Chin. Ann. Math.
33B(6), 2012, 815–822
DOI: 10.1007/s11401-012-0751-y

Chinese Annals of Mathematics, Series B
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A Criterion of Normality Concerning Holomorphic Functions Whose Derivative Omit a Function II*

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Abstract The authors discuss the normality concerning holomorphic functions and get the following result. Let $\mathcal{F}$ be a family of functions holomorphic on a domain $D \subset \mathbb{C}$, all of whose zeros have multiplicity at least k, where $k \geq 2$ is an integer. Let $h(z) \not\equiv 0$ and ∞ be a meromorphic function on D. Assume that the following two conditions hold for every $f \in \mathcal{F}$:

(a) $f(z) = 0 \Rightarrow |f^{(k)}(z)| < |h(z)|$.

(b) $f^{(k)}(z) \neq h(z)$.

Then $\mathcal{F}$ is normal on D.

Keywords Normal family, Meromorphic functions, Omitted function
2000 MR Subject Classification 30D35

1 Introduction

In [2], the following theorem was proved.

Theorem LY *Let $\mathcal{F}$ be a family of functions holomorphic on a domain $D \subset \mathbb{C}$, all of whose zeros have multiplicity at least k, where $k \geq 2$ is an integer. Let $h(z) \not\equiv 0$ be a holomorphic function on D. Assume also that the following two conditions hold for every $f \in \mathcal{F}$:*

(a) $f(z) = 0 \Rightarrow |f^{(k)}(z)| < |h(z)|$.

(b) $f^{(k)}(z) \neq h(z)$.

Then $\mathcal{F}$ is normal on D.

They also gave a counterexample to show that Theorem LY does not hold for a family of meromorphic functions $\mathcal{F}$ when $k = 2$.

In this paper, we continue to study the above problem and obtain that Theorem LY also holds for the meromorphic function $h(z)$.

Theorem 1.1 *Let $\mathcal{F}$ be a family of functions holomorphic on a domain $D \subset \mathbb{C}$, all of whose zeros have multiplicity at least k, where $k \geq 2$ is an integer. Let $h(z) \not\equiv 0$, and ∞ be a meromorphic function on D. Assume that the following two conditions hold for every $f \in \mathcal{F}$:*

(a) $f(z) = 0 \Rightarrow |f^{(k)}(z)| < |h(z)|$.

(b) $f^{(k)}(z) \neq h(z)$.

Then $\mathcal{F}$ is normal on D.

The following counterexample shows that Theorem 1.1 does not hold for a family of meromorphic functions $\mathcal{F}$.

Manuscript received February 9, 2012. Revised July 18, 2012.
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*Project supported by the National Natural Science Foundation of China (No. 11071074) and the Outstanding Youth Foundation of Shanghai (No. slg10015).

Example 1.1 Let $D = \Delta$ be the unit disc, k be a positive integer and $\mathcal{F} = \{f_n\}$, where

$$f_n(z) = \frac{\left(z - \frac{1}{k+2}\frac{1}{n}\right)^{k+2}}{(k+1)!\left(z - \frac{1}{n}\right)}, \quad h(z) = z.$$

Then each f_n has a single zero, whose multiplicity is $k+2$, and $f_n^{(k)} \neq h$. However, f_n assumes the values 0 and ∞ in any fixed neighborhood of 0 if n is sufficiently large, so $\mathcal{F}$ fails to be equicontinuous at 0. Thus $\mathcal{F}$ is not normal in any neighborhood of 0.

Let us set some notations. D is a domain in $\mathbb{C}$. For $z_0 \in \mathbb{C}$ and $r > 0$, $\Delta(z_0, r) = \{z : |z - z_0| < r\}$ and $\Delta'(z_0, r) = \{z : 0 < |z - z_0| < r\}$. The unit disc will be denoted by Δ, and $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. $f_n(z) \overset{\chi}{\Rightarrow} f(z)$ on D indicates that the sequence $\{f_n(z)\}$ converges to f in the spherical metric, uniformly on compact subsets of D, and $f_n(z) \Rightarrow f(z)$ on D if the convergence is in the Euclidean metric. The spherical derivative of the meromorphic function f at the point z is denoted by $f^{\#}(z)$. Frequently, given a sequence $\{f_n(z)\}_{n=1}^{\infty}$ of functions, we need to extract an appropriate subsequence, and this necessity may recur within a single proof. To avoid the awkwardness of multiple indices, we again denote the extracted subsequence by $\{f_n\}$ (rather than, say, $\{f_{n_k}\}$) and signal this operation by writing "taking a subsequence and renumbering" or simply "renumbering". The same convention applies to sequences of constants.

The structure of the paper is as follows. In Section 2, we state a number of preliminary results. Then, in Section 3, we prove Theorem 1.1.

2 Preliminary Results

The following lemma is taken from [5, p. 259 and 9, pp. 216–217].

Lemma 2.1 (see [3–4, 6–8]) *Let $\mathcal{F}$ be a family of functions meromorphic in a domain D, all of whose zeros have multiplicity at least k, and suppose that there exists an $A \geq 1$, such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then if $\mathcal{F}$ is not normal at $z_0 \in D$, there exist, for each $0 \leq \alpha \leq k$,*

(a) *points $z_n \to z_0$,*
(b) *functions $f_n \in \mathcal{F}$,*
(c) *positive numbers $\rho_n \to 0^+$,*

such that $g_n(\zeta) := \rho_n^{-\alpha} f_n(z_n + \rho_n\zeta) \overset{\chi}{\Rightarrow} g(\zeta)$ on $\mathbb{C}$, where g is a nonconstant meromorphic function on $\mathbb{C}$, such that for every $\zeta \in \mathbb{C}$, $g^{\#}(\zeta) \leq g^{\#}(0) = kA + 1$.

Lemma 2.2 (see [1, pp. 118–119, 122–123]) *Let f be a meromorphic function on $\mathbb{C}$. If $f^{\#}$ is uniformly bounded on $\mathbb{C}$, then the order of f is at most 2. If f is an entire function, then the order of f is at most 1.*

Lemma 2.3 *Let f be an entire function of finite order $\rho(f)$ on $\mathbb{C}$, all of whose zeros have multiplicity at least k, where $k \geq 2$ is an integer and $a \neq 0$ is a constant. Suppose that $\rho(f) \leq 1$ and $f(z)$ satisfies the following two conditions:*

(a) $f(z) = 0 \Rightarrow |f^{(k)}(z)| \leq |a|$,
(b) $f^{(k)}(z) \neq a$.

Then

$$f(z) = \frac{b(z - z_0)^k}{k!},$$

where $b \neq a$ and z_0 are constants.

Proof We separate it into two cases.

Case A f is a transcendental entire function on $\mathbb{C}$.

By $\rho(f^{(k)}) = \rho(f) \leq 1$ and $f^{(k)} \neq a$, we have $f^{(k)}(z) = a + B\exp(Az)$, where $A, B \in \mathbb{C}^*$ are two constants. By calculation,

$$f(z) = \frac{az^k}{k!} + a_{k-1}z^{k-1} + \cdots + a_0 + BA^{-k}\exp(Az),$$

where $a_{k-1}, \cdots, a_0$ are constants.

Since $a \neq 0$, there exist $z_m, z_m \to \infty$, such that $f(z_m) = 0,\ m = 1, 2, \cdots$. By the condition that all zeros of f have multiplicity at least $k\ (\geq 2)$, we have $f'(z_m) = 0$. Setting

$$P(z) = A^{-1} f'(z) - f(z),$$

it is obviously to see that P is a polynomial and $P(z_m) = 0,\ m = 1, 2, \cdots$, so then we have that $P(z) \equiv 0,\ f(z) = C\exp(Az)$, where $C \neq 0$ is a constant, a contradiction.

Case B f is a polynomial.

Then by $f^{(k)} \neq a$, we have $f^{(k)}(z) = b$, where $b \neq a$ is a constant. Since all zeros of f have multiplicity at least $k\ (\geq 2)$, we have

$$f(z) = \frac{b(z - z_0)^k}{k!},$$

where z_0 is a constant.

Lemma 2.4 *Let $\mathcal{F} = \{f_n\}$ be a family of holomorphic functions on $\Delta, \ell\ (\geq 1)$ and $k\ (\geq 1)$ be two integers, and b_n be a sequence of functions analytic on Δ, such that $b_n \Rightarrow 1$ on Δ. If*

$$f_n(z) \neq 0, \quad f_n^{(k)}(z) \neq \frac{b_n(z)}{z^\ell}, \quad z \in \Delta,$$

then $\mathcal{F}$ is normal on Δ.

Proof First, we prove that $\mathcal{F}$ is normal on Δ'. If not the case, there exists a $z_0 \neq 0$, such that $\mathcal{F}$ is not normal at z_0, and by Lemma 2.1 with $\alpha = k$, it is obtained that there exist $z_n \to z_0,\ \rho_n \to 0^+$, such that

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n\zeta)}{\rho_n^k} \Rightarrow g(\zeta) \quad \text{on } \mathbb{C},$$

where g is a nonconstant entire function of order at most 1. Obviously, $g(\zeta) \neq 0$ on $\mathbb{C}$. Since

$$g_n^{(k)}(\zeta) = f_n^{(k)}(z_n + \rho_n\zeta) \neq \frac{b_n(z_n + \rho_n\zeta)}{(z_n + \rho_n\zeta)^\ell},$$

and $b_n(z_n + \rho_n\zeta) \Rightarrow 1$, it follows from Hurwitz's theorem that either $g^{(k)}(\zeta) \neq \frac{1}{z_0^\ell}$ or $g^{(k)}(\zeta) \equiv \frac{1}{z_0^\ell}$. The latter does not hold, since g does not assume zero. But then $g \neq 0$, and $g^{(k)} \neq \frac{1}{z_0^\ell}$, so that g is a constant by Hayman's theorem.

Next, we show that $\mathcal{F}$ is normal on $z = 0$. If not in this case, we may assume that $\mathcal{F}$ is not normal at $z = 0$. Then, by taking a subsequence and renumbering, we have

$$f_n \Rightarrow \infty, \quad \text{on } \ \Delta'.$$

Since $f_n \neq 0$ on Δ, by the minimum principle, we have

$$f_n \Rightarrow \infty, \quad \text{on } \ \Delta,$$

a contradiction. This completes the proof of the lemma.

Lemma 2.5 *Let $\{f_n\}$ be a sequence of functions holomorphic on a domain $D \subset \mathbb{C}$, all of whose zeros have multiplicity at least k, and $\{h_n\}$ be a sequence of functions analytic on D, such that $h_n(z) \Rightarrow h(z)$ on D, where $h(z) \neq 0$ for $z \in D$ and $k \geq 2$ is an integer. Suppose that, for each $n, f_n(z) = 0 \Rightarrow |f_n^{(k)}(z)| < |h_n(z)|$ and $f_n^{(k)}(z) \neq h_n(z)$. Then $\{f_n\}$ is normal on D.*

Proof Suppose to the contrary that there exists a $z_0 \in D$, such that $\{f_n\}$ is not normal at z_0. The convergence of $\{h_n\}$ to h implies that, in some neighborhood of z_0, we have $f_n(z) = 0 \Rightarrow |f_n^{(k)}(z)| \leq |h(z_0)| + 1$ (for large enough n). Thus we can apply Lemma 2.1 with $\alpha = k$ and $A = |h(z_0)| + 1$. So we can take an appropriate subsequence of $\{f_n\}$ (denoted also by

$\{f_n\}$ after renumbering), together with points $z_n \to z_0$ and positive numbers $\rho_n \to 0^+$, such that

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n^k} \stackrel{\chi}{\Rightarrow} g(\zeta), \quad \text{on } \mathbb{C},$$

where g is a nonconstant entire function, all of whose zeros have multiplicity at least k and $g^{\#}(\zeta) \le g^{\#}(0) = k(|h(z_0)| + 1) + 1$.

We claim that $g = 0 \Rightarrow |g^{(k)}| \le |h(z_0)|$ and $g^{(k)} \ne h(z_0)$.

In fact, if there exists a $\zeta_0 \in \mathbb{C}$, such that $g(\zeta_0) = 0$, then since $g(\zeta) \not\equiv 0$, there exist ζ_n, $\zeta_n \to \zeta_0$, such that if n is sufficiently large,

$$g_n(\zeta_n) = \frac{f_n(z_n + \rho_n \zeta_n)}{\rho_n^k} = 0.$$

Thus $f_n(z_n + \rho_n \zeta_n) = 0$, so that $|f_n^{(k)}(z_n + \rho_n \zeta_n)| < |h_n(z_n + \rho_n \zeta_n)|$, i.e., $|g_n^{(k)}(\zeta_n)| < |h_n(z_n + \rho_n \zeta_n)|$. Since $|g^{(k)}(\zeta_0)| = \lim\limits_{n\to\infty} |g_n^{(k)}(\zeta_n)| \le |h(z_0)|$, we have established the first part of the claim.

Now, suppose that there exists a $\zeta_0 \in \mathbb{C}$, such that $g^{(k)}(\zeta_0) = h(z_0)$. If $g^{(k)}(\zeta) \equiv h(z_0)$, and all zeros of g have multiplicity at least k, we can get $g(\zeta) = \frac{h(z_0)}{k!}(\zeta - \zeta_0)^k$. Then we have $g^{\#}(0) \le k(|h(z_0)| + 1)$, which contradicts $g^{\#}(0) = k(|h(z_0)| + 1) + 1$. Thus $g^{(k)}$ is not constant, so by Hurwitz's theorem, there exist ζ_n, $\zeta_n \to \zeta_0$, such that

$$f_n^{(k)}(z_n + \rho_n \zeta_n) - h_n(z_n + \rho_n \zeta_n) = g_n^{(k)}(\zeta_n) - h_n(z_n + \rho_n \zeta_n) = 0,$$

which contradicts $f_n^{(k)} \ne h_n$. This completes the proof of the claim.

By Lemma 2.3,

$$g(\zeta) = \frac{b}{k!}(\zeta - \zeta_0)^k,$$

where $\zeta_0 \in \mathbb{C}$ and $b \ne h(z_0)$ are constants. We have $g^{\#}(0) \le k(|b|+1)$. By the above claim, since $g(\zeta_0) = 0$, we can get $|g^{(k)}(\zeta_0)| = |b| \le |h(z_0)|$. Then we have $g^{\#}(0) \le k(|b|+1) \le k(|h(z_0)|+1)$, a contradiction. The lemma is proved.

3 Proof of Theorem 1.1

By Theorem LY, it suffices to prove that $\mathcal{F}$ is normal at the points where h has poles. Consider $z_0 \in D$, such that $h(z_0) = \infty$. Without loss of generality, we can assume that $z_0 = 0$, and then

$$h(z) = \frac{b(z)}{z^\ell}, \tag{3.1}$$

where $\ell\ (\ge 1)$ is an integer, $b(z) \ne 0$ is an analytic function in $\Delta(0, \delta)$, and we can also assume that $b(0) = 1$. We take a subsequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$, and want to prove that $\{f_n\}$ is normal at $z = 0$. Suppose by negation that $\{f_n\}$ is not normal at $z = 0$. Since $\{f_n\}$ is normal in $\Delta'(0, \delta)$, we can assume (after renumbering) that $f_n \Rightarrow F$ on $\Delta'(0, \delta)$. If $F(z) \not\equiv \infty$, then it is a holomorphic function. Hence by the maximum principle, F extends to be analytic also at $z = 0$, so $f_n \Rightarrow F$ on $\Delta(0, \delta)$. Then we are done. Hence, we assume that

$$f_n(z) \Rightarrow \infty, \quad \text{on } \ \Delta'(0, \delta). \tag{3.2}$$

We sperate it into two cases.

Case A Suppose that $\ell \ge k + 1$.

Define $\mathcal{F}_1 = \{F_n = z^\ell f_n : n \in \mathbb{N}\}$. It suffices to prove that $\mathcal{F}_1$ is normal in $\Delta(0, \delta)$. Indeed, if (after renumbering) $z^\ell f_n(z) \Rightarrow H(z)$ on $\Delta(0, \delta)$, then since $z^\ell \ne 0$ in $\Delta'(0, \delta)$, it follows from (3.2) that $H(z) \equiv \infty$ in $\Delta'(0, \delta)$, and thus $H(z) \equiv \infty$ also in $\Delta(0, \delta)$. In particular, $z^\ell f_n(z) \ne 0$ on each compact subset of $\Delta(0, \delta)$, which implies $f_n(z) \ne 0$ on each compact subset of $\Delta(0, \delta)$ for large enough n. Then by the minimum principle, it follows from (3.2) that $f_n(z) \Rightarrow \infty$ on

$\Delta(0,\delta)$, which implies the normality of $\mathcal{F}$. So suppose to the contrary that $\mathcal{F}_1$ is not normal at $z=0$. By Lemma 2.1 and the assumptions of Theorem 1.1, there exist (after renumbering) points $z_n \to 0$, $\rho_n \to 0^+$ and a nonconstant meromorphic function on $\mathbb{C}$, $g(\zeta)$, such that

$$g_n(\zeta) = \frac{F_n(z_n+\rho_n\zeta)}{\rho_n^k} = \frac{(z_n+\rho_n\zeta)^\ell f_n(z_n+\rho_n\zeta)}{\rho_n^k} \overset{\chi}{\Rightarrow} g(\zeta), \quad \text{on } \mathbb{C}, \tag{3.3}$$

all of whose zeros have multiplicity at least k and

$$\text{for every } \zeta \in \mathbb{C}, \quad g^{\#}(\zeta) \le g^{\#}(0) = kA+1, \tag{3.4}$$

where $A > 1$ is a constant. Here we have used Lemma 2.1 with $\alpha = k$. Note that $g_n(\zeta) = 0$ implies $|g_n^{(k)}(\zeta)| < |b(z_n+\rho_n\zeta)|$, and thus A can be chosen to be any number, such that $A \ge 1$. After renumbering, we can assume that $\{\frac{z_n}{\rho_n}\}_{n=1}^{\infty}$ converges. We separate it into two subcases.

Case AI

$$\frac{z_n}{\rho_n} \to \infty. \tag{3.5}$$

Then

$$\begin{aligned} F_n^{(k)}(z) &= \sum_{j=0}^{k} \binom{k}{j} (z^\ell)^{(k-j)} f^{(j)}(z) \\ &= z^\ell f^{(k)}(z) + \sum_{j=0}^{k-1} \binom{k}{j} \ell(\ell-1)\cdots(\ell-k+j+1) z^{\ell-k+j} f^{(j)}(z). \end{aligned}$$

Claim 3.1 (1) $g(\zeta) = 0 \Rightarrow |g^{(k)}(\zeta)| \le 1$. (2) $g^{(k)}(\zeta) \ne 1$.

Proof Observe that from (3.3), we have that g is an entire function. Suppose that $g(\zeta_0) = 0$. Since $g(\zeta) \not\equiv 0$, there exist $\zeta_n \to \zeta_0$, such that $g_n(\zeta_n) = 0$, and thus $f_n(z_n+\rho_n\zeta_n) = 0$. By assumption, we then have $f_n^{(j)}(z_n+\rho_n\zeta_n) = 0$ and $|f_n^{(k)}(z_n+\rho_n\zeta_n)| < |h(z_n+\rho_n\zeta_n)|$, where $j = 2,3,\cdots,k-1$. Thus $|g_n^{(k)}(\zeta_n)| < |b(z_n+\rho_n\zeta_n)|$. Letting $n\to\infty$, we obtain $|g^{(k)}(\zeta_0)| \le 1$.

If there exists a $\zeta_0 \in \mathbb{C}$, such that $g^{(k)}(\zeta_0) = 1$, then there exists a neighborhood $U = U(\zeta_0)$ of ζ_0, such that the functions $g_n^{(j)}$ are analytic on U for sufficiently large n, $j = 0,1,\cdots,k+1$. Obviously,

$$\begin{aligned} g_n^{(k)}(\zeta) &= F_n^{(k)}(z_n+\rho_n\zeta) \\ &= (z_n+\rho_n\zeta)^\ell f_n^{(k)}(z_n+\rho_n\zeta) \\ &\quad + \sum_{j=0}^{k-1} \binom{k}{j} \ell(\ell-1)\cdots(\ell-k+j+1)(z_n+\rho_n\zeta)^{\ell-k+j} f_n^{(j)}(z_n+\rho_n\zeta). \end{aligned}$$

By Leibniz's formula, we have that

$$\begin{aligned} f_n^{(j)}(z) &= \Big(\frac{F_n(z)}{z^\ell}\Big)^{(j)} = \sum_{s=0}^{j} \binom{j}{s} \rho_n^{k-j+s} g_n^{(j-s)}\Big(\frac{z-z_n}{\rho_n}\Big)\Big(\frac{1}{z^\ell}\Big)^{(s)}, \\ \Big(\frac{1}{z^\ell}\Big)^{(s)} &= \frac{(-1)^s \ell(\ell+1)\cdots(\ell+s-1)}{z^{\ell+s}}. \end{aligned}$$

Since $\frac{\rho_n}{z_n+\rho_n\zeta} \Rightarrow 0$ on $\mathbb{C}$, we have

$$(z_n+\rho_n\zeta)^{\ell-k+j} f_n^{(j)}(z_n+\rho_n\zeta) = \sum_{s=0}^{j} C_s \rho_n^{k-j+s} (z_n+\rho_n\zeta)^{j-k-s} g_n^{(j-s)}(\zeta) \Rightarrow 0,$$

on $\mathbb{C}\backslash\{\text{the poles of } g\}$, where $j = 0, 1, \cdots, k-1$ and C_s are constants. Now

$$\frac{(z_n+\rho_n\zeta)^\ell f_n^{(k)}(z_n+\rho_n\zeta)}{b(z_n+\rho_n\zeta)} \Rightarrow g^{(k)}(\zeta),$$

on $\mathbb{C}\backslash\{\text{the poles of } g\}$.

So $\frac{f_n^{(k)}(z_n+\rho_n\zeta)}{h(z_n+\rho_n\zeta)}$ converges locally and uniformly to $g^{(k)}(\zeta)$ on U. By (3.4), we deduce that $g^{(k)}(\zeta) \not\equiv 1$. Thus there exist $\zeta_n \to \zeta_0$, such that $\frac{f_n^{(k)}(z_n+\rho_n\zeta_n)}{h(z_n+\rho_n\zeta_n)} = 1$ and

$$f_n^{(k)}(z_n+\rho_n\zeta_n) = h(z_n+\rho_n\zeta_n), \tag{3.6}$$

which contradicts the condition (b) of Theorem 1.1. The claim is proved.

Also by Lemma 2.3, we have

$$g(\zeta) = \frac{b}{k!}(\zeta-\zeta_0)^k,$$

where $\zeta_0 \in \mathbb{C}$ and $b \neq 1$ are constants. We have $g^{\#}(0) \leq k(|b|+1)$. By the above claim, since $g(\zeta_0) = 0$, we can get $|g^{(k)}(\zeta_0)| = |b| \leq 1$. However, $g^{\#}(0) = kA+1$, A can be chosen to be any number, such that $A \geq 1$, and here we can assume $A = |b|+1$. Then, we get $g^{\#}(0) < kA+1$, a contradiction.

Case AII

$$\frac{z_n}{\rho_n} \to \alpha \in \mathbb{C}. \tag{3.7}$$

As before, we have $g(\zeta_0) = 0 \Rightarrow |g^{(k)}(\zeta_0)| \leq 1$. Now let

$$G_n(\zeta) = \frac{f_n(\rho_n\zeta)}{\rho_n^{k-\ell}}.$$

From (3.3) and (3.7), we have

$$G_n(\zeta) \Rightarrow G(\zeta) = \frac{g(\zeta-\alpha)}{\zeta^\ell}, \quad \text{on } \mathbb{C}.$$

Indeed,

$$\frac{f_n(\rho_n\zeta)}{\rho_n^{k-\ell}} = \frac{(\rho_n\zeta)^\ell f_n(\rho_n\zeta)}{\rho_n^k}\frac{1}{\zeta^\ell} = \frac{F_n\big(z_n+\rho_n\big(\zeta-\frac{z_n}{\rho_n}\big)\big)}{\rho_n^k}\frac{1}{\zeta^\ell}.$$

Since g has a zero of order at least ℓ at $\zeta = -\alpha$,

$$G(0) \neq \infty. \tag{3.8}$$

We claim that $G^{(k)}(\zeta) \neq \frac{1}{\zeta^\ell}$. Indeed, suppose that $G^{(k)}(\zeta_0) = \frac{1}{\zeta_0^\ell}$. Then G is holomorphic at ζ_0, and

$$G_n^{(k)}(\zeta) - \rho_n^\ell h(\rho_n\zeta) = \rho_n^\ell(f_n^{(k)}(\rho_n\zeta) - h(\rho_n\zeta)) \neq 0.$$

Since $\rho_n^\ell h(\rho_n\zeta) \to \frac{1}{\zeta^\ell}$, we have by Hurwitz's theorem that

$$G^{(k)}(\zeta) \equiv \frac{1}{\zeta^\ell}.$$

Since G is a holomorphic function on $\mathbb{C}$, $G^{(k)}(\zeta) \equiv \frac{1}{\zeta^\ell}$ cannot occur.

Thus, $G^{(k)}(\zeta) \neq \frac{1}{\zeta^\ell}$. Since G is an entire function with the order $\rho_G \leq 1$, $G^{(k)}$ is also an entire function with the order $\rho_{G^{(k)}} \leq 1$. If G is transcendental, then

$$G^{(k)}(\zeta) = \frac{1}{\zeta^\ell} - \frac{\exp(P(\zeta))}{\zeta^\ell},$$

where $P(\zeta) = A\zeta + B,\ A$ and B are two constants. Since $\zeta = 0$ is not the pole of $G^{(k)}(\zeta)$, we have $P(0) = 0$, and then $B = 0$. Thus

$$G^{(k)}(\zeta) = \frac{1 - \exp(A\zeta)}{\zeta^\ell}.$$

Since $\ell \geq k+1$, we also have that $\zeta = 0$ is the pole of $G^{(k)}$, so does G, a contradiction.

Then, G is a polynomial, and so does $G^{(k)}$, which implies

$$G^{(k)}(\zeta) = \frac{1}{\zeta^\ell} + \frac{A}{P(\zeta)},$$

where $A \neq 0$ is a constant and $P(\zeta)$ is a polynomial. Since G has no poles, we have $G^{(k)} \equiv 0$. G is a polynomial with degree at most $k-1$, which contradicts the fact that all zeros of G have multiplicity at least k. Thus G is a constant, and we can assume that $G \equiv c$.

If $c = 0$, then $G \equiv 0$ and $g \equiv 0$, a contradiction.

If $c \neq 0$, then $f_n(0) \to \infty$ (Otherwise, we may assume that $f_n(0)$ are bounded, and then $G_n(0) = \rho_n^{\ell-k} f_n(0) \to 0$, a contradiction).

Since $\{f_n\}$ is not normal at $z = 0$, there exists (after renumbering) a sequence $z_n^* \to 0$, such that

$$f_n(z_n^*) = 0. \tag{3.9}$$

Otherwise, there is some δ' $(0 < \delta' < \delta)$ such that (before renumbering) $f_n(z) \neq 0$ in $\Delta(0, \delta')$. Since $f_n(z) \Rightarrow \infty$ on $\Delta'(0, \delta)$, by the minimum principle, we have that $f_n(z) \Rightarrow \infty$ on $\Delta(0, \delta)$, which is a contradiction to the non-normality of $\{f_n\}$ at $z = 0$. Without loss of generality, we may assume that z_n^* is the zero of f_n of the smallest modulus. Since $G_n(\zeta) = \rho_n^{\ell-k} f_n(\rho_n \zeta) \to c\ (\neq 0)$, $\frac{z_n^*}{\rho_n} \to \infty$. Let $G_n^*(\zeta) = (z_n^*)^{\ell-k} f_n(z_n^* \zeta)$. Obviously, all zeros of G_n^* have multiplicity at least k, and $G_n^* = 0 \Rightarrow |G_n^*| < \left|\frac{b(z_n^*\zeta)}{\zeta^\ell}\right|$. Also, by calculation,

$$G_n^{*\,(k)}(\zeta) = (z_n^*)^\ell f_n(z_n^* \zeta) \neq \frac{b(z_n^* \zeta)}{\zeta^\ell}.$$

Since $G_n^*(\zeta) \neq 0$ on Δ, it follows from Lemma 2.4 that $\{G_n^*\}$ is normal on Δ. By Lemma 2.5, $\{G_n^*\}$ is normal on $\mathbb{C}\backslash\{0\}$. Thus $\{G_n^*\}$ is normal on $\mathbb{C}$. Taking subsequences and renumbering, we may assume that $G_n^* \Rightarrow G^*$ on $\mathbb{C}$, where G^* is holomorphic. But

$$G_n^*(0) = \left(\frac{z_n^*}{\rho_n}\right)^{\ell-k} G_n(0) \to \infty,$$

a contradiction.

Case B Suppose that $1 \leq \ell \leq k$.

If $\mathcal{F}$ is not normal at $z = 0$, then by Lemma 2.1 (with $\alpha = k - \ell$), there exist $f_n \in \mathcal{F},\ z_n \to 0$ and $\rho_n \to 0^+$, such that

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n^{k-\ell}} \Rightarrow g(\zeta)$$

on $\mathbb{C}$, where g is a nonconstant entire function, all of whose zeros have multiplicity at least k. Thus

$$g_n^{(k)}(\zeta) = \rho_n^\ell f_n^{(k)}(z_n + \rho_n \zeta) \neq \left(\frac{\rho_n}{z_n + \rho_n \zeta}\right)^\ell b(z_n + \rho_n \zeta).$$

Again we consider two subcases.

Case BI Suppose that $\frac{z_n}{\rho_n} \to \infty$. Consider

$$\varphi_n(\zeta) = \frac{f_n(z_n + z_n\zeta)}{z_n^{k-\ell}},$$

and then

$$\varphi_n^{(k)}(\zeta) = (z_n)^\ell f_n^{(k)}(z_n + z_n\zeta) \neq (z_n)^\ell \frac{b(z_n + z_n\zeta)}{(z_n + z_n\zeta)^\ell} = \frac{b(z_n + z_n\zeta)}{(1+\zeta)^\ell}.$$

Since all zeros of φ_n have multiplicity at least k and $\varphi_n = 0 \Rightarrow |\varphi_n^{(k)}| < \left|\frac{b(z_n+z_n\zeta)}{(1+\zeta)^\ell}\right|$, by Lemma 2.5, we have that $\{\varphi_n\}$ is normal on Δ. On the other hand, since $g^{(k-\ell)}$ is nonconstant (otherwise, g is a polynomial with degree at most $k-\ell$, which contradicts the fact that all zeros of g have multiplicity at least k), there exist $\zeta_1, \zeta_2 \in \mathbb{C}$, such that $g^{(k-\ell)}(\zeta_1) \neq g^{(k-\ell)}(\zeta_2)$. We have for $i = 1, 2$,

$$\begin{aligned} g^{(k-\ell)}(\zeta_j) &= \lim_{n\to\infty} f_n^{(k-\ell)}(z_n + \rho_n\zeta_j) = \lim_{n\to\infty} f_n^{(k-\ell)}\Big(z_n + z_n\Big(\frac{\rho_n}{z_n}\zeta_j\Big)\Big) \\ &= \lim_{n\to\infty} \varphi_n^{(k-\ell)}\Big(\frac{\rho_n}{z_n}\zeta_j\Big). \end{aligned}$$

Since $\frac{\rho_n}{z_n}\zeta_j \to 0$, as $n \to \infty$, the family $\{\varphi_n^{(k-\ell)}(\zeta)\}$ is not equicontinuous at 0 and hence cannot be normal on Δ, so does $\{\varphi_n(\zeta)\}$, a contradiction.

Case BII Suppose that $\frac{z_n}{\rho_n} \to \alpha$, a finite complex number. Then

$$0 \neq g_n^{(k)}(\zeta) - \Big(\frac{\rho_n}{z_n + \rho_n\zeta}\Big)^\ell b(z_n + \rho_n\zeta) \Rightarrow g^{(k)}(\zeta) - \frac{1}{(\alpha+\zeta)^\ell},$$

on $\mathbb{C}\backslash\{-\alpha\}$. Thus, either $g^{(k)}(\zeta) - \frac{1}{(\alpha+\zeta)^\ell} \equiv 0$ or $g^{(k)}(\zeta) - \frac{1}{(\alpha+\zeta)^\ell} \neq 0$. Since g is an entire function, the first alternative obviously cannot hold. Thus $g^{(k)}(\zeta) \neq \frac{1}{(\alpha+\zeta)^\ell}$. By the similar method of Case AII, we can prove that g is a polynomial. By the fundamental theorem of algebra, it is also impossible. Theorem 1.1 is proved.

Acknowledgement The authors are very grateful to the referees for their valuable suggestions.

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