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Convergence of Gaussian Quadrature Formulas for Power Orthogonal Polynomials*

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Abstract In classical theorems on the convergence of Gaussian quadrature formulas for power orthogonal polynomials with respect to a weight w on $\mathbf{I} = (a, b)$, a function $G \in \mathbf{S}(w) := \{f : \int_{\mathbf{I}} |f(x)|w(x)\mathrm{d}x < \infty\}$ satisfying the conditions $G^{(2j)}(x) \geq 0$, $x \in (a, b)$, $j = 0, 1, \cdots$, and growing as fast as possible as $x \to a+$ and $x \to b-$, plays an important role. But to find such a function G is often difficult and complicated. This implies that to prove convergence of Gaussian quadrature formulas, it is enough to find a function $G \in \mathbf{S}(w)$ with $G \geq 0$ satisfying

$$\sup_n \sum_{k=1}^{n} \lambda_{0kn} G(x_{kn}) < \infty$$

instead, where the x_{kn}'s are the zeros of the nth power orthogonal polynomial with respect to the weight w and λ_{0kn}'s are the corresponding Cotes numbers. Furthermore, some results of the convergence for Gaussian quadrature formulas involving the above condition are given.

Keywords Convergence, Gaussian quadrature formula, Freud weight
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1 Introduction and Main Results

Let w be a weight on $\mathbf{I} := (a, b)$, $-\infty \leq a < b \leq \infty$, for which the moment problem possesses a unique solution. Denote by N and N_e the sets of positive and even positive integers, respectively. R stands for the set of real numbers. For each $n \in N$, let $m_{kn} \in N_e$, $k = 1, 2, \cdots$, n and $N_n = \sum\limits_{k=1}^{n} m_{kn} - 1$. We always assume that $m = \max\limits_{\substack{1\leq k\leq n\\ n\in N}} m_{kn} < \infty$. Let $\mathbf{S}(w) := \{f : \int_{\mathbf{I}} |f(x)|w(x)\mathrm{d}x < \infty\}$. The letters c, $c_1, \cdots$ stand for positive constants, which may be different at different occurrences, even in subsequent formulas, unless otherwise indicated. Moreover, $C_n \sim D_n$ means that there are two constants c_1 and c_2 such that $c_1 \leq \frac{C_n}{D_n} \leq c_2$ for the relevant range of n.

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Definition 1.1 (see [1, p. 61]) *Let w be a weight on $\mathbf{I}$ and let $f \in \mathbf{S}(w)$. Assume that there exist $s \in N_e$ and $C_0 \geq 1$ such that*

$$|f(x)| \leq C_0(1 + x^s), \quad x \in \mathbf{I}. \tag{1.1}$$

Then we write $f \in \mathbf{S}_0(w)$.

Definition 1.2 (see [5, Definition 1.1]) *Let $w = \mathrm{e}^{-Q}$, where $Q \in C(R)$ is even, $Q'(x) > 0$, $x \in (0, \infty)$, $Q'' \in C(0, \infty)$, and for some $A, B > 0$,*

$$A \leq \frac{(xQ'(x))'}{Q'(x)} \leq B, \quad x \in (0, \infty).$$

Then we write $w \in \mathcal{F}'$.

Further, assume that $A > 1$, $Q(0) = 0$ and $Q' \in C[0, \infty)$. In this case, we write $w \in \mathcal{F}^*$.

A function $f : (c, d) \to (0, \infty)$ is said to be quasi-increasing (or quasi-decreasing) if there exists $C > 0$ such that

$$f(x) \leq (\text{or} \geq)\ Cf(y), \quad c < x \leq y < d.$$

Definition 1.3 (see [3, p. 10]) *Let $a < 0 < b$. Assume that $w = \mathrm{e}^{-Q}$, where $Q : \mathbf{I} \to [0, \infty)$ satisfies the following properties:*

(a) *$Q' \in C(\mathbf{I})$ and $Q(0) = 0$.*

(b) *Q' is non-decreasing in $\mathbf{I}$.*

(c) *We have*

$$\lim_{t \to a+} Q(t) = \lim_{t \to b-} Q(t) = \infty.$$

(d) *The function*

$$T(t) := \frac{tQ'(t)}{Q(t)}, \quad t \neq 0$$

is quasi-decreasing in $(a, 0)$ and quasi-increasing in $(0, b)$, respectively. Moreover,

$$T(t) \geq \Lambda > 1, \quad t \in \mathbf{I} \setminus \{0\}.$$

(e) *There exists $\epsilon_0 \in (0, 1)$ such that for all $y \in \mathbf{I} \setminus \{0\}$,*

$$T(y) \sim T\Big(y\Big[1 - \frac{\epsilon_0}{T(y)}\Big]\Big).$$

Then we write $w \in \mathcal{F}$.

Definition 1.4 (see [3, pp. 11–12]) Let $w \in \mathcal{F}$.

(f) *Assume that there exist $C, \epsilon > 0$ such that for all $x \in \mathbf{I} \setminus \{0\}$,*

$$\int_{x - \epsilon|x|/T(x)}^{x} \frac{|Q'(t) - Q'(x)|}{|t - x|^{3/2}} \mathrm{d}t \leq C|Q'(x)|\Big[\frac{T(x)}{|x|}\Big]^{\frac{1}{2}}.$$

Then we write $w \in \mathcal{F}\left(\mathrm{Lip}\frac{1}{2}\right)$.

The numbers $a_{-t} := a_{-t}(Q) < 0 < a_t := a_t(Q)$, $t > 0$ are defined by the equations

$$t = \frac{1}{\pi} \int_{a_{-t}}^{a_t} \frac{xQ'(x)}{[(x - a_{-t})(a_t - x)]^{\frac{1}{2}}} \mathrm{d}x$$

and

$$0 = \frac{1}{\pi}\int_{a_{-t}}^{a_t} \frac{Q'(x)}{[(x-a_{-t})(a_t-x)]^{\frac{1}{2}}}\mathrm{d}x.$$

For $w \in \mathcal{F}$ or $\mathcal{F}'$, and $t > 0$, we define

$$\begin{aligned}
\delta_t :&= \frac{1}{2}(a_t + |a_{-t}|),\\
\eta_{\pm t} :&= \Big[tT(a_{\pm t})\sqrt{\frac{|a_{\pm t}|}{\delta_t}}\Big]^{-\frac{2}{3}},\\
\sigma_t(x) :&= \frac{1}{\pi^2}[(x-a_{-t})(a_t-x)]^{\frac{1}{2}}\int_{a_{-t}}^{a_t}\frac{[Q'(t)-Q'(x)]\mathrm{d}t}{(t-x)[(t-a_{-t})(a_t-t)]^{\frac{1}{2}}}
\end{aligned}$$

and

$$\varphi_t(x) := \begin{cases} \dfrac{|x-a_{-2t}||x-a_{2t}|}{t\sqrt{[|x-a_{-t}|+|a_{-t}|\eta_{-t}][|x-a_t|+a_t\eta_t]}}, & x \in [a_{-t}, a_t],\\ \varphi_t(a_t), & x \in (a_t, b),\\ \varphi_t(a_{-t}), & x \in (a, a_{-t}). \end{cases}$$

Definition 1.5 (see [6, Definition 9.28, p. 169]) *The generalized Jacobi weight W is defined by*

$$\begin{aligned}
&W(x) = \prod_{i=1}^{r}|x-t_i|^{p_i},\ |x|<1, \quad W(x) = 0,\ |x| \geq 1,\\
&-1 = t_r < t_{r-1} < \cdots < t_1 = 1\ (r \geq 2), \quad p_i > -1,\ i = 1, 2, \cdots, r.
\end{aligned}$$

The polynomial with $x_{kn} = x_{kn}(w),\ k = 1, 2, \cdots, n,$

$$\begin{aligned}
&P(w;x) = (x-x_{1n})(x-x_{2n})\cdots(x-x_{nn}),\\
&a = x_{n+1,n} < x_{nn} < x_{n-1,n} < \cdots < x_{1n} < x_{0n} = b
\end{aligned}$$

is said to be the nth power orthogonal polynomial, if it is a solution of the extremal problem

$$\int_a^b \prod_{k=1}^{n}(x-x_{kn})^{m_{kn}}w(x)\mathrm{d}x = \min_{a\leq y_n\leq y_{n-1}\leq\cdots\leq y_1\leq b}\int_a^b\prod_{k=1}^{n}(x-y_k)^{m_{kn}}w(x)\mathrm{d}x.$$

It is well known that the solution of the above extremal problem admits the Gaussian quadrature formula

$$\int_a^b f(x)w(x)\mathrm{d}x = \sum_{k=1}^{n}\sum_{j=0}^{m_{kn}-2}\lambda_{jkn}f^{(j)}(x_{kn}), \tag{1.2}$$

which is exact for all $f \in \mathbf{P}_{N_n}$, where $\lambda_{jkn} := \lambda_{jkn}(w)$ are called the Cotes numbers.

For $f \in \mathbf{S}(w)$,

$$Q_n(w;f) := \sum_{k=1}^{n}\lambda_{0kn}(w)f(x_{kn}(w)). \tag{1.3}$$

Our particular interest is the convergence

$$\lim_{n\to\infty} Q_n(w;f) = \int_a^b f(x)w(x)\mathrm{d}x. \tag{1.4}$$

We have the classical result of Shohat (see [1, Theorem 1.6, p. 93])

Theorem 1.1 (see [1, Theorem 1.6, p. 93]) *Let w be a weight on $\mathbf{I} = R$. Let $m_{kn} \equiv 2$ and $f \in \mathbf{S}(w)$. Assume that G has all derivatives on $\mathbf{I}$ and satisfies that*

$$G^{(2j)}(x) \geq 0, \quad x \in \mathbf{I}, \ j = 0, 1, \cdots \tag{1.5}$$

and

$$\lim_{x\to a+} \frac{f(x)}{G(x)} = \lim_{x\to b-} \frac{f(x)}{G(x)} = 0. \tag{1.6}$$

Then (1.4) holds.

According to an inequality of Markov, the condition (1.5) implies

$$\sum_{k=1}^{n} \lambda_{0kn} G(x_{kn}) \leq \int_R G(x)w(x)\mathrm{d}x. \tag{1.7}$$

A crucial and difficult problem is to find an entire functions $G \in \mathbf{S}(w)$ which satisfies (1.5) and grows as fast as possible as $x \to a+$ and $x \to b-$. Lubinsky [4] gives such a function for Freud weight $w \in \mathcal{F}'$,

$$G_Q(x) = \sum_{n=0}^{\infty} \Big(\frac{x}{a_n}\Big)^{2n} n^{-\frac{1}{2}} w(a_n)^{-1}. \tag{1.8}$$

Theorem 1.2 (see [4, Theorem 1, Corollary 2 and (63)]) *Let $w \in \mathcal{F}'$, $\epsilon > 0$, and*

$$\psi(x) = x^{-1-\epsilon} \ \textit{or} \ x^{-1}(\ln x)^{-1-\epsilon}, \cdots. \tag{1.9}$$

Then, for

$$Q^*(x) = Q(x) + \ln \psi(x), \tag{1.10}$$

we have $G_{Q^*} \in \mathbf{S}(w)$,

$$G_{Q^*}^{(2j)}(x) \geq 0, \quad x \geq 1, \quad j = 0, 1, \cdots \tag{1.11}$$

and

$$G_{Q^*}(x) \sim \exp(Q(x))\psi(x), \quad x \to \infty. \tag{1.12}$$

Let

$$\begin{aligned} d_{kn} &:= x_{kn} - x_{k+1,n}, && k = 0, 1, \cdots, n, \\ \overline{d}_{kn} &= \begin{cases} d_{1n}, & k = 1, \\ d_{n-1,n}, & k = n, \\ \max\{d_{k-1,n}, d_{kn}\}, & k = 2, \cdots, n-1 \end{cases} \end{aligned}$$

and

$$D_n := \max_{1\le k\le n} \overline{d}_{kn}. \tag{1.13}$$

Zhou (see [12, Theorem 3.1]) extends Theorem 1.1 to the generalized Gaussian quadrature for power orthogonal polynomials.

Theorem 1.3 (see [12, Theorem 3.1]) *Let w be a weight on $\mathbf{I} = R$ and let $f \in \mathbf{S}(w)$. Assume that G has all derivatives on $\mathbf{I}$ and for some constant $C > 0$,* (1.5)–(1.6) *and*

$$|G^{(j)}(x)| \le CG(x), \quad x \in \mathbf{I}, \quad j = 1, 2, \cdots, m-2 \tag{1.14}$$

hold. If

$$\lim_{n\to\infty} D_n = 0, \tag{1.15}$$

then the relation (1.4) *holds.*

Here the condition (1.5) implies

$$\sum_{k=1}^{n} \sum_{j=0}^{m_{kn}-2} \lambda_{jkn} G^{(j)}(x_{kn}) \le \int_a^b G(x)w(x)\mathrm{d}x, \tag{1.16}$$

which together with (1.14)–(1.15) yields

$$\limsup_{n\to\infty} \sum_{k=1}^{n} \lambda_{0kn} G(x_{kn}) \le \int_a^b G(x)w(x)\mathrm{d}x. \tag{1.17}$$

As pointed out by Nevai in [7, p. 120], for unbounded functions, the quadrature sums need not be uniformly bounded, even if the corresponding integral is bounded. However, if f is dominated by a function $G \in \mathbf{S}(w)$ whose even order derivatives are nonnegative, then, by (1.7) and (1.16), the associated quadrature sums are always uniformly bounded. But to find a function $G \in \mathbf{S}(w)$ satisfying (1.5) is often difficult and complicated. Furthermore, to find a function $G \in \mathbf{S}(w)$ satisfying both (1.5) and (1.14) is more difficult and complicated in general. For example, the functions (1.1) and

$$G(x) = \mathrm{e}^x, \quad x \in [0, \infty)$$

do have these properties.

We observe that to prove (1.4), it need not use (1.7) or (1.17) and is enough to use

$$\limsup_{n\to\infty} \sum_{k=1}^{n} \lambda_{0kn} G(x_{kn}) < \infty,$$

or equivalently,

$$\sup_n \sum_{k=1}^{n} \lambda_{0kn}(w) G(x_{kn}) = C_1 < \infty. \tag{1.18}$$

In this regard, following the main idea of Lubinsky in [4] with modifications, we will give some results of the convergence for Gaussian quadrature formulas involving the condition (1.18) instead of (1.5) and (1.14). The following result will play a basic role.

Theorem 1.4 *Let w be a weight on $\mathbf{I}$. Assume that the relation* (1.4) *holds for all $f \in \mathbf{S}_0(w)$ and there exists a function $G \in \mathbf{S}(w)$ with $G \geq 0$ satisfying* (1.18). *If $f \in \mathbf{S}(w)$ satisfies* (1.6), *then* (1.4) *holds.*

This theorem shows that to prove (1.4) for $f \in \mathbf{S}(w)$, it suffices to do two things:

(a) Prove that the relation (1.4) holds for all $f \in \mathbf{S}_0(w)$.

(b) Find a function $G \in \mathbf{S}(w)$ with $G \geq 0$ satisfying (1.6) and (1.18).

To do the first thing, let $a \leq c < d \leq b$ satisfy

$$\begin{cases} c = a, & a > -\infty, \\ c > a, & a = -\infty, \end{cases} \quad \begin{cases} d = b, & d < \infty, \\ d < b, & d = \infty, \end{cases} \tag{1.19}$$

and put

$$D_n(c,d) := \max_{x_{kn} \in (c,d)} \overline{d}_{kn}$$

and

$$R_n(c,d) := \begin{cases} 0, & (c,d) = \mathbf{I}, \\ \max\limits_{x_{kn} \in \mathbf{I} \setminus (c,d)} \dfrac{\overline{d}_{kn}}{|x_{kn}|}, & \text{otherwise}. \end{cases}$$

The following two theorems deal with the first thing.

Theorem 1.5 *Let w be a weight on $\mathbf{I}$. Further, when $m > 2$, assume that for some fixed interval (c,d),*

$$\lim_{n \to \infty} D_n(c,d) = 0 \tag{1.20}$$

and

$$\lim_{n \to \infty} R_n(c,d) = 0. \tag{1.21}$$

Then (1.4) *holds for all $f \in \mathbf{S}_0(w)$.*

Theorem 1.6 *Let w be a weight on $\mathbf{I}$. Further, when $m > 2$ assume that* (1.20) *holds for every $(c,d) \subset \mathbf{I}$ satisfying* (1.19) *and*

$$\sup_n D_n = D < \infty. \tag{1.22}$$

Then for $f \in \mathbf{S}_0(w)$,

$$\lim_{n \to \infty} \sum_{k=1}^{n} \lambda_{jkn} f^{(j)}(x_{kn}) = 0, \quad 1 \leq j \leq m-2, \tag{1.23}$$

and the relation (1.4) *holds.*

Combining Theorems 1.4 and 1.6, we can get the following theorem.

Theorem 1.7 *Let the assumptions of Theorem 1.6 hold. Let $f \in \mathbf{S}(w)$ and $G \in \mathbf{S}(w)$ with $G \geq 0$ satisfying* (1.6) *and* (1.18). *Then* (1.4) *holds.*

Now we will find $G \in \mathbf{S}(w)$ with $G \geq 0$ satisfying (1.18). For $0 < \epsilon < 1$, write

$$\begin{cases} \psi_\infty(x) = (3+|x|)^{-1-\epsilon} \text{ or } (3+|x|)^{-1}[\ln(3+|x|)]^{-1-\epsilon}, \cdots, \\ \psi_a(x) = (x-a)^{\epsilon-1}, \\ \psi_b(x) = (b-x)^{\epsilon-1} \end{cases} \tag{1.24}$$

and

$$\psi(x) = \begin{cases} \psi_\infty(x), & -a = b = \infty, \\ \psi_a(x)\psi_\infty(x), & -\infty < a < b = \infty, \\ \psi_b(x)\psi_\infty(x), & -\infty = a < b < \infty, \\ \psi_a(x)\psi_b(x), & -\infty < a < b < \infty. \end{cases} \tag{1.25}$$

We choose

$$G(x) = u(x)^{-1}\psi(x), \quad u(x) \geq w(x), \quad \text{a.e. } x \in \mathbf{I}. \tag{1.26}$$

Thus $G \geq 0$ and

$$\int_{\mathbf{I}} G(x)w(x)\mathrm{d}x \leq \int_{\mathbf{I}} \psi(x)\mathrm{d}x < \infty,$$

that is, $G \in \mathbf{S}(w)$. Then we have the following result which provides a way of proving (1.18).

Theorem 1.8 *Let w be a weight on $\mathbf{I}$. Let $\mathbf{K}_{1n}$ and $\mathbf{K}_{2n}$ be two disjoint subsets of the set $\{1, 2, \cdots, n\}$ with $\mathbf{K}_{1n} \cup \mathbf{K}_{2n} = \{1, 2, \cdots, n\}$. Let G be given by* (1.26). *Assume that for certain positive numbers C_2 and C_3,*

$$\lambda_{0kn} \leq C_2 u(x_{kn}) \min\{d_{kn}, d_{k-1,n}\}, \quad k \in \mathbf{K}_{1n} \tag{1.27}$$

and

$$\sup_n \sum_{k \in \mathbf{K}_{2n}} \lambda_{0kn}(w)G(x_{kn}) \leq C_3. \tag{1.28}$$

Then (1.18) *holds.*

Applying Theorem 1.8, we can obtain the following three theorems, the last one of which needs some modifications.

Theorem 1.9 *Let $w \in \mathcal{F}(\mathrm{Lip}\frac{1}{2})$ and $m_{kn} \equiv 2$. Then for $G(x) = w(x)^{-1}\psi(x)$,* (1.18) *holds. Furthermore, if $f \in \mathbf{S}(w)$ satisfies* (1.6)*, then* (1.4) *holds.*

Let

$$u(x) = \prod_{i=1}^{r} \Big[|x - t_i| + \frac{1}{n}\Big]^{q_i}, \tag{1.29}$$

where

$$q_i = \begin{cases} p_i, & i = 1, r, \\ p_i, & p_i < 0,\ 2 \leq i \leq r-1, \\ 0, & \text{otherwise}. \end{cases} \tag{1.30}$$

Theorem 1.10 *Let W be a generalized Jacobi weight on* $(-1,1)$ *and let* $m_{kn} \equiv m$. *Assume that* $G \in \mathbf{S}(W)$ *with* $G \geq 0$ *having the form* (1.26), *where* ψ *and* u *are given by* (1.25) *and* (1.29) *with* $-a = b = 1$, *respectively. If* $f \in \mathbf{S}(w)$ *satisfies* (1.6) *with* $-a = b = 1$, *then* (1.4) *holds.*

Theorem 1.11 *Let* $w \in \mathcal{F}^*$, $m_{kn} \equiv m$, *and*

$$G(x) = \frac{1}{[1+|x|^{B(\frac{2}{3}m-1)+1}]w(x)}. \tag{1.31}$$

Further when $m > 2$, *assume* $A \geq \frac{3}{2}$. *If* $f \in \mathbf{S}(w)$ *satisfies* (1.6), *then* (1.4) *holds.*

We shall give some auxiliary lemmas in Section 2 and the proofs of the theorems in Section 3.

2 Auxiliary Lemmas

Lemma 2.1 (see [8, Theorem 4.1.3, p. 43]) *Let w be a weight on* $\mathbf{I}$. *If* $m_k - j \in N_e$, $j < i < m_k$ *and* $1 \leq k \leq n$, *then*

$$|\lambda_{ikn}| \leq \frac{j!}{i!}\bar{d}_{kn}^{\,i-j}\lambda_{jkn}. \tag{2.1}$$

Since by Theorem 4.1.2 in [8, p. 42]

$$\lambda_{0kn} > 0, \quad k = 1, 2, \cdots, n, \tag{2.2}$$

according to Theorem 1.1 in [1, p. 89], we can obtain the following lemma.

Lemma 2.2 *Let w be a weight on* $\mathbf{I}$. *If the relation* (1.4) *holds for every polynomial, then the relation* (1.4) *holds for every* $f \in \mathbf{S}_0(w)$.

For $m_{kn} \equiv m$, the functions $\lambda_{jn}(w;x)$, $j = 0, 1, \cdots, m-1$ are defined to be the Christoffel type functions with respect to a weight w; in particular, $\lambda_n(w;x) := \lambda_{0n}(w;x)$ is the classical Christoffel function (see [8, Definition 5.1.1, pp. 75–76]).

Lemma 2.3 (see [3, Theorem 1.13, p. 20, Theorem 11.4, p. 315]) *Let* $w \in \mathcal{F}(\mathrm{Lip}\frac{1}{2})$, $m_{kn} \equiv 2$ *and* $C > 0$. *Then*

$$\lambda_n(w;x) \sim w(x)\varphi_n(x), \quad x \in [a_{-n}(1 + C\eta_{-n}), a_n(1 + C\eta_n)] \tag{2.3}$$

and

$$d_{kn} \leq c\varphi_n(x_{kn}), \quad k = 1, 2, \cdots, n-1. \tag{2.4}$$

Lemma 2.4 (see [6, Theorem 6.3.28, p. 120] and [8, Theorem 5.3, p. 97]) *Let* $m_{kn} \equiv m$ *and* $w \sim W$, *a.e., where W is defined in Definition* 1.5. *Then with the constants associated with the symbol* $\sim$ *depending on w and m,*

$$\lambda_{jn}(w;x) \sim \frac{1}{n}W_n(x)\Delta_n(x)^j, \quad x \in [-1,1], \ m - j \in N_e, \tag{2.5}$$

where

$$W_n(x) = \Big[(1-x)^{\frac{1}{2}} + \frac{1}{n}\Big]^{2p_1+1}\Big[(1+x)^{\frac{1}{2}} + \frac{1}{n}\Big]^{2p_r+1}\prod_{i=2}^{r-1}\Big[|x-t_i| + \frac{1}{n}\Big]^{p_i}.$$

Lemma 2.5 (see [6, Theorem 9.22, pp. 166–167] and [10, Theorem 1.1]) *Let the assumptions of Lemma 2.4 hold. Then, with the constants associated with the symbol $\sim$ depending on w and m,*

$$\theta_{k+1,n} - \theta_{kn} \sim \frac{1}{n}, \quad k = 0, 1, \cdots, n, \tag{2.6}$$

where

$$x_{kn} = \cos\theta_{kn}, \quad 0 \le \theta_{kn} \le \pi.$$

Lemma 2.6 (see [2, Theorem 1.1] and [9, Theorem 1.3]) *Let $w \in \mathcal{F}^*$, $d > 0$, $m_{kn} \equiv m$ and $m - j \in N_e$. Then for $x \in R$,*

$$\lambda_{jn}(w^m, m, x) \ge \begin{cases} c\left(\dfrac{a_n}{n}\right)^{j+1} w(x)^m \phi_n(x)^{-\frac{1}{2}}, & j = 0, \\ c\left(\dfrac{a_n}{n}\right)^{j+1} w(x)^m, & \textit{otherwise}, \end{cases} \tag{2.7}$$

and for $|x| \le a_n(1 + dn^{-\frac{2}{3}})$,

$$\lambda_{jn}(w^m, m, x) \le c\Big(\frac{a_n}{n}\Big)^{j+1} w(x)^m \phi_n(x)^{\frac{1-m}{2}}, \tag{2.8}$$

where

$$\phi_n(x) := \phi_n(Q, x) := \max\Big\{n^{-\frac{2}{3}}, 1 - \frac{|x|}{a_n(Q)}\Big\} = \max\Big\{n^{-\frac{2}{3}}, 1 - \frac{|x|}{a_n}\Big\}.$$

Lemma 2.7 (see [3, Theorem 1.19, pp. 22–23] and [11, Theorem 1.1]) *Let $w \in \mathcal{F}^*$ and $m_{kn} \equiv m$. Then for $1 \le k \le n-1$,*

$$x_{kn} - x_{k+1,n} \le c\frac{a_n}{n}\phi_n(x_{kn})^{-\frac{1}{2}} \tag{2.9}$$

and

$$x_{kn} - x_{k+1,n} \ge \begin{cases} c\dfrac{a_n}{n}\phi_n(x_{kn})^{-\frac{1}{2}}, & m = 2, \\ c\dfrac{a_n}{n}\phi_n(x_{kn})^{\frac{m-2}{2}}, & m \ge 4. \end{cases} \tag{2.10}$$

Lemma 2.8 (see [3, Theorem 5.7, pp. 125–126]) *Let $w \in \mathcal{F}(\mathrm{Lip}\frac{1}{2})$ and $C_4 > 0$. Then if there exists t_0 such that for $t > t_0$, $x, y \in \mathbf{I}$ and*

$$|y - x| \le C_4\varphi_t(x), \tag{2.11}$$

we have

$$\varphi_t(x) \sim \varphi_t(y). \tag{2.12}$$

Lemma 2.9 (see [2, Lemma 5.1]) *Let $w \in \mathcal{F}^*$. Then*

$$Q'(1)x^{A-1} \le Q'(x) \le Q'(1)x^{B-1}, \quad x \in [1, \infty), \tag{2.13}$$

$$\frac{Q'(1)}{A}x^A \le Q(x) \le \frac{Q'(1)}{B}x^B, \qquad x \in [0, \infty), \tag{2.14}$$

$$a_1 n^{\frac{1}{B}} \le a_n \le a_1 n^{\frac{1}{A}}. \tag{2.15}$$

3 Proof of Theorems

3.1 Proof of Theorem 1.4

Given an arbitrary number $\epsilon > 0$, with the help of (1.6), we may choose a subinterval (c, d) satisfying (1.19), such that

$$|f(x)| \le \epsilon G(x), \quad x \in \mathbf{I} \setminus (c, d). \tag{3.1}$$

Put

$$f_{c,d}(x) = \begin{cases} f(x), & x \in (c, d), \\ 0, & x \in \mathbf{I} \setminus (c, d). \end{cases}$$

Clearly, $f_{c,d} \in \mathbf{S}_0(w)$. By the assumptions of Theorem 1.4,

$$\begin{aligned} &\lim_{n\to\infty} \sum_{x_{kn} \in (c,d)} \lambda_{0kn} f(x_{kn}) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{0kn} f_{c,d}(x_{kn}) \\ &= \int_a^b f_{c,d}(x) w(x) \mathrm{d}x \\ &= \int_c^d f(x) w(x) \mathrm{d}x. \end{aligned} \tag{3.2}$$

On the other hand, by (1.18) and (3.1),

$$\begin{aligned} \Big| \sum_{x_{kn} \in \mathbf{I} \setminus (c,d)} \lambda_{0kn} f(x_{kn}) \Big| &\le \sum_{x_{kn} \in \mathbf{I} \setminus (c,d)} \lambda_{0kn} \, |f(x_{kn})| \\ &\le \epsilon \sum_{k=1}^{n} \lambda_{0kn} G(x_{kn}) \le C_1 \epsilon. \end{aligned} \tag{3.3}$$

Since ϵ is arbitrary, (1.4) follows from (3.2)–(3.3).

3.2 Proof of Theorem 1.5

For the case when $\mathbf{I} = R$, this theorem is given in [8, Theorem 4.6.1, p. 67]. So, to prove the present theorem, it needs only to put

$$w^*(x) = \begin{cases} w(x), & x \in \mathbf{I}, \\ 0, & x \in R \setminus \mathbf{I}. \end{cases}$$

Then applying that theorem in [8], we obtain the present one.

3.3 Proof of Theorem 1.6

If $-\infty < a < b < \infty$, then by definition, $R_n(c, d) = 0$ and the relation (1.4) follows from Theorem 1.5. Meanwhile, by (1.4), (1.20) and (2.1), for $1 \le j \le m - 2$,

$$\lim_{n\to\infty} \Big| \sum_{k=1}^{n} \lambda_{jkn} f(x_{kn}) \Big| \le \lim_{n\to\infty} D_n(c, d)^j \sum_{k=1}^{n} \lambda_{0kn} |f(x_{kn})| = 0.$$

This proves (1.23).

Now let $a = -\infty$ or $b = \infty$. Choose

$$r = s + m, \quad c_0 = r(r-1)\cdots(r-m+3), \quad c_1 = [D(1+c_0)]^{-1}, \quad c_2 = C_0(1 + c_1^{-r})$$

and

$$G(x) = c_2[1 + (c_1 x)^r], \quad x \in \mathbf{I}, \tag{3.4}$$

where s, C_0 and D are given by (1.1) and (1.22), respectively. For $1 \le j \le m-2$,

$$|G^{(j)}(x)| = c_1^j c_2 r(r-1)\cdots(r-j+1)|c_1 x|^{r-j} \le c_0 c_1^j c_2[1 + (c_1 x)^r] = c_0 c_1^j G(x).$$

Therefore

$$|G^{(j)}(x)| \le c_0 c_1^j G(x), \quad x \in \mathbf{I},\ j = 0, 1, \cdots, m-2. \tag{3.5}$$

Since G is a polynomial, for n large enough such that $N_n \ge r$, we have

$$\int_a^b G(x)w(x)\mathrm{d}x = \sum_{k=1}^{n} \sum_{j=0}^{m_{kn}-2} \lambda_{jkn} G^{(j)}(x_{kn}). \tag{3.6}$$

On the other hand, by (1.22), (2.1) and (3.5),

$$\begin{aligned}
\Big|\sum_{k=1}^{n} \sum_{j=1}^{m_{kn}-2} \lambda_{jkn} G^{(j)}(x_{kn})\Big| &\le c_0 \sum_{k=1}^{n} \sum_{j=1}^{m_{kn}-2} (c_1 D)^j \lambda_{0kn} G(x_{kn}) \\
&\le c_0 \Big[\sum_{j=1}^{m-2} (c_1 D)^j\Big] \sum_{k=1}^{n} \lambda_{0kn} G(x_{kn}) \\
&= c_0 \frac{c_1 D[1 - (c_1 D)^{m-2}]}{1 - c_1 D} \sum_{k=1}^{n} \lambda_{0kn} G(x_{kn}) \\
&= [1 - (1 + c_0)^{2-m}] \sum_{k=1}^{n} \lambda_{0kn} G(x_{kn}),
\end{aligned}$$

which, together with (3.6), gives

$$\sum_{k=1}^{n} \lambda_{0kn} G(x_{kn}) \le (1 + c_0)^{m-2} \int_a^b G(x)w(x)\mathrm{d}x.$$

Hence (1.18) holds.

Let us prove (1.23). It follows from (1.1) and (3.4) that

$$|f(x)| \le \frac{c_2}{1 + c_1^{-r}}(1 + x^s) = c_2\Big[\frac{1}{1 + c_1^{-r}} + \frac{c_1^{-s}}{1 + c_1^{-r}}(c_1 x)^s\Big].$$

Since $r > s$, we see $\frac{c_1^{-s}}{(1+c_1^{-r})} \le 1$. Hence the above inequalities yield

$$|f(x)| \le G(x), \quad x \in \mathbf{I}. \tag{3.7}$$

Meanwhile, since $r > s$, for an arbitrary number $\epsilon > 0$, there is a subinterval (c,d) satisfying (1.19) such that (3.1) holds. By virtue of (1.18), (1.22), (2.1) and (3.1), we see that for $1 \le j \le m-2$,

$$\begin{aligned}\Big|\sum_{k=1}^{n}\lambda_{jkn}f(x_{kn})\Big| &\le \sum_{x_{kn}\in(c,d)}\lambda_{jkn}\,|f(x_{kn})| + \sum_{x_{kn}\in\mathbf{I}\setminus(c,d)}\lambda_{jkn}\,|f(x_{kn})| \\ &\le D_n(c,d)^j \sum_{x_{kn}\in(c,d)}\lambda_{0kn}G(x_{kn}) + D^j\epsilon \sum_{x_{kn}\in\mathbf{I}\setminus(c,d)}\lambda_{0kn}G(x_{kn}) \\ &\le \left[D_n(c,d)^j + D^j\epsilon\right]\sum_{k=1}^{n}\lambda_{0kn}G(x_{kn}) \\ &\le C_1\left[D_n(c,d)^j + D^j\epsilon\right],\end{aligned}$$

which, together with (1.20), implies

$$\limsup_{n\to\infty}\Big|\sum_{k=1}^{n}\lambda_{jkn}f(x_{kn})\Big| \le C_1D^j\epsilon.$$

Since ϵ is arbitrary, we obtain (1.23).

Now for $f(x) = x^p,\ p \in N$, (1.1) with $s = 2p$ holds. Meanwhile, for $N_n \ge p$ we have (1.2), which by (1.23) implies (1.4). This shows that (1.4) holds for every polynomial. Applying Lemma 2.2, the relation (1.4) holds for all $f \in \mathbf{S}_0(w)$.

3.4 Proof of Theorem 1.7

Apply Theorem 1.4 and Theorem 1.6.

3.5 Proof of Theorem 1.8

Let $\mathbf{I}_{kn} = (x_{k+1,n}, x_{kn}),\ k = 0,1,\cdots,n$. For $k \in \mathbf{K}_{1n}$, by (1.25)–(1.27),

$$\begin{aligned}\lambda_{0kn}G(x_{kn}) &\le C_2\psi(x_{kn})\min\{d_{k-1,n}, d_{kn}\} \\ &= C_2\min\Big\{\int_{\mathbf{I}_{kn}}\psi(x_{kn})\mathrm{d}x, \int_{\mathbf{I}_{k-1,n}}\psi(x_{kn})\mathrm{d}x\Big\}.\end{aligned} \tag{3.8}$$

With the help of (1.28), it is enough to show

$$\sup_n \sum_{k\in\mathbf{K}_{1n}}\lambda_{0kn}G(x_{kn}) < \infty.$$

To this end we separate the proof into four cases.

Case 1 $-a = b = \infty$. In this case, $\psi(x)$ is increasing on $(-\infty,0]$ and decreasing on $[0,\infty)$, respectively. So $\psi(x)$ is increasing on $\mathbf{I}_{k-1,n}$, if $x_{k-1,n} \le 0$ and decreasing on $\mathbf{I}_{kn}$, if $x_{k+1,n} \ge 0$, respectively. Thus

$$\psi(x_{kn}) \le \psi(x), \quad x \in \mathbf{I}_{k-1,n} \text{ and } x_{k-1,n} \le 0, \text{ or } x \in \mathbf{I}_{kn} \text{ and } x_{k+1,n} \ge 0.$$

Hence by (3.8),

$$\lambda_{0kn}G(x_{kn}) \le \begin{cases} C_2\displaystyle\int_{\mathbf{I}_{k-1,n}}\psi(x)\mathrm{d}x, & x_{k-1,n} \le 0, \\ C_2\displaystyle\int_{\mathbf{I}_{kn}}\psi(x)\mathrm{d}x, & x_{k+1,n} \ge 0. \end{cases}$$

Then

$$\begin{aligned}
\sum_{k\in\mathbf{K}_{1n}} \lambda_{0kn} G(x_{kn}) &= \sum_{\substack{k\in\mathbf{K}_{1n}\\ x_{k-1,n}\le 0}} \lambda_{0kn} G(x_{kn}) + \sum_{\substack{k\in\mathbf{K}_{1n}\\ x_{k+1,n}\ge 0}} \lambda_{0kn} G(x_{kn}) \\
&\quad + \sum_{\substack{k\in\mathbf{K}_{1n}\\ x_{k+1,n}<0<x_{k-1,n}}} \lambda_{0kn} G(x_{kn}) \\
&\le C_2\Big[\sum_{\substack{k\in\mathbf{K}_{1n}\\ x_{k-1,n}\le 0}} \int_{\mathbf{I}_{k-1,n}} \psi(x)\mathrm{d}x + \sum_{\substack{k\in\mathbf{K}_{1n}\\ x_{k+1,n}\ge 0}} \int_{\mathbf{I}_{kn}} \psi(x)\mathrm{d}x \\
&\quad + \sum_{\substack{k\in\mathbf{K}_{1n}\\ x_{k+1,n}<0<x_{k-1,n}}} \psi(x_{kn})\mathrm{d}_{kn}\Big] \\
&\le C_2\Big[2\int_a^b \psi(x)\mathrm{d}x + \psi(0) \sum_{\substack{k\in\mathbf{K}_{1n}\\ x_{k+1,n}<0<x_{k-1,n}}} \mathrm{d}_{kn}\Big] \le c < \infty,
\end{aligned}$$

because the last sum is equal to $x_{k-1,n} - x_{k+1,n}$ satisfying $x_{k+1,n} < 0 < x_{k-1,n}$, which must be uniformly finite .

Case 2 $-\infty < a < b = \infty$. In this case $\psi(x) \le (x-a)^{\epsilon-1}$ and hence

$$\begin{aligned}
\sum_{k\in\mathbf{K}_{1n}} \lambda_{0kn} G(x_{kn}) &\le C_2 \sum_{k\in\mathbf{K}_{1n}} \int_{\mathbf{I}_{kn}} (x_{kn}-a)^{\epsilon-1}\mathrm{d}x \\
&\le C_2 \sum_{k\in\mathbf{K}_{1n}} \int_{\mathbf{I}_{kn}} (x-a)^{\epsilon-1}\mathrm{d}x \\
&\le C_2 \int_a^b (x-a)^{\epsilon-1}\mathrm{d}x.
\end{aligned}$$

Case 3 $-\infty = a < b < \infty$. In this case $\psi(x) \le (b-x)^{\epsilon-1}$ and hence

$$\sum_{k\in\mathbf{K}_{1n}} \lambda_{0kn} G(x_{kn}) \le C_2 \sum_{k\in\mathbf{K}_{1n}} \int_{\mathbf{I}_{k-1,n}} (b-x)^{\epsilon-1}\mathrm{d}x \le C_2 \int_a^b (b-x)^{\epsilon-1}\mathrm{d}x.$$

Case 4 $-\infty < a < b < \infty$. In this case letting $h = \frac{b-a}{2}$ and $d = \frac{b+a}{2}$, we have

$$\psi(x) \le \begin{cases} [h(x-a)]^{\epsilon-1}, & x \le d, \\ [h(b-x)]^{\epsilon-1}, & x > d. \end{cases}$$

So

$$\begin{aligned}
&\sum_{k\in\mathbf{K}_{1n}} \lambda_{0kn} G(x_{kn}) \\
&\qquad \le C_2\Big[\sum_{\substack{k\in\mathbf{K}_{1n}\\ x_{kn}\le d}} \psi(x_{kn})\mathrm{d}_{kn} + \sum_{k\in\mathbf{K}_{1n}, x_{kn}>d} \psi(x_{kn})\mathrm{d}_{k-1,n}\Big] \\
&\qquad \le c\Big[\sum_{\substack{k\in\mathbf{K}_{1n}\\ x_{kn}\le d}} \int_{\mathbf{I}_{kn}} (x-a)^{\epsilon-1}\mathrm{d}x + \sum_{k\in\mathbf{K}_{1n}, x_{kn}>d} \int_{\mathbf{I}_{k-1,n}} (b-x)^{\epsilon-1}\mathrm{d}x\Big] \\
&\qquad \le c\int_a^b \Big[(x-a)^{\epsilon-1} + (b-x)^{\epsilon-1}\Big]\mathrm{d}x.
\end{aligned}$$

This completes the proof.

3.6 Proof of Theorem 1.9

The relation (1.27) with $u = w$ and $\mathbf{K}_{1n} = \{1, 2, \cdots, n\}$ follows immediately from (2.3)–(2.4) and (2.12). Then applying Theorem 1.8, we obtain (1.18).

Meanwhile, for a polynomial f and $N_n \geq \deg f$, we have

$$\sum_{k=1}^{n} \lambda_{0kn} f(x_{kn}) = \int_{\mathbf{I}} f(x) w(x) \mathrm{d}x.$$

Thus (1.4) is true for this polynomial f. Applying Lemma 2.2, we conclude that (1.4) holds for all $f \in \mathbf{S}_0(w)$. Then according to Theorem 1.4, the relation (1.4) holds for the given function f.

3.7 Proof of Theorem 1.10

We observe that

$$\begin{aligned}
&\Big[(1-x)^{\frac{1}{2}} + \frac{1}{n}\Big]^{2p_1+1} \Big[(1+x)^{\frac{1}{2}} + \frac{1}{n}\Big]^{2p_r+1} \\
&= \Big[(1-x)^{\frac{1}{2}} + \frac{1}{n}\Big] \Big[(1+x)^{\frac{1}{2}} + \frac{1}{n}\Big] \Big[(1-x)^{\frac{1}{2}} + \frac{1}{n}\Big]^{2p_1} \Big[(1+x)^{\frac{1}{2}} + \frac{1}{n}\Big]^{2p_r} \\
&\leq cn\Delta_n(x) \Big[(1-x) + \frac{1}{n}\Big]^{p_1} \Big[(1+x) + \frac{1}{n}\Big]^{p_r}.
\end{aligned}$$

Hence

$$W_n(x) \leq cn\Delta_n(x) \prod_{i=1}^{r} \Big[|x - t_i| + \frac{1}{n}\Big]^{p_i}.$$

Meanwhile, by Lemma 4.5.6 in [8, p. 66], the relation (2.6) means

$$d_{kn} \sim \Delta_n(x_{kn}), \quad k = 0, 1, \cdots, n. \tag{3.9}$$

Then by (2.5) and (3.9),

$$\lambda_{0kn} \leq c\Delta_n(x) \prod_{i=1}^{r} \Big[|x - t_i| + \frac{1}{n}\Big]^{p_i} \leq c \prod_{i=1}^{r} \Big[|x - t_i| + \frac{1}{n}\Big]^{p_i} \min\{d_{kn}, d_{k-1,n}\}. \tag{3.10}$$

Now choose G satisfying (1.26), where u is given by (1.29)–(1.30). Then $G \in \mathbf{S}(w)$. Moreover, by (1.29)–(1.30) and (3.10), the relation (1.27) with $\mathbf{K}_{1n} = \{1, 2, \cdots, n\}$ is true. Applying Theorems 1.6 and 1.8, we obtain (1.4) and (1.18).

3.8 Proof of Theorem 1.11

By (1.31), (2.8) and (2.10)

$$\begin{aligned}
\lambda_{0kn} G(x_{kn}) &\leq c \frac{a_n}{n} W(x_{kn})^m \phi_n(x_{kn})^{\frac{1-m}{2}} G(x_{kn}) \\
&\leq cW(x_{kn})^m G(x_{kn}) \phi_n(x_{kn})^{\frac{3}{2}-m} \mathrm{d}_{kn} \\
&\leq c \frac{\phi_n(x_{kn})^{\frac{3}{2}-m}}{1 + x_{kn}^{B(\frac{2m}{3}-1)+1}} \mathrm{d}_{kn} \\
&\leq c \int_{\mathbf{I}_{kn}} \frac{\phi_n(x_{kn})^{\frac{3}{2}-m}}{1 + x_{kn}^{B(\frac{2m}{3}-1)+1}} \mathrm{d}x.
\end{aligned}$$

By the definition of ϕ_n we see that for $x_{k+1,n} \geq 0$, $\phi_n(x_{kn}) \leq \phi_n(x_{k+1,n})$. Hence the above inequalities yield

$$\lambda_{0kn} G(x_{kn}) \leq c \int_{\mathbf{I}_{kn}} \frac{\phi_n(x_{k+1,n})^{\frac{3}{2}-m}}{1 + x_{kn}^{B(\frac{2m}{3}-1)+1}} \mathrm{d}x. \tag{3.11}$$

Since the functions $\phi_n(x)^{\frac{3}{2}-m}$ and $\frac{1}{1+x^{B(\frac{2m}{3}-1)+1}}$ are increasing and decreasing on $[0, \infty)$, respectively, we have that for $x_{k+1,n} \geq 0$,

$$\phi_n(x_{k+1,n})^{\frac{3}{2}-m} \leq \phi_n(x)^{\frac{3}{2}-m}, \quad x \in \mathbf{I}_{kn}$$

and

$$\frac{1}{1 + x_{kn}^{B(\frac{2m}{3}-1)+1}} \leq \frac{1}{1 + x^{B(\frac{2m}{3}-1)+1}}, \quad x \in \mathbf{I}_{kn}.$$

Therefore, the inequality (3.11) gives

$$\lambda_{0kn} G(x_{kn}) \leq c \int_{\mathbf{I}_{kn}} \frac{\phi_n(x)^{\frac{3}{2}-m}}{1 + x^{B(\frac{2m}{3}-1)+1}} \mathrm{d}x$$

which implies ($E[t]$ stands for the integral part of the number t)

$$\begin{aligned}\sum_{k=1}^{n} \lambda_{0kn} G(x_{kn}) &\leq 2 \sum_{x_{k+1,n} \geq 0} \lambda_{0kn} G(x_{kn}) + \lambda_{0,E[\frac{n}{2}],n} G(x_{E[\frac{n}{2}],n}) \\ &\leq c \sum_{x_{k+1,n} \geq 0} \int_{\mathbf{I}_{kn}} \frac{\phi_n(x)^{\frac{3}{2}-m}}{1 + x^{B(\frac{2m}{3}-1)+1}} \mathrm{d}x + c D_n \phi_n(0)^{\frac{3}{2}-m} \\ &\leq c \int_0^\infty \frac{\phi_n(x)^{\frac{3}{2}-m}}{1 + x^{B(\frac{2m}{3}-1)+1}} \mathrm{d}x + c D_n := c S_n + c D_n.\end{aligned}$$

By (2.13)–(2.15), for $a_n \geq 1$,

$$a_n \leq \frac{BQ(a_n)}{Q'(a_n)} \leq \frac{BQ(a_n)}{Q'(1) a_n^{A-1}} \leq \frac{cBn}{Q'(1) a_n^{A-1}}.$$

Hence

$$a_n \leq \left[\frac{cB}{Q'(1)}\right]^{\frac{1}{A}} n^{\frac{1}{A}}.$$

Since $A \geq \frac{3}{2}$, we get $a_n \leq c n^{\frac{2}{3}}$. Noticing that $\phi_n(x) \geq n^{-\frac{2}{3}}$, we conclude

$$d_{kn} \leq c a_n n^{-\frac{2}{3}} \leq c$$

which proves (1.22). To prove (1.18), we have to show that

$$S = \sup_n S_n < \infty.$$

To this end we separate S_n into two parts

$$S_n = \int_{0 \leq x \leq \frac{a_n}{2}} \frac{\phi_n(x)^{\frac{3}{2}-m}}{1 + x^{B(\frac{2m}{3}-1)+1}} \mathrm{d}x + \int_{x > \frac{a_n}{2}} \frac{\phi_n(x)^{\frac{3}{2}-m}}{1 + x^{B(\frac{2m}{3}-1)+1}} \mathrm{d}x := S_{1n} + S_{2n}.$$

It is easy to see that

$$S_{1n} \le 2^{m-\frac{3}{2}} \int_{0\le x\le \frac{a_n}{2}} \frac{1}{1+x^{B(\frac{2m}{3}-1)+1}} \mathrm{d}x \le 2^{m-\frac{3}{2}} \int_0^\infty \frac{1}{x^{B(\frac{2m}{3}-1)+1}} \mathrm{d}x < \infty.$$

To estimate S_{2n}, we use (2.15) to get

$$\begin{aligned} S_{2n} &\le cn^{\frac{2m}{3}-1} \int_{x>\frac{a_n}{2}} \frac{1}{1+x^{B(\frac{2m}{3}-1)+1}} \mathrm{d}x \le cn^{\frac{2m}{3}-1} \int_{x>\frac{a_n}{2}} \frac{1}{x^{B(\frac{2m}{3}-1)+1}} \mathrm{d}x \\ &= cn^{\frac{2m}{3}-1} a_n^{B(\frac{2m}{3}-1)} \le c. \end{aligned}$$

This proves (1.18).

By means of (2.9) we get (1.20). Then applying Theorem 1.6, we conclude that (1.4) holds for all $f \in \mathbf{S}_0(w)$. Further, using Theorem 1.4, we see that (1.4) holds for all $f \in \mathbf{S}(w)$ satisfying (1.6).

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