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Rigid Properties of Quasi-almost-Einstein Metrics*

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Abstract In this paper, quasi-almost-Einstein metrics on complete manifolds are studied. Two examples are given and several formulas are established. With the help of these formulas, the author proves rigid results on compact or noncompact manifolds, in which some basic tools, such as the weighted volume comparison theorem and the weak maximum principle at infinity, are used. A lower bound estimate for the scalar curvature is also obtained.

Keywords Quasi-almost-Einstein metric, Potential function, Gradient estimate, Rigid property
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1 Introduction

Let f be a smooth real-value function on an n-dimensional complete manifold M. When considering the weighted measure $\mathrm{d}\mu = \mathrm{e}^{-f}\,\mathrm{d}x$, we always use the m-dimensional Bakry-Émery curvature

$$\mathrm{Ric}_{f,m} = \mathrm{Ric} + \mathrm{Hess}\ f - \frac{\mathrm{d}f \otimes \mathrm{d}f}{m-n}$$

to replace the Ricci curvature, where $m \geq n$ and $m = n$ if and only if f is a constant (see [1–2]). There is an active interest in the study of the weighted measure under conditions about the m-dimensional Bakry-Émery curvature (see [3–4]).

We call a metric g m-dimensional quasi-Einstein with potential function f, if for some constant λ,

$$\mathrm{Ric}_{f,m} = \lambda \mathrm{g}. \tag{1.1}$$

This definition can be found in [5–7]. A quasi-Einstein metric becomes Einstein when the potential function is constant. Quasi-Einstein metrics were studied in some literature, and we can refer to [5–10] and the references therein. Some rigid results for quasi-Einstein metrics are obtained in [7]. The scalar curvature estimates, L^2_f-spectrum estimates and diameter estimates for quasi-Einstein metrics are considered in [8], [9] and [10], respectively.

An ∞-dimensional quasi-Einstein metric satisfies

$$\mathrm{Ric} + \mathrm{Hess}\ f = \lambda \mathrm{g} \tag{1.2}$$

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for some constant λ, which is a gradient Ricci soliton. Many people consider the generalization of the gradient Ricci soliton in different directions, and we can refer to [5, 11–12] and the references therein. In particular, the authors of [12] introduced the gradient Ricci almost soliton, which is a natural extension of the gradient Ricci soliton. A gradient Ricci almost soliton means that (1.2) holds for the potential function f and some smooth soliton function λ. Inspired by the gradient Ricci almost soliton, we propose an extension of the quasi-Einstein metric.

Definition 1.1 *We call a metric* g *m-dimensional quasi-almost-Einstein with potential function f, if* (1.1) *holds for some smooth soliton function λ. Moreover, we say that a quasi-almost-Einstein metric is shrinking, steady or expanding, if λ is positive, null or negative, respectively. If λ has no definitive sign, the quasi-almost-Einstein metric will be called indefinite.*

In Section 2, we give two examples of quasi-almost-Einstein metrics on some product manifolds. These two examples show that a certain flexibility on the quasi-almost-Einstein metric is allowed, and the reason is that the smooth soliton function λ is not necessarily constant.

We generalize the formulas in [7] for the quasi-Einstein metric to the quasi-almost-Einstein metric in Section 3. With the help of the moving frame in a local orthonormal coframe, we derive the expressions of $\triangle|\nabla f|^2$, ∇R and $\triangle R$, respectively, where R is the scalar curvature. It is proved in [7] that for a quasi-Einstein metric with potential function f and constant λ, there exists some constant μ, so that

$$R + \frac{m-n-1}{m-n}|\nabla f|^2 + (m-2n)\lambda = \mu \,\mathrm{e}^{\frac{2}{m-n}f}, \tag{1.3}$$

which generalizes a classical identity

$$R + |\nabla f|^2 - f = \mu \tag{1.4}$$

for the gradient Ricci soliton. We also derive a formula similar to (1.3) for a quasi-almost-Einstein metric satisfying $\lambda(x) = F(f(x))$.

For a gradient Ricci soliton, by adding constant μ to f, we can assume that $\mu = 0$ in (1.4), which means that μ can be neglected when we study a gradient Ricci soliton. From this observation, the authors of [13–14] proved that there does not exist a nontrivial expanding gradient Ricci soliton on a closed manifold. In [7], the author proves some rigid results for quasi-Einstein metrics on closed manifolds by using (1.3). If we add an integral condition to λ, we can get a rigid result for a closed quasi-almost-Einstein metric with $\lambda \le 0$. Moreover, when $\lambda = F(f)$, we can also derive a rigid result by using a formula similar to (1.3). We do these in Section 4.

The weak maximum principle at infinity is a basic tool in studying quasi-almost-Einstein metrics on noncompact manifolds. In Section 5, we prove a weighted volume monotone formula when the m-dimensional Bakry-Émery curvature is bounded from below. This monotone formula implies the weak maximum principle at infinity. In this section, we also introduce the weighted Laplacian comparison theorem (see [2, 4, 7]), which will be used in this paper.

We prove two rigid results for quasi-almost-Einstein metrics on noncompact manifolds in Section 6. The first one states that an expanding quasi-almost-Einstein metric is trivial if the potential function f satisfies a certain integral condition. The second one is a generalization of a rigid result for the quasi-Einstein metric.

In [15–16], the authors study the lower bound estimates of the scalar curvature for gradient Ricci solitons. These estimates are very useful in studying the geometry of gradient Ricci solitons (see [16–18]). The authors of [5] got some estimates of the scalar curvature for closed quasi-Einstein metrics. Recently, the author of [8] has obtained lower bound estimates of the scalar curvature for noncompact quasi-Einstein metrics. In Section 7, we prove a lower bound estimate of the scalar curvature for a noncompact quasi-almost-Einstein metric with $\lambda \le 0$.

2 Examples of Quasi-almost-Einstein Metric

In this section, we construct product manifolds with quasi-almost-Einstein metrics. We begin with a product manifold $M = \mathbb{R} \times N^{n-1}$ endowed with the warped product metric

$$\mathrm{d}s_M^2 = \mathrm{d}t^2 + \varphi^2(t)\mathrm{d}s_N^2,$$

where $\varphi(t) : \mathbb{R} \to [0, +\infty]$ is a smooth function and $\mathrm{d}s_N^2$ is a metric on N. Consider the orthonormal coframe $\{\theta_\alpha, 2 \le \alpha \le n\}$ on N^{n-1}, while $\{\omega_1 = \mathrm{d}t,\ \omega_\alpha = \varphi(t)\theta_\alpha,\ 2 \le \alpha \le n\}$ is the orthonormal coframe on M^n, and we also use $\omega_{M,ij}$ $(1 \le i, j \le n)$ and $\omega_{N,\alpha,\beta}$ $(2 \le \alpha, \beta \le n)$ to denote the connection 1-form on M and N, respectively. The Einstein summation convention will be in force throughout this section. By the first structure equation, we have

$$0 = \mathrm{d}\omega_1 = \omega_{M,1j} \wedge \omega_j, \tag{2.1}$$

$$\mathrm{d}\omega_\alpha = \omega_{M,\alpha j} \wedge \omega_j. \tag{2.2}$$

(2.2) can be rewritten as

$$\frac{\varphi'(t)}{\varphi(t)}\omega_1 \wedge \omega_\alpha + \omega_{N,\alpha\beta} \wedge \omega_\beta = -\omega_1 \wedge \omega_{M,\alpha 1} + \omega_{M,\alpha\beta} \wedge \omega_\beta.$$

Together with (2.1), we have

$$\omega_{M,1\alpha} = -\omega_{M,\alpha 1} = \frac{\varphi'(t)}{\varphi(t)}\omega_\alpha, \tag{2.3}$$

$$\omega_{M,\alpha\beta} = \omega_{N,\alpha\beta}. \tag{2.4}$$

Differentiating (2.3) and (2.4), along with the second structure equations, we conclude that

$$[(\log\varphi(t))'' + ((\log\varphi(t))')^2]\omega_1 \wedge \omega_\alpha = -\frac{1}{2}R_{M,1\alpha ij}\omega_i \wedge \omega_j, \tag{2.5}$$

$$((\log\varphi(t))')^2\omega_\alpha \wedge \omega_\beta + \frac{1}{2}R_{M,\alpha\beta ij}\omega_i \wedge \omega_j = \frac{1}{2}R_{N,\alpha\beta\gamma\theta}\theta_\gamma \wedge \theta_\theta, \tag{2.6}$$

where $R_{M,ijkl}$ and $R_{N,\alpha\beta\gamma\delta}$ denote the Riemannian curvature tensors of M and N, respectively. By (2.5)–(2.6), we conclude that

$$R_{M,1\alpha ij} = \begin{cases} -(\log\varphi(t))'' - ((\log\varphi(t))')^2, & i = 1,\ j = \alpha, \\ (\log\varphi(t))'' + ((\log\varphi(t))')^2, & i = \alpha,\ j = 1, \\ 0, & \text{otherwise}, \end{cases} \tag{2.7}$$

$$R_{M,\alpha\beta ij} = \begin{cases} \varphi^{-2}(t)R_{N,\alpha\beta\gamma\theta} + ((\log\varphi(t))')^2(\delta_{\alpha\theta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\theta}), & i = \gamma,\ j = \theta, \\ 0, & \text{otherwise}. \end{cases} \tag{2.8}$$

If we use $R_{N,\alpha\beta}$ to denote the Ricci curvature tensor on N, then by (2.7)–(2.8), the Ricci curvature tensor of M can be expressed as follows:

$$R_{M,1i} = -(n-1)[(\log\varphi(t))'' + ((\log\varphi(t))')^2]\delta_{1i}, \tag{2.9}$$

$$R_{M,\alpha\beta} = \varphi^{-2}(t)R_{N,\alpha\beta} - [(\log\varphi(t))'' + (n-1)((\log\varphi(t))')^2]\delta_{\alpha\beta}. \tag{2.10}$$

Example 2.1 For $m > n$, we assume that N is an Einstein manifold with

$$R_{N,\alpha\beta} = -m\delta_{\alpha\beta}. \tag{2.11}$$

Choose

$$f(t,x) = f(t) = -(m-n)\log\sinh t$$

and

$$\varphi(t) = \sinh t.$$

A few calculations show that

$$-(n-1)[(\log\varphi(t))'' + ((\log\varphi(t))')^2] + f''(t) - \frac{(f'(t))^2}{m-n} = -(m-1), \tag{2.12}$$

$$\varphi^{-2}(t)R_{N,\alpha\alpha} - [(\log\varphi(t))'' + (n-1)((\log\varphi(t))')^2] + f'(t)(\log\varphi(t))' = -(m-1). \tag{2.13}$$

(2.9)–(2.13) show that

$$R_{M,ij} + f_{ij} - \frac{f_i f_j}{m-n} = \lambda g_{ij}$$

holds for $\lambda = -(m-1)$. Hence, the product manifold $M = \mathbb{R}\times N^{n-1}$ is an m-dimensional quasi-Einstein with the potential function $f = -(m-n)\log\sinh t$ and the constant $\lambda = -(m-1)$.

Example 2.2 For $m > n \geq 3$, we assume that N is an $(n-1)$-dimensional Einstein manifold with

$$R_{N,\alpha\beta} = -\frac{m-2}{(m-n)(n-2)}\delta_{\alpha\beta}, \tag{2.14}$$

and $M = [0,+\infty)\times N^{n-1}$ is a product manifold endowed with the warped product metric

$$\mathrm{d}s_M^2 = \mathrm{d}t^2 + \varphi^2(t)\mathrm{d}s_N^2,$$

where

$$\varphi(t) = \frac{t}{n-2}.$$

Let

$$f(t) = (n-2)\log t.$$

A few calculations show that

$$-(n-1)[(\log\varphi(t))'' + ((\log\varphi(t))')^2] + f''(t) - \frac{(f'(t))^2}{m-n} = -\frac{(m-2)(n-2)}{(m-n)t^2}, \tag{2.15}$$

$$\begin{aligned}&\varphi^{-2}(t)R_{N,\alpha\alpha} - [(\log\varphi(t))'' + (n-1)((\log\varphi(t))')^2] + f'(t)(\log\varphi(t))' \\ &= -\frac{(m-2)(n-2)}{(m-n)t^2}.\end{aligned} \tag{2.16}$$

(2.9)–(2.10) and (2.14)–(2.16) show that

$$R_{M,ij} + f_{ij} - \frac{f_i f_j}{m-n} = \lambda g_{ij}$$

holds for

$$\lambda = -\frac{(m-2)(n-2)}{(m-n)t^2}.$$

Hence, the product manifold $M = [0, +\infty) \times N^{n-1}$ is an m-dimensional quasi-almost-Einstein with the potential function $f = (n-2)\log\frac{t}{n-2}$ and the soliton function $\lambda = -\frac{m-2}{(m-n)(n-2)t^2}$.

3 Some Formulas

In this section, we generalize the formulas in [7] for the quasi-Einstein metric to the quasi-almost-Einstein metric. The main idea comes from [19]. In what follows, in order to perform computations, we shall use the method of moving frame referring to a local orthonormal frame $\{e_i, i = 1, 2, \cdots, n\}$. As before, the Einstein summation convention will be in force. We firstly prove the following generalized Bochner's formula.

Lemma 3.1 *If* g *is an m-dimensional quasi-almost-Einstein metric with potential function f and smooth soliton function λ, then we have*

$$\frac{1}{2}\triangle|\nabla f|^2 = |\nabla^2 f|^2 - \mathrm{Ric}(\nabla f, \nabla f) + \frac{2}{m-n}|\nabla f|^2\triangle f - (n-2)\nabla\lambda\cdot\nabla f. \tag{3.1}$$

Proof The following Ricci identity is well-known:

$$f_{iji} = (\triangle f)_j + R_{ij}f_i. \tag{3.2}$$

Hence

$$\frac{1}{2}\triangle|\nabla f|^2 = f_{ij}^2 + f_{iji}f_j = |\nabla^2 f|^2 + \nabla f\cdot\nabla\triangle f + \mathrm{Ric}(\nabla f, \nabla f). \tag{3.3}$$

By taking covariant derivative of (1.1), we deduce

$$R_{ij,k} + f_{ijk} - \frac{1}{m-n}(f_{ik}f_j + f_i f_{jk}) = \lambda_k\delta_{ij}. \tag{3.4}$$

Tracing (3.4) and using the contracted second Bianchi identity

$$R_{ij,i} = \frac{1}{2}R_j \tag{3.5}$$

leads to

$$\frac{1}{2}R_j + f_{kjk} - \frac{1}{m-n}(\triangle f f_j + f_k f_{jk}) = \lambda_j. \tag{3.6}$$

By (3.2), we have

$$\frac{1}{2}\nabla R\cdot\nabla f + \nabla f\cdot\nabla\triangle f + \mathrm{Ric}(\nabla f, \nabla f) - \frac{1}{m-n}\Big[\triangle f|\nabla f|^2 + \frac{1}{2}\nabla f\cdot\nabla|\nabla f|^2\Big] = \nabla\lambda\cdot\nabla f. \tag{3.7}$$

We trace the quasi-almost-Einstein equation (1.1) to obtain

$$R+\triangle f-\frac{1}{m-n}|\nabla f|^2=n\lambda, \tag{3.8}$$

which means that

$$\nabla R\cdot\nabla f+\nabla f\cdot\nabla\triangle f-\frac{1}{m-n}\nabla|\nabla f|^2\cdot\nabla f=n\nabla\lambda\cdot\nabla f. \tag{3.9}$$

By (3.7) and (3.9), we have

$$\nabla\triangle f\cdot\nabla f=\frac{2}{m-n}\triangle f|\nabla f|^2-2\mathrm{Ric}(\nabla f,\nabla f)-(n-2)\nabla\lambda\cdot\nabla f. \tag{3.10}$$

(3.1) follows due to (3.3) and (3.10).

Lemma 3.2 *If* g *is an* m*-dimensional quasi-almost-Einstein metric with potential function* f *and smooth soliton function* λ*, then*

$$\frac{1}{2}\nabla R=(n-1)\nabla\lambda-\frac{n-1}{m-n}\lambda\nabla f+\frac{m-n-1}{m-n}\mathrm{Ric}(\nabla f)+\frac{1}{m-n}R\nabla f. \tag{3.11}$$

Proof By (1.1), we have

$$\nabla|\nabla f|^2=2\lambda\nabla f+\frac{2}{m-n}|\nabla f|^2\nabla f-2\mathrm{Ric}(\nabla f). \tag{3.12}$$

By (3.2), (3.6), (3.8) and (3.12), we have

$$\begin{aligned}\nabla R&=2\nabla\lambda-2(\nabla\triangle f+\mathrm{Ric}(\nabla f))+\frac{2}{m-n}\Big(\triangle f\nabla f+\frac{1}{2}\nabla|\nabla f|^2\Big)\\&=2\nabla\lambda-2\Big(n\nabla\lambda+\frac{1}{m-n}\nabla|\nabla f|^2-\nabla R\Big)-2\mathrm{Ric}(\nabla f)\\&\quad+\frac{2}{m-n}\Big(n\lambda+\frac{1}{m-n}|\nabla f|^2-R\Big)\nabla f+\frac{1}{m-n}\nabla|\nabla f|^2\\&=2(1-n)\nabla\lambda+2\nabla R-2\mathrm{Ric}(\nabla f)+\frac{2n}{m-n}\lambda\nabla f+\frac{2}{(m-n)^2}|\nabla f|^2\nabla f\\&\quad-\frac{2}{m-n}R\nabla f-\frac{1}{m-n}\Big[2\lambda\nabla f+\frac{2}{m-n}|\nabla f|^2\nabla f-2\mathrm{Ric}(\nabla f)\Big].\end{aligned}$$

Then (3.11) follows.

Corollary 3.1 *If* g *is an* m*-dimensional quasi-almost-Einstein metric with potential function* f *and soliton function* $\lambda=F(f)$*, where* $F(t)$ *is a smooth function, then there exists a constant* μ*, so that*

$$R+\frac{m-n-1}{m-n}|\nabla f|^2=\Big[2(n-1)G'(f)+\frac{2(m-2)}{m-n}G(f)+\mu\Big]\mathrm{e}^{\frac{2}{m-n}f}, \tag{3.13}$$

where

$$G(t)=\int F(t)\mathrm{e}^{-\frac{2}{m-n}t}\,\mathrm{d}t. \tag{3.14}$$

Proof Due to (1.1), (3.11) can be rewritten as

$$\begin{aligned}\nabla R &= 2(n-1)\nabla\lambda + \frac{2(m-2n)}{m-n}\lambda\nabla f + \frac{2}{m-n}R\nabla f \\ &\quad + \frac{2(m-n-1)}{(m-n)^2}|\nabla f|^2\nabla f - \frac{m-n-1}{m-n}\nabla|\nabla f|^2.\end{aligned} \tag{3.15}$$

By (3.15), the fact that $\nabla\lambda = F'(f)\nabla f$ and the definition of G, we deduce that

$$\nabla\Big[\Big(R + \frac{m-n-1}{m-n}|\nabla f|^2\Big)\mathrm{e}^{-\frac{2}{m-n}f} - 2(n-1)G'(f) - \frac{2(m-2)}{m-n}G(f)\Big] = 0,$$

so

$$\Big(R + \frac{m-n-1}{m-n}|\nabla f|^2\Big)\mathrm{e}^{-\frac{2}{m-n}f} - 2(n-1)G'(f) - \frac{2(m-2)}{m-n}G(f)$$

is constant, and (3.13) follows.

Remark 3.1 If λ is constant, we recover Theorem 2.2 in [7].

Lemma 3.3 *If* g *is an m-dimensional quasi-almost-Einstein metric with potential function f and smooth soliton function λ, then*

$$\begin{aligned}\frac{1}{2}\triangle R &= (n-1)\triangle\lambda - \frac{2(n-1)}{m-n}\nabla\lambda\cdot\nabla f + \frac{m-n+2}{2(m-n)}\nabla R\cdot\nabla f \\ &\quad - \frac{m-n-1}{m-n}\Big|\mathrm{Ric} - \frac{1}{n}R\mathrm{g}\Big|^2 - \frac{m-1}{(m-n)n}(R-n\lambda)\Big(R - \frac{n(n-1)}{m-1}\lambda\Big).\end{aligned} \tag{3.16}$$

Proof By (3.5) and (3.11), we get

$$\begin{aligned}\frac{1}{2}\triangle R &= (n-1)\triangle\lambda - \frac{n-1}{m-n}\nabla\lambda\cdot\nabla f + \frac{m-n+1}{2(m-n)}\nabla R\cdot\nabla f \\ &\quad - \frac{n-1}{m-n}\lambda\triangle f + \frac{m-n-1}{m-n}R_{ij}f_{ij} + \frac{1}{m-n}R\triangle f.\end{aligned} \tag{3.17}$$

Plugging (1.1), (3.8) and (3.11) into (3.17) yields

$$\begin{aligned}\frac{1}{2}\triangle R &= (n-1)\triangle\lambda - \frac{2(n-1)}{m-n}\nabla\lambda\cdot\nabla f + \frac{m-n+2}{2(m-n)}\nabla R\cdot\nabla f \\ &\quad - \frac{n(n-1)}{m-n}\lambda^2 + \frac{m+n-2}{m-n}\lambda R - \frac{R^2}{m-n} - \frac{m-n-1}{m-n}R_{ij}^2,\end{aligned}$$

which means (3.16).

4 Rigid Properties on Closed Manifolds

In this section, we prove two rigid properties for quasi-almost-Einstein metrics on closed manifolds. The next theorem states that a quasi-almost-Einstein metric should be trivial, if f satisfies some integral condition.

Theorem 4.1 *Let* g *be an m-dimensional quasi-almost-Einstein metric with potential function f and soliton function $\lambda \le 0$ on a closed manifold M^n. If*

$$\int_M \nabla f\cdot\nabla\lambda\mathrm{e}^{-(1+\frac{2}{m-n})f}\,\mathrm{d}x \le 0, \tag{4.1}$$

then g *is trivial in the sense that* f *is constant. Moreover,* λ *is constant if* $n \geq 3$*, and now* g *is Einstein.*

Proof By (3.10) and the fact that

$$\mathrm{Ric}(\nabla f, \nabla f) = \lambda|\nabla f|^2 + \frac{1}{m-n}|\nabla f|^4 - \frac{1}{2}\nabla|\nabla f|^2 \cdot \nabla f, \tag{4.2}$$

we have

$$\begin{aligned}&(2-n)\nabla\lambda \cdot \nabla f - \nabla\triangle f \cdot \nabla f + \nabla|\nabla f|^2 \cdot \nabla f\\ &= 2\lambda|\nabla f|^2 + \frac{2}{m-n}|\nabla f|^4 - \frac{2}{m-n}\triangle f|\nabla f|^2.\end{aligned} \tag{4.3}$$

Integrating (4.3) on M leads to

$$\begin{aligned}&-\int_M \nabla\triangle f \cdot \nabla f \mathrm{e}^{\alpha f}\,\mathrm{d}x + \int_M \nabla|\nabla f|^2 \cdot \nabla f \mathrm{e}^{\alpha f}\,\mathrm{d}x\\ &\leq \frac{2}{m-n}\int_M |\nabla f|^4 \mathrm{e}^{\alpha f}\,\mathrm{d}x - \frac{2}{m-n}\int_M \triangle f|\nabla f|^2 \mathrm{e}^{\alpha f}\,\mathrm{d}x,\end{aligned} \tag{4.4}$$

where

$$\alpha = -\frac{m-n+2}{m-n}.$$

Integrating by parts leads to

$$\int_M \nabla|\nabla f|^2 \cdot \nabla f \mathrm{e}^{\alpha f}\,\mathrm{d}x = -\int_M (\triangle f|\nabla f|^2 + \alpha|\nabla f|^4)\mathrm{e}^{\alpha f}\,\mathrm{d}x, \tag{4.5}$$

$$\int_M \nabla\triangle f \cdot \nabla f \mathrm{e}^{\alpha f}\,\mathrm{d}x = -\int_M [(\triangle f)^2 + \alpha\triangle f|\nabla f|^2]\mathrm{e}^{\alpha f}\,\mathrm{d}x. \tag{4.6}$$

Plugging (4.5)–(4.6) into (4.4) yields

$$\int_M \Big[(\triangle f)^2 + \Big(\alpha - 1 + \frac{2}{m-n}\Big)\triangle f|\nabla f|^2 - \Big(\alpha + \frac{2}{m-n}\Big)|\nabla f|^4\Big]\mathrm{e}^{\alpha f}\,\mathrm{d}x \leq 0$$

or

$$\int_M (\triangle f - |\nabla f|^2)^2 \mathrm{e}^{\alpha f}\,\mathrm{d}x \leq 0,$$

which means

$$\triangle f = |\nabla f|^2.$$

Hence, f is constant by the maximum principle. Moreover, if $n \geq 3$, g is Einstein by the Schur Theorem in [20].

It is pointed out in [21] that any expanding or steady gradient Ricci solitons on closed manifolds should be trivial. The same result for quasi-Einstein metrics on closed manifolds is proved in [6–7], which can be deduced directly from Theorem 4.1.

Corollary 4.1 *Let* g *be an* m*-dimensional expanding or steady quasi-Einstein metric with potential function* f *on a closed manifold* M*. Then* g *is trivial in the sense that* f *is constant.*

Theorem 4.2 *Let* g *be an* m*-dimensional quasi-almost-Einstein metric with potential function* f *and soliton function* $\lambda = F(f)$ *on a closed manifold* M^n *with* $n \geq 3$, *where* $F(t)$ *is a smooth function. If*

$$\Big(G(t) + \frac{m-n}{2(m-2)}\mu\Big) \exp\Big[\frac{2(m-2)}{(n-2)(m-n)}t\Big] \tag{4.7}$$

is a monotone function about t, *where* μ *and* $G(t)$ *are defined in Corollary* 3.1, *then* g *is trivial in the sense that* f *is constant. Moreover,* g *is Einstein.*

Proof By (3.8) and (3.13), we have

$$\triangle f - |\nabla f|^2 = \Big[-(n-2)G'(f) - \frac{2(m-2)}{m-n}G(f) - \mu\Big]\mathrm{e}^{\frac{2}{m-n}f}. \tag{4.8}$$

Differentiating (4.7) shows that

$$-(n-2)G'(f) - \frac{2(m-2)}{m-n}G(f) - \mu$$

is nonpositive or nonnegative. Through integrating (4.8) against the measure $\mathrm{e}^{-f}\mathrm{d}x$, it is easy to see that

$$-(n-2)G'(f) - \frac{2(m-2)}{m-n}G(f) - \mu = 0,$$

which shows immediately that f is constant and g is Einstein.

Remark 4.1 Theorem 4.2 can be regarded as a generalization of Theorem 2.3 in [7]. In fact, rigid results for integer-dimensional closed steady or expanding quasi-Einstein metrics were proved in [6].

Remark 4.2 As pointed out in [1], a finite-dimensional shrinking quasi-Einstein metric is automatically compact. A finite-dimensional quasi-almost-Einstein metric is also automatically compact, if soliton function λ has a positive lower bound.

5 Weak Maximal Principle at Infinity

The maximum principle is a basic tool in geometric analysis (see [22–24]). In this section, we introduce a monotone formula for the weighted volume, and state the weighted Laplacian comparison theorem. Then we prove the weak maximum principle at infinity for some quasi-almost-Einstein metrics on complete noncompact manifolds.

Lemma 5.1 *Let* (M, g) *be an* n*-dimensional complete manifold,* f *be a real value smooth function on* M, *and* $\triangle_\mu = \triangle - \nabla f \cdot \nabla$ *be the weighted Laplacian. We also assume that the* m*-dimensional Bakry-Émery curvature on* M *is bounded by*

$$\mathrm{Ric}_{f,m} \geq -(m-1)K,$$

where $K = K(r(x)) \geq 0$ *is a function depending on* $r(x) = \mathrm{dist}(O, x)$, *and* $O \in M$ *is a fixed point. If we use* $\mu(B_R)$ *to denote the* $\mathrm{d}\mu = \mathrm{e}^{-f}\mathrm{d}x$ *measure of the geodesic ball* B_R *centered at* O *with radius* R, *then*

$$\frac{\mu(B_R)}{\int_0^R \psi^{m-1}(s)\mathrm{d}s}$$

is a monotone decreasing function about R*, where* $\psi(r)$ *satisfies the following equation:*

$$\psi''(r) = K(r)\psi(r), \quad \psi(0) = 0, \quad \psi'(0) = 1. \tag{5.1}$$

Proof Consider the geodesic sphere

$$S(O, r) = \{x \in M,\ \mathrm{dist}(O, x) = r\}.$$

We use H and II to denote the mean curvature and the second fundamental form of $S(O, r)$, respectively. Let (r, θ) be the geodesic coordinate around O. Then

$$\mathrm{d}x = J(r, \theta)\mathrm{d}r\mathrm{d}\theta,$$

where $J(r, \theta)$ is the Jacobian. By [25], we know

$$\frac{\partial}{\partial r} \log J = H. \tag{5.2}$$

The following is the well-known Riccati equation:

$$\frac{\partial H}{\partial r} = -\mathrm{Ric}(\nabla r, \nabla r) - |\mathrm{II}|^2.$$

Hence

$$\frac{\partial H}{\partial r} \le -\mathrm{Ric}(\nabla r, \nabla r) - \frac{H^2}{n-1}. \tag{5.3}$$

By (5.2)–(5.3), we get

$$H_\mu = H - \frac{\partial f}{\partial r} = \frac{\partial}{\partial r} \log(\mathrm{e}^{-f} J) = \frac{\partial J_\mu}{\partial r},$$

where H_μ and J_μ are called weighted mean curvature and weighted Jacobian, respectively. By the fact that

$$\mathrm{Hess} f\,(\nabla r, \nabla r) = \frac{\partial^2 f}{\partial r^2},$$

we can compute as follows:

$$\begin{aligned}\frac{\partial H_\mu}{\partial r} &\le -\frac{\partial^2 f}{\partial r^2} - \mathrm{Ric}(\nabla r, \nabla r) - \frac{H^2}{n-1}\\ &\le -\frac{(\frac{\partial f}{\partial r})^2}{m-n} + (m-1)K(r) - \frac{(H_\mu + \frac{\partial f}{\partial r})^2}{n-1}\\ &\le -\frac{H_\mu^2}{m-1} + (m-1)K(r).\end{aligned} \tag{5.4}$$

Note that

$$\lim_{r \searrow 0} r H_\mu = n - 1 \le m - 1.$$

By (5.4) and the Sturm-Liouville comparison theorem (see [3]), we conclude that

$$H_\mu(r) \le a_K(r),$$

where $a_K(r)$ satisfies the following Riccati equation:

$$\begin{cases} \dfrac{\partial a_K}{\partial r} = (m-1)K(r) - \dfrac{a_K^2}{m-1}, \\ \lim\limits_{r\searrow 0} r a_K = m-1. \end{cases} \tag{5.5}$$

It is easy to testify that $a_K(r) = (m-1)\frac{\psi'}{\psi}$ is a solution to (5.5) if $\psi(r)$ solves (5.1). Hence, outside of $\mathrm{Cut}(O)$,

$$\frac{J_\mu(r,\theta)}{J_\mu(s,\theta)} \le \Big(\frac{\psi(r)}{\psi(s)}\Big)^{m-1}$$

holds for $0 < s < r$. The rest part of the proof can be found in [25].

It is well-known that $\triangle r = H$ (see [25]), which means that $\triangle_\mu r = H_\mu$. Hence, the above proof implies the weighted Laplacian comparison theorem, which can also be found in [2–4].

Lemma 5.2 *Let* (M, g) *be an* n*-dimensional complete manifold,* f *be a real value smooth function on* M*, and* $\triangle_\mu = \triangle - \nabla f \cdot \nabla$ *be the weighted Laplacian. We also assume that the* m*-dimensional Bakry-Émery curvature on* M *is bounded by*

$$\mathrm{Ric}_{f,m} \ge -(m-1)K(r).$$

Then, at $x \notin \mathrm{Cut}(O)$*, we have*

$$\triangle_\mu r \le a_K(r),$$

where $a_K(r)$ *solves* (5.5)*. In particular, if* $K(r) = K > 0$ *is constant, then*

$$\begin{aligned} \triangle_\mu r &\le (m-1)\sqrt{K} \coth \sqrt{K} r \\ &\le \frac{m-1}{r}(1 + \sqrt{K} r). \end{aligned}$$

Let us introduce the weak maximum principle at infinity for the weighted Laplacian $\triangle_\mu$, which is discussed in [24] and was used in [7, 12, 26].

Definition 5.1 *We say that the weak maximum principle at infinity for* $\triangle_\mu$ *holds, if given a* C^2 *function* u*,*

$$\sup_M u = u^* < +\infty,$$

and then there exists a sequence $\{x_n\} \subset M$*, such that*

$$u(x_n) > u^* - \frac{1}{n} \quad \textit{and} \quad \triangle_\mu u(x_n) \le \frac{1}{n}.$$

The following result states that the weak maximum principle at infinity holds for a quasi-almost-Einstein metric when the soliton function satisfies a lower bound condition.

Lemma 5.3 *Let* g *be a quasi-almost-Einstein metric with potential function* f*. If the soliton function* λ *satisfies*

$$\lambda(x) \ge -(m-1) r^2(x) \tag{5.6}$$

for $r(x) = \mathrm{dist}(O, x)$ *large enough, then the weak maximum principle at infinity for the weighted Laplacian* $\triangle_\mu$ *holds on* M*.*

Proof Consider the Riccati equation

$$\frac{\partial a(r)}{\partial r} = (m-1)r^2 - \frac{a^2(r)}{m-1}.$$

Let $a(r) = (m-1)r + b(r)$. Then

$$\frac{\partial b(r)}{\partial r} + 2rb(r) + (m-1) \leq 0$$

or

$$\frac{\partial}{\partial r}\Big[\mathrm{e}^{r^2} b(r) + (m-1)\int_1^r \mathrm{e}^{s^2}\mathrm{d}s\Big] \leq 0.$$

Hence $b(r) \leq C$ for r large enough, which means that $a(r) \leq (m-1)r + C$. Then,

$$\frac{\partial}{\partial r}\log\psi(r) = \frac{1}{m-1}a(r) \leq r + C.$$

Hence, for r large enough,

$$\psi(r) \leq C\mathrm{e}^{\frac{1}{2}r^2},$$

and then

$$\int_1^r \psi^{m-1}(s)\ \mathrm{d}s \leq Cr\mathrm{e}^{\frac{m-1}{2}r^2}$$

or

$$\log\int_1^r \psi^{m-1}(s)\ \mathrm{d}s \leq C[1+r^2].$$

By Lemma 5.1, we conclude that

$$\log\mu(B_r) \leq C[1+r^2]. \tag{5.7}$$

Let us recall a result given in [24], which states that if a complete weighted manifold satisfies the volume growth condition

$$\int_1^\infty \frac{r}{\log\mu(B_r)}\mathrm{d}r = +\infty, \tag{5.8}$$

then the weak maximum principle at infinity for the weighted Laplacian $\triangle_\mu$ holds. Lemma 5.3 follows from (5.7)–(5.8).

6 Rigid Results on Noncompact Manifolds

In this section, we prove two rigid results for quasi-almost-Einstein metrics on complete noncompact manifolds. The following lemma is inspired by Theorem 4.1 of independent interest.

Lemma 6.1 *Let* g *be an m-dimensional quasi-almost-Einstein metric with potential function f and smooth soliton function $\lambda \leq 0$ on a complete noncompact manifold M. We assume that $\nabla f \cdot \nabla\lambda \leq 0$ and*

$$R^{-2}\int_{B_{2R}/B_R} |\nabla f|^2 \exp\Big(-\frac{m-n+2}{m-n}f\Big)\mathrm{d}x \to 0, \tag{6.1}$$

as $R \to \infty$, where B_R denotes the geodesic ball centered at O with radius R. Then e^f *is a harmonic function, i.e.,* $\triangle\mathrm{e}^f = 0$ *on M.*

Proof Consider a smooth function $\theta(t) : [0, +\infty) \to [0, 1]$,

$$\theta(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 0, & t \ge 2, \end{cases} \tag{6.2}$$

so that

$$-10\sqrt{\theta} \le \theta' \le 0. \tag{6.3}$$

For $R > 0$, let

$$\varphi(x) = \theta\Big(\frac{r(x)}{R}\Big)$$

be a cut-off function, where $r(x)$ is the distance function determined by $O \in M$. Then

$$0 \le \varphi \le 1, \quad |\nabla\varphi|(x) \le \frac{C}{R},$$

and $\varphi(x) = 1$ on B_R, $\varphi(x) = 0$ outside of B_{2R}. Let

$$\alpha = -\frac{m-n+2}{m-n}.$$

By (4.3) and the fact that $\lambda \le 0,\ \nabla\lambda \cdot \nabla f \le 0$, we conclude that

$$\begin{aligned} &-\int_M \nabla\triangle f \cdot \nabla f \varphi \mathrm{e}^{\alpha f}\ \mathrm{d}x + \int_M \nabla|\nabla f|^2 \cdot \nabla f \varphi \mathrm{e}^{\alpha f}\ \mathrm{d}x \\ &\le \frac{2}{m-n}\int_M |\nabla f|^4 \varphi \mathrm{e}^{\alpha f}\ \mathrm{d}x - \frac{2}{m-n}\int_M \triangle f |\nabla f|^2 \varphi \mathrm{e}^{\alpha f}\ \mathrm{d}x. \end{aligned} \tag{6.4}$$

Integrating by parts yields

$$\int_M \nabla|\nabla f|^2 \cdot \nabla f \varphi \mathrm{e}^{\alpha f}\ \mathrm{d}x = -\int_M |\nabla f|^2 (\triangle f \varphi + \alpha|\nabla f|^2\varphi + \nabla f \cdot \nabla\varphi)\mathrm{e}^{\alpha f}\ \mathrm{d}x, \tag{6.5}$$

$$\int_M \nabla\triangle f \cdot \nabla f \varphi \mathrm{e}^{\alpha f}\ \mathrm{d}x = -\int_M [(\triangle f)^2\varphi + \alpha\triangle f|\nabla f|^2\varphi + \triangle f \nabla f \cdot \nabla\varphi]\mathrm{e}^{\alpha f}\ \mathrm{d}x. \tag{6.6}$$

Substituting (6.5) and (6.6) into (6.4), we have

$$\begin{aligned} &\int_M [(\triangle f)^2 - 2\triangle f|\nabla f|^2 + |\nabla f|^4]\varphi \mathrm{e}^{\alpha f}\ \mathrm{d}x \\ &\le \int_M [\triangle f \nabla f \cdot \nabla\varphi - |\nabla f|^2 \nabla f \cdot \nabla\varphi]\mathrm{e}^{\alpha f}\ \mathrm{d}x \\ &\le \Big(\int_M (\triangle f - |\nabla f|^2)^2 \varphi \mathrm{e}^{\alpha f}\ \mathrm{d}x\Big)^{\frac{1}{2}} \Big(\int_{B_{2R}/B_R} \frac{|\nabla f \cdot \nabla\varphi|^2}{\varphi} \mathrm{e}^{\alpha f}\ \mathrm{d}x\Big)^{\frac{1}{2}}. \end{aligned} \tag{6.7}$$

By the fact that

$$|\nabla f \cdot \nabla\varphi| \le |\nabla f||\nabla\varphi| \le \frac{\theta'}{R}|\nabla f|,$$

we get

$$\int_{B_R} [\triangle f - |\nabla f|^2]^2 \mathrm{e}^{\alpha f}\ \mathrm{d}x \le \int_M [\triangle f - |\nabla f|^2]^2 \varphi \mathrm{e}^{\alpha f}\ \mathrm{d}x \le CR^{-2}\int_{B_{2R}/B_R} |\nabla f|^2 \mathrm{e}^{\alpha f}\ \mathrm{d}x.$$

Letting $R \to \infty$ leads to

$$\int_M [\triangle f - |\nabla f|^2]^2 \mathrm{e}^{\alpha f}\ \mathrm{d}x = 0.$$

Hence Lemma 6.1 follows.

Theorem 6.1 *Let* g *be an m-dimensional expanding quasi-almost-Einstein metric with potential function f and smooth soliton function λ on a complete noncompact manifold M. We assume that $\nabla f\cdot\nabla\lambda\leq 0$ and* (6.1) *is right. Then* g *is trivial in the sense that f is constant. Moreover, λ is constant when $n\geq 3$, and now* g *is Einstein.*

Proof Lemma 6.1 implies that

$$\triangle f=|\nabla f|^2,$$

and together with (4.3), we conclude that

$$0\leq(2-n)\nabla\lambda\cdot\nabla f=2\lambda|\nabla f|^2\leq 0.$$

Note that $\lambda<0$, so $|\nabla f|^2\equiv 0$ and Theorem 6.1 follows.

Remark 6.1 It is pointed out in [7, 27] that for a steady quasi-Einstein metric, the constant μ in (1.3) is null if and only if g is Ricci flat. Theorem 6.1 seems to be new even for the quasi-Einstein metric.

By using the weak maximum principle at infinity, we also get a rigid result for an m-dimensional expanding quasi-almost-Einstein metric on a noncompact manifold.

Theorem 6.2 *Let* g *be an m-dimensional quasi-almost-Einstein metric with potential function f and smooth soliton function λ on a complete noncompact manifold M. We assume that λ satisfies*

$$\lambda(x)\geq-(m-1)r^2(x)\tag{6.8}$$

for $r(x)=\mathrm{dist}(O,x)$ large enough. If

$$\nabla\lambda\cdot\nabla f\leq 0$$

and

$$\sup_M|\nabla f|^2<-\frac{(m-n)^2(m-1)\lambda_{\sup}}{m},\tag{6.9}$$

where

$$\lambda_{\sup}=\sup_{x\in M}\lambda(x)<0,$$

then f is constant. Moreover, λ is constant if $n\geq 3$, and now g *is Einstein.*

Proof By (1.1) and Lemma 3.1, we have

$$\begin{aligned}\frac12\triangle|\nabla f|^2&=|\nabla^2 f|^2-\mathrm{Ric}(\nabla f,\nabla f)+\frac{2}{m-n}|\nabla f|^2\triangle f-(n-2)\nabla\lambda\cdot\nabla f\\&=|\nabla^2 f|^2-\frac{1}{m-n}|\nabla f|^4-\lambda|\nabla f|^2+\frac{2}{m-n}|\nabla f|^2\triangle f\\&\quad+\frac12\nabla|\nabla f|^2\cdot\nabla f-(n-2)\nabla\lambda\cdot\nabla f,\end{aligned}$$

which means that

$$\frac12\triangle_\mu|\nabla f|^2\geq|\nabla^2 f|^2-\frac{1}{m-n}|\nabla f|^4-\lambda|\nabla f|^2+\frac{2}{m-n}|\nabla f|^2\triangle f.$$

Note that

$$\begin{aligned}|\nabla^2 f|^2 + \frac{2}{m-n}|\nabla f|^2 \triangle f &\geq \frac{1}{n}(\triangle f)^2 + \frac{2}{m-n}|\nabla f|^2 \triangle f \\ &\geq -\frac{n}{(m-n)^2}|\nabla f|^4.\end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{2}\triangle_\mu |\nabla f|^2 &\geq -\lambda|\nabla f|^2 - \frac{m}{(m-n)^2}|\nabla f|^4 \\ &\geq -\lambda_{\sup}|\nabla f|^2 - \frac{m}{(m-n)^2}|\nabla f|^4. \end{aligned} \tag{6.10}$$

By (6.8) and Lemma 5.3, we know that the weak maximum principle at infinity holds for $\triangle_\mu$. The fact that $\sup\limits_M |\nabla f|^2 < +\infty$ means that there exists a sequence $\{x_k\} \subset M$, such that

$$|\nabla f|^2(x_k) \geq \sup_M |\nabla f|^2 - \frac{1}{k}$$

and

$$\triangle_\mu |\nabla f|^2(x_k) \leq \frac{1}{k}.$$

(6.10) implies that

$$\frac{m}{(m-n)^2}|\nabla f|^4 + \lambda_{\sup}|\nabla f|^2 + \frac{1}{2k} \geq 0$$

holds at x_k. By (6.9), we conclude that for k large enough,

$$|\nabla f|^2(x_k) \leq \frac{(m-n)^2}{2m}\Big[-\lambda_{\sup} - \sqrt{\lambda_{\sup}^2 - \frac{2m}{k(m-n)^2}}\ \Big],$$

which means that

$$\sup_M |\nabla f|^2 - \frac{1}{k} \leq \frac{(m-n)^2}{2m}\Big[-\lambda_{\sup} - \sqrt{\lambda_{\sup}^2 - \frac{2m}{k(m-n)^2}}\ \Big].$$

Letting $k \to \infty$ leads to

$$\sup_M |\nabla f|^2 \leq 0.$$

Hence f is constant.

Remark 6.2 If $m = \infty$, we recover Theorem 3.6 in [7].

7 Lower Bound of Scalar Curvature

In [8], the author gets the lower bound estimate of scalar curvature for noncompact quasi-Einstein metrics. In this section, we prove the lower bound estimate of the scalar curvature for a quasi-almost-Einstein metric with soliton function satisfying $\lambda = F(f)$ on a noncompact manifold. We firstly give a gradient estimate for f of independent interest.

Lemma 7.1 *Let* g *be an m-dimensional quasi-almost-Einstein metric with potential function f and smooth soliton function $\lambda = F(f)$ on a complete noncompact manifold M. If*

$$\lambda_{\inf} = \inf_{x \in M} \lambda(x) > -\infty$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}[H(t)\mathrm{e}^{\frac{2}{m-n}t}] \geq -\nu \tag{7.1}$$

for some constant $\nu > 0$, where

$$H(t) = -(n-2)G'(t) - \frac{2(m-2)}{m-n}G(t) - \mu \tag{7.2}$$

with $G(t)$ being defined in (3.14)*, then*

$$|\nabla f|^2(x) \leq (m-n)\Big[-\lambda_{\inf} + \frac{\nu}{2}\Big]$$

holds for any $x \in M$.

Proof Consider a smooth function $\theta(t) : [0, +\infty) \to [0, 1]$,

$$\theta(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 0, & t \geq 2, \end{cases}$$

so that

$$-10\theta^{\frac{1}{2}} \leq \theta' \leq 0, \quad \theta'' \geq -10. \tag{7.3}$$

For some constant $R_0 > 0$, define the smooth cut-off function $\varphi : M \to \mathbb{R}$ by

$$\varphi(x,t) = \theta\Big(\frac{r(x)}{R_0}\Big).$$

Then

$$\nabla\varphi = \frac{\theta'\nabla r}{R_0} \tag{7.4}$$

and

$$\begin{aligned} \triangle_\mu\varphi &= \triangle\varphi - \nabla\varphi \cdot \nabla f \\ &= \frac{\theta''}{R_0^2} + \frac{\theta'\triangle_\mu r}{R_0} \\ &\geq \frac{\theta''}{R_0^2} + \frac{(m-1)\theta'\Big(1 + \sqrt{-\frac{\lambda_{\inf}}{m-1}}R_0\Big)}{R_0^2}, \end{aligned} \tag{7.5}$$

where we have used Lemma 5.2. Let

$$Q = \varphi|\nabla f|^2.$$

Then

$$\triangle_\mu Q = \triangle_\mu\varphi|\nabla f|^2 + 2\nabla\varphi \cdot \nabla|\nabla f|^2 + \varphi\triangle_\mu|\nabla f|^2.$$

By (1.1), (3.2)–(3.3) and (7.1), we have

$$\begin{aligned}\triangle_\mu|\nabla f|^2 &= \triangle|\nabla f|^2 - \nabla|\nabla f|^2\cdot\nabla f\\ &= 2|\nabla^2 f|^2 + 2\mathrm{Ric}(\nabla f,\nabla f) + 2\nabla f\cdot\nabla\triangle f - \nabla|\nabla f|^2\cdot\nabla f\\ &= 2|\nabla^2 f|^2 + 2\Big[\frac{1}{m-n}\nabla f\otimes\nabla f - \mathrm{Hess} f + \lambda \mathrm{g}\Big](\nabla f,\nabla f)\\ &\quad + 2\nabla f\cdot\nabla[|\nabla f|^2 + H(f)\mathrm{e}^{\frac{2}{m-n}f}] - \nabla f\cdot\nabla|\nabla f|^2\\ &= 2|\nabla^2 f|^2 + 2\lambda|\nabla f|^2 + \frac{2}{m-n}|\nabla f|^4 + \Big(H'(f) + \frac{2}{m-n}H(f)\Big)\mathrm{e}^{\frac{2}{m-n}f}|\nabla f|^2\\ &\geq 2\lambda_{\inf}|\nabla f|^2 + \frac{2}{m-n}|\nabla f|^4 - \nu|\nabla f|^2.\end{aligned}$$

Note that

$$\nabla Q = \nabla\varphi|\nabla f|^2 + \varphi\nabla|\nabla f|^2$$

or

$$\nabla|\nabla f|^2 = \frac{\nabla Q}{\varphi} - \frac{Q\nabla\varphi}{\varphi^2}.$$

Hence

$$\triangle_\mu Q \geq \frac{\triangle_\mu\varphi Q}{\varphi} + \frac{2\nabla Q\cdot\nabla\varphi}{\varphi} - \frac{2|\nabla\varphi|^2 Q}{\varphi^2} + 2\lambda_{\inf}Q + \frac{2Q^2}{(m-n)\varphi} - \nu Q.$$

We assume that Q achieves its maximal value at x_0. Then

$$\nabla Q = 0 \quad\text{and}\quad \triangle_\mu Q \leq 0$$

hold at x_0. By (7.3)–(7.5), we conclude that at x_0,

$$\begin{aligned}0 &\geq \frac{\theta'' Q}{R_0^2} + \frac{(m-1)\theta'\Big(1+\sqrt{-\frac{\lambda_{\inf}}{m-1}}R_0\Big)}{R_0^2}Q - \frac{2|\theta'|^2 Q}{\theta R_0^2} + 2\lambda_{\inf}\theta Q + \frac{2Q^2}{m-n} - \nu\theta Q\\ &\geq \frac{2Q^2}{m-n} - \Big[-2\lambda_{\inf} + \frac{210 + 10(m-1)\Big(1+\sqrt{\frac{\lambda_{\inf}}{m-1}}R_0\Big)}{R_0^2} + \nu\Big]Q.\end{aligned}$$

So for $x\in B_{R_0}$,

$$\begin{aligned}|\nabla f|^2(x) &= |\nabla f|^2(x)\varphi(x) = Q(x) \leq Q(x_0)\\ &\leq (m-n)\Big[-\lambda_{\inf} + \frac{105 + 5(m-1)\Big(1+\sqrt{\frac{\lambda_{\inf}}{m-1}}R_0\Big)}{R_0^2} + \frac{\nu}{2}\Big].\end{aligned}$$

Lemma 7.1 follows by letting $R_0\to\infty$.

The following result is a corollary of Lemma 7.1, which can be seen as a generalization of Theorem 3.3 in [7].

Corollary 7.1 *Let* g *be an m-dimensional quasi-almost-Einstein metric with potential function f and smooth soliton function $\lambda = F(f)$ on a complete noncompact manifold M. If*

$$\frac{\mathrm{d}}{\mathrm{d}t}[H(t)\mathrm{e}^{\frac{2}{m-n}t}] \geq -2\lambda_{\inf},$$

then f is constant. Moreover, g *is Einstein when $n\geq 3$.*

Based on the gradient estimate of f, we can prove a lower bound estimate of scalar curvature R. The following is the main result in this section.

Theorem 7.1 *Let* g *be an* m*-dimensional quasi-almost-Einstein metric with potential function* f *and smooth soliton function* $\lambda = F(f)$ *on a complete noncompact manifold* M*. We also assume that*

$$\lambda_{\inf} = \inf_{x \in M} \lambda(x) > -\infty$$

and

$$\lambda_{\sup} = \sup_{x \in M} \lambda(x) \le 0.$$

If $m - n \ge 1$*,* $F'(t) \le 0$*, and* $H(t)$ *satisfies* (7.1) *for some constant* $\nu > 0$*, then*

$$R(x) \ge 2(n-1)\lambda(x) - (n-2)\lambda_{\inf} - \frac{1}{2}[A + \sqrt{A^2 - 4B}\,] \tag{7.6}$$

holds for any $x \in M$*, where*

$$A = 2(n-2)(\lambda_{\sup} - \lambda_{\inf}) + \frac{n(m-n)}{m-1}\lambda_{\sup} \tag{7.7}$$

and

$$B = \frac{n(m-n)(n-2)}{m-1}\lambda_{\inf}(\lambda_{\sup} - \lambda_{\inf}). \tag{7.8}$$

Proof Consider a smooth function $\theta(t) : [0, +\infty) \to [0, 1]$,

$$\theta(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 0, & t \ge 2, \end{cases}$$

so that $\theta(t)$ satisfies (7.3). For $R_0 > 0$, define a smooth cut-off function $\varphi : M \to \mathbb{R}$ by

$$\varphi(x, t) = \theta\Big(\frac{r(x)}{R_0}\Big).$$

Let

$$S = R - 2(n-1)\lambda + (n-2)\lambda_{\inf}.$$

Then (3.16) is equivalent to

$$\begin{aligned} \frac{1}{2}\triangle S = {} & \frac{m-n+2}{2(m-n)}\nabla S \cdot \nabla f + (n-1)\nabla\lambda \cdot \nabla f - \frac{m-n-1}{m-n}\Big|\mathrm{Ric} - \frac{1}{n}R\mathrm{g}\Big|^2 \\ & - \frac{m-1}{(m-n)n}(S + (n-2)(\lambda - \lambda_{\inf}))\Big(S - (n-2)\lambda_{\inf} + \frac{(n-1)(2m-n-2)}{m-1}\lambda\Big). \end{aligned} \tag{7.9}$$

Let $N = \varphi S$. Then

$$\nabla S = -\frac{N\nabla\varphi}{\varphi^2} + \frac{\nabla N}{\varphi}.$$

By (7.9), we have

$$\begin{aligned}\triangle_\mu N &= \triangle_\mu \varphi S + 2\nabla\varphi\cdot\nabla S + \varphi\triangle_\mu S\\ &= \triangle_\mu\varphi S + 2\nabla\varphi\cdot\nabla S + \varphi(\triangle S - \nabla S\cdot\nabla f)\\ &= \frac{\triangle_\mu\varphi N}{\varphi} - \frac{2|\nabla\varphi|^2 N}{\varphi^2} + \frac{2\nabla\varphi\cdot\nabla N}{\varphi} - \frac{2\nabla f\cdot\nabla\varphi N}{(m-n)\varphi}\\ &\quad + \frac{2\nabla f\cdot\nabla N}{m-n} - \frac{2(m-n-1)\varphi}{m-n}\Big|\mathrm{Ric} - \frac{1}{n}R\mathrm{g}\Big|^2 + 2(n-1)\varphi\nabla\lambda\cdot\nabla f\\ &\quad - \frac{2(m-1)\varphi}{(m-n)n}\Big[\frac{N}{\varphi} + (n-2)(\lambda-\lambda_{\inf})\Big]\Big[\frac{N}{\varphi} - (n-2)\lambda_{\inf} + \frac{(n-1)(2m-n-2)}{m-1}\lambda\Big].\end{aligned}$$

We assume that for $R_0 > 0$ large enough, the minimal value of N on B_{R_0} can be achieved at x_0 and $N(x_0) < 0$. Then

$$\nabla N = 0 \quad \text{and} \quad \triangle_\mu N \geq 0$$

hold at x_0. Hence at x_0,

$$\begin{aligned}&[N + (n-2)\varphi(\lambda-\lambda_{\inf})]\Big[N - (n-2)\lambda_{\inf}\varphi + \frac{(n-1)(2m-n-2)}{m-1}\lambda\varphi\Big]\\ &\leq \frac{(m-n)n}{2(m-1)}\Big[\triangle_\mu\varphi N - \frac{2|\nabla\varphi|^2 N}{\varphi} - \frac{2\nabla f\cdot\nabla\varphi N}{m-n}\Big],\end{aligned} \tag{7.10}$$

where we have used the fact that

$$\nabla\lambda\cdot\nabla f = F'(f)|\nabla f|^2 \leq 0.$$

For $R_0 > 0$ large enough, we define

$$\sigma(R_0) = \frac{\inf\{S(x) \mid x\in B_{R_0}\}}{\inf\{S(x) \mid x \in B_{2R_0}\}}.$$

It is easy to testify that

$$N(x_0) = \varphi(x_0)S(x_0) \leq \inf\{S(x) \mid x \in B_{R_0}\}$$

and

$$N(x_0) = \varphi(x_0)S(x_0) \geq \varphi(x_0)\inf\{S(x) \mid x\in B_{2R_0}\}.$$

By the assumption that $\inf\{S(x) \mid x \in B_{R_0}\} < 0$, we have

$$\sigma(R_0) \leq \varphi(x_0) \leq 1. \tag{7.11}$$

By (7.10) and Lemma 5.2, we conclude that at x_0,

$$\begin{aligned}&[N + (n-2)\varphi(\lambda-\lambda_{\inf})]\Big[N - (n-2)\varphi\lambda_{\inf} + \frac{(n-1)(2m-n-2)}{m-1}\lambda\varphi\Big]\\ &\leq \frac{(m-n)n}{2(m-1)}\Big[\frac{\theta'' + (m-1)\theta'(1+\sqrt{-\frac{\lambda_{\inf}}{m-1}}R_0)}{R_0^2} - \frac{2|\nabla f||\theta'|}{(m-n)R_0} - \frac{2|\theta'|^2}{\theta R_0^2}\Big]N.\end{aligned}$$

(7.3) and Lemma 7.1 implies that

$$\begin{aligned}
&[N+(n-2)\varphi(\lambda-\lambda_{\inf})]\Big[N-(n-2)\varphi\lambda_{\inf}+\frac{(n-1)(2m-n-2)}{m-1}\lambda\varphi\Big]\\
&\le \frac{C_1+C_2R_0}{R_0^2}N, \qquad (7.12)
\end{aligned}$$

where C_1, C_2 are constants independent of R_0. Due to (7.11) and the fact that $N(x_0)\le 0,\ \lambda(x_0)\le 0$, we can estimate at x_0 that

$$\begin{aligned}
&[N+(n-2)\varphi(\lambda-\lambda_{\inf})]\Big[N-(n-2)\varphi\lambda_{\inf}+\frac{(n-1)(2m-n-2)}{m-1}\lambda\varphi\Big]\\
&= N^2+\Big[\Big(n-2+\frac{(n-1)(2m-n-2)}{m-1}\Big)\lambda-2(n-2)\lambda_{\inf}\Big]\varphi N\\
&\quad+(n-2)(\lambda-\lambda_{\inf})\Big(\frac{(n-1)(2m-n-2)}{m-1}\lambda-(n-2)\lambda_{\inf}\Big)\varphi^2\\
&\ge N^2+\Big[2(n-2)(\lambda_{\sup}-\lambda_{\inf})+\frac{n(m-n)}{m-1}\lambda_{\sup}\Big]\varphi N\\
&\quad+\frac{n(m-n)(n-2)}{m-1}\lambda_{\inf}(\lambda_{\sup}-\lambda_{\inf})\varphi^2\\
&\ge N^2+2(n-2)(\lambda_{\sup}-\lambda_{\inf})\sigma(R_0)N+\frac{n(m-n)}{m-1}\lambda_{\sup}N\\
&\quad+\frac{n(m-n)(n-2)}{m-1}\lambda_{\inf}(\lambda_{\sup}-\lambda_{\inf}).
\end{aligned}$$

This inequality together with (7.12) shows that

$$\begin{aligned}
&N^2+2(n-2)(\lambda_{\sup}-\lambda_{\inf})\sigma(R_0)N+\frac{n(m-n)}{m-1}\lambda_{\sup}N\\
&\le -\frac{n(m-n)(n-2)}{m-1}\lambda_{\inf}(\lambda_{\sup}-\lambda_{\inf})+\frac{C_1+C_2R_0}{R_0^2}N. \qquad (7.13)
\end{aligned}$$

Hence, for $x\in B_{R_0}$,

$$S(x)=\varphi(x)S(x)=N(x)\ge N(x_0)\ge\frac{1}{2}[-A(R_0)-\sqrt{A^2(R_0)-4B}\,], \qquad (7.14)$$

where

$$A(R_0)=-\frac{C_1+C_2R_0}{R_0^2}+2(n-2)(\lambda_{\sup}-\lambda_{\inf})\sigma(R_0)+\frac{n(m-n)}{m-1}\lambda_{\sup}$$

and B is defined in (7.8). Note that $0\le\sigma(R_0)\le 1$, and (7.14) means that S is bounded from below. Hence

$$\lim_{R_0\to\infty}\sigma(R_0)=1,$$

which means that

$$\lim_{R_0\to\infty}A(R_0)=2(n-2)(\lambda_{\sup}-\lambda_{\inf})+\frac{n(m-n)}{m-1}\lambda_{\sup}=A.$$

By (7.14), we deduce that for $x\in M$,

$$S(x)\ge\frac{1}{2}[-A-\sqrt{A^2-4B}\,]$$

and (7.6) follows.

The following estimate for the quasi-Einstein metric was proved in [8], which is a natural corollary of Theorem 7.1.

Corollary 7.2 *Let* g *be a quasi-Einstein metric with potential function f and constant $\lambda \leq 0$ on a complete noncompact manifold M. If $m - n \geq 1$, then*

$$R(x) \geq n\lambda \tag{7.15}$$

holds for any $x \in M$.

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