On Univalence of the Power Deformation $z\left(\frac{f(z)}{z}\right)^c$ *

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Abstract The authors mainly concern the set U_f of $c \in \mathbb{C}$ such that the power deformation $z\left(\frac{f(z)}{z}\right)^c$ is univalent in the unit disk $|z| < 1$ for a given analytic univalent function $f(z) = z + a_2 z^2 + \cdots$ in the unit disk. It is shown that U_f is a compact, polynomially convex subset of the complex plane $\mathbb{C}$ unless f is the identity function. In particular, the interior of U_f is simply connected. This fact enables us to apply various versions of the λ-lemma for the holomorphic family $z\left(\frac{f(z)}{z}\right)^c$ of injections parametrized over the interior of U_f. The necessary or sufficient conditions for U_f to contain 0 or 1 as an interior point are also given.

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1 Introduction

Let $\mathcal{A}$ be the class of analytic functions on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We denote by $\mathcal{A}_0$ its subclass consisting of functions h normalized by $h(0) = 1$. The set $\mathcal{A}_0^\times$ of invertible elements in $\mathcal{A}_0$ with respect to pointwise multiplication is nothing but the set of non-vanishing functions in $\mathcal{A}_0$. For $h \in \mathcal{A}_0^\times$, define $\operatorname{Log} h$ to be the analytic branch of $\log h$ on $\mathbb{D}$ determined by the condition $\operatorname{Log} h(0) = 0$. The set of functions f in $\mathcal{A}$ with the representation $f(z) = zh(z)$ for some $h \in \mathcal{A}_0$ (resp. $h \in \mathcal{A}_0^\times$) will be designated by $\mathcal{A}_1$ (resp. $\mathcal{ZF}$). In other words, $f \in \mathcal{A}$ belongs to $\mathcal{A}_1$ if and only if $f(0) = 0, f'(0) = 1$; and $f \in \mathcal{A}$ belongs to $\mathcal{ZF}$ if and only if $f \in \mathcal{A}_1$ and $f(z) \neq 0$ for $0 < |z| < 1$. We denote by $\mathcal{S}$ the set of univalent functions in $\mathcal{A}_1$. Note that $\mathcal{S}$ is contained in $\mathcal{ZF}$.

In [4], the authors investigated the power deformation

$$K_c[f](z) = z\left(\frac{f(z)}{z}\right)^c$$

for $f \in \mathcal{ZF}$ and $c \in \mathbb{C}$. Here and in what follows, the power h^c will be defined as $\exp(c \operatorname{Log} h)$ for $h \in \mathcal{A}_0^\times$ and $c \in \mathbb{C}$. We determined the sets $[\mathcal{M}, \mathcal{N}]_K = \{c \in \mathbb{C} : K_c[\mathcal{N}] \subset \mathcal{M}\}$ for various subclasses $\mathcal{M}, \mathcal{N}$ of $\mathcal{S}$ in [4]. In this paper, we focus our attention on the set

$$U_f = [\{f\}, \mathcal{S}]_K = \{c \in \mathbb{C} : K_c[f] \text{ is univalent on } \mathbb{D}\}$$

for $f \in \mathcal{ZF}$. For instance, for the Koebe function $\kappa(z) = \frac{z}{(1-z)^2}$, we have $U_\kappa = \{c : |c - \frac{1}{2}| \leq \frac{1}{2}\}$ (see the remark right after the proof of Theorem 1.1 in [4]). By the property $K_c \circ K_{c'} = K_{cc'}$, we have the relation $U_{K_c[f]} = c^{-1} \cdot U_f$ (see [4, Lemma 2.1]).

Note that U_f has an interior point when f is a starlike univalent function or, more generally, a spirallike function (see [4]). We are motivated, in part, by the fact that the interior $\operatorname{Int} U_f$ serves as a parameter region of the holomorphic family $K_c[f]$ of injections on $\mathbb{D}$. Therefore, we could relate the present study to the theory of quasiconformal mappings and Teichmüller spaces.

We recall here the notion of holomorphic motions. A holomorphic motion of a subset E of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ over a domain D with a base point c_0 is a map $F : D \times E \to \widehat{\mathbb{C}}$ with the following three properties:

(1) $F(c, \cdot) : E \to \widehat{\mathbb{C}}$ is injective for each $c \in D$.

(2) $F(\cdot, z) : D \to \widehat{\mathbb{C}}$ is holomorphic for each $z \in E$.

(3) $F(c_0, z) = z$ for $z \in E$.

This simple notion appeared only recently in a paper by Mañé, Sad and Sullivan [5] to study complex dynamics, and afterwards, it found many applications in various branches of complex analysis. We summarize necessary results concerning holomorphic motions in Section 3.

In the present paper, we will show the following theorems.

Theorem 1.1 *Suppose that $f \in \mathcal{ZF}$ is not the identity function. Then U_f is a compact, polynomially convex set in $\mathbb{C}$ with $0 \in U_f$.*

Note that $U_f = \mathbb{C}$ when f is the identity function. We recall here that a compact set E in $\mathbb{C}$ is polynomially convex if and only if $\mathbb{C} \setminus E$ is connected (see [3, Chapter VII, Proposition 5.3] for instance). The latter condition is also known as a characterization of the Runge property in dimension one. In particular, we see that each connected component of the interior $\operatorname{Int} U_f$ is simply connected.

Theorem 1.2 *Let D be a connected component of $\operatorname{Int} U_f$ for a non-identity function $f \in \mathcal{ZF}$. Then the family of univalent functions $f_c = K_c[f]$ over $c \in D$ is quasiconformally homogeneous in $\mathbb{C}$: more precisely, for each pair of points c_0 and c_1 in D, there exists a $\tanh d_D(c_0, c_1)$-quasiconformal automorphism g of $\mathbb{C}$, such that $f_{c_1} = g \circ f_{c_0}$ on $\mathbb{D}$.*

Here, d_D denotes the hyperbolic distance in D induced by the hyperbolic metric of constant curvature -4. For instance, $d_{\mathbb{D}}(0, z) = \frac{1}{2} \log \frac{1+|z|}{1-|z|} = \operatorname{arctanh}|z|$. A mapping $g : D_1 \to D_2$ between domains D_1 and D_2 in $\widehat{\mathbb{C}}$ is called k-quasiconformal, if g is a homeomorphism with locally integrable partial derivatives on $D_1 \setminus \{\infty, g^{-1}(\infty)\}$, such that $|\partial_{\bar{z}} g| \leq k|\partial_z g|$ a.e. in D_1 for a constant $0 \leq k < 1$.

The quasiconformal homogeneity in $\mathbb{C}$ implies, for instance, that f_{c_1} is bounded on $\mathbb{D}$ precisely when so is f_{c_0} for $c_0, c_1 \in D$.

A key step to prove the last theorem is the fact that $F(c, z) = f_c(z)$ satisfies conditions (1) and (2) in the definition of holomorphic motions. We note that condition (3) is also satisfied when $0 \in D \subset \operatorname{Int} U_f$. However, there is no guarantee that the holomorphic family f_c of injections over D contains f itself. This happens when $1 \in D \subset \operatorname{Int} U_f$. If $\{0, 1\} \subset D$, then f is a quasiconformal deformation of the identity mapping, and therefore f extends to a quasiconformal automorphism of $\mathbb{C}$. Here we have conditions for these situations.

Theorem 1.3 *Let f be a non-identity function in $\mathcal{ZF}$.*

(1) $0 \in \operatorname{Int} U_f$ if and only if the function $\frac{zf'(z)}{f(z)}$ is bounded on $\mathbb{D}$.

(2) Suppose that $1 \in \operatorname{Int} U_f$. Then $f \in \mathcal{S}$ and the function $\frac{zf'(z)}{f(z)}$ is bounded away from 0 on $\mathbb{D}$.

The converse is not true in general in the second assertion of the last theorem (see Lemma 2.1 and Example 2.1 below). Though we have not found so far a sufficient condition that is general enough, we have several geometric conditions for f to have the property $1 \in \operatorname{Int} U_f$. For instance, it suffices to assume that f is starlike and is of order α for some $\alpha > 0$. See [4] for details.

We note here that $\operatorname{Int} U_f$ might be empty. On the other hand, $\operatorname{Int} U_f$ may have many components. We will show the following result.

Theorem 1.4 *There does exist a function $f \in \mathcal{S}$, such that $\operatorname{Int} U_f$ consists of at least two connected components.*

We briefly describe the organization of the present note. In Section 2, we prove Theorems 1.1 and 1.3. There, key ingredients are an idea of Žuravlev [10] and a fundamental relation in (2.3) (see also [4]) between a set LU_f containing U_f and the variability region $V(f)$ of $\frac{zf'(z)}{f(z)}$. Section 3 is a short section giving a version of the λ-lemma and a proof of Theorem 1.2. In Section 4, we prove Theorem 1.4 and give a couple of related results. To prove the theorem, we prepare a univalence criterion (see Lemma 4.1), which may be of independent interest.

2 Proofs of Theorems 1.1 and 1.3

Univalence is a global property of a function so that it is not easy to check. Therefore, it is helpful to consider local univalence instead as in [4]. Recall that an analytic function f is locally univalent at z_0 if and only if $f'(z_0) \neq 0$. For a function f in $\mathcal{ZF}$, we set

$$LU_f = \{c \in \mathbb{C} : K_c[f] \text{ is locally univalent on } \mathbb{D}\}.$$

Obviously, $U_f \subset LU_f$.

We now set $f_c = K_c[f]$ for brevity. A simple computation gives us the relation

$$\frac{zf_c'(z)}{f_c(z)} = 1 - c + c\frac{zf'(z)}{f(z)}. \tag{2.1}$$

Hence, for a point $z_0 \in \mathbb{D}$, $f_c'(z_0) = 0$ if and only if $\frac{z_0 f'(z_0)}{f(z_0)} = \frac{c-1}{c}$, equivalently, $c = T\left(\frac{z_0 f'(z_0)}{f(z_0)}\right)$, where

$$T(w) = \frac{1}{1 - w}. \tag{2.2}$$

In this way, we have the fundamental relation

$$LU_f = \mathbb{C} \setminus T(V(f)), \tag{2.3}$$

where $V(f)$ is the image of $\mathbb{D}$ under the function $\frac{zf'(z)}{f(z)}$, namely

$$V(f) = \left\{\frac{zf'(z)}{f(z)} : z \in \mathbb{D}\right\}.$$

We need to recall the Grunsky theorem to prove polynomial convexity of U_f. The reader may refer to [7] for details. The Grunsky coefficients b_{jk} of $f \in \mathcal{A}_1$ are defined by expansion in the form

$$\log \frac{\frac{1}{f(z)} - \frac{1}{f(w)}}{\frac{1}{z} - \frac{1}{w}} = -\sum_{j,k=1}^{\infty} b_{jk} z^j w^k$$

of double power series convergent in $|z| < \delta$ and $|w| < \delta$ for small enough $\delta > 0$. Indeed, we can take ρ as δ, when $f(z)$ is univalent on the disk $|z| < \rho$. The Grunsky theorem says that f is univalent on $\mathbb{D}$ if and only if

$$\left| \sum_{j,k=1}^{N} b_{jk} x_j x_k \right| \leq \sum_{j=1}^{N} \frac{|x_j|^2}{j}$$

for any positive integer N and any vector $(x_1, \cdots, x_N) \in \mathbb{C}^N$.

We are now ready to prove Theorems 1.1 and 1.3.

Proof of Theorem 1.1 Let $f \in \mathcal{ZF}$ be a non-identity function. Then $\frac{zf'(z)}{f(z)}$ is not constant (and is thus an open mapping). Therefore, $V(f)$ is an open neighbourhood of 1, which implies that $T(V(f))$ is an open neighbourhood of ∞. Now the relation (2.3) yields that LU_f is a compact subset of $\mathbb{C}$. Since $U_f \subset LU_f$, we conclude that U_f is bounded. The Hurwitz theorem implies that U_f is closed. Hence, U_f is compact.

We next show that U_f is polynomially convex by employing the idea of Žuravlev [10]. Suppose, to the contrary, that $\mathbb{C} \setminus U_f$ has a bounded component Δ. Then we note that $\partial \Delta \subset U_f$ and $\Delta \cap U_f = \varnothing$. We denote by $b_{jk}(c)$ the Grunsky coefficients of the function $f_c = K_c[f]$. Then $b_{jk}(c)$ is a holomorphic function in c for each pair of j and k. (Indeed, it is not difficult to see that $b_{jk}(c)$ is a polynomial in c.) By the Grunsky theorem, for each $(x_1, \cdots, x_N) \in \mathbb{C}^N$, the inequality

$$\left| \sum_{j,k=1}^{N} b_{jk}(c) x_j x_k \right| \leq \sum_{j=1}^{N} \frac{|x_j|^2}{j} \tag{2.4}$$

holds for $c \in \partial \Delta \subset U_f$. By the maximum modulus principle for analytic functions, we see that the inequality (2.4) still holds for all $c \in \Delta$. Therefore, by the converse part of the Grunsky theorem, we conclude that f_c is univalent for $c \in \Delta$. This means that $\Delta \subset U_f$, which is a contradiction.

The assertion $0 \in U_f$ is trivial. The proof is now completed.

Proof of Theorem 1.3 Let $f \in \mathcal{ZF}$ be a non-identity function. First, we assume that $0 \in \operatorname{Int} U_f$. Then, $0 \in \operatorname{Int} LU_f$, which implies that 0 is an exterior point of $T(V(f))$. Since $T(\infty) = 0$, we have that $V(f)$ is bounded.

Conversely, we assume that $\frac{zf'(z)}{f(z)}$ is bounded. Then, by (2.1), the range of $\frac{zf_c'(z)}{f_c(z)}$ shrinks to the point 1, when c approaches 0. In particular, $\operatorname{Re}\left[\frac{zf_c'(z)}{f_c(z)}\right] > 0$ for the sufficiently small c. In this case, f_c is a starlike univalent function. Therefore, a neighbourhood of 0 is contained in U_f. Thus the first part of the theorem is confirmed.

Finally, we assume that $1 \in \operatorname{Int} U_f$. Then $1 \in U_f$, namely, f is univalent. Since $U_f \subset LU_f$, it is enough to show the following lemma to complete the proof.

Lemma 2.1 *For a function $f \in \mathcal{ZF}$, $1 \in \text{Int } LU_f$ if and only if $\frac{f(z)}{zf'(z)}$ is bounded.*

Proof In view of the relation (2.3), we see that $1 \in \text{Int } LU_f$ precisely when 1 is an exterior point of $T(V(f))$. Since $T(0) = 1$, the last condition means that 0 is an exterior point of $V(f)$, namely, $\frac{zf'(z)}{f(z)}$ is bounded away from 0. Now this proof is completed.

In general, the sets U_f and LU_f are different and the converse of the second half of Theorem 1.3 does not hold as the following example exhibits.

Example 2.1 Let $f(z) = \frac{e^{\pi z}-1}{\pi}$. Then it is easy to check that $f \in \mathcal{S}$. A simple computation gives us that $\frac{f(z)}{zf'(z)} = \frac{e^{-\pi z}-1}{(-\pi z)}$, which is obviously bounded on $\mathbb{D}$. However, $f_c(z) = K_c[f](z) = z^{1-c}(e^{\pi z}-1)^c$ is not univalent for each $c > 1$. Indeed, since $\arg f_c(z) = (1-c)\arg z + c\arg(e^{\pi z} - 1)$ for $c > 0$, one has

$$\arg f_c(\mathrm{i}) = (1 - c)\frac{\pi}{2} + c\pi = \frac{\pi}{2}(1 + c),$$

where $\mathrm{i} = \sqrt{-1}$. In particular, $\arg f_c(\mathrm{i}) > \pi$ for $c > 1$. This implies that $f_c(\mathbb{D}^+)$ intersects the negative real axis $(-\infty, 0)$ for $c > 1$. Here, $\mathbb{D}^+ = \{z \in \mathbb{D} : \text{Im } z > 0\}$. Take a point $z_c \in \mathbb{D}^+$ so that $f_c(z_c) \in (-\infty, 0)$ for $c > 1$. In view of the symmetric property that $\overline{f_c(\overline{z})} = f_c(z)$, one has $f_c(z_c) = f_c(\overline{z_c})$, which implies $c \notin U_f$ for $c > 1$. Therefore, we conclude that $1 \in U_f \setminus \text{Int } U_f$.

3 Proof of Theorem 1.2

In recent years, holomorphic motions have been intensively studied in various contexts, and a number of deep results and interesting applications have been found. Among them, we need the following assertions, which can be found, for instance, in a paper by Astala and Martin [1].

Lemma 3.1 *Let $F : D \times E \to \widehat{\mathbb{C}}$ be a holomorphic motion of $E \subset \widehat{\mathbb{C}}$ over a simply connected domain $D \subset \mathbb{C}$ with the basepoint c_0. Then F extends to a holomorphic motion $\widehat{F}$ of $\widehat{\mathbb{C}}$ over D. Moreover, $\widehat{F}$ is jointly continuous on $D \times \widehat{\mathbb{C}}$ and $\widehat{F}(c, \cdot)$ is $\tanh d_D(c_0, c)$-quasiconformal automorphism of $\widehat{\mathbb{C}}$ for each $c \in D$.*

We are in a position to prove Theorem 1.2.

Proof of Theorem 1.2 Recall that D is a connected component of $\text{Int } U_f$ for a non-identity function $f \in \mathcal{ZF}$. Let $f_c = K_c[f]$ for $c \in D$. By Theorem 1.1, D is simply connected. Fix $c_0 \in D$ and consider the function $F(c, w) = (f_c \circ f_{c_0}^{-1})(w)$ for $c \in D$ and $w \in f_{c_0}(\mathbb{D})$ and $F(c, \infty) = \infty$ for $c \in D$. Then this gives a holomorphic motion of $E = f_{c_0}(\mathbb{D}) \cup \{\infty\}$ over D with the basepoint c_0. We now use the above lemma to obtain an extension $\widehat{F}$ of F to $D \times \widehat{\mathbb{C}}$. Then $g = \widehat{F}(c, \cdot)$ gives a $\tanh d_D(c_0, c)$-quasiconformal automorphism of $\mathbb{C}$, such that $f_c = g \circ f_{c_0}$ on $\mathbb{D}$. Thus this proof is completed.

4 Proof of Theorem 1.4 and Concluding Remarks

In order to construct such an example as in Theorem 1.4, we will make use of the following univalence criterion, which may be of independent interest.

Lemma 4.1 *There exists a positive number m, such that the condition $e^{-m} < |\frac{zf'(z)}{f(z)}| < e^m$ on $|z| < 1$ implies univalence of f on $\mathbb{D}$ for $f \in \mathcal{A}$.*

Proof Fix an arbitrary positive number m and let $\alpha = \frac{2\mathrm{i}m}{\pi}$. Then the function

$$q(z) = \left(\frac{1+z}{1-z}\right)^\alpha$$

is a universal covering projection of $\mathbb{D}$ onto the annulus $\mathrm{e}^{-m} < |w| < \mathrm{e}^m$ with $q(0) = 1$. Therefore, the assumption means that the function $p(z) = \frac{zf'(z)}{f(z)}$ is subordinate to $q(z)$; in other words, $p = q \circ \omega$ for a function $\omega \in \mathcal{A}$ with $|\omega(z)| \le |z|$. The situation is the same as that in the proof of Theorem 1.1 in [9] except for the exponent α, which is assumed to be a positive number there. Thus we follow the argument in [9].

We first observe that the inequality

$$\left|\log\frac{1+z}{1-z}\right| = \left|\sum_{n=1}^\infty \frac{2z^{2n-1}}{2n-1}\right| \le \sum_{n=1}^\infty \frac{2|z|^{2n-1}}{2n-1} = \log\frac{1+|z|}{1-|z|}$$

holds. Hence, letting $w = \log(\frac{1+z}{1-z})$ and $W = \log(\frac{1+|z|}{1-|z|})$, we have

$$|q(z) - 1| = |\mathrm{e}^{\alpha w} - 1| = \left|\sum_{n=1}^\infty \frac{(\alpha w)^n}{n!}\right| \le \sum_{n=1}^\infty \frac{(|\alpha|W)^n}{n!} = \mathrm{e}^{|\alpha|W} - 1 = Q(|z|) - 1,$$

where $Q(z) = \left(\frac{1+z}{1-z}\right)^{|\alpha|}$. By using this, we have the inequality $\left|\frac{f''(z)}{f'(z)}\right| \le \frac{F''(|z|)}{F'(|z|)}$ for $|z| < 1$ in the same way as in [9], where $F \in \mathcal{A}_1$ is defined by the relation $\frac{zF'(z)}{F(z)} = Q(z)$. In particular, we have

$$\sup_{z \in \mathbb{D}}(1 - |z|^2)\left|\frac{f''(z)}{f'(z)}\right| \le \sup_{z \in \mathbb{D}}(1 - |z|^2)\left|\frac{F''(z)}{F'(z)}\right|.$$

The right-hand term is estimated by $6|\alpha|$ from the above (see [9]). Therefore, when $m \le \frac{\pi}{12}$, we have

$$\sup_{z \in \mathbb{D}}(1 - |z|^2)\left|\frac{f''(z)}{f'(z)}\right| \le 6|\alpha| = \frac{12}{\pi}m \le 1.$$

Becker's theorem (see [2]) now implies univalence of f. Thus the lemma is proved with the choice $m = \frac{\pi}{12}$.

We made a crude estimate above. Therefore, $\frac{\pi}{12}$ is not the sharp constant. As we will see below, m cannot be taken so that $m > \frac{\pi}{2}$. It may be an interesting problem to find (or to estimate) the best possible value of m in the lemma. Since the problem is out of our scope in this note, we will treat this problem in a separate paper.

We now prove Theorem 1.4.

Proof of Theorem 1.4 Let m be the number which appears in Lemma 4.1. Let $f \in \mathcal{A}_1$ be the function determined by the relation $\frac{zf'(z)}{f(z)} = \left(\frac{1+z}{1-z}\right)^{\frac{\mathrm{i}m}{\pi}}$. Then

$$V(f) = \{w \in \mathbb{C} : \mathrm{e}^{-\frac{m}{2}} < |w| < \mathrm{e}^{\frac{m}{2}}\}.$$

Since $V(f)$ separates 0 from ∞, the image $T(V(f))$ under the Möbius transformation T given in (2.2) separates 1 from 0. Since $T(V(f)) = \widehat{\mathbb{C}} \setminus LU_f \subset \widehat{\mathbb{C}} \setminus U_f$ by (2.3), it is enough to see that $\{0, 1\} \subset \operatorname{Int} U_f$ to obtain a desired example. The first assertion of Theorem 1.3 implies $0 \in \operatorname{Int} U_f$ because $\frac{zf'(z)}{f(z)}$ is bounded. On the other hand, in view of (2.1), the range of $\frac{zf_c'(z)}{f_c(z)}$ stays in the annulus $\mathrm{e}^{-m} < |w| < \mathrm{e}^m$ for c close enough to 1, where $f_c = K_c[f]$ for $c \in \mathbb{C}$.

Therefore, Lemma 4.1 implies that f_c is univalent on $\mathbb{D}$ when $|c - 1|$ is small enough. This means that $1 \in \operatorname{Int} U_f$. Now the program of the proof is completed.

The same technique used in the above proof yields the following proposition.

Proposition 4.1 *Let A be a compact, polynomially convex subset of $\mathbb{C}$ with $0 \in A$. Then there exists a function f in $\mathcal{ZF}$, such that $LU_f = A$.*

Proof Let $\Omega = T^{-1}(\widehat{\mathbb{C}} \setminus A)$. Then the polynomial convexity of A implies that Ω is a domain (a connected non-empty open set) in $\widehat{\mathbb{C}}$. Note also that $1 \in \Omega \subset \mathbb{C}$. Suppose first that A consists of at most two points, namely, $A = \{0\}$ or $A = \{0, T(a)\}$ for some $a \neq 1, \infty$. Choose $f \in \mathcal{ZF}$, so that $\frac{zf'(z)}{f(z)} = (\frac{2z}{1-z})^2$ or $(1-a)(\frac{1+z}{1-z})^3 + a$. Then $V(f) = \Omega$ and thus $LU_f = A$ by (2.3). We now assume that A contains at least three points. Then, thanks to the uniformization theorem, we can take a holomorphic universal covering projection p of $\mathbb{D}$ onto Ω with $p(0) = 1$. If we take $f \in \mathcal{ZF}$ so that $\frac{zf'(z)}{f(z)} = p(z)$, we have $V(f) = \Omega$. Therefore, $LU_f = \mathbb{C} \setminus T(V(f)) = A$ by (2.3).

We mention necessary conditions for univalence of f in terms of its power deformations. Prawitz [8] extended Gronwall's area theorem in the following way (see also [6]). Let $F(\zeta) = \zeta + b_0 + \frac{b_1}{\zeta} + \frac{b_2}{\zeta^2} + \cdots$ be a non-vanishing univalent meromorphic function in $|\zeta| > 1$. When

$$\left(\frac{F(\zeta)}{\zeta}\right)^\lambda = \sum_{n=0}^{\infty} D_n(\lambda)\zeta^{-n}, \quad |\zeta| > 1,$$

the inequality

$$\sum_{n=0}^{\infty}(\lambda - n)|D_n(\lambda)|^2 = \lambda + \sum_{n=1}^{\infty}(\lambda - n)|D_n(\lambda)|^2 \geq 0$$

holds for each $\lambda > 0$. This result can be translated into our setting in a simple way.

Theorem 4.1 (A Variant of Prawitz's Area Theorem) *Let $f \in \mathcal{S}$ and $K_c[f](z) = z + \sum_{n=2}^{\infty} a_n(c)z^n$ for $c \in \mathbb{C}$. Then the inequality*

$$\sum_{n=2}^{\infty}(n - 1 - \lambda)|a_n(-\lambda)|^2 \leq \lambda$$

holds for each $\lambda > 0$.

Proof Let $F(\zeta) = \frac{1}{f(\frac{1}{\zeta})}$. We note that

$$\left(\frac{F(\zeta)}{\zeta}\right)^\lambda = \left(\frac{f(\frac{1}{\zeta})}{\frac{1}{\zeta}}\right)^{-\lambda} = 1 + \sum_{n=1}^{\infty} a_{n+1}(-\lambda)\zeta^{-n}.$$

Prawitz's area theorem now yields the required inequality.

We note that the coefficient $a_n(c)$ is a polynomial in c for each n. (This is true for a general $f \in \mathcal{ZF}$.) For instance, when $\frac{zf'(z)}{f(z)} = \left(\frac{1+z}{1-z}\right)^\alpha$, $\alpha = \frac{2\mathrm{i}m}{\pi}$, we have $f_c(z) = z + 2c\alpha z^2 + c(1 + 2c)\alpha^2 z^3 + \cdots$. We now suppose that f is univalent in $\mathbb{D}$. Taking only $a_2(-\lambda)$-term in the last theorem, we obtain the inequality

$$4\lambda^2(1 - \lambda)|\alpha|^2 \leq \lambda,$$

which is equivalent to $4\lambda(1-\lambda)|\alpha|^2 \leq 1$. Letting $\lambda = \frac{1}{2}$, we have $|\alpha| \leq 1$. In this way, we showed that $m \leq \frac{\pi}{2}$ is necessary for univalence of f. We could improve this upper bound if we increase the number of terms in the summation with the cost of calculation amount.

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