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The Average Errors for Hermite Interpolation on the 1-Fold Integrated Wiener Space

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Abstract For the weighted approximation in L_p-norm, the authors determine the weakly asymptotic order for the p-average errors of the sequence of Hermite interpolation based on the Chebyshev nodes on the 1-fold integrated Wiener space. By this result, it is known that in the sense of information-based complexity, if permissible information functionals are Hermite data, then the p-average errors of this sequence are weakly equivalent to those of the corresponding sequence of the minimal p-average radius of nonadaptive information.

Keywords Chebyshev polynomial, Hermite interpolation, Weighted L_p-norm, 1-Fold integrated Wiener space
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1 Introduction

Let F be a real separable Banach space equipped with a probability measure μ on the Borel sets of F. Let H be another normed space, such that F is continuously embedded in H. $\|\cdot\|$ denotes the norm in H. Any $A : F \to H$, such that $f \mapsto \|f - A(f)\|$ is a measurable mapping, is called an approximation operator (or just approximation). The p-average error of A is defined as

$$e_p(A, \|\cdot\|, F, \mu) = \Big(\int_F \|f - A(f)\|^p \mu(\mathrm{df}) \Big)^{\frac{1}{p}}.$$

Denote

$$F_0 = \{f \in C[0,1] : f(0) = 0\}.$$

For every $f \in F_0$, set

$$\|f\|_C := \max_{0 \le t \le 1} |f(t)|.$$

Then $(F_0, \|\cdot\|_C)$ becomes a separable Banach space. Denote by $\mathcal{B}(F_0)$ the Borel class of $(F_0, \|\cdot\|_C)$, and by ω_0 the Wiener measure on $\mathcal{B}(F_0)$ (see [1]). For $g \in F_0$, let

$$(T_1 g)(t) = \int_0^t g(u)\mathrm{d}u.$$

Then

$$T_1 g \in F_1 = \{f \in C^{(1)}[0,1] : f^{(k)}(0) = 0,\ k = 0, 1\}.$$

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It is well-known that T_1 is a bijective mapping from F_0 to F_1. The 1-fold integrated Wiener measure ω_1 on F_1 is defined by induced measure $\omega_1 = T_1\omega_0$, i.e., for $A \subset F_1$,

$$\omega_1(A) = \omega_0(\{g : T_1 g \in A\}). \tag{1.1}$$

Let

$$F = \{f \in C^{(1)}[-1,1] : f^{(k)}(-1) = 0,\ k = 0,1\}.$$

The 1-fold integrated Wiener measure ω on F is defined as follows: for $A \subset F$,

$$\omega(A) = \omega_1(\{g(t) = f(2t-1) : f \in A\}). \tag{1.2}$$

For $\varrho \in L_1[-1,1]$, $\varrho \geq 0$, the weighted L_p-norm of $f \in C[-1,1]$ is defined by

$$\|f\|_{p,\varrho} = \Big(\int_{-1}^{1} |f(t)|^p \cdot \varrho(t)\mathrm{d}t\Big)^{\frac{1}{p}},$$

and we simply write $\|\cdot\|_p$ if $\varrho(t) = 1$. Let

$$x_k = x_{kn} = \cos\frac{2k-1}{2n}\pi, \quad k = 1, \cdots, n$$

be the zeros of

$$T_n(x) = \cos n\theta, \quad x = \cos\theta,$$

the n-th degree Chebyshev polynomial of the first kind. In this case, the well-known Lagrange interpolation polynomial is given by (see [2])

$$L_n(f,x) = \sum_{k=1}^{n} f(x_k)\ell_k(x), \tag{1.3}$$

where

$$\ell_k(x) = \frac{T_n(x)}{T_n'(x_k)(x-x_k)} = \frac{(-1)^{k+1}\sqrt{1-x_k^2}\,T_n(x)}{n(x-x_k)}, \quad k = 1, \cdots, n. \tag{1.4}$$

The well-known Hermite-Fejér interpolation polynomial is given by (see [3])

$$H_n(f,x) = \sum_{k=1}^{n} f(x_k)h_k(x), \tag{1.5}$$

where

$$h_k(x) = (1 - x x_k)\Big(\frac{T_n(x)}{n(x-x_k)}\Big)^2 \geq 0, \quad \sum_{k=1}^{n} h_k(x) = 1. \tag{1.6}$$

From [4], we know that if $\varrho \in L_1[-1,1]$, $\varrho > 0$, and ϱ is continuous on $(-1,1)$, then for $1 \leq p < \infty$, we have

$$e_p\left(L_n, \|\cdot\|_{p,\varrho}, F, \omega\right) \asymp n^{-\frac{3}{2}}, \tag{1.7}$$

and for $\varrho(x) = \frac{1}{\sqrt{1-x^2}}$, we have

$$e_2\left(H_n, \|\cdot\|_{2,\varrho}, F, \omega\right) \asymp n^{-1}. \tag{1.8}$$

Here and in the following notation, $a_n \asymp b_n$ for sequences $\{a_n\}$ and $\{b_n\}$ of positive numbers means the existence of a constant C independent of n, such that $\frac{a_n}{C} \leq b_n \leq Ca_n$, and C may be different in the different expressions. From (1.7) and (1.8), we know that the sequence of Lagrange interpolation defined by (1.3) is a suboptimal approximation operators sequence for standard information on the 1-fold integrated Wiener space, but the corresponding sequence of Hermite-Fejér interpolation is not. However, from [5], we know that both the Lagrange interpolation sequence and the Hermite-Fejér interpolation sequence defined by (1.3) and (1.5) are suboptimal approximation operators sequences for standard information on the Wiener space. Similar results can be find in [6–8]. These results show that the average errors of interpolation operators in different probability spaces can have completely different properties. Noticing that the Hermite data is a kind of information which is widely used in practice, we will consider the average errors of Hermite interpolation based on $\{x_k\}_{k=1}^n$.

If $f \in C^{(1)}[-1,1]$, then it is known that the Hermite interpolation polynomial $G_n(f,x)$, which is of degree at most $2n-1$ and satisfies the conditions

$$G_n(f,x_k) = f(x_k), \quad G_n'(f,x_k) = f'(x_k), \quad k = 1,2,\cdots,n, \tag{1.9}$$

is given by (see [9])

$$G_n(f,x) = \sum_{k=1}^{n} f(x_k)h_k(x) + \sum_{k=1}^{n} f'(x_k)\sigma_k(x), \tag{1.10}$$

where

$$\sigma_k(x) = (x-x_k)\ell_k^2(x), \quad k = 1,\cdots,n. \tag{1.11}$$

From [9], it follows that if $p_n(x)$ is an algebraic polynomial of degree at most $2n-1$, then

$$p_n(x) = G_n(p_n,x) = \sum_{k=1}^{n} p_n(x_k)h_k(x) + \sum_{k=1}^{n} p_n'(x_k)\sigma_k(x). \tag{1.12}$$

In this paper, we obtain the following theorem.

Theorem 1.1 *Let $G_n(f,x)$ be defined by* (1.10)*. Then for an arbitrary $\varrho \in L_1[-1,1]$, $\varrho > 0$, if ϱ is continuous on $(-1,1)$ and $1 \leq p < \infty$, we have*

$$e_p\left(G_n, \|\cdot\|_{p,\varrho}, F, \omega\right) \asymp n^{-\frac{3}{2}}.$$

Remark 1.1 Let us recall some fundamental notions about the information-based complexity in the average case setting. Let F be a set with a probability measure μ, and G be a normed linear space with norm $\|\cdot\|$. Let S be a measurable mapping from F into G, which is called a solution operator. Let N be a measurable mapping from F into $\mathbb{R}^n$, and ϕ be a measurable mapping from $\mathbb{R}^n$ into G, which are called an information operator and an algorithm, respectively. The p-average error of the approximation $\phi \circ N$ with respect to the measure μ is defined by

$$e_p(S, N, \phi, \|\cdot\|, \mu, F) := \Big(\int_F \|S(f) - \phi(N(f))\|^p \mu(\mathrm{d}f)\Big)^{\frac{1}{p}},$$

and the p-average radius of information N with respect to μ is defined by

$$r_p(S, N, \|\cdot\|, \mu, F) := \inf_{\phi} e_p(S, N, \phi, \|\cdot\|, \mu, F),$$

where ϕ ranges over the set of all algorithms. Furthermore, let Λ denote a class of permissible information functionals L, and denote by $\mathcal{N}_n^{\Lambda}$ the set of nonadaptive information operators N from Λ of cardinality n, i.e.,

$$N(f) = (L_1(f), L_2(f), \cdots, L_n(f)), \quad L_i \in \Lambda,\ i = 1, \cdots, n.$$

Let

$$r_p(n, S, \Lambda, \|\cdot\|, \mu, F) = \inf_{N \in \mathcal{N}_n^{\Lambda}} r_p(S, N, \|\cdot\|, \mu, F)$$

denote the n-th minimal average radius of nonadaptive information in the class Λ.

For example, if F, ω are defined as above, S is the identity mapping I and Λ consists of Hermite data, then by [1, p. 108], we know

$$r_p(n, I, \Lambda, \|\cdot\|_p, \omega, F) \asymp n^{-\frac{3}{2}}, \quad 1 \le p < \infty.$$

It is easy to understand that $G_n(f, x)$ can be viewed as a composition of a nonadaptive information operator of cardinality $2n$ from Λ and an algorithm. From Theorem 1.1, we know

$$e_p\left(G_n, \|\cdot\|_p, F, \omega\right) \asymp r_p\left(2n, I, \Lambda, \|\cdot\|_p, \omega, F\right), \quad 1 \le p < \infty.$$

2 Some Lemmas

From [4], we obtain the following lemma.

Lemma 2.1 *Let $s \ge t$. Then*

$$\int_F f(s) f(t) \omega(\mathrm{d}f) = \frac{(1+t)^3}{24} + \frac{(s-t)(1+t)^2}{16}, \tag{2.1}$$

$$\int_F f'(t) f(s) \omega(\mathrm{d}f) = \frac{(1+t)^2}{16} + \frac{(s-t)(1+t)}{8}, \tag{2.2}$$

$$\int_F f(t) f'(s) \omega(\mathrm{d}f) = \frac{(1+t)^2}{16}, \tag{2.3}$$

$$\int_F f'(x) f'(y) \omega(\mathrm{d}f) = \frac{2 + x + y - |x-y|}{16}. \tag{2.4}$$

The following lemma is well-known.

Lemma 2.2 *If $p_n(x)$ is an algebraic polynomial of degree at most $n-1$, then*

$$p_n(x)=\sum_{k=1}^{n} p_n(x_k)\ell_k(x). \tag{2.5}$$

From (2.5), we conclude that if $p_n(x)$ is an algebraic polynomial of degree at most $n-1$, then

$$p_n'(x)=\sum_{k=1}^{n} p_n(x_k)\ell_k'(x). \tag{2.6}$$

3 Proof of Theorem 1.1

From [1, p. 108], we obtain the lower estimate. Now we will consider the upper estimate. From (1.4) and a simple computation, we obtain

$$\ell_k(x_k)=1, \quad \ell_k'(x_k)=\frac{x_k}{2\,(1-x_k^2)}, \quad k=1,\cdots,n, \tag{3.1}$$

and for $j\neq k$,

$$\ell_k(x_j)=0, \quad \ell_k'(x_j)=\frac{(-1)^{k+j}\sqrt{1-x_k^2}}{(x_j-x_k)\sqrt{1-x_j^2}}. \tag{3.2}$$

From (1.9), (1.12), (3.1)–(3.2), it follows that

$$L_n(f,x)-G_n(f,x)=\sum_{j=1}^{n}\Big[\sum_{k=1}^{n} f(x_k)\ell_k'(x_j)-f'(x_j)\Big]\sigma_j(x). \tag{3.3}$$

Therefore, we obtain

$$f(x)-G_n(f,x)=f(x)-L_n(f,x)+\sum_{j=1}^{n}\Big[\sum_{k=1}^{n} f(x_k)\ell_k'(x_j)-f'(x_j)\Big]\sigma_j(x). \tag{3.4}$$

For $x\in[-1,1]$, we have

$$\begin{aligned}C(x)&=\int_F\Big[\sum_{j=1}^{n}\Big[\sum_{k=1}^{n} f(x_k)\ell_k'(x_j)-f'(x_j)\Big]\sigma_j(x)\Big]^2\omega(\mathrm{d}f)\\&=\sum_{s=1}^{n}\sum_{j=1}^{n}\sigma_s(x)\sigma_j(x)\int_F\Big[\sum_{k=1}^{n} f(x_k)\ell_k'(x_s)-f'(x_s)\Big]\\&\quad\cdot\Big[\sum_{m=1}^{n} f(x_m)\ell_m'(x_j)-f'(x_j)\Big]\omega(\mathrm{d}f).\end{aligned} \tag{3.5}$$

By (1.4) and (1.11), we know that for $1\le j,s\le n$,

$$\sigma_s(x)\sigma_j(x)=\frac{T_n^2(x)}{n^2}(-1)^{s+j}\sqrt{1-x_s^2}\sqrt{1-x_j^2}\,\ell_s(x)\ell_j(x). \tag{3.6}$$

In a similar way to the proof of (4.5) and (7.1) in [4], we know that for $1 \le j, s \le n$,

$$\begin{aligned}
&\int_F \sum_{k=1}^n f(x_k)\ell_k'(x_s) \sum_{m=1}^n f(x_m)\ell_m'(x_j)\omega(\mathrm{d}f) \\
&= \frac{2+x_s+x_j}{16} + \frac{1}{96}\sum_{k=1}^n\sum_{m=1}^n |x_k - x_m|^3 \ell_k'(x_s)\ell_m'(x_j), \qquad (3.7)
\end{aligned}$$

$$\sum_{k=1}^n \int_F f(x_k)f'(x_j)\omega(\mathrm{d}f)\ell_k'(x_s) = \frac{1+x_s}{8} - \sum_{k=1}^{j-1}\frac{(x_k-x_j)^2}{16}\ell_k'(x_s), \qquad (3.8)$$

$$\sum_{m=1}^n \int_F f(x_m)f'(x_s)\omega(\mathrm{d}f)\ell_m'(x_j) = \frac{1+x_j}{8} - \sum_{m=1}^{s-1}\frac{(x_m-x_s)^2}{16}\ell_m'(x_j). \qquad (3.9)$$

From (2.4) and (3.5)–(3.9), it follows that

$$C(x) = \frac{T_n^2(x)}{n^2}H(x), \qquad (3.10)$$

where

$$\begin{aligned}
H(x) &= \sum_{s=1}^n\sum_{j=1}^n \frac{(-1)^{s+j}\sqrt{1-x_s^2}\sqrt{1-x_j^2}}{96}\ell_s(x)\ell_j(x)\sum_{k=1}^n\sum_{m=1}^n |x_k-x_m|^3\ell_k'(x_s)\ell_m'(x_j) \\
&\quad + \sum_{s=1}^n\sum_{j=1}^n \frac{(-1)^{s+j}\sqrt{1-x_s^2}\sqrt{1-x_j^2}}{8}\ell_s(x)\ell_j(x)\sum_{k=1}^{j-1}(x_k-x_j)^2\ell_k'(x_s) \\
&\quad - \sum_{s=1}^n\sum_{j=1}^n \frac{(-1)^{s+j}\sqrt{1-x_s^2}\sqrt{1-x_j^2}\,|x_s-x_j|}{16}\ell_s(x)\ell_j(x). \qquad (3.11)
\end{aligned}$$

It is easy to know that $H(x)$ is an algebraic polynomial of degree at most $2n-2$. Then from (1.12), we conclude that

$$H(x) = \sum_{l=1}^n H(x_l)h_l(x) + \sum_{l=1}^n H'(x_l)\sigma_l(x). \qquad (3.12)$$

By (3.1)–(3.2) and (7.10), (7.12) in [4], we know that for an arbitrary $1 \le l \le n$,

$$\begin{aligned}
H(x_l) &= \frac{1-x_l^2}{96}\sum_{k=1}^n\sum_{m=1}^n |x_k-x_m|^3\ell_k'(x_l)\ell_m'(x_l) + \frac{1-x_l^2}{8}\sum_{k=1}^{l-1}(x_k-x_l)^2\ell_k'(x_l) \\
&= O\Big(\frac{1}{n}\Big). \qquad (3.13)
\end{aligned}$$

Here and in the following, the notation $A(x) = O(B(x))$ for sequences or functions $A(x)$ and $B(x)$ means the existence of a constant C independent of x, such that $|A(x)| \le CB(x)$, and C may be different in the different expressions. By (1.6) and (3.13), we obtain

$$\sum_{l=1}^n H(x_l)h_l(x) = O\Big(\frac{1}{n}\Big). \qquad (3.14)$$

From (3.1)–(3.2), it follows that for an arbitrary $1 \le l \le n$,

$$\begin{aligned} H'(x_l) &= \frac{1}{48}\sum_{j=1}^{n}(-1)^{l+j}\sqrt{1-x_l^2}\sqrt{1-x_j^2}\sum_{k=1}^{n}\sum_{m=1}^{n}|x_k-x_m|^3\ell_k'(x_l)\ell_m'(x_j)\ell_j'(x_l) \\ &\quad+\frac{1}{8}\sum_{j=1}^{n}(-1)^{l+j}\sqrt{1-x_l^2}\sqrt{1-x_j^2}\sum_{k=1}^{j-1}(x_k-x_j)^2\ell_k'(x_l)\ell_j'(x_l) \\ &\quad+\frac{1}{8}\sum_{s=1}^{n}(-1)^{l+s}\sqrt{1-x_l^2}\sqrt{1-x_s^2}\sum_{k=1}^{l-1}(x_k-x_l)^2\ell_k'(x_s)\ell_s'(x_l) \\ &\quad-\frac{1}{8}\sum_{j=1}^{n}(-1)^{l+j}\sqrt{1-x_l^2}\sqrt{1-x_j^2}|x_l-x_j|\ell_j'(x_l) \\ &= \frac{1}{48}(\mathrm{M}_1+6\mathrm{M}_2+6\mathrm{M}_3+6\mathrm{M}_4). \end{aligned} \tag{3.15}$$

Exchanging the sum order, we obtain

$$\begin{aligned} \mathrm{M}_1 &= \sum_{k=1}^{n}\sum_{m=1}^{n}|x_k-x_m|^3\ell_k'(x_l)\sum_{j=1}^{n}(-1)^{l+j}\sqrt{1-x_l^2}\sqrt{1-x_j^2}\,\ell_m'(x_j)\ell_j'(x_l) \\ &= \sum_{k=1}^{n}|x_k-x_l|^3\ell_k'(x_l)\sum_{j=1}^{n}(-1)^{l+j}\sqrt{1-x_l^2}\sqrt{1-x_j^2}\,\ell_l'(x_j)\ell_j'(x_l) \\ &\quad+\sum_{k=1}^{n}\sum_{m\ne l}|x_k-x_m|^3\ell_k'(x_l)\sum_{j=1}^{n}(-1)^{l+j}\sqrt{1-x_l^2}\sqrt{1-x_j^2}\,\ell_m'(x_j)\ell_j'(x_l) \\ &= \mathrm{L}_1+\mathrm{L}_2. \end{aligned} \tag{3.16}$$

By (3.1)–(3.2), (3.16) as well as (7.17) in [4] and

$$1-x_l^2 \ge \sin^2\frac{\pi}{2n} \ge \frac{1}{n^2},$$

we obtain

$$\mathrm{L}_1 = \Big(\frac{x_l^2}{4(1-x_l^2)}-\sum_{j\ne l}\frac{(-1)^{l+j}\sqrt{1-x_l^2}\sqrt{1-x_j^2}}{(x_j-x_l)^2}\Big)\sum_{k=1}^{n}|x_k-x_l|^3\ell_k'(x_l) = O\Big(\frac{1}{n}\Big). \tag{3.17}$$

We will consider L_2. For an arbitrary $m \ne l$, from (3.1)–(3.2), it follows that

$$\begin{aligned} &\sum_{j=1}^{n}(-1)^{l+j}\sqrt{1-x_l^2}\sqrt{1-x_j^2}\,\ell_m'(x_j)\ell_j'(x_l) \\ =&\frac{x_l}{2}\ell_m'(x_l)+\frac{(-1)^{l+m}x_m\sqrt{1-x_l^2}}{2\sqrt{1-x_m^2}}\,\ell_m'(x_l)+\sum_{j\ne m,l}\frac{(-1)^{j+m}\sqrt{1-x_j^2}\sqrt{1-x_m^2}}{(x_j-x_m)(x_l-x_j)}. \end{aligned} \tag{3.18}$$

From (2.6), we know

$$\sum_{k=1}^{n}\ell_k'(x) = 0. \tag{3.19}$$

By (3.2), (3.19) and the identity

$$\frac{1}{(x_j-x_m)(x_l-x_j)}=\frac{1}{(x_l-x_m)(x_j-x_m)}+\frac{1}{(x_l-x_m)(x_l-x_j)},$$

we obtain

$$\begin{aligned}
&\sum_{j\neq m,l}\frac{(-1)^{j+m}\sqrt{1-x_j^2}\sqrt{1-x_m^2}}{(x_j-x_m)(x_l-x_j)}\\
&=(-1)^{l+m+1}\sqrt{1-x_l^2}\sqrt{1-x_m^2}\,\ell'_m(x_l)\sum_{j\neq l,m}\ell'_j(x_m)+(1-x_l^2)\ell'_m(x_l)\sum_{j\neq l,m}\ell'_j(x_l)\\
&=(-1)^{l+m}\sqrt{1-x_l^2}\sqrt{1-x_m^2}\,\ell'_m(x_l)(\ell'_l(x_m)+\ell'_m(x_m))-(1-x_l^2)\ell'_m(x_l)(\ell'_l(x_l)+\ell'_m(x_l))\\
&=\Big(\frac{1-x_l^2}{x_m-x_l}+\frac{(-1)^{l+m}x_m\sqrt{1-x_l^2}}{2\sqrt{1-x_m^2}}-\frac{x_l}{2}-\frac{(-1)^{l+m}\sqrt{1-x_l^2}\sqrt{1-x_m^2}}{x_l-x_m}\Big)\ell'_m(x_l). \qquad (3.20)
\end{aligned}$$

From (3.16), (3.18) and (3.20), we conclude that

$$\begin{aligned}
\mathrm{L}_2=&\sum_{m\neq l}\frac{(-1)^{l+m}\sqrt{1-x_l^2}(x_lx_m-1)}{\sqrt{1-x_m^2}(x_l-x_m)}\ell'_m(x_l)\sum_{k=1}^n|x_k-x_m|^3\ell'_k(x_l)\\
&+\sum_{k=1}^n\sum_{m\neq l}\frac{1-x_l^2}{x_m-x_l}|x_k-x_m|^3\ell'_k(x_l)\ell'_m(x_l)=\mathrm{E}_1+\mathrm{E}_2. \qquad (3.21)
\end{aligned}$$

By (2.6), we obtain

$$\sum_{k=1}^n(x_k-x_m)^3\ell'_k(x_l)=3(x-x_m)^2|_{x=x_l}=3(x_m-x_l)^2. \qquad (3.22)$$

From (3.22), we know

$$\sum_{k=1}^n|x_k-x_m|^3\ell'_k(x_l)=2\sum_{k=1}^{m-1}(x_k-x_m)^3\ell'_k(x_l)-3(x_m-x_l)^2. \qquad (3.23)$$

From (3.21)–(3.23), it follows that

$$\begin{aligned}
\mathrm{E}_1=&\,2\sum_{m\neq l}\frac{(-1)^{l+m}\sqrt{1-x_l^2}(x_lx_m-1)}{\sqrt{1-x_m^2}(x_l-x_m)}\,\ell'_m(x_l)\sum_{k=1}^{m-1}(x_k-x_m)^3\ell'_k(x_l)\\
&-3\sum_{m\neq l}\frac{(-1)^{l+m}\sqrt{1-x_l^2}(x_lx_m-1)}{\sqrt{1-x_m^2}(x_l-x_m)}(x_m-x_l)^2\ell'_m(x_l)\\
=&\,2\mathrm{G}_1-3\mathrm{G}_2. \qquad (3.24)
\end{aligned}$$

Combining (3.1)–(3.2), (7.2) in [4] and the identity

$$x_k-x_m=(x_k-x_l)+(x_l-x_m),$$

we obtain

$$\mathrm{G}_1=\sum_{m\neq l}\frac{(-1)^m(x_lx_m-1)}{\sqrt{1-x_m^2}(x_l-x_m)}\,\ell'_m(x_l)\sum_{k=1,k\neq l}^{m-1}(-1)^{k+1}(x_k-x_m)^2\sqrt{1-x_k^2}$$

$$
\begin{aligned}
&+\sum_{m=1}^{n}\frac{(-1)^{l+m}\sqrt{1-x_l^2}(x_lx_m-1)}{\sqrt{1-x_m^2}}\,\ell_m'(x_l)\sum_{k=1}^{m-1}(x_k-x_m)^2\ell_k'(x_l)\\
&+\frac{(-1)^l x_l}{2\sqrt{1-x_l^2}}\sum_{k=1}^{l-1}(-1)^{k+1}(x_k-x_l)\sqrt{1-x_k^2}\\
=&\sum_{m\neq l}\frac{(-1)^m(x_mx_l-1)}{(x_l-x_m)\sqrt{1-x_m^2}}\,\ell_m'(x_l)\sum_{k=1}^{m-1}(-1)^{k+1}(x_k-x_m)^2\sqrt{1-x_k^2}+\sum_{m=l+1}^{n}(x_mx_l-1)\\
&+\sum_{m=1}^{n}\frac{(-1)^{l+m}(x_mx_l-1)\sqrt{1-x_l^2}}{\sqrt{1-x_m^2}}\,\ell_m'(x_l)\sum_{k=1}^{m-1}(x_k-x_m)^2\ell_k'(x_l)+O\Big(\frac{1}{n}\Big)\\
=&\ \mathrm{G}_{11}+\mathrm{G}_{12}+\mathrm{G}_{13}+O\Big(\frac{1}{n}\Big). \qquad (3.25)
\end{aligned}
$$

From

$$\sin x>\frac{2x}{\pi},\quad 0<x<\frac{\pi}{2}$$

and

$$\sin x\le\sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2}\le 2\sin\frac{x+y}{2},\quad 0\le x,y\le\pi,$$

we verify that

$$\sum_{k\neq l}\frac{1-x_k^2}{|x_k-x_l|^2}=\sum_{k\neq l}\frac{\sin^2\frac{(2k-1)\pi}{2n}}{|\sin\frac{(k+l-1)\pi}{2n}\sin\frac{(k-l)\pi}{2n}|^2}\le\sum_{k\neq l}\frac{4n^2}{|k-l|^2}=O(n^2).$$

From (4.16) in [4], the above relation and

$$1-x_k^2\ge\frac{1}{n^2},\quad k=1,2,\cdots,n,$$

it follows that

$$
\begin{aligned}
\mathrm{G}_{11}&=\frac{2\pi^3}{n^3}\sum_{m\neq l}\frac{(-1)^m x_m(x_mx_l-1)(1-x_m^2)}{(x_l-x_m)^2\sqrt{1-x_l^2}}+O\Big(\frac{1}{n^5\sqrt{1-x_l^2}}\Big)\sum_{m\neq l}\frac{1-x_lx_m}{(x_l-x_m)^2}\\
&=O\Big(\frac{1}{n^3\sqrt{1-x_l^2}}\Big)\sum_{m\neq l}\frac{1-x_m^2}{(x_l-x_m)^2}=O\Big(\frac{1}{n\sqrt{1-x_l^2}}\Big). \qquad (3.26)
\end{aligned}
$$

From (3.25), (3.15) and the identity

$$(x_k-x_m)^2=(x_k-x_l)(x_k-x_m)+(x_k-x_l)(x_l-x_m)+(x_l-x_m)^2,$$

we obtain

$$
\begin{aligned}
\mathrm{G}_{13}+3\mathrm{M}_2&=\sum_{m=1}^{n}\frac{(-1)^{l+m}(x_mx_l+2-3x_m^2)\sqrt{1-x_l^2}}{\sqrt{1-x_m^2}}\,\ell_m'(x_l)\sum_{k=1}^{m-1}(x_k-x_m)^2\ell_k'(x_l)\\
&=\sum_{m=1}^{n}\frac{(-1)^m(x_mx_l+2-3x_m^2)}{\sqrt{1-x_m^2}}\ell_m'(x_l)\sum_{k=1}^{m-1}(-1)^{k+1}(x_k-x_m)\sqrt{1-x_k^2}\\
&\quad+\sum_{m=1}^{n}\frac{(-1)^m(x_mx_l+2-3x_m^2)}{\sqrt{1-x_m^2}}(x_l-x_m)\ell_m'(x_l)\sum_{k=1}^{m-1}(-1)^{k+1}\sqrt{1-x_k^2}
\end{aligned}
$$

$$
\begin{aligned}
&+2\sum_{m=l+1}^{n}(x_m x_l+2-3x_m^2)\\
&+\sum_{m=1}^{n}\frac{(-1)^{l+m}(x_m x_l+2-3x_m^2)\sqrt{1-x_l^2}}{\sqrt{1-x_m^2}}(x_l-x_m)^2\ell_m'(x_l)\sum_{k=1}^{m-1}\ell_k'(x_l)\\
=\;&\mathrm{N}_1+\mathrm{N}_2+\mathrm{N}_3+\mathrm{N}_4.
\end{aligned}\tag{3.27}
$$

From (3.2), (3.19) (3.27) and (7.2) in [4], the relation

$$\tan x = x+O(x^3),\quad 0\le x\le\frac{3\pi}{8},$$

and a direct computation, we obtain

$$
\begin{aligned}
\mathrm{N}_1&=\frac{1}{4}\sum_{m=1}^{n}\frac{x_m x_l+2-3x_m^2}{\sqrt{1-x_m^2}}\Big(2x_m^2\tan\frac{\pi}{2n}-(2x_m^2-1)\tan\frac{\pi}{n}\Big)\ell_m'(x_l)\\
&=\frac{\pi}{2n}\sum_{m=1}^{n}(1-x_l^2)\sqrt{1-x_m^2}\,\ell_m'(x_l)+\frac{\pi}{4n}\sum_{m=1}^{n}(3x_m+2x_l)(x_l-x_m)\sqrt{1-x_m^2}\,\ell_m'(x_l)\\
&\quad+O\Big(\frac{1}{n^3}\Big)\sum_{m\neq l}\frac{\sqrt{1-x_l^2}}{|x_l-x_m|}+O\Big(\frac{1}{n^3}\Big)\sum_{m=1}^{n}\frac{1}{\sqrt{1-x_l^2}}\\
&=\frac{\pi}{2n}\sum_{m=1}^{n}(1-x_l^2)\sqrt{1-x_m^2}\,\ell_m'(x_l)+\frac{\pi}{4n}\sum_{m=1}^{n}\frac{(-1)^{l+m}(3x_m+2x_l)(1-x_m^2)}{\sqrt{1-x_l^2}}+O\Big(\frac{1}{n}\Big)\\
&=\frac{\pi}{2n}\sum_{m=1}^{n}(1-x_l^2)\Big(\sqrt{1-x_m^2}-\sqrt{1-x_l^2}\Big)\ell_m'(x_l)+\frac{\pi}{2n}(1-x_l^2)^{\frac{3}{2}}\sum_{m=1}^{n}\ell_m'(x_l)\\
&\quad-\frac{(-1)^l\pi}{4n\sqrt{1-x_l^2}}\Big(\frac{3(1+(-1)^n)}{8\cos\frac{\pi}{2n}}-\frac{3(1+(-1)^n)}{8\cos\frac{3\pi}{2n}}-\frac{1-(-1)^n}{2}x_l+\frac{1-(-1)^n}{2\cos\frac{\pi}{n}}x_l\Big)+O\Big(\frac{1}{n}\Big)\\
&=\frac{(-1)^l\pi\sqrt{1-x_l^2}}{4n}\sum_{m\neq l}\frac{(-1)^m(x_l+x_m)\sqrt{1-x_m^2}}{\sqrt{1-x_m^2}+\sqrt{1-x_l^2}}+O\Big(\frac{1}{n}\Big).
\end{aligned}\tag{3.28}
$$

For an arbitrary l, it is easy to verify that $\frac{(x_l+x)\sqrt{1-x^2}}{\sqrt{1-x^2}+\sqrt{1-x_l^2}}$ has at most 6 monotone intervals and $\Big|\frac{(x_l+x_m)\sqrt{1-x_m^2}}{\sqrt{1-x_m^2}+\sqrt{1-x_l^2}}\Big|\le 2$. Combining it with the Leibniz theorem of alternating series, we conclude that

$$\Big|\sum_{m\neq l}\frac{(-1)^m(x_l+x_m)\sqrt{1-x_m^2}}{\sqrt{1-x_m^2}+\sqrt{1-x_l^2}}\Big|\le 12.$$

Therefore, (3.28) and the above relation give

$$\mathrm{N}_1=O\Big(\frac{1}{n}\Big).\tag{3.29}$$

By a direct computation, we obtain

$$\sum_{k=1}^{m-1}(-1)^{k+1}\sqrt{1-x_k^2}=\frac{(-1)^m}{2}\Big(\sqrt{1-x_m^2}-x_m\tan\frac{\pi}{2n}\Big).\tag{3.30}$$

From (2.6), (3.27), (3.30) and a direct computation, it follows that

$$
\begin{aligned}
\mathrm{N}_2 &= \frac{1}{2}\sum_{m=1}^{n}\frac{(x_m x_l+2-3x_m^2)}{\sqrt{1-x_m^2}}(x_l-x_m)\Big(\sqrt{1-x_m^2}-x_m\tan\frac{\pi}{2n}\Big)\ell_m'(x_l)\\
&= \frac{1}{2}\sum_{m=1}^{n}(x_m x_l+2-3x_m^2)(x_l-x_m)\ell_m'(x_l)-\frac{(-1)^l\tan\frac{\pi}{2n}}{2\sqrt{1-x_l^2}}\sum_{m=1}^{n}(-1)^m x_m(x_m x_l+2-3x_m^2)\\
&= \frac{1}{2}[(xx_l+2-3x^2)(x_l-x)]'|_{x=x_l}\\
&\quad+\frac{(-1)^l\tan\frac{\pi}{2n}}{16\sqrt{1-x_l^2}}\Big(2x_l+2(-1)^{n+1}x_l+2\frac{1+(-1)^{n+1}}{\cos\frac{\pi}{n}}x_l-\frac{1+(-1)^n}{\cos\frac{\pi}{2n}}-3\frac{1+(-1)^n}{\cos\frac{3\pi}{2n}}\Big)\\
&= -(1-x_l^2)+O\Big(\frac{1}{n\sqrt{1-x_l^2}}\Big).
\end{aligned}
\tag{3.31}
$$

Exchanging the sum order similar to (4.13)–(4.15) in [4], we obtain

$$
\begin{aligned}
\mathrm{N}_4 &= \sum_{k=1}^{n}\ell_k'(x_l)\sum_{m=k+1}^{n}(x_m x_l+2-3x_m^2)(x_l-x_m)\\
&= \sum_{k=1}^{n}\ell_k'(x_l)\sum_{m=k+1}^{n}\Big[\Big(x_l^2+\frac{1}{4}\Big)\cos\frac{2m-1}{2n}\pi-2x_l\cos\frac{m-1}{n}\pi+\frac{3\cos\frac{3(2m-1)}{2n}\pi}{4}\Big]\\
&= -\sum_{k=1}^{n}\ell_k'(x_l)\Big[\Big(x_l^2+\frac{1}{4}\Big)\frac{\sin\frac{k}{n}\pi}{2\sin\frac{\pi}{2n}}-x_l\frac{\sin\frac{2k}{n}\pi}{\sin\frac{\pi}{n}}+\frac{3\sin\frac{3k}{n}\pi}{8\sin\frac{3\pi}{2n}}\Big]\\
&= -\frac{1}{2}\sum_{k=1}^{n}\ell_k'(x_l)\sqrt{1-x_k^2}\Big[\Big(x_l^2+\frac{1}{4}\Big)\cot\frac{\pi}{2n}-4x_l x_k\cot\frac{\pi}{n}+\frac{3(4x_k^2-1)}{4}\cot\frac{3\pi}{2n}\Big]\\
&\quad-\frac{1}{2}\sum_{k=1}^{n}\ell_k'(x_l)\Big[\Big(x_l^2+\frac{1}{4}\Big)x_k-2x_l(2x_k^2-1)+\frac{3(4x_k^2-3x_k)}{4}\Big]=\mathrm{N}_{41}+\mathrm{N}_{42}.
\end{aligned}
\tag{3.32}
$$

By (2.6), we know

$$
\mathrm{N}_{42}=-\frac{1}{2}\Big[\Big(x_l^2+\frac{1}{4}\Big)x-2x_l(2x^2-1)+\frac{3(4x^3-3x)}{4}\Big]'\Big|_{x=x_l}=1-x_l^2. \tag{3.33}
$$

From

$$
\cot x=\frac{1}{x}-\frac{x}{3}+\frac{x^3}{15}+O(x^5),\quad 0\le x\le\frac{\pi}{4},
$$

it follows that

$$
\begin{aligned}
\mathrm{N}_{41} &= -\frac{n}{\pi}\sum_{k=1}^{n}(x_l-x_k)^2\sqrt{1-x_k^2}\,\ell_k'(x_l)-\frac{\pi}{6n}\sum_{k=1}^{n}\sqrt{1-x_k^2}\,\ell_k'(x_l)(x_l^2-2-8x_l x_k+9x_k^2)\\
&\quad-\frac{\pi^3}{120n^3}\sum_{k=1}^{n}\sqrt{1-x_k^2}\,\ell_k'(x_l)(x_l^2-20-32x_l x_k+81x_k^2)+O\Big(\frac{1}{n^5}\Big)\sum_{k=1}^{n}|\ell_k'(x_l)|.
\end{aligned}
\tag{3.34}
$$

By a direct computation and

$$
\cos x=1+O(x^2),\quad 0<x<\frac{\pi}{4},
$$

we obtain

$$\begin{aligned}
&\sum_{k=1}^{n}(x_l-x_k)^2\sqrt{1-x_k^2}\,\ell_k'(x_l)\\
=&\frac{(-1)^{l+1}}{\sqrt{1-x_l^2}}\sum_{k=1}^{n}(-1)^{k+1}\Big[\frac{x_l}{2}\Big(1-\cos\frac{2k-1}{n}\pi\Big)-\frac{\cos\frac{2k-1}{2n}\pi-\cos\frac{3(2k-1)}{2n}\pi}{4}\Big]\\
=&\frac{(-1)^{l+1}}{\sqrt{1-x_l^2}}\Big[\frac{x_l(1-(-1)^n)}{4}\Big(1-\frac{1}{\cos\frac{\pi}{n}}\Big)-\frac{1+(-1)^n}{8}\Big(\frac{1}{\cos\frac{\pi}{2n}}-\frac{1}{\cos\frac{3\pi}{2n}}\Big)\Big]\\
=&O\Big(\frac{1}{n^2\sqrt{1-x_l^2}}\Big). \qquad (3.35)
\end{aligned}$$

From the proof of (3.29) and a direct computation, we conclude that

$$\begin{aligned}
&\sum_{k=1}^{n}\sqrt{1-x_k^2}\,\ell_k'(x_l)(x_l^2-2-8x_lx_k+9x_k^2)\\
=&\frac{(-1)^{l+1}}{\sqrt{1-x_l^2}}\sum_{k=1}^{n}(-1)^k(9x_k+x_l)(1-x_k^2)-2(1-x_l^2)\sum_{k=1}^{n}\sqrt{1-x_k^2}\,\ell_k'(x_l)\\
=&\frac{(-1)^{l}}{\sqrt{1-x_l^2}}\Big[\frac{x_l(1-(-1)^n)}{4}\Big(1-\frac{1}{\cos\frac{\pi}{n}}\Big)-\frac{9(1+(-1)^n)}{8}\Big(\frac{1}{\cos\frac{\pi}{2n}}-\frac{1}{\cos\frac{3\pi}{2n}}\Big)\Big]+O(1)\\
=&O\Big(\frac{1}{\sqrt{1-x_l^2}}\Big). \qquad (3.36)
\end{aligned}$$

By the same method, we obtain

$$\sum_{k=1}^{n}\sqrt{1-x_k^2}\,\ell_k'(x_l)(x_l^2-20-32x_lx_k+81x_k^2)=O\Big(\frac{1}{\sqrt{1-x_l^2}}\Big). \qquad (3.37)$$

From Markov inequality and

$$\|\ell_k(x)\|_\infty\le 2,\quad k=1,\cdots,n,$$

it follows that

$$\sum_{k=1}^{n}\Big|\ell_k'(x_l)\Big|\le 2\sum_{k=1}^{n}n^2=2n^3. \qquad (3.38)$$

From (3.34)–(3.38), we get

$$\mathrm{N}_{41}=O\Big(\frac{1}{n\sqrt{1-x_l^2}}\Big). \qquad (3.39)$$

By (3.2) and a simple computation, we obtain

$$\mathrm{G}_2=\sum_{m\ne l}(x_mx_l-1)\,. \qquad (3.40)$$

From (3.1) and (3.2), we conclude that

$$\mathrm{E}_2=\frac{x_l}{2\sqrt{1-x_l^2}}\sum_{m\ne l}(-1)^{l+m+1}\sqrt{1-x_m^2}|x_l-x_m|$$

$$+\sum_{k\neq l}\sum_{m\neq l}\frac{(-1)^{k+m}\sqrt{1-x_k^2}\sqrt{1-x_m^2}|x_k-x_m|^3}{(x_l-x_k)^2(x_m-x_l)}$$
$$=\mathrm{E}_{21}+\mathrm{E}_{22}. \tag{3.41}$$

A simple computation leads to

$$\mathrm{E}_{21}=\frac{x_l}{4\sqrt{1-x_l^2}}\Big(2x_l^2\tan\frac{\pi}{2n}-(2x_l^2-1)\tan\frac{\pi}{n}\Big)=O\Big(\frac{1}{n}\Big). \tag{3.42}$$

From (7.18) in [4], we know

$$\mathrm{E}_{22}=O\Big(\frac{1}{n}\Big). \tag{3.43}$$

In a similar way to (3.20), we have

$$\begin{aligned}\mathrm{M}_3&=\sum_{m=1}^{l-1}(x_m-x_l)^2\sum_{j=1}^{n}(-1)^{l+j}\sqrt{1-x_l^2}\sqrt{1-x_j^2}\,\ell_m'(x_j)\ell_j'(x_l)\\&=\sum_{m=1}^{l-1}(x_m-x_l)^2\Big(\frac{1-x_l^2}{x_m-x_l}+\frac{(-1)^{l+m}(x_lx_m-1)\sqrt{1-x_l^2}}{(x_l-x_m)\sqrt{1-x_m^2}}\Big)\ell_m'(x_l)\\&=(-1)^l\sqrt{1-x_l^2}\sum_{m=1}^{l-1}(-1)^{m+1}\sqrt{1-x_m^2}+\sum_{m=1}^{l-1}(x_lx_m-1)\\&=\frac{\sqrt{1-x_l^2}}{2}\Big(\sqrt{1-x_l^2}-x_l\tan\frac{\pi}{2n}\Big)+\sum_{m=1}^{l-1}(x_lx_m-1)\\&=\frac{1-x_l^2}{2}+\sum_{m=1}^{l-1}(x_lx_m-1)+O\Big(\frac{1}{n}\Big).\end{aligned} \tag{3.44}$$

By (3.2) and a simple computation, we obtain

$$\mathrm{M}_4=\sum_{j=1}^{l-1}\left(1-x_j^2\right)-\sum_{j=l+1}^{n}\left(1-x_j^2\right). \tag{3.45}$$

From (3.25), (3.27), (3.40), (3.44)–(3.45), we obtain

$$\begin{aligned}&2\mathrm{G}_{12}+2\mathrm{N}_3-3\mathrm{G}_2+6\mathrm{M}_3+6\mathrm{M}_4\\&=2\sum_{m=l+1}^{n}(x_mx_l-1)+4\sum_{m=l+1}^{n}(x_mx_l+2-3x_m^2)-3\sum_{m\neq l}(x_mx_l-1)\\&\quad+6\Big(\frac{1-x_l^2}{2}+\sum_{m=1}^{l-1}(x_lx_m-1)+O\Big(\frac{1}{n}\Big)\Big)+6\Big(\sum_{m=1}^{l-1}(1-x_m^2)-\sum_{m=l+1}^{n}(1-x_m^2)\Big)\\&=3\sum_{m\neq l}(x_mx_l-1)+6\sum_{m\neq l}(1-x_m^2)+3(1-x_l^2)+O\Big(\frac{1}{n}\Big)\\&=O\Big(\frac{1}{n}\Big).\end{aligned} \tag{3.46}$$

From (3.15)–(3.46), we obtain

$$H'(x_l) = O\Big(\frac{1}{n\sqrt{1-x_l^2}}\Big). \tag{3.47}$$

By (1.4), (1.11), (3.47) and the well-known estimate

$$\sum_{k=1}^{n} |\ell_k(x)| = O(\ln(n+1)), \tag{3.48}$$

we know

$$\sum_{l=1}^{n} |H'(x_l)\sigma_l(x)| = O\Big(\frac{1}{n^2}\Big)\sum_{l=1}^{n} |\ell_l(x)| = O\Big(\frac{\ln(n+1)}{n^2}\Big) = o\Big(\frac{1}{n}\Big). \tag{3.49}$$

From (3.5), (3.10), (3.12), (3.14) and (3.49), we obtain

$$C(x) = O\Big(\frac{1}{n^3}\Big). \tag{3.50}$$

By [1, p. 107], we know

$$e_p^p(G_n, \|\|_{p,\varrho}, F, \omega) = v_p \int_{-1}^{1} \Big(\int_F |f(x) - G_n(f,x)|^2 \omega(\mathrm{d}f)\Big)^{\frac{p}{2}} \varrho(x)\mathrm{d}x, \tag{3.51}$$

where v_p is the p-th absolute moment of the standard normal distribution. By (1.7), (3.4), (3.50)–(3.51), we obtain the desired upper estimate.

References

[1] Klaus, R., Average-case Analysis of Numerical Problems, Springer-Verlag, New York, 2000.

[2] Erdös, P. and Feldheim, E., Sur le mode de convergence pour 1′interpolation de Lagrange, *C. R. Acad Sci. Paris Sèr I. Math.*, **203**, 1936, 913–915.

[3] Fejér, L., Lagrangesche interpolation und zugehörigen konjugierten punkte, *Math. Ann.*, **106**, 1932, 1–55.

[4] Xu, G. Q., The simultaneous approximation average errors for interpolation polynomials on the 1-fold integrated Wiener space (in Chinese), *Sci. Sin. Math.*, **41**(5), 2011, 407–426. DOI: 10.1360/012010-304

[5] Xu, G. Q., The average errors for Lagrange interpolation and Hermite-Fejér interpolation on the Wiener space (in Chinese), *Acta Math. Sinica*, **50**(6), 2007, 1281–1296.

[6] Xu, G. Q. and Du, Y. F., The average errors for quasi-Hermite-Fejér interpolation on the Wiener space, *Sci. China Ser. A*, **53**(6), 2010, 1841–1852.

[7] Xu, G. Q., The average errors for Lagrange interpolation on the Wiener space, *Acta Math. Sin.* (*Engl. Ser.*), DOI:10.1007/s10114-012-0242-9

[8] Xu, G. Q. and Du, Y. F., The average errors for Trigonometric polynomial operators on the Brownian bridge measure (in Chinese), *Acta Math. Sinica*, **52**(3), 2009, 523–534.

[9] Varma, A. K. and Prasad, J., An analogue of a problem of P. Erös and E. Feldheim on L_p convergence of interpolatory processes, *J. Approx. Theory*, **56**, 1989, 225–240.