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Some Univalence Conditions for a General Integral Operator*

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Abstract The authors consider the classes of the univalent functions denoted by $\mathcal{SH}(\beta)$, $\mathcal{SP}$ and $\mathcal{SP}(\alpha,\beta)$. On these classes, the univalence conditions for a general integral operator are studied.

Keywords Analytic function, Integral operator, Univalence condition, General Schwarz lemma
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1 Introduction

Let $\mathcal{A}$ denote the class of all functions of the form $f(z)=z+\sum\limits_{k=2}^{\infty}a_kz^k$, which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ and satisfy the following usual normalization condition: $f(0)=f'(0)-1=0$. Also, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions f which are univalent in $\mathbb{U}$ (see [1–2, 5]).

In [8], Stankiewicz and Wisniowska introduced the class of univalent functions $\mathcal{SH}(\beta)$ $(\beta>0)$ defined by

$$\Big|\frac{zf'(z)}{f(z)}-2\beta(\sqrt{2}-1)\Big|<\mathrm{Re}\Big\{\sqrt{2}\frac{zf'(z)}{f(z)}\Big\}+2\beta(\sqrt{2}-1) \tag{1.1}$$

for all $z\in\mathbb{U}$.

Also, in [7], Ronning introduced the class of univalent functions $\mathcal{SP}$ defined by

$$\Big|\frac{zf'(z)}{f(z)}-1\Big|<\mathrm{Re}\Big(\frac{zf'(z)}{f(z)}\Big) \tag{1.2}$$

for all $z\in\mathbb{U}$.

The geometric interpretation of the relation (1.2) is that the class $\mathcal{SP}$ is the class of all functions $f\in\mathcal{S}$ for which the expression $\frac{zf'(z)}{f(z)}$ $(z\in\mathbb{U})$ takes all values in the parabolic region

$$\Omega=\{\omega:\ |\omega-1|\le\mathrm{Re}\,\omega\}=\{\omega=u+\mathrm{i}v:\ v^2\le 2u-1\}.$$

In [6], Ronning introduced the class of univalent functions $\mathcal{SP}(\alpha,\beta)$, $\alpha>0$, $\beta\in[0,1)$, as the class of all functions $f\in\mathcal{S}$ which have the property

$$\Big|\frac{zf'(z)}{f(z)}-(\alpha+\beta)\Big|\le\mathrm{Re}\frac{zf'(z)}{f(z)}+\alpha-\beta \tag{1.3}$$

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for all $z \in \mathbb{U}$.

Geometric interpretation: $f \in \mathcal{SP}(\alpha, \beta)$ if and only if $\frac{zf'(z)}{f(z)}$ $(z \in \mathbb{U})$ takes all values in the parabolic region

$$\Omega_{\alpha,\beta} = \{\omega : \ |\omega - (\alpha + \beta)| \leq \text{Re } \omega + \alpha - \beta\} = \{\omega = u + \text{i}v : \ v^2 \leq 4\alpha(u - \beta)\}.$$

In the proofs of our main results (see Theorems 2.1–2.3), we need the following univalence criterion. The univalence criterion asserted by Theorem 1.1, is a generalization of Ahlfor's and Becker's univalence criterions, and was proved by Pescar [4].

Theorem 1.1 (cf. [4]) *Let β be a complex number, $\text{Re}\,\beta > 0$, c be a complex number, $|c| \leq 1$, $c \neq -1$, and $f(z) = z + \cdots$ be a regular function in U. If*

$$\Big| c|z|^{2\beta} + (1 - |z|^{2\beta}) \frac{zf''(z)}{\beta f'(z)} \Big| \leq 1$$

for all $z \in U$, then the function

$$F_\beta(z) = \Big(\beta \int_0^z t^{\beta - 1} f'(t) \text{d}t\Big)^{\frac{1}{\beta}} = z + \cdots$$

is regular and univalent in U.

Here, in our investigation, we consider the integral operator studied by Breaz and Breaz [3]

$$\begin{aligned} &I_n(z) = \Big(\beta \int_0^z t^{\beta - 1} \prod_{i=1}^n \Big(\frac{f_i(t)}{t}\Big)^{\frac{1}{\alpha_i}} \text{d}t\Big)^{\frac{1}{\beta}}, \\ &f_i \in \mathcal{A}, \quad \alpha_i, \beta \in \mathbb{C} - \{0\} \quad \text{for all } i \in \{1, 2, \cdots, n\}. \end{aligned} \tag{1.4}$$

In the present paper, we propose to investigate further univalence conditions involving the general family of integral operators defined by (1.4).

2 Main Results

Theorem 2.1 *Let the functions $f_i \in \mathcal{A}$ for all $i \in \{1, 2, \cdots, n\}$, and α_i, β be complex numbers, $\text{Re}\,\beta > 0$ and $M_i \geq 1$ with*

$$\text{Re}\,\beta \geq \sum_{i=1}^n \frac{1}{|\alpha_i|}(\sqrt{2}M_i + 4\lambda(\sqrt{2} - 1) + 1)$$

for all $i \in \{1, 2, \cdots, n\}$.

If $f_i \in \mathcal{SH}(\lambda)$, $\lambda > 0$ and

$$\begin{aligned} &\text{Re}\Big(\frac{zf_i'(z)}{f_i(z)}\Big) \leq M_i, \quad z \in \mathbb{U}, \\ &|c| \leq 1 - \frac{1}{\text{Re}\,\beta} \sum_{i=1}^n \frac{1}{|\alpha_i|}(\sqrt{2}M_i + 4\lambda(\sqrt{2} - 1) + 1) \end{aligned} \tag{2.1}$$

for all $i \in \{1, 2, \cdots, n\}$, then the integral operator $I_n(z)$ defined by (1.4) is in the class $\mathcal{S}$.

Proof We define the function

$$h(z) = \int_0^z \prod_{i=1}^n \Big(\frac{f_i(t)}{t}\Big)^{\frac{1}{\alpha_i}} \text{d}t, \tag{2.2}$$

and then calculate the derivatives of the first and second orders for $h(z)$.

From (2.2), we obtain

$$h'(z) = \prod_{i=1}^{n} \Big(\frac{f_i(z)}{z}\Big)^{\frac{1}{\alpha_i}},$$

$$h''(z) = \sum_{i=1}^{n} \frac{1}{\alpha_i} \Big(\frac{f_i(z)}{z}\Big)^{\frac{1-\alpha_i}{\alpha_i}} \Big(\frac{z f_i'(z) - f_i(z)}{z^2}\Big) \prod_{\substack{k=1 \\ k\neq i}}^{n} \Big(\frac{f_k(z)}{z}\Big)^{\frac{1}{\alpha_k}}.$$

After the calculus, we obtain that

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \frac{1}{\alpha_i}\Big(\frac{z f_i'(z)}{f_i(z)} - 1\Big), \tag{2.3}$$

which readily shows that

$$\begin{aligned}\Big|c|z|^{2\beta} + (1-|z|^{2\beta})\frac{zh''(z)}{\beta h'(z)}\Big| &= \Big|c|z|^{2\beta} + (1-|z|^{2\beta})\frac{1}{\beta}\sum_{i=1}^{n}\frac{1}{\alpha_i}\Big(\frac{z f_i'(z)}{f_i(z)} - 1\Big)\Big| \\ &\le |c| + \frac{1}{|\beta|}\sum_{i=1}^{n}\frac{1}{|\alpha_i|}\Big(\Big|\frac{z f_i'(z)}{f_i(z)}\Big| + 1\Big) \\ &\le |c| + \frac{1}{|\beta|}\sum_{i=1}^{n}\frac{1}{|\alpha_i|}\Big[\Big|\frac{z f_i'(z)}{f_i(z)} - 2\lambda(\sqrt{2}-1)\Big| + 2\lambda(\sqrt{2}-1) + 1\Big].\end{aligned}$$

Because $f_i \in \mathcal{SH}(\lambda)$ $(\lambda > 0)$ and $\mathrm{Re}\big(\frac{z f_i'(z)}{f_i(z)}\big) \le M_i$ for all $i \in \{1, 2, \cdots, n\}$, we obtain that

$$\begin{aligned}\Big|c|z|^{2\beta} + (1-|z|^{2\beta})\frac{zh''(z)}{\beta h'(z)}\Big| &\le |c| + \frac{1}{|\beta|}\sum_{i=1}^{n}\frac{1}{|\alpha_i|}\Big[\Big(\sqrt{2}\mathrm{Re}\Big(\frac{z f_i'(z)}{f_i(z)}\Big) + 2\lambda(\sqrt{2}-1)\Big) \\ &\quad + 2\lambda(\sqrt{2}-1) + 1\Big] \\ &\le |c| + \frac{1}{|\beta|}\sum_{i=1}^{n}\frac{1}{|\alpha_i|}\Big(\sqrt{2}M_i + 4\lambda\Big(\sqrt{2}-1\Big) + 1\Big) \\ &\le |c| + \frac{1}{\mathrm{Re}\,\beta}\sum_{i=1}^{n}\frac{1}{|\alpha_i|}\Big(\sqrt{2}M_i + 4\lambda\Big(\sqrt{2}-1\Big) + 1\Big),\end{aligned}$$

which in the light of the hypothesis (2.1) implies

$$\Big|c|z|^{2\beta} + (1-|z|^{2\beta})\frac{zh''(z)}{\beta h'(z)}\Big| \le 1, \qquad z \in \mathbb{U}.$$

Applying Theorem 1.1, for the function $h(z)$, we obtain that the integral operator $I_n(z)$ defined by (1.4) is in the class $\mathcal{S}$.

Setting $n = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 2.1, we obtain the following result.

Corollary 2.1 *Let the function* $f \in \mathcal{A}$, *and* α, β *be complex numbers,* $\mathrm{Re}\,\beta > 0$ *and* $M \ge 1$ *with*

$$\mathrm{Re}\,\beta \ge \frac{1}{|\alpha|}(\sqrt{2}M + 4\lambda(\sqrt{2}-1) + 1).$$

If $f \in \mathcal{SH}(\lambda)$ $(\lambda > 0)$ *and*

$$\operatorname{Re}\Big(\frac{zf'(z)}{f(z)}\Big) \le M, \quad z \in \mathbb{U},$$
$$|c| \le 1 - \frac{1}{\operatorname{Re}\beta}\Big(\frac{1}{|\alpha|}(\sqrt{2}M + 4\lambda(\sqrt{2}-1)+1)\Big),$$

then the integral operator

$$I(z) = \Big(\beta \int_0^z t^{\beta-1}\Big(\frac{f(t)}{t}\Big)^{\frac{1}{\alpha}} \mathrm{d}t\Big)^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

Theorem 2.2 *Let the functions* $f_i \in \mathcal{A}$ *for all* $i \in \{1, 2, \cdots, n\}$, *and* β, α_i *be complex numbers,* $\operatorname{Re}\beta > 0$ *and* $M_i \ge 1$ *with* $\operatorname{Re}\beta \ge \sum\limits_{i=1}^{n} \frac{1}{|\alpha_i|}(M_i + 2)$ *for all* $i \in \{1, 2, \cdots, n\}$.

If $f_i \in \mathcal{SP}$ *and*

$$\operatorname{Re}\Big(\frac{zf_i'(z)}{f_i(z)}\Big) \le M_i, \quad z \in \mathbb{U},$$
$$|c| \le 1 - \frac{1}{\operatorname{Re}\beta}\sum_{i=1}^{n}\frac{1}{|\alpha_i|}(M_i + 2) \tag{2.4}$$

for all $i \in \{1, 2, \cdots, n\}$, *then the integral operator* $I_n(z)$ *defined by* (1.4) *is in the class* $\mathcal{S}$.

Proof By the proof of Theorem 2.1, we have

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n}\frac{1}{\alpha_i}\Big(\frac{zf_i'(z)}{f_i(z)} - 1\Big). \tag{2.5}$$

Thus, we obtain

$$\Big|c|z|^{2\beta} + (1-|z|^{2\beta})\frac{zh''(z)}{\beta h'(z)}\Big| = \Big|c|z|^{2\beta} + (1-|z|^{2\beta})\frac{1}{\beta}\sum_{i=1}^{n}\frac{1}{\alpha_i}\Big(\frac{zf_i'(z)}{f_i(z)} - 1\Big)\Big|$$
$$\le |c| + \frac{1}{|\beta|}\sum_{i=1}^{n}\frac{1}{|\alpha_i|}\Big(\Big|\frac{zf_i'(z)}{f_i(z)}\Big| + 1\Big).$$

Because $f_i \in \mathcal{SP}$ for all $i \in \{1, 2, \cdots, n\}$, we get

$$\Big|c|z|^{2\beta} + (1-|z|^{2\beta})\frac{zh''(z)}{\beta h'(z)}\Big| \le |c| + \frac{1}{|\beta|}\sum_{i=1}^{n}\frac{1}{|\alpha_i|}\Big(\Big(\Big|\frac{zf_i'(z)}{f_i(z)} - 1\Big| + 1\Big) + 1\Big)$$
$$\le |c| + \frac{1}{|\beta|}\sum_{i=1}^{n}\frac{1}{|\alpha_i|}\Big(\operatorname{Re}\Big(\frac{zf_i'(z)}{f_i(z)}\Big) + 2\Big)$$
$$\le |c| + \frac{1}{|\beta|}\sum_{i=1}^{n}\frac{1}{|\alpha_i|}(M_i + 2)$$
$$\le |c| + \frac{1}{\operatorname{Re}\beta}\sum_{i=1}^{n}\frac{1}{|\alpha_i|}(M_i + 2)$$

$(z \in \mathbb{U})$, which in the light of the hypothesis (2.4) implies

$$|c|z|^{2\beta} + (1-|z|^{2\beta})\frac{zh''(z)}{\beta h'(z)}| \le 1, \quad z \in \mathbb{U}.$$

Applying Theorem 1.1 for the function $h(z)$, we obtain that the integral operator $I_n(z)$ defined by (1.4) is in the class $\mathcal{S}$.

Setting $n = 1$, $f_1 = f$, $\alpha_1 = \alpha$ in Theorem 2.2, we obtain the following result.

Corollary 2.2 *Let the function* $f \in \mathcal{A}$, *and* β, α *be complex numbers,* $\operatorname{Re}\beta > 0$ *and* $M \geq 1$ *with* $\operatorname{Re}\beta \geq \frac{M+2}{|\alpha|}$. *If* $f \in \mathcal{SP}$ *and*

$$\operatorname{Re}\Big(\frac{zf'(z)}{f(z)}\Big) \leq M, \quad z \in \mathbb{U},$$
$$|c| \leq 1 - \frac{1}{\operatorname{Re}\beta}\Big(\frac{M+2}{|\alpha|}\Big),$$

then the integral operator

$$I(z) = \Big(\beta \int_0^z t^{\beta-1}\Big(\frac{f(t)}{t}\Big)^{\frac{1}{\alpha}} \mathrm{d}t\Big)^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

Theorem 2.3 *Let the functions* $f_i \in \mathcal{A}$ *for all* $i \in \{1, 2, \cdots, n\}$, *and* β, α_i *be complex numbers,* $\operatorname{Re}\beta > 0$ *and* $M_i \geq 1$ *with* $\operatorname{Re}\beta \geq \sum\limits_{i=1}^{n} \frac{1}{|\alpha_i|}(M_i + 2\alpha + 1)$ *for all* $i \in \{1, 2, \cdots, n\}$ *and* $\alpha > 0$.

If $f_i \in \mathcal{SP}(\alpha, \lambda)$, $\alpha > 0$, $\lambda \in [0, 1)$ *and*

$$\operatorname{Re}\Big(\frac{zf_i'(z)}{f_i(z)}\Big) \leq M_i, \quad z \in \mathbb{U},$$
$$|c| \leq 1 - \frac{1}{\operatorname{Re}\beta}\sum_{i=1}^{n}\frac{1}{|\alpha_i|}(M_i + 2\alpha + 1) \tag{2.6}$$

for all $i \in \{1, 2, \cdots, n\}$, *then the integral operator* $I_n(z)$ *defined by* (1.4) *is in the class* $\mathcal{S}$.

Proof By (2.3), we deduce that

$$\begin{aligned}\Big|c|z|^{2\beta} + (1-|z|^{2\beta})\frac{zh''(z)}{\beta h'(z)}\Big| &= \Big|c|z|^{2\beta} + (1-|z|^{2\beta})\frac{1}{\beta}\sum_{i=1}^{n}\frac{1}{\alpha_i}\Big(\frac{zf_i'(z)}{f_i(z)} - 1\Big)\Big| \\ &\leq |c| + \frac{1}{|\beta|}\sum_{i=1}^{n}\frac{1}{|\alpha_i|}\Big(\Big|\frac{zf_i'(z)}{f_i(z)}\Big| + 1\Big) \\ &\leq |c| + \frac{1}{|\beta|}\sum_{i=1}^{n}\frac{1}{|\alpha_i|}\Big(\Big|\frac{zf_i'(z)}{f_i(z)} - (\alpha+\lambda)\Big| + \alpha + \lambda + 1\Big).\end{aligned}$$

Because $f_i \in \mathcal{SP}(\alpha, \lambda)$, $\alpha > 0$, $\lambda \in [0, 1)$ and $\operatorname{Re}\big(\frac{zf_i'(z)}{f_i(z)}\big) \leq M_i$ for all $i \in \{1, 2, \cdots, n\}$, we obtain that

$$\begin{aligned}\Big|c|z|^{2\beta} + (1-|z|^{2\beta})\frac{zh''(z)}{\beta h'(z)}\Big| &\leq |c| + \frac{1}{|\beta|}\sum_{i=1}^{n}\frac{1}{|\alpha_i|}\Big[\Big(\operatorname{Re}\Big(\frac{zf_i'(z)}{f_i(z)}\Big) + \alpha - \lambda\Big) + \alpha + \lambda + 1\Big] \\ &\leq |c| + \frac{1}{|\beta|}\sum_{i=1}^{n}\frac{1}{|\alpha_i|}(M_i + 2\alpha + 1) \\ &\leq |c| + \frac{1}{\operatorname{Re}\beta}\sum_{i=1}^{n}\frac{1}{|\alpha_i|}(M_i + 2\alpha + 1)\end{aligned}$$

$(z \in \mathbb{U})$, which in the light of the hypothesis (2.6) implies

$$\left| c|z|^{2\beta} + (1 - |z|^{2\beta}) \frac{zh''(z)}{\beta h'(z)} \right| \le 1, \quad z \in \mathbb{U}.$$

Applying Theorem 1.1 for the function $h(z)$, we obtain that the integral operator $I_n(z)$ defined by (1.4) is in the class $\mathcal{S}$.

Setting $n = 1$, $f_1 = f$, $\alpha_1 = \gamma$ in Theorem 2.3, we obtain the following result.

Corollary 2.3 *Let the function $f \in \mathcal{A}$, β, γ be complex numbers,* $\mathrm{Re}\,\beta > 0$ *and* $M \ge 1$ *with* $\mathrm{Re}\beta \ge \frac{1}{|\gamma|}(M + 2\alpha + 1)$ *for $\alpha > 0$. If $f \in \mathcal{SP}(\alpha, \lambda)$, $\alpha > 0$, $\lambda \in [0, 1)$ and*

$$\mathrm{Re}\Big(\frac{zf'(z)}{f(z)}\Big) \le M, \quad z \in \mathbb{U},$$
$$|c| \le 1 - \frac{1}{\mathrm{Re}\,\beta}\Big(\frac{M + 2\alpha + 1}{|\gamma|}\Big),$$

then the integral operator

$$I(z) = \Big(\beta \int_0^z t^{\beta-1} \Big(\frac{f(t)}{t}\Big)^{\frac{1}{\gamma}} \mathrm{d}t\Big)^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

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