Y \in C^{n+1}$, we have

$$Y = \Big(\frac{\langle u, u\rangle - 1}{2}, u, \frac{\langle u, u\rangle + 1}{2}\Big). \tag{5.10}$$

Then x determines a submanifold $u : \mathbf{M} \to \mathbb{R}^n_p$ with the first fundamental form

$$\mathrm{I} = \langle \mathrm{d}u, \mathrm{d}u\rangle = \langle \mathrm{d}Y, \mathrm{d}Y\rangle = g,$$

which implies that

$$\Delta_{\mathbf{M}} = \Delta, \quad \kappa_{\mathbf{M}} = \kappa = \mp\frac{1}{m}. \tag{5.11}$$

From (5.5) and (5.6), we have

$$\Delta Y = -m(2\lambda Y + \mathbf{c}) = (m, \mathbf{0}, m), \quad \Delta_{\mathbf{M}} u = \mathbf{0}.$$

It is implied by (3.27) that $H^\alpha = 0$, i.e., u is a stationary submanifold in $\mathbb{R}^n_p$. In this case, x is conformally equivalent to the image of σ of a stationary submanifold with a constant scalar curvature in $\mathbb{R}^n_p$.

Case 2 $\boldsymbol{\langle \mathbf{c}, \mathbf{c}\rangle = -2\lambda = -r^2}$ **$(r > 0)$**

By use of an isometric transformation of $\mathbb{R}^{n+2}_{p+1}$, if it is necessary, we assume

$$\mathbf{c} = (\mathbf{0}, r). \tag{5.12}$$

Letting

$$Y = \Big(\frac{u}{r}, x_{n+2}\Big), \tag{5.13}$$

by (5.7), we have

$$x_{n+2} = \frac{1}{r}. \tag{5.14}$$

So

$$Y = \frac{(u, 1)}{r}, \quad \langle u, u \rangle = 1. \tag{5.15}$$

Then x determines a submanifold $u : \mathbf{M} \to \mathbb{S}^n_p$ with

$$\frac{\mathrm{I}}{r^2} = \frac{\langle \mathrm{d}u, \mathrm{d}u \rangle}{r^2} = \langle \mathrm{d}Y, \mathrm{d}Y \rangle = g,$$

which implies that

$$r^2 \Delta_{\mathbf{M}} = \Delta, \quad \kappa_{\mathbf{M}} = \frac{\kappa}{r^2} = \text{constant}.$$

From (5.5) and (5.6), we have

$$\Delta Y = -m(2\lambda Y + \mathbf{c}) = (-mr^2 u, 0), \quad \Delta_{\mathbf{M}} u = -mu.$$

It is implied by (3.35) that $H^\alpha = 0$, i.e., u is a stationary submanifold in $\mathbb{S}^n_p$. In this case, x is conformally equivalent to the image of σ_+ of a stationary submanifold with the constant scalar curvature in $\mathbb{S}^n_p$.

Case 3 $\quad \langle \mathbf{c}, \mathbf{c} \rangle = -2\lambda = r^2 \ (r > 0)$

By use of an isometric transformation of $\mathbb{R}^{n+2}_{p+1}$, if it is necessary, we assume

$$\mathbf{c} = (-r, \mathbf{0}). \tag{5.16}$$

Letting

$$Y = \left(x_1, \frac{u}{r} \right), \tag{5.17}$$

by (5.7), we have

$$x_1 = \frac{1}{r}. \tag{5.18}$$

So

$$Y = \frac{(1, u)}{r}, \quad \langle u, u \rangle = -1. \tag{5.19}$$

Then x determines a submanifold $u : \mathbf{M} \to \mathbb{H}^n_p$ with

$$\frac{\mathrm{I}}{r^2} = \frac{\langle \mathrm{d}u, \mathrm{d}u \rangle}{r^2} = \langle \mathrm{d}Y, \mathrm{d}Y \rangle = g,$$

which implies that

$$r^2 \Delta_{\mathbf{M}} = \Delta, \quad \kappa_{\mathbf{M}} = \frac{\kappa}{r^2} = \text{constant}.$$

From (5.5) and (5.6), we have

$$\Delta Y = -m(2\lambda Y + \mathbf{c}) = (mr^2 u, 0), \quad \Delta_{\mathbf{M}} u = mu.$$

It is implied by (3.35) that $H^\alpha = 0$, i.e., u is a stationary submanifold in $\mathbb{H}^n_p$. In this case, x is conformally equivalent to the image of σ_- of a stationary submanifold with a constant scalar curvature in $\mathbb{H}^n_p$.

So, combining Proposition 5.1 and Remark 5.1, we get the following theorem.

Theorem 5.1 *Any conformal isotropic submanifold in $\mathbb{Q}^n_p$ is conformally equivalent to a stationary submanifold with a constant scalar curvature in $\mathbb{R}^n_p$, $\mathbb{S}^n_p$ or $\mathbb{H}^n_p$.*

Acknowledgements The first author would like to express his gratitude to Prof. Changping Wang for his warm-hearted inspiration and the support of the Beijing International Center for Mathematical Research. The authors would like to thank the referees for their useful comments.

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