Chin. Ann. Math.
33B(6), 2012, 831–840
DOI: 10.1007/s11401-012-0749-5

Chinese Annals of Mathematics, Series B

Numerical Algorithm Based on Quintic Nonpolynomial Spline for Solving Third-Order Boundary Value Problems Associated with Draining and Coating Flows

Pankaj Kumar SRIVASTAVA[1] Manoj KUMAR[2]

Abstract A numerical algorithm is developed for the approximation of the solution to certain boundary value problems involving the third-order ordinary differential equation associated with draining and coating flows. The authors show that the approximate solutions obtained by the numerical algorithm developed by using nonpolynomial quintic spline functions are better than those produced by other spline and domain decomposition methods. The algorithm is tested on two problems associated with draining and coating flows to demonstrate the practical usefulness of the approach.

Keywords Third-order boundary value problem, Spline functions, Nonpolynomial quartic spline, Nonpolynomial quintic spline, Draining and coating flows
2000 MR Subject Classification 65D07, 65L10

1 Introduction

In recent years, the third-order boundary value problems have been considered in many ways [1–3]. In this paper, nonpolynomial quintic spline functions are used to obtain a numerical solution to the third-order two-point boundary value problems associated with draining and coating flows of the following form:

$$y''' = f(x, y), \quad x \in [a, b] \tag{1.1}$$

with the boundary conditions

$$y(a) - A_1 = y^{(1)}(a) - A_2 = y^{(1)}(b) - A_3 = 0, \tag{1.2}$$

where A_i $(i = 1, 2, 3)$ are finite real constants.

To find the approximate solution to this type of problem, a few other methods were developed, and El-Danaf [4] solved this by using the quartic nonpolynomial spline. Al Said et al. [5] solved a system of third-order two-point boundary value problems of the type

$$u''' = \begin{cases} f(x), & a \le x \le c, \\ p(x)u(x) + f(x) + r, & c \le \mathrm{x} \le d, \\ f(x), & d \le \mathrm{x} \le b \end{cases} \tag{1.3}$$

Manuscript received November 29, 2011. Revised July 23, 2012.
[1]Department of Mathematics, Jaypee Institute of Information Technology, Noida, Uttar Pradesh 201301, India. E-mail: pankaj.srivastava@jiit.ac.in
[2]Department of Mathematics, Motilal Nehru National Institute of Technology, Allahabad, Uttar Pradesh 211004, India. E-mail: manoj@mnnit.ac.in

with the boundary conditions

$$u(a) = \alpha, \quad u'(a) = \beta_1, \quad u'(b) = \beta_2, \tag{1.4}$$

and the continuity conditions of u, u' and u'' at c and d. Here, f and p are continuous functions on $[a, b]$ and $[c, d]$, respectively. The parameters r, α, β_1 and β_2 are real finite constants.

In [5], Al Said et al. used cubic spline functions to derive some consistency relations which are then used to develop a numerical technique for solving the problem (1.3). This method is of order two, and gives better numerical results than those produced by the third-order quintic spline collocation method discussed by Noor [6]. Finally, this paper is devoted to the convergence analysis of the method.

Noor et al. [7] generated the second order method based on quartic splines. They solved the system of the third-order boundary value problem of the type (1.3) with the boundary conditions (1.4). In [7], the authors developed a new numerical method of solving a system of type (1.3) associated with third-order obstacle problems by using the quartic splines. They used the penalty function technique to show that the obstacle problems can be characterized by a system of differential equations. Further, they developed the numerical method of solving third-order obstacle boundary value problems.

Khan et al. [8] solved a third-order linear or non-linear boundary value problem of the type

$$y'''(x) = f(x, y), \quad a \le x \le b \tag{1.5}$$

with

$$y(a) = k_1, \quad y'(a) = k_2, \quad y(b) = k_3, \tag{1.6}$$

by deriving a fourth-order method with the use of quintic splines. They presented the formulation of their method for third-order linear and non-linear BVPs. To retain the pentadiagonal structure of the coefficient matrix, they derived fourth-order boundary equations. In this paper, the methods discussed are tested by the authors on two problems from [9], and absolute errors of the analytical solutions are calculated. The results confirm the theoretical analysis of the methods. For the sake of comparisons, the authors also tabulated the results of the method of Caglar [10]. All computations are performed in double-precision arithmetic to reduce the round-off errors to a minimum.

Ul-Islam et al. [11] solved the system of the third-order boundary value problem of the type (1.3). In [11], the authors used non-polynomial spline functions that have a polynomial and trigonometric part to develop a new numerical method for obtaining smooth approximations of the solution to such a system of third-order differential equations. The new method is of order two for arbitrary α and β, such that $\alpha + \beta = \frac{1}{2}$. The spline function proposed in this paper has the form

$$T_n = \text{Span}\{1, x, x^2, \cos(kx), \sin(kx)\},$$

where k is the frequency of the trigonometric part of the splines function, which can be real or purely imaginary and will be used to raise the accuracy of the method. In this paper, they developed the new non-polynomial spline method solving (1.3).

Ul-Islam et al. [12] solved (1.3) by applying quartic non-polynomial spline functions

$$T_n = \text{Span}\{1, x, x^2, \cos(kx), \sin(kx)\},$$

where k is the frequency of the trigonometric part of the spline function, consisting of polynomial and trigonometric parts. This new method is of order two for arbitrary α and β, such that $\alpha + \beta = \frac{1}{2}$. Frequency k of the trigonometric part can be real or purely imaginary and will be used to raise the accuracy of the method.

The above fact is evident when the correlation between the basis of polynomial splines and the basis of non-polynomial splines is investigated in the following manner:

$$\begin{aligned} T_n &= \mathrm{Span}\{1, x, x^2, \cos(kx), \sin(kx)\} \\ &= \Big\{1, x, x^2, \frac{6}{k^3}(kx - \sin(kx)), \frac{24}{k^4}\Big(\cos(kx) - 1 + \frac{(kx)^2}{2}\Big)\Big\}. \end{aligned} \tag{1.7}$$

From (1.7), it follows that

$$\lim_{k\to 0} T_4 = \{1, x, x^2, x^3, x^4\}.$$

The main idea is to use the conditions of continuity to get the recurrence relation for the system (1.3)–(1.4).

Gao et al. [13] solved the system of the third-order obstacle boundary value problem by finding u, such that

$$\begin{cases} -u''' \geq f, & \text{on } \Omega = [0,1], \\ u \geq \psi, & \text{on } \Omega = [0,1], \\ [-u''' - f][u - \psi] = 0, & \text{on } \Omega = [0,1], \\ u(0) = 0, \quad u'(0) = 0, \quad u'(1) = 0, \end{cases} \tag{1.8}$$

where $f(x)$ is a continuous function, and $\varphi(x)$ is the obstacle function. The authors studied the problem (1.8) in the frame work of the general variational inequality. By using the technique of [14], the problem (1.8) can be finally reduced to the following system of third-order differential equations:

$$u''' = \begin{cases} f & \text{for } 0 \leq x \leq \dfrac{1}{4} \text{ and } \dfrac{3}{4} \leq x \leq 1, \\ u + f - 1 & \text{for } \dfrac{1}{4} \leq x \leq \dfrac{3}{4} \end{cases} \tag{1.9}$$

with the boundary conditions

$$u(0) = 0, \quad u'(0) = 0, \quad u'(1) = 0, \tag{1.10}$$

and the continuity conditions of u, u' and u''' at $x = \frac{1}{4}$ and $\frac{3}{4}$.

In this way, in [13], the authors first transformed the concerned boundary problems into initial value problems, and then used quartic B-splines to solve them.

In the present paper, our main objective is to use the non-polynomial quintic spline functions (see [15–18]) with a polynomial and trigonometric part to develop a new numerical method for obtaining smooth approximations of the solution to the third-order differential equations of the system with the form (1.1) associated with draining and coating flows. Here, algorithms are developed and the approximate solutions obtained by these algorithms are compared with the solutions obtained by the nonpolynomial quartic spline (see [4]) and the polynomial quintic spline (see [8]). This paper is organized as follows. In Section 2, we discuss physical description of the differential equations based on draining and coating flows. In Section 3, we give a brief introduction of the nonpolynomial quintic spline. In Section 4, we give a brief derivation of this non-polynomial quintic spline. We present the spline relations used for the discretization of the given system (1.1). In Section 5, we present our numerical method for a system of third-order boundary value problems, and the development of boundary conditions, the truncation error and class of the method are discussed. In Section 6, numerical evidences are included to compare and demonstrate the efficiency of the methods, and we show that our algorithm performs better than nonpolynomial quartic spline methods and polynomial quintic spline methods. Finally, in Section 7, we conclude this paper with some remarks.

2 Physical Description of the Third-Order Differential Equations for Draining and Coating Flows

The third-order ordinary differential equation is a model for a viscous fluid draining over a wet surface for a thin film flowing on an inclined plane with an opening at the bottom of the plane (see [1–2]). In the study of draining and coating flows, the thin films flowing on an inclined plane with an opening (a gap) at the bottom of the plane, representing an outlet, can be modeled as a third-order ordinary differential equation (see [19]).

The film is subject to the gravitational force and a constant surface shear stress at the film surface. Such a problem can be regarded as a special draining-type problem, in which there is a flow bifurcation from one side of the domain (inlet flux) to the outlet and the other side of the domain. The thin film flow is modeled as being governed by the two-dimensional Stokes system of equations, and an investigation for the effects of surface tension, the surface shear stress, and the geometry variation on the film flow into an outlet is attempted. In the present paper, we have considered the problem of draining and coating flows, which can be modeled as the third-order ordinary differential equation of the type (1.1) with the boundary condition (1.2).

3 Nonpolynomial Quintic Spline

A polynomial quintic spline function (generally, called a quintic spline function) $S_\triangle(x)$, interpolating to a function $u(x)$ defined on $[a, b]$, is such that

(i) In each subinterval $[x_{j-1}, x_j]$, $S_\triangle(x)$ is a polynomial of degree at most five.

(ii) The first, second, third and fourth derivatives of $S_\triangle(x)$ are continuous on $[a, b]$.

To deal effectively with such problems, we introduce spline functions containing a parameter τ. These are non-polynomial splines defined by the solution to a differential equation in each subinterval. Arbitrary constants are being chosen to satisfy certain smoothness conditions at the joints. These splines belong to the class C^2 and reduce to polynomial splines as the parameter $\tau \to 0$. The exact form of the spline depends upon the manner in which the parameter is introduced. We have studied parametric spline functions: the spline under compression, the spline under tension and the adaptive spline. A number of spline relations have been obtained for subsequent use.

A function $S_\triangle(x, \tau)$ of the class $C^4[a, b]$, which interpolates $u(x)$ at the mesh points $\{x_j\}$, depends on a parameter τ, and reduces to an ordinary quintic spline $S_\triangle(x)$ in $[a, b]$ as $\tau \to 0$, is termed as a parametric quintic spline function. The three parametric quintic splines derived from the quintic spline by introducing the parameter in three different ways are termed as the parametric quintic spline-I, the parametric quintic spline-II and the adaptive quintic spline, respectively.

The spline function proposed in this paper has the following forms:

$$\begin{aligned}&\text{span}\,\{1, x, x^2, x^3, \sin|\tau|x, \cos|\tau|x\}\\&\text{span}\,\{1, x, x^2, x^3, \sinh|\tau|x, \cosh|\tau|x\},\\&\text{span}\,\{1, x, x^2, x^3, x^4, x^5\}.\end{aligned}$$

with $\tau = 0$.

The above fact is evident, when the correlation between the basis of polynomial splines and the basis of non-polynomial splines is investigated in the following manner:

$$\begin{aligned}T_5 &= \text{span}\{1, x, x^2, x^3, \sin(\tau x), \cos(\tau x)\},\\&= \text{span}\Big\{1, x, x^2, x^3, \frac{24}{\tau^4}\Big(\cos(\tau x) - 1 + \frac{(\tau x)^2}{2}\Big),\ \frac{120}{\tau^5}\Big(\sin(\tau x) - (\tau x) + \frac{(\tau x)^3}{6}\Big)\Big\},\end{aligned}$$

From the above equation, it follows that $\lim\limits_{\tau\to 0} T_5 = \{1, x, x^2, x^3, x^4, x^5\}$, where τ is the frequency of the trigonometric part of the splines function, which can be real or purely imaginary and will be used to raise the accuracy of the method. This approach has the advantage over finite difference methods, and provides continuous approximation not only for $y(x)$, but also for y', y'' and higher derivatives at every point of the range of integration. Also, the C^∞-differentiability of the trigonometric part of non-polynomial splines compensates for the loss of smoothness inherited by polynomial splines in this paper.

4 Development of the Main Recurrence Relation (Derivation of the Method)

In order to develop the numerical method for approximating solution to the differential equation (1.1), we consider a uniform mesh $\triangle$ with nodal points x_i on $[a, b]$, such that

$$\begin{aligned}\triangle:\quad & a = x_0 < x_1 < x_2 < x_3 \cdots < x_N = b,\\& x_i = a + ih, \quad i = 0, 1, 2, \cdots, N,\end{aligned}$$

where $h = \frac{b-a}{N}$.

Let us consider a nonpolynomial quintic spline function $S_\triangle(x)$ of class $C^4[a, b]$, which interpolates $y(x)$ at the mesh points x_i $(i = 0, 1, 2, \cdots, N)$, depends on a parameter τ and reduces to an ordinary quintic spline $S_\triangle(x)$ in $[a, b]$ as $\tau \to 0$.

For each segment $[x_i,\ x_{i+1}]$ $(i = 0, 1, 2, \cdots, N-1)$, the non-polynomial $S_\triangle(x)$ defined by

$$\begin{aligned}S_\triangle(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3&\\ + e_i \sin(\tau(x - x_i)) + f_i \cos(\tau(x - x_i)),&\quad i = 0, 1, 2, \cdots, N-1,\end{aligned}\tag{4.1}$$

where a_i, b_i, c_i, d_i, e_i and f_i are constants and τ is an arbitrary parameter.

Let y_i be an approximation to $y(x_i)$, obtained by the segment $S_\triangle(x)$ of the mixed splines function passing through the points $(x_i,\ y_i)$ and $(x_{i+1},\ y_{i+1})$. To obtain the necessary conditions for the coefficients introduced in (4.1), we require not only that $S_\triangle(x)$ satisfy interpolatory conditions at x_i and x_{i+1}, but also that the continuity of the first, second and third derivatives at the common nodes (x_i, y_i) be fulfilled.

To derive the expressions for the coefficients of (4.1) in terms of y_i, y_{i+1}, D_i, D_{i+1}, T_i, T_{i+1}, F_i and F_{i+1}, we first denote

$$\begin{aligned} S_\triangle(x_i) &= y_i, \quad & S_\triangle(x_{i+1}) &= y_{i+1}, \\ S'_\triangle(x_i) &= D_i, \quad & S'_\triangle(x_{i+1}) &= D_{i+1}, \\ S^{(3)}_\triangle(x_i) &= T_i, \quad & S^{(3)}_\triangle(x_{i+1}) &= T_{i+1}, \\ S^{(4)}_\triangle(x_i) &= F_i, \quad & S^{(4)}_\triangle(x_{i+1}) &= F_{i+1}. \end{aligned} \tag{4.2}$$

From algebraic manipulation, we get the following expressions:

$$\begin{aligned} a_i &= y_i - \frac{F_i}{\tau^4}, \\ b_i &= D_i - \frac{F_{i+1} - F_i \cos\theta}{\tau^3 \sin\theta}, \\ c_i &= \frac{y_{i-1} - 2y_i + y_{i+1}}{2h^2}, \\ d_i &= \frac{\tau[T_{i+1} - T_i \cos\theta] - F_i \sin\ \theta}{6(1-\cos\theta)}, \\ e_i &= \frac{F_{i+1} - F_i \cos\theta}{\tau^4 \sin\theta}, \\ f_i &= \frac{F_i}{\tau^4}, \end{aligned} \tag{4.3}$$

where $\theta = \tau h$ and $i = 0, 1, 2, \cdots, N-1$. The proposed differential equation

$$y^{(3)} + f(x)y = g(x), \quad x \in [a, b]$$

with boundary conditions (1.2), may be discretized by

$$T_i + f_i y_i = g_i, \quad 0 < x < 1. \tag{4.4}$$

Using the continuity of the first and third derivatives at (x_i, y_i), that is, $S'_{\triangle-i}(x_i) = S'_{\triangle i}(x_i)$ and $S'''_{\triangle-i}(x_i) = S'''_{\triangle i}(x_i)$, we obtain the following relations:

$$\begin{aligned} (\alpha+\beta)\frac{y_{i-2} - 3y_{i-1} + 3y_i - y_{i+1}}{h^3} &= \frac{1}{h^2}[h\ F_{i-1} - T_{i-1} - T_i] - [\alpha T_i - \beta T_{i-1}], \\ T_{i-1} - 2T_i + T_{i+1} &= h\ (\alpha+\beta)[F_{i+1} - F_{i-1}], \quad i = 1, \cdots, N-1. \end{aligned} \tag{4.5}$$

The operator Λ is defined by

$$\Lambda w_i = p(w_{i+2} + w_{i-2}) + q(w_{i+1} + w_{i-1}) + s w_i$$

for any function w evaluated at the mesh points. Then, we have the following relations connecting y and its derivatives:

$$\Lambda T_i = \frac{1}{h^3}[(\alpha+\beta)(y_{i+2} + y_{i-2}) + (2\alpha - 4\beta)(y_{i+1} - y_{i-1})], \tag{4.6a}$$

$$\Lambda F_i = \frac{1}{h^4}\delta^4 y_i, \tag{4.6b}$$

where

$$p=\alpha_1+\frac{\alpha}{6},\quad q=2\Big[\frac{1}{6}(2\alpha+\beta)-(\alpha_1-\beta_1)\Big],$$
$$s=2\Big[\frac{1}{6}(\alpha+4\beta)+(\alpha_1-2\beta_1)\Big],$$
$$\alpha=\Big(\frac{1}{\theta^2}\Big)(\theta\ \csc\theta-1),\quad \beta=\Big(\frac{1}{\theta^2}\Big)(1-\theta\ \cot\theta),$$
$$\alpha_1=\frac{1}{\theta^2}\Big(\frac{1}{6}-\alpha\Big),\quad \beta_1=\frac{1}{\theta^2}\Big(\frac{1}{3}-\beta\Big),$$
$$\theta=\tau h,\quad T_i=S'''_{\triangle}(x_i),$$
$$F_i=S^{4_\triangle}(x_i).$$

5 Description of the Method and Development of Boundary Conditions

At the mesh point x_i, the proposed differential equation

$$y^{(3)}+f(x)y=g(x),\quad x\in[a,b] \tag{5.1}$$

with the boundary conditions (1.2), may be discretized by

$$T_i+f_iy_i=g_i, \tag{5.2}$$

where $T_i=S'''_{\triangle}(x_i),\ g_i=g(x_i),\ f_i=f(x_i)$ and $y_i=y(x_i)$.

Substituting the spline relation (4.6a) into (5.3), we have

$$\begin{aligned}&(-\alpha-\beta-ph^2f_{i-2})y_{i-2}+(2\alpha-4\beta-h^2qf_{i-1})y_{i-1}-(sh^2f_i)y_i\\&+(-2\alpha+4\beta-h^2qf_{i+1})y_{i+1}+(-\alpha-\beta-ph^2f_{i+2})y_{i+2}\\=&-h^3(pg_{i-2}+qg_{i-1}+sg_i+qg_{i+1}+pg_{i+2}),\quad i=2,\cdots,N-2.\end{aligned} \tag{5.3}$$

To obtain the unique solution, we need two equations associated with (5.3). So we use the following boundary conditions:

(a) To obtain the second-order boundary formula, we define

$$\begin{aligned}&(\alpha+\beta)[y_1-3y_2+3y_3-y_4]\\=&h(-y_2'''-y_3''')-h^3[\alpha y_3'''-\beta y_2'''],\quad i=1,\\&(\alpha+\beta)[y_{N-3}-3y_{N-2}+3y_{N-1}-y_N]\\=&h(-y_{N-2}'''-y_{N-1}''')-h^3[\alpha y_{N-1}'''-\beta y_{N-2}'''],\quad i=N-1\end{aligned}$$

for any $\alpha$and β, $\alpha+\beta=\frac{1}{2}$.

Using (5.1), we have

$$\begin{aligned}&(\alpha+\beta)y_1-(3(\alpha+\beta)+hf_2+\beta h^3f_2)y_2\\&-(3(\alpha+\beta)-hf_3+\alpha h^3f_3)y_3-(\alpha+\beta)y_4\\=&[-h+\beta h^3]g_2-[h+\alpha h^3]g_3,\\&(\alpha+\beta)y_{N-3}-(3(\alpha+\beta)+hf_{N-2}+\beta h^3f_{N-2})y_{N-2}\\&-(3(\alpha+\beta)-hf_{N-1}+\alpha h^3f_{N-1})y_{N-1}-(\alpha+\beta)y_N\\=&[-h+\beta h^3]g_{N-2}-[h+\alpha h^3]g_{N-1}.\end{aligned}$$

(b) To obtain the fourth-order boundary formula, we define

$$(\alpha+\beta)[y_1-3y_2+3y_3-y_4]=\frac{h}{2}(-y_2'''-y_3''')-\frac{h^3}{8}[\alpha y_3'''-\beta y_2'''],\quad i=1,$$
$$(\alpha+\beta)[y_{N-3}-3y_{N-2}+3y_{N-1}-y_N]=\frac{h}{2}(-y_{N-2}'''-y_{N-1}''')-\frac{h^3}{8}[\alpha y_{N-1}'''-\beta y_{N-2}'''],\quad i=N-1$$

for any α and β, $\alpha+\beta=\frac{1}{2}$.

Using (5.1), we have

$$\begin{aligned}
&(\alpha+\beta)y_1-\Big(3(\alpha+\beta)+\frac{h}{2}f_2+\beta\frac{h^3}{8}f_2\Big)y_2\\
&-\Big(3(\alpha+\beta)-\frac{h}{2}f_3+\alpha\frac{h^3}{8}f_3\Big)y_3-(\alpha+\beta)y_4\\
=&\Big[-\frac{h}{2}+\beta\frac{h^3}{8}\Big]g_2-\Big[\frac{h}{2}+\alpha\frac{h^3}{8}\Big]g_3,\\
&(\alpha+\beta)y_{N-3}-\Big(3(\alpha+\beta)+\frac{h}{2}f_{N-2}+\beta\frac{h^3}{8}f_{N-2}\Big)y_{N-2}\\
&-\Big(3(\alpha+\beta)-\frac{h}{2}f_{N-1}+\alpha\frac{h^3}{8}f_{N-1}\Big)y_{N-1}-(\alpha+\beta)y_N\\
=&\Big[-\frac{h}{2}+\beta\frac{h^3}{8}\Big]g_{N-2}-\Big[\frac{h}{2}+\alpha\frac{h^3}{8}\Big]g_{N-1}.
\end{aligned}$$

By expanding (5.3) in the Taylor series about x_i, we obtain the following local truncation error:

$$\begin{aligned}
T_i=&\Big[\frac{1}{6}\ (9\alpha-3\beta)-(4p+q)\Big]h^4y_i^{(4)}+\Big[\frac{1}{1806}\ (33\alpha-3\beta)-\frac{1}{12}(16p+q)\Big]h^6y_i^{(6)}\\
&+\Big[\frac{1}{131040}\ (1613\alpha+27\beta)-\frac{1}{360}(4p+q)\Big]h^8y_i^{(8)}+O(h^9)
\end{aligned}$$

for any α and β provided that $\alpha+\beta=\frac{1}{2}$.

Remark 5.1 The second-order method for $\alpha=\frac{1}{4}$, $\beta=\frac{1}{4}$ and $p=0.04063489941134321703$, $q=0.25412730690212937985$, $s=0.41047570631347259688$, gives $T_i=O(h^4)$.

Remark 5.2 The fourth-order method for $\alpha=\frac{1}{6}$, $\beta=\frac{1}{3}$, $p=\frac{1}{120}$, $q=\frac{26}{120}$ and $s=\frac{66}{120}$, gives $T_i=O(h^6)$.

6 Numerical Examples

We now give two numerical examples considered by El-Danaf [4] and Arshad Khan et al. [8]. They solved a couple of problems associated with draining and coating flows by using the nonpolynomial quartic spline and the polynomial quintic spline, respectively. In this paper, we solve the same problems by using the nonpolynomial quintic spline to obtain a healthy comparison of these methods.

Example 6.1 Consider the boundary value problem

$$y^{(3)}+y=(7-x^2)\cos x+(x^2-6x-1)\sin x,\tag{6.1}$$
$$y(0)=y^{(1)}(0)+1=y^{(1)}(1)+2\sin 1=0.\tag{6.2}$$

The analytic solution to (6.1)–(6.2) is

$$y(x) = (x^2 - 1)\sin x. \tag{6.3}$$

Nonpolynomial Quintic Spline Solution to Example 6.1 The maximum absolute errors by our algorithms (of the second-order and the fourth-order) and by the nonpolynomial quartic spline method (see [4]) for Example 6.1 are presented in Table 1.

Table 1 The maximum absolute errors for Example 6.1.

N	Max. Absolute Error (see [4])	Max. absolute error (our second-order method)	Max. absolute error (our fourth-order method)
8	1.6501E−4	4.5396E−7	2.1572E−8
16	9.8380E−6	1.1045E−7	1.9481E−9
32	5.8773E−7	3.1625E−8	1.8236E−10
64	3.5687E−8	7.2238E−9	2.0663E−11
128	2.1968E−9	1.7254E−9	2.3917E−12

Example 6.2 Consider the boundary value problem

$$y^{(3)} - x\, y = (-3 - 5x - 2x^2 + x^3)\mathrm{e}^x, \tag{6.4}$$

$$y(0) = y^{(1)}(0) - 1 = y^{(1)}(1) + \mathrm{e}^1 = 0. \tag{6.5}$$

The analytic solution to (6.4)–(6.5) is

$$y(x) = x(1 - x)\mathrm{e}^x. \tag{6.6}$$

Nonpolynomial Quintic Spline Solution to Example 6.2 The maximum absolute errors by our algorithms (of the second-order and the fourth-order) and by the polynomial quintic spline method (see [8]) for Example 6.2 are presented in Table 2.

Table 2 The maximum absolute errors for Example 6.2.

N	Max. Absolute Error (see [8])	Max. absolute error (our second-order method)	Max. absolute error (our fourth-order method)
8	1.8421E−6	2.6421E−7	9.4915E−8
16	1.0454E−7	5.6054E−8	5.1425E−9
32	6.3221E−9	1.65136E−9	3.7768E−10

7 Discussion and Conclusion

In this paper, we used a nonpolynomial Quintic spline function to develop numerical algorithms of a system of third-order boundary value problems associated with draining and coating flows. Here, the result obtained by our algorithm is better than those obtained by the nonpolynomial quartic spline and the polynomial quintic spline, which are compared in Tables 1 and 2. The approximate solutions obtained by the present algorithms are very encouraging and are better than those produced by other collocation, finite difference and spline methods. These algorithms are also more accurate and reduce the cost of computation in comparison with other methods.

Acknowledgement The authors would like to express their gratitude to the anonymous referees for their valuable suggestions.

References

[1] Kumar, M. and Singh, N., Computer simulation of third-order nonlinear singular problems arising in draining and coating flows, *Internat. J. Numer. Methods Fluids*, 2011, to appear. Doi: 10.1002/fld.2635

[2] Bai, Z. B., Existence of solutions for some third-order boundary problems, *Electr. J. Diff. Eq.*, **25**, 2008, 1–6.

[3] Anderson, D. R. and Davis, J. M., Multiple solutions and eigenvalues for third order right focal boundary-value problems, *J. Math. Anal. Appl.*, **267**, 2002, 135–157.

[4] El-Danaf, T. S., Quartic nonpolynomial spline solutions for third order two-point boundary value problem, *World Acad. Sci., Eng. Technology*, **45**, 2008, 453–456.

[5] Al-Said, E. A. and Noor, M. A., Cubic splines method for a system of third-order boundary-value problems, *Appl. Math. Comput.*, **142**, 2003, 195–204.

[6] Noor, M. A. and Khalifa, A. K., A numerical approach for odd-order obstacle problems, *Internat. J. Comput. Math.*, **54**, 1994, 109–116.

[7] Noor, M. A. and Al-Said, E. A., Quartic splines solutions of third-order obstacle problems, *Appl. Math. Comput.*, **153**, 2004, 307–316.

[8] Khan, A. and Aziz, T., The numerical solution of third-order boundary-value problems using quintic splines, *Appl. Math. Comput.*, **137**, 2003, 253–260.

[9] Tirmizi, S. I. A., On numerical solution of third-order boundary-value problems, *Commun. Appl. Numer. Math.*, **7**, 1991, 309–313.

[10] Caglar, H. N., Caglar, S. H. and Twizell, E. H., The numerical solution of third-order boundary-value problems with fourth-degree B-spline functions, *Internat. J. Comput. Math.*, **71**, 1999, 373–381.

[11] Ul-Islam, S., Khan, M. A., Tirmizi, I. A., et al., Non-polynomial spline approach to the solution of a system of third-order boundary-value problems, *Appl. Math. Comput.*, **168**, 2005, 152–163.

[12] Ul-Islam, S, Tirmizi, I. A. and Khan, M. A., Quartic non-polynomial spline approach to the solution of a system of third-order boundary-value problems, *J. Math. Anal. Appl.*, **335**(2), 2007, 1095–1104.

[13] Gao, F. and Chi, C. M., Solving third-order obstacle problems with quartic B-splines, *Appl. Math. Comput.*, **180**, 2006, 270–274.

[14] Tonti, E., Variational formulation for every nonlinear problem, *Internat. J. Eng. Sci.*, **22**, 1984, 1343–1371.

[15] Srivastava, P. K., Kumar, M. and Mohapatra, R. N., Quintic nonpolynomial spline method for the solution of a second-order boundary-value problem with engineering applications, *Comput. Math. Appl.*, **62**, 2011, 1707–1714.

[16] Kumar, M. and Srivastava, P. K., Computational techniques for solving differential equations by quadratic, quartic and octic Spline, *Adv. Eng. Software*, **39**, 2008, 646–653.

[17] Kumar, M. and Srivastava, P. K., Computational techniques for solving differential equations by cubic, quintic and sextic spline, *Internat. J. Comput. Meth. Eng. Sci. Mech.*, **10**(A201), 2009, 108–115.

[18] Rashidinia, J., Jalilian, R. and Mohammadi, R., Non-polynomial spline methods for the solution of a system of obstacle problems, *Appl. Math. Comput.*, **188**(2), 2007, 1984–1990.

[19] Shuaib, N. H., Power, H. and Hibberd, S., BEM solution of thin film flows on an inclined plane with a bottom outlet, *Eng. Anal. Boundary Elements*, **33**(3), 2009, 388–398.