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Cubature Formula for Spherical Basis Function Networks*

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Abstract Some mathematical models in geophysics and graphic processing need to compute integrals with scattered data on the sphere. Thus cubature formula plays an important role in computing these spherical integrals. This paper is devoted to establishing an exact positive cubature formula for spherical basis function networks. The authors give an existence proof of the exact positive cubature formula for spherical basis function networks, and prove that the cubature points needed in the cubature formula are not larger than the number of the scattered data.

Keywords Cubature formula, Spherical basis function, Scattered data
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1 Introduction

Let $\mathbb{S}^d$ be the unit sphere embedded into the $(d+1)$-dimensional Euclidean space $\mathbb{R}^{d+1}$ with rotation invariant measure $\mathrm{d}\omega$. Given a spherical integral $I(f) := \int_{\mathbb{S}^d} f(x)\mathrm{d}\omega(x)$, if there exist cubature points $y_1, \cdots, y_N \in \mathbb{S}^d$, and cubature weights $a_1, \cdots, a_N \in \mathbb{R}$, such that

$$I(f) = \sum_{k=1}^{N} a_k f(y_k) \tag{1.1}$$

for all f belonging to a class of functions $\mathcal{F}$, then (1.1) is called an exact cubature formula for $\mathcal{F}$ on the sphere. Instead of (1.1), if for arbitrary $f \in \mathcal{F}$, there holds

$$\sum_{k=1}^{N} a_k f(y_k) \to I(f), \quad \text{when } N \to \infty, \tag{1.2}$$

then we call (1.2) an approximate cubature formula for $\mathcal{F}$ on the sphere.

Cubature formulas on the sphere were extensively used in scientific applications, such as [1–2] for graphic processing, and [3–4] for partial differential equations in geophysics. There are some investigations on both exact cubature and approximate cubature formulas on the sphere (see [5–8]). For an exact cubature formula on the sphere, one usually considers the space of spherical polynomials as the underlying space. For example, in the seminal paper [8], Mhaskar

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et al. established an exact positive cubature for spherical polynomials, and proved that if the number of the cubature points $\{x_i\}_{i=1}^N$ and the degree of the spherical polynomials satisfy some extra assumptions, then there exists a set of cubature weights $\{w_i\}_{i=1}^N \subset \mathbb{R}_+$, such that for all spherical polynomials P_n with degree at most n, there holds

$$I(P_n) = \sum_{i=1}^{N} w_i P_n(x_i), \quad N \sim n^d. \tag{1.3}$$

In [9], Brown and Dai made an improvement of the above result and extended it to the compact two-point manifolds. Similar studies can be found in [10–11].

The main purpose for the study of approximate cubature formula on the sphere is to deduce the relation between the rate of convergence in (1.2) and the number of cubature points. One usually takes the native spaces corresponding to some spherical basis functions (SBFs), i.e., positive definite radial basis functions on the sphere as the underlying spaces. We refer the readers to Hesse and Sloan [5–7] and Brauchart and Hesse [12] etc. for the spherical Sobolev space, and Sun and Chen [13] for the native space of a general SBF.

On the other hand, a popular method to deal with spherical scattered data is to construct the required approximation or interpolation from spaces of SBFs, which are kernels located at points in a discrete set $X := \{x_i\}_{i=1}^n \subset \mathbb{S}^d$. A continuous function $\phi : [-1, 1] \to \mathbb{R}$ is an SBF on $\mathbb{S}^d$, if in its expansion in the well-known Legendre polynomials, the Fourier-Legendre coefficients $\{\widehat{\phi}(l)\}$ of ϕ are all positive (see the next section for details). To do with spherical data, the approximation or interpolation space is usually taken as

$$\Phi_n := \Big\{ g(x) : g(x) = \sum_{i=1}^{n} c_i \phi(x_i \cdot x) \Big\}, \tag{1.4}$$

where $x \cdot y$ denotes the usual "dot" product. Following the usage common in the neural network community, we will say that a function $g \in \Phi_n$ is a spherical basis function network (SBFN).

As mentioned above, except for the spherical polynomial space, cubature formulas in other underlying spaces are approximate. Thus it is natural to raise the question: If the underlying space is not the spherical polynomial space, are there any exact cubature formulas? In this paper, we give an affirmative answer to the question. We will prove that there exists an exact cubature formula for SBFN defined in (1.4). It will be shown that for arbitrary $g \in \Phi_n$, there exist at most n cubature points and cubature weights, such that the exact positive cubature formula holds.

2 SBF and Exact Cubature Formula

For any integer $k \geq 0$, the restriction to $\mathbb{S}^d$ of a homogeneous harmonic polynomial of degree k is called a spherical harmonic of degree k. The class of all spherical harmonics of degree k is denoted by $\mathbb{H}_k^d$, and the class of all spherical harmonics of degree $k \leq n$ is denoted by Π_n^d. Of course, $\Pi_n^d = \bigoplus_{k=0}^{n} \mathbb{H}_k^d$, and it comprises the restriction to $\mathbb{S}^d$ of all algebraic polynomials in

$(d+1)$ variables of a total degree not exceeding n. The dimension of $\mathbb{H}_k^d$ is given by

$$d_k^d := \dim \mathbb{H}_k^d = \begin{cases} \dfrac{2k+d-1}{k+d-1}\dbinom{k+d-1}{k}, & k \geq 1, \\ 1, & k = 0, \end{cases}$$

and that of Π_n^d is $\sum\limits_{k=0}^{n} d_k^d = d_n^{d+1} \leq C_1 n^d$. Here and hereafter, C_1 and C are the positive constants depending only on d.

The addition theorem establishes a connection between spherical harmonics of degree k and the Legendre polynomial P_k^{d+1} (see [3]),

$$\sum_{l=1}^{d_k^d} Y_{k,l}(x)Y_{k,l}(y) = \frac{d_k^d}{\Omega_d} P_k^{d+1}(x \cdot y), \tag{2.1}$$

where $Y_{k,j}$ $(k \in \mathbb{N},\ j = 1, \cdots, d_k^d)$ is an arbitrary orthonormal basis of $\mathbb{H}_k^d$, and P_k^{d+1} is the Legendre polynomial with degree k and dimension $(d+1)$. The Legendre polynomial P_k^{d+1} can be normalized, such that $P_k^{d+1}(1) = 1$, and it satisfies the orthogonality relations

$$\int_{-1}^{1} P_k^{d+1}(t) P_j^{d+1}(t)(1-t^2)^{\frac{d-2}{2}} \mathrm{d}t = \frac{\Omega_d}{\Omega_{d-1} \mathrm{d}_k^d} \delta_{k,j},$$

where $\delta_{k,j}$ is the usual Kronecker symbol.

Consider a function ϕ in $C([-1,1])$. We always assume that ϕ has the following expansion in the orthogonal set of Legendre polynomials:

$$\phi(x \cdot y) = \sum_{k=0}^{\infty} \widehat{\phi}(k) \frac{d_k^d}{\Omega_d} P_k^d(x \cdot y), \tag{2.2}$$

where

$$\Omega_d := \int_{\mathbb{S}^d} \mathrm{d}\omega = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}$$

is the volume of $\mathbb{S}^d$ and $\widehat{\phi}(k)$ is the kth Legendre-Fourier coefficient of ϕ.

Positive definite functions on spheres were introduced and characterized by Schoenberg [14], i.e., a function ϕ is called positive definite, if for every set X of scattered data, the matrix $[\phi(x_j \cdot x_k)]$ is positive semidefinite. It was shown in [14] that ϕ is positive definite, if and only if the Legendre-Fourier coefficients satisfy $\widehat{\phi}(l) \geq 0$ for all l and $\sum\limits_{l=0}^{\infty} \widehat{\phi}(l) \mathrm{d}_l^d < \infty$. If in addition, $\widehat{\phi}(l) > 0$, then $[\phi(x_j \cdot x_k)]$ is a positive definite matrix and one can use shifts of ϕ to interpolate any function $f \in C(\mathbb{S}^d)$ on X. We will say that ϕ is an SBF in this case.

SBFN plays an important role in the study of scattered data interpolation (see [15–18]). Because the matrix $[\phi(x_j \cdot x_k)]$ corresponding to any X is positive definite (and hence invertible), one can always use an interpolant of the form $\sum\limits_{j=1}^{n} \alpha_j(\phi(x \cdot x_j)$ to solve the interpolation problem for scattered data:

$$\sum_{j=1}^{n} \alpha_j \phi(x \cdot x_j) = f(x_j).$$

Therefore, it is urgent to study more important properties about SBFNs, and the exact positive cubature formula is one of these important properties. The following theorem gives an existence proof of the exact positive cubature formula for SBFN. Its proof can be given in the next section.

Theorem 2.1 *Let Φ_n be defined in (1.4). If ϕ is an SBF and $g \in \Phi_n$, then there exists a set of points $\{y_i\}_{i=1}^r \subset \mathbb{S}^d$ and a set of real numbers $\{a_i\}_{i=1}^r \subset \mathbb{R}_+$ satisfying $\sum\limits_{i=1}^r |a_i| = \Omega_d$ with $1 \le r \le n$, such that*

$$\int_{\mathbb{S}^d} g(x)\mathrm{d}\omega(x) = \sum_{i=1}^r a_i g(y_i).$$

Remark 2.1 It will be seen in the next section that the positive definition of the activation function ϕ can be relaxed. In fact, we can obtain the same result by setting $\widehat{\phi}(1) \neq 0$, where $\widehat{\phi}(1)$ is defined in (2.2).

3 Proof of the Main Result

To prove Theorem 2.1, the following two lemmas will play key roles. The first one is a corollary of Hahn-Banaha Theorem which can be found in [19].

Lemma 3.1 *Let $M \subset \mathbb{R}^n$ be closed and convex. Let $x \in \mathbb{R}^n$, $x \notin M$. Then there exist $0 \neq a \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$, such that $a \cdot x < \beta$ and $a \cdot z > \beta$ for all $z \in M$.*

The second one is the well-known Carathedory Lemma, which can be found in [20].

Lemma 3.2 (Caratheodory Lemma) *If $A \subset \mathbb{R}^n$, then every point from the convex hull of A can be written as a convex linear combination of at most $(n+1)$ points of A.*

Proof of Theorem 2.1 Since ϕ is continuous on the interval $[-1, 1]$, then $\Phi_n \subset C(\mathbb{S}^d)$. So, it follows from the finite dimensionality of Φ_n that the dimension of

$$H := \{g \in \Phi_n : \|g\|_{\mathbb{S}^d} = 1\}$$

is $(n-1)$, where $\|\cdot\|_{\mathbb{S}^d}$ denotes the uniform norm on $\mathbb{S}^d$. Furthermore, for arbitrary $g \in H$, we have

$$\int_{\mathbb{S}^d} g(x)\mathrm{d}\omega(x) = \sum_{i=1}^n c_i \int_{\mathbb{S}^d} \phi(x_i \cdot x)\mathrm{d}\omega(x) = \sum_{i=1}^n c_i \Omega_{d-1} \int_{-1}^1 \phi(t)(1-t^2)^{\frac{d-2}{2}}\mathrm{d}t.$$

Since $\phi(\cdot\,,\cdot)$ is a positive definition,

$$\int_{-1}^1 \phi(t)(1-t^2)^{\frac{d-2}{2}}\mathrm{d}t > 0.$$

Then it is easy to deduce that the dimension of the set

$$\Big\{g \in H : \int_{\mathbb{S}^d} g(x)\mathrm{d}\omega(x) = \Omega_d\Big\}$$

is $(n-2)$. Therefore, there exists an $h \in \Phi_n$, such that

$$\|h\|_{\mathbb{S}^d} = 1, \quad \int_{\mathbb{S}^d} h(x)\mathrm{d}\omega(x) = \Omega_d. \tag{3.1}$$

Let

$$G_0 := \Big\{ g \in \Phi_n : \int_{\mathbb{S}^d} g(x)\mathrm{d}\omega(x) = 0 \Big\}.$$

Thus $G_0 \ni g = \sum\limits_{i=1}^{n} c_i \phi(x_i \cdot x)$ satisfies $\sum\limits_{i=1}^{n} c_i = 0$. Hence,

$$G_0 = \Big\{ \sum_{i=1}^{n-1} c_i \phi(x_i \cdot x) - \sum_{i=1}^{n-1} c_i \phi(x_n \cdot x) : c_i \in \mathbb{R},\ 1 \le i \le n-1 \Big\},$$

and the dimension of G_0 is $(n-1)$.

Now we prove that for all $g_0 \in G_0$, there holds

$$\|h\|_{\mathbb{S}^d} \le \|h + g_0\|_{\mathbb{S}^d}. \tag{3.2}$$

Indeed, if there exists a $g_0^* \in G_0$, such that

$$\|h + g_0^*\|_{\mathbb{S}^d} < \|h\|_{\mathbb{S}^d} = 1,$$

then $g_1 := h + g_0^*$ satisfies $\|g_1\|_{\mathbb{S}^d} < 1$. Therefore

$$\int_{\mathbb{S}^d} g_1(x)\mathrm{d}\omega(x) \le \int_{\mathbb{S}^d} |g_1(x)|\mathrm{d}\omega(x) \le \|g_1\|_{\mathbb{S}^d} \int_{\mathbb{S}^d} \omega_d < \Omega_d.$$

On the other hand, by the definition of G_0, we have $\int_{\mathbb{S}^d} g_0^*(x)\mathrm{d}\omega(x) = 0$. Thus

$$\int_{\mathbb{S}^d} g_1(x)\mathrm{d}\omega(x) = \int_{\mathbb{S}^d} h(x)\mathrm{d}\omega(x) + \int_{\mathbb{S}^d} g_0^*(x)\mathrm{d}\omega(x) = \Omega_d.$$

This is impossible. Hence, (3.2) holds for all $g_0 \in G_0$.

Let $\{\psi_i\}_{i=1}^{n-1}$ be a basis of G_0. Define $T : \mathbb{S}^d \to \mathbb{R}^{n-1}$, and

$$T(x) := h(x)(\psi_1(x), \cdots, \psi_{n-1}(x)).$$

If we set

$$E := E(x) := \left\{ x \in \mathbb{S}^d : |h(x)| = \|h\|_{\mathbb{S}^d} \right\},$$

then the origin $(0, \cdots, 0)$ is in the convex hull of

$$T(E) := \{T(x) : x \in E\}.$$

Otherwise, it follows from Lemma 3.1 that there exist $b_1, \cdots, b_{n-1} \in \mathbb{R}$, such that for all $x \in E$, there holds

$$\sum_{i=1}^{n-1} b_i h(x)\psi_i(x) > 0.$$

If we denote $\psi^* := \sum\limits_{i=1}^{n-1} b_i \psi_i \in \Phi_n$, then

$$h(x)\psi^*(x) > 0.$$

The above inequality together with the fact that E is compact yields that there exists a $\delta > 0$ and an open set D which contains E, such that

$$h(x)\psi^*(x) > \delta > 0, \quad x \in D. \tag{3.3}$$

Since D is an open set, $D^c := \mathbb{S}^d - D$ is compact. Therefore, it follows from the definition of E that there is an $\alpha > 0$, such that

$$|h(x)| < (1-\alpha)\|h\|_{\mathbb{S}^d}, \quad x \in D^c. \tag{3.4}$$

Thus, for a sufficiently small $\varepsilon > 0$, (3.3) yields that for arbitrary $x \in D$, there holds

$$\begin{aligned}|h(x) - \varepsilon\psi^*(x)|^2 &= |h(x)|^2 - 2\varepsilon h(x)\psi^*(x) + \varepsilon^2|\psi^*(x)|^2 \\ &\leq \|h\|_{\mathbb{S}^d}^2 + \varepsilon^2\|\psi^*\|_{\mathbb{S}^d} - 2\varepsilon\delta < \|h\|_{\mathbb{S}^d}^2.\end{aligned}$$

Since (3.4) yields that for arbitrary $x \in D^c$ there holds

$$|h(x) - \varepsilon\psi^*(x)| \leq |h(x)| + \varepsilon|\psi^*(x)| < (1-\alpha)\|h\|_{\mathbb{S}^d} + \varepsilon\|\psi^*\|_{\mathbb{S}^d} < \|h\|_{\mathbb{S}^d}$$

for arbitrary $x \in \mathbb{S}^d$, there holds

$$\|h - \varepsilon\psi^*\|_{\mathbb{S}^d} \leq \|h\|_{\mathbb{S}^d},$$

which contradicts (3.2). Thus the origin is in the convex hull of $T(E)$.

On the other hand, by Lemma 3.2, we can deduce that there exists a set of points $\{y_i\}_{i=1}^r \subset \mathbb{E}$ and a set of numbers $\{c_i\}_{i=1}^r$ $(1 \leq r \leq n)$ satisfying $c_i > 0$ and $\sum\limits_{i=1}^r c_i = 1$, such that

$$\sum_{i=1}^r c_i h(y_i)\psi_j(y_i) = 0, \quad j = 1, \cdots, n-1.$$

Since $\{\psi\}_{i=1}^{n-1}$ is a basis of G_0, for arbitrary $g_0 \in G_0$, we obtain

$$\sum_{i=1}^r c_i h(y_i) g_0(y_i) = \sum_{j=1}^{n-1} b_j \sum_{i=1}^r c_i h(y_i)\psi_j(y_i) = 0. \tag{3.5}$$

Noting that $g_2 := \int_{\mathbb{S}^d} h(x)\mathrm{d}\omega(x)g - \int_{\mathbb{S}^d} g(x)\mathrm{d}\omega(x)h \in G_0$ for all $g \in \Phi_n$, it follows from (3.5) that

$$\begin{aligned}&\int_{\mathbb{S}^d} g(x)\mathrm{d}\omega(x)\sum_{i=1}^r c_i(h(y_i))^2 - \int_{\mathbb{S}^d} h(x)\mathrm{d}\omega(x)\sum_{i=1}^r c_i g(y_i)h(y_i) \\ &= \sum_{i=1}^r c_i\Big(\int_{\mathbb{S}^d} g(x)\mathrm{d}\omega(x)h - \int_{\mathbb{S}^d} h(x)\mathrm{d}\omega(x)g\Big)(y_i)h(y_i) \\ &= \sum_{i=1}^r c_i g_2(y_i)h(y_i) = 0.\end{aligned}$$

Hence,

$$\int_{\mathbb{S}^d} g(x)\mathrm{d}\omega(x) = \frac{\int_{\mathbb{S}^d} h(x)\mathrm{d}\omega(x) \sum\limits_{i=1}^{r} c_i g(y_i) h(y_i)}{\sum\limits_{i=1}^{r} c_i (h(y_i))^2}.$$

Moreover, (3.1) yields that $\int_{\mathbb{S}^d} h(x)\mathrm{d}\omega(x) = \Omega_d$. Then

$$\int_{\mathbb{S}^d} g(x)\mathrm{d}\omega(x) = \sum_{i=1}^{r} \Omega_d \frac{c_i h(y_i)}{\sum\limits_{i=1}^{r} c_i (h(y_i))^2} g(y_i).$$

Since $y_i \in E$, there holds $|h(y_i)| = 1$. If we set

$$a_i := \Omega_d \frac{c_i h(y_i)}{\sum\limits_{i=1}^{r} c_i (h(y_i))^2},$$

then we have

$$\sum_{i=1}^{r} |a_i| = \Omega_d.$$

This completes the proof of Theorem 2.1.

References

[1] Lam, P. M., Ho, T. Y., Leung, C. S., et al., All-frequency lighting with multiscale spherical radial basis functions, *IEEE Trans. Visual. Comput. Graph.,* **16**, 2010, 43–56.

[2] Tsai, Y. T., Chang, C. C., Jiang Q. Z., et al., Importance sampling of products from illumination and BRDF using spherical radial basis functions, *Visual. Comput.,* **24**, 2008, 817–826.

[3] Freeden, W., Gervens, T. and Schreiner, M., Constructive Approximation on the Sphere, Calderon Press, Oxford, 1998.

[4] Freeden, W., Michel, V. and Nutz, H., Satellite-to-satellite tracking and satellite gravity gradiometry (Advanced techniques for high-resolution geopotential field determination), *J. Engin. Math.,* **43**, 2002, 19–56.

[5] Hesse, K. and Sloan, I. H., Worst-case errors in a Sobolev space setting for cubature over the sphere S^2, *Bull. Austral. Math. Soc.,* **71**, 2005, 81–105.

[6] Hesse, K. and Sloan, I. H., Optimal lower bounds for cubature error on the sphere S^2, *J. Complex,* **21**, 2005, 790–803.

[7] Hesse, K., A lower bound for the worst-case cubature error on spheres of arbitrary dimension, *Numer. Math.,* **103**, 2006, 413–433.

[8] Mhaskar, H. N., Narcowich, F. J. and Ward, J. D., Spherical Marcinkiewicz-Zygmund inequalities and positive quadrature, *Math. Comput.,* **70**, 2000, 1113–1130.

[9] Brown, G. and Dai, F., Approximation of smooth functions on compact two-point homogeneous spaces, *J. Funct. Anal.,* **220**, 2005, 401–423.

[10] Dai, F. and Wang, H. P., Positive cubature formulas and Marcinkiewicz-Zygmund inequalities on spherical caps, *Constr. Approx.,* **31**, 2010, 1–36.

[11] Dai, F., Multivariate polynomial inequalities with respect to doubling weights and A_∞ weights, *J. Approx. Theory,* **235**, 2006, 137–170.

[12] Brauchart, J. S. and Hesse, K., Numerical integration over spheres of arbitrary dimension, *Constr. Approx.*, **25**, 2007, 41–71.

[13] Sun, X. and Chen, Z., Spherical basis functions and uniform distribution of points on spheres, *J. Approx. Theory,* **151**, 2008, 186–207.

[14] Schoenberg, I. J., Positive definite functions on spheres, *Duke Math. J.*, **9**, 1942, 96–108.

[15] Jetter, K., Stöckler, J. and Ward, J. D., Error estimates for scattered data interpolation, *Math. Comput.,* **68**, 1999, 743–747.

[16] Levesley, J. and Sun, X., Approximation in rough native spaces by shifts of smooth kernels on spheres, *J. Approx. Theory,* **133**, 2005, 269–283.

[17] Narcowich, F. J. and Ward, J. D., Scattered data interpolation on spheres: Error estimates and locally supported basis functions, *SIAM J. Math. Anal.,* **33**, 2002, 1393–1410.

[18] Narcowich, F. J., Sun, X., Ward, J. D., et al., Direct and inverse Sobolev error estimates for scattered data interpolation via spherical basis functions, *Found. Comput. Math.,* **7**, 2007, 369–370.

[19] Rudin, W., Functional Analysis, McGraw-Hill, New York, 1973.

[20] Borwein, P. B. and Erdélyi, T., Polynomials and Polynomial Inequalities, Graduate Texts in Mathematics, Springer-Verlag, New York, 1995.