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An Upper Bound of Essential Norm of Composition Operator on $H^2(B_n)$*

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Abstract The authors give an upper bound of the essential norm of a composition operator on $H^2(B_n)$, which involves the counting function in the higher dimensional value distribution theory defined by S. S. Chern. A criterion is also given to assure that the composition operator on $H^2(B_n)$ is bounded or compact.

Keywords Essential norm, Composition operator, Hardy space
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1 Introduction

Let D $(= B_1)$ denote the unit disc of $\mathbb{C}$ and φ be a holomorphic function on D with $\varphi(D) \subset D$. Then, $C_\varphi f = f \circ \varphi$ defines a composition operator C_φ on the space of holomorphic functions in D.

In 1973, Shapiro and Taylor [1] gave the following necessary condition for the compactness of C_φ on $H^2(D)$.

Theorem A *If C_φ is compact on $H^2(D)$, then φ cannot have a finite angular derivative at any point of ∂D.*

In 1987, Shapiro [2] considered the essential norm of the composition operator C_φ on $H^2(D)$ and gave the following necessary and sufficient condition which involves the Nevanlinna counting function of φ.

Theorem B *Let $\|C_\varphi\|_e$ denote the essential norm of C_φ, regarded as an operator on $H^2(D)$. Then*

$$\|C_\varphi\|_e^2 = \limsup_{|w|\to 1^-} \frac{N_\varphi(w)}{(-\log |w|)},$$

where

$$N_\varphi(w) = \sum_{z\in\varphi^{-1}(w)} -\log |z|$$

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is the Nevanlinna counting function, and $\varphi^{-1}(w)$ *denotes the sequence of* φ*-preimages of* w *with each point repeated in the sequence according to its multiplicity. In particular,* C_φ *is compact on* $H^2(D)$ *if and only if*

$$\lim_{|w|\to 1^-}\frac{N_\varphi(w)}{-\log|w|}=0.$$

We note that Theorem B gives a precise estimate of the essential norm of C_φ and makes no extra assumptions about φ (it only need to be a holomorphic self-map of D).

Now, we consider the case of several complex variables. Let B_n be the unit ball of $\mathbb{C}^n$ and $\varphi(z)=(\varphi_1(z),\cdots,\varphi_n(z))$ be a holomorphic self-map of B_n. We consider the composition operator C_φ acting on the classical Hardy space $H^2(B_n)$. In B_n, many self-maps induce unbounded composition operators on $H^2(B_n)$. It is hard to give some sufficient condition for the boundedness of C_φ, and even the strong non-degeneracy requirement that φ is univalent together with the smoothness requirement that φ is analytic in $\overline{B}_n$, is not sufficient to guarantee that C_φ is bounded.

In [3], MacCluer and Shapiro showed the following theorem.

Theorem C *Let* $\varphi: B_n\to B_n$ *be univalent, such that*

$$\Omega_\varphi(z)=\frac{\left\|\left(\frac{\partial\varphi}{\partial z}\right)\right\|^2}{\left|\det\left(\frac{\partial\varphi}{\partial z}\right)\right|^2}$$

is bounded in B_n*, where* $(\frac{\partial\varphi}{\partial z})$ *is the Jacobi matrix of the map* φ*, and*

$$\left\|\left(\frac{\partial\varphi}{\partial z}\right)\right\|^2=\sum_{i,j=1}^{n}\left|\frac{\partial\varphi_i}{\partial z_j}\right|^2.$$

Then, C_φ *is bounded on* $H^2(B_n)$*. Furthermore,* C_φ *is compact on* $H^2(B_n)$ *if and only if* φ *has no finite angular derivative at any point of* ∂B_n*.*

We claim that if $\Omega_\varphi(z)$ is a well-defined function on B_n, then $\det\left(\frac{\partial\varphi}{\partial z}\right)$ must be a nowhere zero function on B_n.

Actually, consider

$$V=\left\{z\in B_n \,\Big|\, \det\left(\frac{\partial\varphi}{\partial z}\right)=0\right\}.$$

If $V\neq\emptyset$, then V is an analytic variety with codimension 1. For any regular point $a\in V$, there exists a neighborhood U and a holomorphic function h on U, such that

$$\det\left(\frac{\partial\varphi}{\partial z}\right)=\lambda h^k \quad \text{on } U,$$

where λ is a nowhere zero holomorphic function. On the other hand, we have that, for any i,j, $\frac{\partial\varphi_i}{\partial z_j}=a_{ij}h^{k_{ij}}$ with $k_{ij}\geq k$. It means that

$$\begin{aligned}\det\left(\frac{\partial\varphi}{\partial z}\right)&=\sum_{i_1,\cdots,i_n}\delta_{i_1\cdots i_n}\frac{\partial\varphi_1}{\partial z_{i_1}}\cdots\frac{\partial\varphi_n}{\partial z_{i_n}}\\&=\sum_{i_1,\cdots,i_n}\delta_{i_1\cdots i_n}a_{1i_1}\cdots a_{ni_n}h^{k_{1i_1}+\cdots+k_{ni_n}},\end{aligned}$$

where $k_{1i_1}+\cdots+k_{ni_n}\geq nk$. It is impossible when $n>1$.

Furthermore, if $\varphi\in C^1(\overline{B}_n)$, then the fact that $\Omega_\varphi(z)$ is bounded in B_n induces that $\det\left(\frac{\partial\varphi}{\partial z}\right)\neq 0$ on $\overline{B}_n$. Otherwise, assume that there exists an $a\in\partial B_n$ such that $\lim\limits_{z\to a}\det\left(\frac{\partial\varphi}{\partial z}\right)=0$. Since

$$\frac{\left|\left(\dfrac{\partial\varphi_i}{\partial z_j}\right)\right|^2}{\left|\det\left(\dfrac{\partial\varphi}{\partial z}\right)\right|^2}\leq\Omega_\varphi(z)$$

is bounded on B_n for any i,j, we have

$$\left|\left(\frac{\partial\varphi_i}{\partial z_j}\right)\right|\leq M\left|\det\left(\frac{\partial\varphi}{\partial z}\right)\right|$$

for some $M>0$. Hence,

$$\begin{aligned}\left|\det\left(\frac{\partial\varphi}{\partial z}\right)\right|&=\Big|\sum_{i_1,\cdots,i_n}\delta_{i_1\cdots i_n}\frac{\partial\varphi_1}{\partial z_{i_1}}\cdots\frac{\partial\varphi_n}{\partial z_{i_n}}\Big|\\&\leq\sum_{i_1,\cdots,i_n}\Big|\frac{\partial\varphi_1}{\partial z_{i_1}}\Big|\cdots\Big|\frac{\partial\varphi_n}{\partial z_{i_n}}\Big|\\&\leq M^n n!\left|\det\left(\frac{\partial\varphi}{\partial z}\right)\right|^n.\end{aligned}$$

For $n>1$, it is impossible as $z\to a$.

In this paper, we give a partial generalization of Theorem B as follows.

Theorem 1.1 *Let $\varphi(z)=(\varphi_1(z),\cdots,\varphi_n(z)):B_n\to B_n$ be a holomorphic map and C_φ be the composition operator on $H^2(B_n)$. Assume that $a\leq\Omega_\varphi(z)\leq b$ on B_n with $a,b\in\mathbb{R}^+$. If*

$$\limsup_{|w|\to 1^-}\frac{N_\varphi(w)}{1-|w|}=c<+\infty,$$

then C_φ is a bounded operator and the essential norm

$$\|C_\varphi\|_e^2\leq 2b(n-1)\limsup_{|w|\to 1^-}\frac{N_\varphi(w)}{1-|w|}.$$

Furthermore, C_φ is compact on $H^2(B_n)$ if

$$\lim_{|w|\to 1^-}\frac{N_\varphi(w)}{1-|w|}=0.$$

Here, $N_\varphi(w)$ is the counting function in the higher dimensional value distribution theory defined by S. S. Chern [4].

Corollary 1.2 *Let $\varphi:B_n\to B_n$ be a holomorphic map. Assume that $\varphi\in C^1(\overline{B}_n)$ and $\det(\frac{\partial\varphi}{\partial z})$ is a nowhere zero holomorphic function on $\overline{B}_n$. If*

$$\limsup_{|w|\to 1^-}\frac{N_\varphi(w)}{1-|w|}=c<+\infty,$$

then C_φ is a bounded operator. In particular, C_φ is a compact operator as $c=0$.

Obviously, under the assumption $\varphi\in C^1(\overline{B}_n)$, "$\det(\frac{\partial\varphi}{\partial z})$ is nowhere zero on $\overline{B}_n$" is equivalent to "$0<a\leq\Omega_\varphi(z)\leq b$ on B_n".

2 Some Notations and Green Formula

Denote by $B_n(r)=\{z\in\mathbb{C}^n \mid |z|<r\}$ the ball of $\mathbb{C}^n$ with radius r. Let $B_n=B_n(1)$ be the unit ball and $B_n(r)=rB_n$. Set $\partial B_n(r)=\{z\in\mathbb{C}^n \mid |z|=r\}$.

Let $\mathrm{d}\tau$ be the Euclidean volume element of $\mathbb{C}^n=\mathbb{R}^{2n}$ with

$$\int_{B_n(r)}\mathrm{d}\tau=\frac{\pi^n}{n!}r^{2n}.$$

We have

$$\mathrm{d}\tau=r^{2n-1}\mathrm{d}r\wedge\mathrm{d}\sigma,$$

where $\mathrm{d}\sigma$ is the induced volume element on ∂B_n. Let $\mathrm{d}\sigma_r=r^{2n-1}\mathrm{d}\sigma$ be the volume element of $\partial B_n(r)$ with

$$\int_{\partial B_n(r)}\mathrm{d}\sigma_r=\frac{2\pi^n}{(n-1)!}r^{2n-1}.$$

Let f be a holomorphic function on B_n. f is said to be in the Hardy space $H^2(B_n)$ provided that

$$\begin{aligned}\|f\|^2&=\sup_{0<r<1}\frac{(n-1)!}{2\pi^n}\int_{\partial B_n(r)}|f|^2\mathrm{d}\sigma\\&=\sup_{0<r<1}\frac{(n-1)!}{2\pi^n}\int_{\partial B_n}|f(r\xi)|^2\mathrm{d}\sigma_\xi<\infty.\end{aligned}$$

Assume that φ is a holomorphic self-map of B_n, and C_φ is the composition operator on $H^2(B_n)$ with the norm

$$\|C_\varphi\|=\sup_{f\neq0}\frac{\|f\circ\varphi\|}{\|f\|}.$$

In order to estimate $\|f\circ\varphi\|$, we need the following well-known Green formula.

Green Formula Let U and V be C^2 real functions on $\overline{D}\subset\mathbb{R}^m$, where D is a domain with a smooth boundary ∂D. Then

$$\int_D(U\triangle V-V\triangle U)\mathrm{d}\tau=\int_{\partial D}\Big(U\frac{\partial V}{\partial n}-V\frac{\partial U}{\partial n}\Big)\mathrm{d}\sigma,$$

where $\mathrm{d}\tau$ is the volume form on $\mathbb{R}^m$, $\mathrm{d}\sigma$ is the induced volume form on ∂D, and $\frac{\partial V}{\partial n}\big(\frac{\partial U}{\partial n}\big)$ is the outward normal derivative of $V(U)$ on ∂D.

We consider the following real function:

$$\Phi_n(x)=\begin{cases}\log\dfrac{1}{x}, & n=1,\\ \dfrac{1}{x^{2n-2}}, & n>1,\end{cases}\qquad x>0.$$

Then

$$G_n(z)=\Phi_n(|z|)=\begin{cases}\log\dfrac{1}{|z|}, & n=1,\\ \dfrac{1}{|z|^{2n-2}}, & n>1\end{cases}$$

is the Green function on B_n with the pole at 0.

Using the Green formula for

$$U = G_1(z) - \Phi_1(r_0) = \log \frac{r_0}{|z|}$$

and

$$V = |f \circ \varphi|^2 \quad \text{on } B_1(r) \setminus B_1(\varepsilon)$$

with

$$r \to 1^- \quad \text{and} \quad \varepsilon \to 0^+,$$

Shapiro gave the following estimate of $\|f \circ \varphi\|$,

$$\|f \circ \varphi\|^2 = \frac{2}{\pi} \int_{B_1} \Big(\log \frac{1}{|z|}\Big) |f'|^2 |\varphi'|^2 \mathrm{d}\tau_z + |f(\varphi(0))|^2, \quad f \in H^2(B_1). \tag{2.1}$$

For the higher dimensional case, we have the following proposition.

Proposition 2.1 *Let $(w_1, \cdots, w_n) = \varphi(z_1, \cdots, z_n)$ be a holomorphic self-map of B_n with $n > 1$, and $f(w) \in H^2(B_n)$. Then*

$$\|f \circ \varphi\|^2 = \frac{(n-2)!}{\pi^n} \int_{B_n} \Big(\frac{1}{|z|^{2n-2}} - 1\Big) \mathrm{grad} f \cdot \Big(\frac{\partial \varphi}{\partial z}\Big) \cdot \overline{\Big(\frac{\partial \varphi}{\partial z}\Big)}^{\mathrm{T}} \cdot \overline{\mathrm{grad} f}^{\mathrm{T}} \mathrm{d}\tau_z + |f(\varphi(0))|^2. \tag{2.2}$$

Proof Consider $D = B_n(r_0) \setminus B_n(\varepsilon)$ with $\frac{1}{2} \le r_0 < 1$ and $0 < \varepsilon \le \frac{1}{4}$. Let

$$U = G_n(z) - \Phi_n(r_0) = \frac{1}{|z|^{2n-2}} - \frac{1}{|r_0|^{2n-2}}$$

and

$$V = |f \circ \varphi|^2.$$

We have

$$\begin{aligned}
&\int_{B_n(r_0)\setminus B_n(\varepsilon)} (U \triangle V - V \triangle U) \mathrm{d}\tau_z \\
&= 4 \int_{B_n(r_0)\setminus B_n(\varepsilon)} \Big(\frac{1}{|z|^{2n-2}} - \frac{1}{|r_0|^{2n-2}}\Big) \sum_{k=1}^n \frac{\partial^2}{\partial z_k \partial \overline{z}_k} |f \circ \varphi|^2 \mathrm{d}\tau_z,
\end{aligned}$$

where $\mathrm{d}\tau_z$ is the volume form. Let $\varepsilon \to 0^+$ and $r_0 \to 1^-$. Hence,

$$\begin{aligned}
\int_{B_n(r_0)\setminus B_n(\varepsilon)} (U \triangle V - V \triangle U) \mathrm{d}\tau_z &\to 4 \int_{B_n} \Big(\frac{1}{|z|^{2n-2}} - 1\Big) \sum_{k=1}^n \frac{\partial^2}{\partial z_k \partial \overline{z}_k} |f \circ \varphi|^2 \mathrm{d}\tau_z, \\
\int_{\partial B_n(r_0)} \Big(U \frac{\partial V}{\partial n} - V \frac{\partial U}{\partial n}\Big) \mathrm{d}\sigma_{r_0 z} &= -\int_{\partial B_n(r_0)} |f \circ \varphi|^2 \Big(-(2n-2) \frac{1}{r_0^{2n-1}}\Big) \mathrm{d}\sigma_{r_0 z} \\
&= (2n-2) \int_{\partial B_n(r_0)} |f \circ \varphi|^2 \mathrm{d}\sigma_z \\
&\to (2n-2) \frac{2\pi^n}{(n-1)!} \|f \circ \varphi\|^2, \quad r_0 \to 1^-, \\
-\int_{\partial B_n(\varepsilon)} \Big(U \frac{\partial V}{\partial n} - V \frac{\partial U}{\partial n}\Big) \mathrm{d}\sigma_{\varepsilon z} &= -\int_{\partial B_n(\varepsilon)} \Big(\frac{1}{\varepsilon^{2n-2}} - \frac{1}{r_0^{2n-2}}\Big) \frac{\partial V}{\partial n} \mathrm{d}\sigma_{\varepsilon z} \\
&\quad + \int_{\partial B_n(\varepsilon)} |f \circ \varphi|^2 \Big(-(2n-2) \frac{1}{\varepsilon^{2n-1}}\Big) \mathrm{d}\sigma_{\varepsilon z},
\end{aligned}$$

where $\mathrm{d}\sigma_{\varepsilon z} = \varepsilon^{2n-1}\mathrm{d}\sigma_{1z}$. We have

$$-\int_{\partial B_n(\varepsilon)} \Big(\frac{1}{\varepsilon^{2n-2}} - \frac{1}{r_0^{2n-2}}\Big)\frac{\partial V}{\partial n}\varepsilon^{2n-1}\mathrm{d}\sigma_z \to 0, \quad \varepsilon \to 0^+$$

and

$$\begin{aligned} &-(2n-2)\int_{\partial B_n(\varepsilon)} |f\circ\varphi|^2 \frac{1}{\varepsilon^{2n-1}}\cdot \varepsilon^{2n-1}\mathrm{d}\sigma_z \\ &\to -(2n-2)\frac{2\pi^n}{(n-1)!}|f(\varphi(0))|^2, \quad \varepsilon \to 0^+. \end{aligned}$$

By the Green formula, we have

$$\|f\circ\varphi\|^2 = \frac{2}{n-1}\cdot\frac{(n-1)!}{2\pi^n}\int_{B_n}\Big(\frac{1}{|z|^{2n-2}} - 1\Big)\sum_{k=1}^n \frac{\partial^2}{\partial z_k\partial\overline{z}_k}|f\circ\varphi|^2\mathrm{d}\tau_z + |f(\varphi(0))|^2.$$

Furthermore,

$$\begin{aligned} &\int_{B_n}\Big(\frac{1}{|z|^{2n-2}} - 1\Big)\sum_{k=1}^n \frac{\partial^2}{\partial z_k\partial\overline{z}_k}|f\circ\varphi|^2\mathrm{d}\tau_z \\ &= \int_{B_n}\Big(\frac{1}{|z|^{2n-2}} - 1\Big)\sum_{k,j,s=1}^n \frac{\partial f}{\partial w_j}\cdot\frac{\partial\varphi_j}{\partial z_k}\cdot\overline{\frac{\partial f}{\partial w_s}\cdot\frac{\partial\varphi_s}{\partial z_k}}\mathrm{d}\tau_z \\ &= \int_{B_n}\Big(\frac{1}{|z|^{2n-2}} - 1\Big)\mathrm{grad}\, f\cdot\Big(\frac{\partial\varphi}{\partial z}\Big)\cdot\overline{\Big(\frac{\partial\varphi}{\partial z}\Big)}^{\mathrm{T}}\cdot\overline{\mathrm{grad}\, f}^{\mathrm{T}}\mathrm{d}\tau_z, \end{aligned}$$

where $\mathrm{grad} f = \big(\frac{\partial f}{\partial w_1}, \cdots, \frac{\partial f}{\partial w_n}\big)$.

Hence,

$$\|f\circ\varphi\|^2 = \frac{(n-2)!}{\pi^n}\int_{B_n}\Big(\frac{1}{|z|^{2n-2}} - 1\Big)\mathrm{grad}\, f\cdot\Big(\frac{\partial\varphi}{\partial z}\Big)\cdot\overline{\Big(\frac{\partial\varphi}{\partial z}\Big)}^{\mathrm{T}}\cdot\overline{\mathrm{grad}\, f}^{\mathrm{T}}\mathrm{d}\tau_z + |f(\varphi(0))|^2.$$

3 Counting Functions in Value Distribution Theory

In the classical Nevanlinna theory for one variable, for a meromorphic function f and $w \in \mathbb{C}\cup\{\infty\} = \mathbb{P}^1(\mathbb{C})$, the Nevanlinna counting function is defined by

$$N_f(r,w) = n_f(0,w)\log r + \int_0^r (n_f(r,w) - n_f(0,w))\frac{\mathrm{d}t}{t},$$

where $n_f(t,w)$ is the number of f taking value w on the closed disc $\overline{B_1(t)}$ with counting multiplicity, and $n_f(0,w)$ is the multiplicity at $z=0$.

It is easy to check that, for $w\in\mathbb{C}$,

$$N_f(r,w) = \mathrm{ord}_0(f-w)\log r + \sum_{\substack{z\in B_1(r)\\ z\neq 0}} \mathrm{ord}_z(f-w)\log\frac{r}{|z|}.$$

Now, we consider an entire function $\varphi: B_1\to B_1$. For any $w\in\mathbb{C}\setminus\{\varphi(0)\}$ and $0<r<1$,

$$N_\varphi(r,w) = \sum_{\substack{z\in B_1(r)\\ z\neq 0}} \mathrm{ord}_z(\varphi - w)\log\frac{r}{|z|} = \sum_{j=1}^{n_\varphi(r,w)}\log\frac{r}{|z_j|}.$$

In [2], Shapiro defined

$$N_\varphi(w) = \lim_{r\to 1^-} N_\varphi(r,w) = \sum_{z\in\varphi^{-1}(w)} \log\frac{1}{|z|},$$

where each point in $\varphi^{-1}(w)$ is repeated in the sequence according to its multiplicity.

Using this counting function, Shapiro gave the following equality:

$$\|f\circ\varphi\|^2 = \frac{2}{\pi}\int_{B_1} N_\varphi(w)|f'|^2 \mathrm{d}\tau_w + |f(\varphi(0))|^2,$$

which gives the connection between composition operators and counting functions (see (1) in Section 2 of [2]).

We now recite the counting function in the higher dimensional value distribution theory introduced by S. S. Chern in 1960 (see [4]).

Let f be a holomorphic map from $\mathbb{C}^n$ into $\mathbb{P}^n(\mathbb{C})$ with reduced representation $f = [f_0 : f_1 : \cdots : f_n]$, where $f_0, f_1, \cdots, f_n$ are holomorphic functions on $\mathbb{C}^n$ without common zeros.

For a point A in $\mathbb{P}^n(\mathbb{C})$ with $f^{-1}(A)\cap B_n(r)$ consisting only of a finite number of points, let $n_f(r,A)$ be the number of times that A is covered by $f(B_n(r))$, counting multiplicities.

For $A\in\mathbb{P}^n(\mathbb{C})\setminus\{f(0)\}$, we define

$$N_f(r,A) = \int_0^r \frac{n_f(t,A)}{t^{2n-1}}\mathrm{d}t.$$

By using a simple computation, we have

$$N_f(r,A) = \frac{1}{2n-2}\sum_{z\in B_n(r)\cap f^{-1}(A)}\Big(\frac{1}{|z|^{2n-2}} - \frac{1}{r^{2n-2}}\Big),$$

where each point is repeated according to its multiplicity.

Now, we consider that $\varphi = (\varphi_1,\cdots,\varphi_n)$ is the holomorphic map from B_n into B_n with $\det\big(\frac{\partial\varphi}{\partial z}\big)\neq 0$ on B_n. Then, φ can be regarded as a holomorphic map from $\mathbb{C}^n$ into $\mathbb{P}^n(\mathbb{C})$ with reduced representation $[1:\varphi_1:\cdots:\varphi_n]$. For any

$$w = (w_1,\cdots,w_n)\in B_n\setminus\{\varphi(0)\}$$

or

$$w = [1:w_1:\cdots:w_n]\in\mathbb{P}^n(\mathbb{C})\setminus\{[1:\varphi_1(0):\cdots:\varphi_n(0)]\},$$

$\varphi^{-1}(w)\cap B_n(r)$ consists of only finite number points, where $0<r<1$. We can consider

$$N_\varphi(r,w) = \frac{1}{2n-2}\sum_{z\in\varphi^{-1}(w)\cap B_n(r)}\Big(\frac{1}{|z|^{2n-2}} - \frac{1}{r^{2n-2}}\Big).$$

Since $N_\varphi(r,w)$ increases with r, let

$$N_\varphi(w) = \lim_{r\to 1^-} N_\varphi(r,w)$$

and

$$N_\varphi(w) = \frac{1}{2n-2}\sum_{z\in\varphi^{-1}(w)}\Big(\frac{1}{|z|^{2n-2}} - 1\Big).$$

4 Essential Norm of C_φ and Some Estimates

The essential norm of a composition operator C_φ is defined as

$$\|C_\varphi\|_e := \inf\{\|C_\varphi - K\| \mid K \text{ is a compact operator}\}.$$

Notice that $\|C_\varphi\|_e = 0$ if and only if C_φ is compact. So, estimates on $\|C_\varphi\|_e$ lead to the conditions for C_φ to be compact.

Proposition 4.1 (see [2, Proposition 5.1]) *Suppose that T is a bounded linear operator on a Hilbert space H. Let $\{K_p\}$ be a sequence of compact self-adjoint operators on H. Write $R_p = I - K_p$. Suppose that $\|R_p\| = 1$ for each p, and $\|R_p x\| \to 0$ for each $x \in H$. Then*

$$\|T\|_e = \lim_{p\to\infty} \|TR_p\|.$$

For any holomorphic function f on B_n, we consider the series representation

$$f(w) = \sum_{s_1,\cdots,s_n} a_{s_1\cdots s_n} w_1^{s_1}\cdots w_n^{s_n} = \sum_{s=0}^{\infty}\sum_{s_1+\cdots+s_n=s} a_{s_1\cdots s_n} w_1^{s_1}\cdots w_n^{s_n}.$$

Set

$$f := K_p f + R_p f$$

with

$$K_p f = \sum_{s=0}^{p}\sum_{s_1+\cdots+s_n=s} a_{s_1\cdots s_n} w_1^{s_1}\cdots w_n^{s_n},$$

$$R_p f = \sum_{s=p+1}^{\infty}\sum_{s_1+\cdots+s_n=s} a_{s_1\cdots s_n} w_1^{s_1}\cdots w_n^{s_n}.$$

Let

$$S_p := \operatorname{span}_{\mathbb{C}}\{w_1^{s_1}\cdots w_n^{s_n} \mid 0 \le s_1+\cdots+s_n \le p\}.$$

Then, S_p is a finite dimensional subspace of $H^2(B_n)$, such that the projective operator $K_p : H^2 \to S_p$ is self-adjoint and compact. $R_p = I - K_p$ is the orthogonal complementary operator of K_p. Since $I = K_p + R_p = K_p^* + R_p^* = K_p + R_p^*$, R_p is also self-adjoint. Obviously, $\|K_p\| = \|R_p\| = 1$. Thus, the hypotheses of Proposition 4.1 are fulfilled. So

$$\|C_\varphi\|_e = \lim_{p\to\infty} \|C_\varphi R_p\|,$$

if C_φ is bounded.

We now give the estimates for $|f|$, $|\operatorname{grad} f|$, $|R_p f|$ and $|\operatorname{grad} R_p f|$.

Lemma 4.1 *Let f be a holomorphic function on $H^2(B_n)$. Then*

$$|f(w)|^2 \le \frac{\|f\|^2}{(1-|w|^2)^n} \tag{4.1}$$

and

$$|\operatorname{grad} f(w)|^2 \le \frac{(2n^2+n)\|f\|^2}{(1-|w|^2)^{n+2}}. \tag{4.2}$$

Proof The proof of (4.1) can be found in [5, Theorem 7.2.5]. Here, we give the proof for completeness.

It is clear that

$$|f(w)| = |\langle f(\zeta), c(\zeta, w)\rangle| \le \|f\| \cdot \|c(\zeta, w)\|,$$

where

$$\begin{aligned} c(\zeta, w) = \frac{1}{(1 - \langle \zeta, w\rangle)^n} &= \sum_{s=0}^{\infty} \frac{(n-1+s)!}{(n-1)!s!} \langle \zeta, w\rangle^s \\ &= \sum_{s=0}^{\infty} \frac{(n-1+s)!}{(n-1)!s!} \sum_{s_1+\cdots+s_n=s} \frac{s!}{s_1!\cdots s_n!} \zeta_1^{s_1}\cdots\zeta_n^{s_n}\overline{w}_1^{s_1}\cdots\overline{w}_n^{s_n} \end{aligned}$$

is the Cauchy kernel for the holomorphic functions on B_n (see [6]).

We compute that

$$\begin{aligned} \|c(\zeta, w)\|^2 &= \sup_{0<r<1} \frac{(n-1)!}{2\pi^n} \int_{\partial B_n} |c(r\zeta, w)|^2 \mathrm{d}\sigma_\zeta \\ &= \sum_{s=0}^{\infty} \Big(\frac{(n-1+s)!}{(n-1)!}\Big)^2 \sum_{s_1+\cdots+s_n=s} \frac{1}{(s_1!\cdots s_n!)^2} \Big(\frac{(n-1)!}{2\pi^n} \\ &\quad \times \int_{\partial B_n} |\zeta_1|^{2s_1}\cdots|\zeta_n|^{2s_n} \mathrm{d}\sigma_\zeta\Big) |w_1|^{2s_1}\cdots|w_n|^{2s_n} \\ &= \sum_{s=0}^{\infty} \frac{(n-1+s)!}{(n-1)!} \sum_{s_1+\cdots+s_n=s} \frac{1}{s_1!\cdots s_n!} |w_1|^{2s_1}\cdots|w_n|^{2s_n} \\ &= \sum_{s=0}^{\infty} \frac{(n-1+s)!}{(n-1)!s!} |w|^{2s} \\ &= \frac{1}{(1-|w|^2)^n}, \end{aligned}$$

where

$$\frac{(n-1)!}{2\pi^n} \int_{\partial B_n} |\zeta_1|^{2s_1}\cdots|\zeta_n|^{2s_n} \mathrm{d}\sigma_\zeta = \frac{(n-1)!s_1!\cdots s_n!}{(n-1+s)!}$$

(see [5, Proposition 1.4.9]).

Hence, we have (4.1).

In order to estimate $|\mathrm{grad}\, f(w)|$, we consider that, for $1 \le i \le n$,

$$\Big|\frac{\partial}{\partial w_i} f(w)\Big| = \Big|\frac{\partial}{\partial w_i} \langle f, c(\zeta, w)\rangle\Big| = \Big|\Big\langle f, \frac{\partial}{\partial \overline{w}_i} c(\zeta, w)\Big\rangle\Big| \le \|f\| \cdot \Big\|\frac{\partial}{\partial \overline{w}_i} c(\zeta, w)\Big\|,$$

where

$$\begin{aligned} \frac{\partial}{\partial \overline{w}_i} c(\zeta, w) &= \frac{n\zeta_i}{(1-\langle \zeta, w\rangle)^{n+1}} \\ &= \sum_{s=0}^{\infty} \frac{(n+s)!}{n!s!} \sum_{s_1+\cdots+s_n=s} \frac{s!}{s_1!\cdots s_n!} n\zeta_i \zeta_1^{s_1}\cdots\zeta_n^{s_n}\overline{w}_1^{s_1}\cdots\overline{w}_n^{s_n}. \end{aligned}$$

We have

$$\Big\|\frac{\partial}{\partial \overline{w}_i}c(\zeta,w)\Big\|^2=\sum_{s=0}^{\infty}\Big(\frac{(n+s)!}{(n-1)!}\Big)^2\sum_{s_1+\cdots+s_n=s}\frac{1}{(s_1!\cdots s_n!)^2}\Big(\frac{(n-1)!}{2\pi^n}\times\int_{\partial B_n}|\zeta_i|^2|\zeta_1|^{2s_1}\cdots|\zeta_n|^{2s_n}\,\mathrm{d}\sigma_\zeta\Big)|w_1|^{2s_1}\cdots|w_n|^{2s_n}.$$

Hence,

$$\begin{aligned}|\mathrm{grad}\, f(w)|^2&=\sum_{i=1}^{n}\Big|\frac{\partial f}{\partial w_i}\Big|^2\le\|f\|^2\Big(\sum_{i=1}^{n}\Big\|\frac{\partial}{\partial \overline{w}_i}c(\zeta,w)\Big\|^2\Big)\\
&\le\|f\|^2\Big(\sum_{s=0}^{\infty}\Big(\frac{(n+s)!}{(n-1)!}\Big)^2\sum_{s_1+\cdots+s_n=s}\frac{1}{(s_1!\cdots s_n!)^2}\Big(\frac{(n-1)!}{2\pi^n}\\
&\quad\times\int_{\partial B_n}\Big(\sum_{i=1}^{n}|\zeta_i|^2\Big)|\zeta_1|^{2s_1}\cdots|\zeta_n|^{2s_n}\,\mathrm{d}\sigma_\zeta\Big)|w_1|^{2s_1}\cdots|w_n|^{2s_n}\Big)\\
&=\|f\|^2\Big(\sum_{s=0}^{\infty}\frac{(n+s)!(n+s)}{(n-1)!s!}\sum_{s_1+\cdots+s_n=s}\frac{s!}{s_1!\cdots s_n!}|w_1|^{2s_1}\cdots|w_n|^{2s_n}\Big)\\
&=\|f\|^2\sum_{s=0}^{\infty}\frac{(n+s)!(n+s)}{(n-1)!s!}|w|^{2s}\\
&=\|f\|^2\Big(n\sum_{s=0}^{\infty}\frac{(n+s)!}{(n-1)!s!}|w|^{2s}+\sum_{s=1}^{\infty}\frac{(n+s)!}{(n-1)!(s-1)!}|w|^{2s}\Big)\\
&\le\|f\|^2\Big(n^2\sum_{s=0}^{\infty}\frac{(n+s)!}{n!s!}|w|^{2s}+n(n+1)\sum_{s=1}^{\infty}\frac{(n+s)!}{(n+1)!(s-1)!}|w|^{2(s-1)}\Big)\\
&=\|f\|^2\Big(\frac{n^2}{(1-|w|^2)^{n+1}}+\frac{n(n+1)}{(1-|w|^2)^{n+2}}\Big)\\
&\le\frac{(2n^2+n)\|f\|^2}{(1-|w|^2)^{n+2}}.\end{aligned}$$

Lemma 4.2 *Let f be a holomorphic function on $H^2(B_n)$. Then*

$$|R_pf(w)|^2\le\frac{(2+p)^{n-1}|w|^{2(p+1)}\|f\|^2}{(1-|w|^2)^n}\tag{4.3}$$

and

$$|\mathrm{grad}R_pf(w)|^2\le\frac{(2n^2+n)(2+p)^{n+1}|w|^{2p}\|f\|^2}{(1-|w|^2)^{n+2}}.\tag{4.4}$$

Proof Since R_p is self-adjoint, we have

$$|R_pf(w)|=|\langle R_pf,c(\zeta,w)\rangle|=|\langle f,R_pc(\zeta,w)\rangle|\le\|f\|\cdot\|R_pc(\zeta,w)\|.$$

Similarly,

$$\|R_pc(\zeta,w)\|^2=\sum_{s=p+1}^{\infty}\frac{(n-1+s)!}{(n-1)!s!}|w|^{2s}$$

$$
\begin{aligned}
&= |w|^{2(p+1)} \sum_{s=p+1}^{\infty} \frac{(n-1+s)!}{(n-1)!s!} |w|^{2(s-p-1)} \\
&= |w|^{2(p+1)} \sum_{t=0}^{\infty} \frac{(n+t+p)!}{(n-1)!(t+p+1)!} |w|^{2t} \\
&= |w|^{2(p+1)} \sum_{t=0}^{\infty} \frac{(n-1+t)!}{(n-1)!t!} |w|^{2t} \frac{t!(n+t+p)!}{(n-1+t)!(t+p+1)!} \\
&\le (2+p)^{n-1} \frac{|w|^{2(p+1)}}{(1-|w|^2)^n},
\end{aligned}
$$

where the last line follows from

$$
\begin{aligned}
\frac{t!(n+t+p)!}{(n-1+t)!(t+p+1)!} &= \frac{(t+p+2)\cdots(n+t+p)}{(t+1)\cdots(n-1+t)} \\
&= \Big(1+\frac{p+1}{t+1}\Big)\cdots\Big(1+\frac{p+1}{n-1+t}\Big) \\
&\le (2+p)^{n-1}.
\end{aligned}
$$

Hence, we have (4.3).

For $1 \le i \le n$,

$$
\begin{aligned}
\Big|\frac{\partial}{\partial w_i} R_p f(w)\Big| &= \Big|\Big\langle R_p f, \frac{\partial}{\partial \overline{w}_i} c(\zeta, w)\Big\rangle\Big| = \Big|\Big\langle f, R_p\Big(\frac{\partial}{\partial \overline{w}_i} c(\zeta, w)\Big)\Big\rangle\Big| \\
&\le \|f\| \cdot \Big\|R_p\Big(\frac{\partial}{\partial \overline{w}_i} c(\zeta, w)\Big)\Big\|.
\end{aligned}
$$

We compute

$$
\begin{aligned}
&\sum_{i=1}^{n} \Big\|R_p\Big(\frac{\partial}{\partial \overline{w}_i} c(\zeta, w)\Big)\Big\|^2 = \sum_{s=p+1}^{\infty} \frac{(n+s)!(n+s)}{(n-1)!s!} |w|^{2s} \\
&\le \sum_{s=p+1}^{\infty} \frac{(n+s)!n}{(n-1)!s!} |w|^{2s} + \sum_{s=p+1}^{\infty} \frac{(n+s)!}{(n-1)!(s-1)!} |w|^{2(s-1)} \\
&= |w|^{2(p+1)} \sum_{s=p+1}^{\infty} \frac{(n+s)!n}{(n-1)!s!} |w|^{2(s-p-1)} + |w|^{2p} \sum_{s=p+1}^{\infty} \frac{(n+s)!}{(n-1)!(s-1)!} |w|^{2(s-1)} \\
&= n^2 |w|^{2(p+1)} \sum_{t=0}^{\infty} \frac{(n+1+t+p)!}{n!(t+p+1)!} |w|^{2t} + n(n+1)|w|^{2p} \sum_{t=0}^{\infty} \frac{(n+t+p+1)!}{(n+1)!(t+p)!} |w|^{2t} \\
&= n^2 |w|^{2(p+1)} \sum_{t=0}^{\infty} \frac{(n+t)!}{n!t!} |w|^{2t} \frac{t!(n+1+t+p)!}{(n+t)!(t+p+1)!} \\
&\quad + n(n+1)|w|^{2p} \sum_{t=0}^{\infty} \frac{(n+t+1)!}{(n+1)!t!} |w|^{2t} \frac{t!(n+t+p+1)!}{(n+1+t)!(t+p)!} \\
&\le \frac{n^2 |w|^{2(p+1)} (2+p)^n}{(1-|w|^2)^{n+1}} + \frac{n(n+1)|w|^{2p}(1+p)^{n+1}}{(1-|w|^2)^{n+2}} \\
&\le \frac{(2n^2+n)(2+p)^{n+1}|w|^{2p}}{(1-|w|^2)^{n+2}}.
\end{aligned}
$$

Therefore,

$$|\mathrm{grad} R_p f(w)|^2 \le \frac{(2n^2+n)(2+p)^{n+1}|w|^{2p}\|f\|^2}{(1-|w|^2)^{n+2}}.$$

5 Proof of Main Result

Recalling (2.2),

$$\|f\circ\varphi\|^2 = \frac{(n-2)!}{\pi^n}\int_{B_n}\Big(\frac{1}{|z|^{2n-2}}-1\Big)\mathrm{grad}\, f\cdot\Big(\frac{\partial\varphi}{\partial z}\Big)\cdot\overline{\Big(\frac{\partial\varphi}{\partial z}\Big)}^{\mathrm{T}}\cdot\overline{\mathrm{grad}\, f}^{\mathrm{T}}\mathrm{d}\tau_z + |f(\varphi(0))|^2,$$

we estimate $\mathrm{grad}\, f\cdot\big(\frac{\partial\varphi}{\partial z}\big)\cdot\overline{\big(\frac{\partial\varphi}{\partial z}\big)}^{\mathrm{T}}\cdot\overline{\mathrm{grad}\, f}^{\mathrm{T}}$. For any $z\in B_n$, there exists a unitary matrix U, such that

$$\Big(\frac{\partial\varphi}{\partial z}\Big)\cdot\overline{\Big(\frac{\partial\varphi}{\partial z}\Big)}^{\mathrm{T}} = U\begin{pmatrix}\lambda_1^2 & & 0\\ & \ddots & \\ 0 & & \lambda_n^2\end{pmatrix}\overline{U}^{\mathrm{T}},$$

where the positive real numbers $\lambda_1^2,\cdots,\lambda_n^2$ are the eigenvalues of $\big(\frac{\partial\varphi}{\partial z}\big)\cdot\overline{\big(\frac{\partial\varphi}{\partial z}\big)}^{\mathrm{T}}$. Hence,

$$\begin{aligned}\mathrm{grad}\, f\cdot\Big(\frac{\partial\varphi}{\partial z}\Big)\cdot\overline{\Big(\frac{\partial\varphi}{\partial z}\Big)}^{\mathrm{T}}\cdot\overline{\mathrm{grad}\, f}^{\mathrm{T}} &= \mathrm{grad}\, fU\begin{pmatrix}\lambda_1^2 & & 0\\ & \ddots & \\ 0 & & \lambda_n^2\end{pmatrix}\overline{\mathrm{grad}\, fU}^{\mathrm{T}}\\ &= T\begin{pmatrix}\lambda_1^2 & & 0\\ & \ddots & \\ 0 & & \lambda_n^2\end{pmatrix}\overline{T}^{\mathrm{T}}\\ &= \lambda_1^2|T_1|^2+\cdots+\lambda_n^2|T_n|^2,\end{aligned} \tag{5.1}$$

where

$$T=(T_1,\cdots,T_n),\quad T_k=\sum_{s=1}^n\frac{\partial f}{\partial w_s}u_{sk}$$

and

$$U=(u_{sk})_{1\le s,k\le n}.$$

For k, $1\le k\le n$, we have

$$|T_k|^2 = \Big|\sum_{s=1}^n\frac{\partial f}{\partial w_s}u_{sk}\Big|^2 \le \sum_{s=1}^n\Big|\frac{\partial f}{\partial w_s}\Big|^2\sum_{s=1}^n|u_{sk}|^2 \le \sum_{s=1}^n\Big|\frac{\partial f}{\partial w_s}\Big|^2 = |\mathrm{grad}\, f|^2. \tag{5.2}$$

By (5.1)–(5.2), we have

$$\begin{aligned}\mathrm{grad}\, f\cdot\Big(\frac{\partial\varphi}{\partial z}\Big)\cdot\overline{\Big(\frac{\partial\varphi}{\partial z}\Big)}^{\mathrm{T}}\cdot\overline{\mathrm{grad}\, f}^{\mathrm{T}} &\le |\mathrm{grad}\, f|^2(\lambda_1^2+\cdots+\lambda_n^2)\\ &= |\mathrm{grad}\, f|^2\cdot\mathrm{Tr}\Big(\frac{\partial\varphi}{\partial z}\Big)\overline{\Big(\frac{\partial\varphi}{\partial z}\Big)}^{\mathrm{T}}\\ &= |\mathrm{grad}\, f|^2\sum_{i,j=1}^n\Big|\frac{\partial\varphi_i}{\partial z_j}\Big|^2.\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_{B_n}\Big(\frac{1}{|z|^{2n-2}}-1\Big)\mathrm{grad}\, f\cdot\Big(\frac{\partial\varphi}{\partial z}\Big)\cdot\overline{\Big(\frac{\partial\varphi}{\partial z}\Big)}^{\mathrm{T}}\cdot\overline{\mathrm{grad}\, f}^{\mathrm{T}}\mathrm{d}\tau_z\\
\le&\int_{B_n}\Big(\frac{1}{|z|^{2n-2}}-1\Big)|\mathrm{grad}\, f|^2\sum_{i,j=1}^{n}\Big|\frac{\partial\varphi_i}{\partial z_j}\Big|^2\mathrm{d}\tau_z\\
=&\int_{B_n}\Big(\frac{1}{|z|^{2n-2}}-1\Big)|\mathrm{grad}\, f|^2\Omega_\varphi(z)\Big|\det\Big(\frac{\partial\varphi}{\partial z}\Big)\Big|^2\mathrm{d}\tau_z\\
\le&\, b\int_{B_n}\Big(\frac{1}{|z|^{2n-2}}-1\Big)|\mathrm{grad}\, f|^2\Big|\det\Big(\frac{\partial\varphi}{\partial z}\Big)\Big|^2\mathrm{d}\tau_z\\
=&\,2(n-1)b\int_{B_n}N_\varphi(w)|\mathrm{grad}\, f|^2\mathrm{d}\tau_w, \qquad (5.3)
\end{aligned}$$

where

$$N_\varphi(w)=\frac{1}{2n-2}\sum_{z\in\varphi^{-1}(w)}\Big(\frac{1}{|z|^{2n-2}}-1\Big)$$

is the Chern's counting function.

Combining (2.2) and (5.3), we have

$$\|f\circ\varphi\|^2\le\frac{2b(n-1)!}{\pi^n}\int_{B_n}N_\varphi(w)|\mathrm{grad}\, f|^2\mathrm{d}\tau_w+|f(\varphi(0))|^2. \qquad (5.4)$$

Next, we show that C_φ is a bounded operator on $H^2(B_n)$.

Firstly, we take $\varphi(z)\equiv z$. Then, by (2.2), we get

$$\begin{aligned}
\|f\|^2&=\frac{(n-2)!}{\pi^n}\int_{B_n}\Big(\frac{1}{|z|^{2n-2}}-1\Big)|\mathrm{grad}\, f|^2\mathrm{d}\tau_z+|f(0)|^2\\
&=\frac{(n-2)!}{\pi^n}\int_{B_n}\Big(\frac{1-|w|^{2n-2}}{|w|^{2n-2}}\Big)|\mathrm{grad}\, f|^2\mathrm{d}\tau_w+|f(0)|^2.
\end{aligned}$$

Since,

$$\frac{1-|w|^{2n-2}}{|w|^{2n-2}}=\frac{(1-|w|)(1+|w|)\sum\limits_{k=0}^{n-2}|w|^{2k}}{|w|^{2n-2}}\ge 1-|w|,$$

we have

$$\|f\|^2\ge\frac{(n-2)!}{\pi^n}\int_{B_n}|\mathrm{grad}\, f|^2(1-|w|)\mathrm{d}\tau_w.$$

It follows that

$$\int_{B_n}|\mathrm{grad}\, f|^2(1-|w|)\mathrm{d}\tau_w\le\frac{\pi^n}{(n-2)!}\|f\|^2. \qquad (5.5)$$

In another case, take $f=w_i$ $(1\le i\le n)$, $f\circ\varphi=\varphi_i$ and $\mathrm{grad}\, f=(0,\cdots,0,\overset{i\text{th}}{1},0,\cdots,0)$. By (2.2), we have

$$\|\varphi_i\|^2=\frac{(n-2)!}{\pi^n}\int_{B_n}\Big(\frac{1}{|z|^{2n-2}}-1\Big)\sum_{k=1}^{n}\Big|\frac{\partial\varphi_i}{\partial z_k}\Big|^2\mathrm{d}\tau_z+|\varphi_i(0)|^2.$$

Thus,

$$
\begin{aligned}
1 \geq \sum_{i=1}^{n} \|\varphi_i\|^2 &= \frac{(n-2)!}{\pi^n} \int_{B_n} \Big(\frac{1}{|z|^{2n-2}} - 1\Big) \sum_{k,i=1}^{n} \Big|\frac{\partial \varphi_i}{\partial z_k}\Big|^2 \mathrm{d}\tau_z + \sum_{i=1}^{n} |\varphi_i(0)|^2 \\
&= \frac{(n-2)!}{\pi^n} \int_{B_n} \Big(\frac{1}{|z|^{2n-2}} - 1\Big) \Omega_\varphi(z) \Big| \det \Big(\frac{\partial \varphi}{\partial z}\Big)\Big|^2 \mathrm{d}\tau_z + \sum_{i=1}^{n} |\varphi_i(0)|^2 \\
&\geq \frac{a(n-2)!}{\pi^n} \int_{B_n} \Big(\frac{1}{|z|^{2n-2}} - 1\Big) \Big| \det \Big(\frac{\partial \varphi}{\partial z}\Big)\Big|^2 \mathrm{d}\tau_z + \sum_{i=1}^{n} |\varphi_i(0)|^2 \\
&= \frac{2a(n-1)!}{\pi^n} \int_{B_n} N_\varphi(w) \mathrm{d}\tau_w + \sum_{i=1}^{n} |\varphi_i(0)|^2,
\end{aligned}
$$

i.e,

$$
\int_{B_n} N_\varphi(w) \mathrm{d}\tau_w \leq \frac{\Big(1 - \sum_{i=1}^{n} |\varphi_i(0)|^2\Big)\pi^n}{2a(n-1)!} \leq \frac{\pi^n}{2a(n-1)!}. \tag{5.6}
$$

If

$$
\limsup_{|w| \to 1^-} \frac{N_\varphi(w)}{1 - |w|} = c,
$$

then, for any fixed $\varepsilon > 0$, we can find an r with $\frac{1}{2} \leq r < 1$, such that

$$
\frac{N_\varphi(w)}{1 - |w|} < c + \varepsilon,
$$

when $|w| \geq r$.

Using (5.4), we obtain

$$
\begin{aligned}
\|f \circ \varphi\|^2 &\leq \frac{2b(n-1)!}{\pi^n} \int_{B_n} N_\varphi(w) |\mathrm{grad}\, f|^2 \mathrm{d}\tau_w + |f(\varphi(0))|^2 \\
&= \frac{2b(n-1)!}{\pi^n} \int_{B_n \setminus rB_n} N_\varphi(w) |\mathrm{grad}\, f|^2 \mathrm{d}\tau_w \\
&\quad + \frac{2b(n-1)!}{\pi^n} \int_{rB_n} N_\varphi(w) |\mathrm{grad}\, f|^2 \mathrm{d}\tau_w + |f(\varphi(0))|^2 \\
&\leq \frac{2b(n-1)!}{\pi^n} \int_{B_n \setminus rB_n} \frac{N_\varphi(w)}{1 - |w|} |\mathrm{grad}\, f|^2 (1 - |w|) \mathrm{d}\tau_w \\
&\quad + \frac{2b(n-1)!}{\pi^n} \int_{rB_n} N_\varphi(w) \frac{(2n^2 + n)\|f\|^2}{(1 - |w|^2)^{n+2}} \mathrm{d}\tau_w + |f(\varphi(0))|^2 \\
&\leq \frac{2b(n-1)!}{\pi^n} (c + \varepsilon) \int_{B_n \setminus rB_n} |\mathrm{grad}\, f|^2 (1 - |w|) \mathrm{d}\tau_w \\
&\quad + \frac{2b(n-1)!}{\pi^n} \frac{(2n^2 + n)\|f\|^2}{(1 - r^2)^{n+2}} \int_{rB_n} N_\varphi(w) \mathrm{d}\tau_w + |f(\varphi(0))|^2 \\
&\leq \frac{2b(n-1)!}{\pi^n} (c + \varepsilon) \frac{\pi^n}{(n-2)!} \|f\|^2 + \frac{4b(n+1)!\|f\|^2}{\pi^n (1 - r^2)^{n+2}} \frac{\pi^n}{2a(n-1)!} + |f(\varphi(0))|^2 \\
&\leq 2b(n-1)(c + \varepsilon)\|f\|^2 + \frac{2bn(n+1)}{a(1 - r^2)^{n+2}} \|f\|^2 + \frac{\|f\|^2}{(1 - |\varphi(0)|^2)^n},
\end{aligned}
$$

where the second inequality is provided by (4.2), and the final two inequalities are provided by (5.5)–(5.6) and (4.1), respectively. Therefore,

$$\|f\circ\varphi\|^2 \le \Big(2b(n-1)(c+\varepsilon)+\frac{2bn(n+1)}{a(1-r^2)^{n+2}}+\frac{1}{(1-|\varphi(0)|^2)^n}\Big)\|f\|^2.$$

So, we prove that C_φ is bounded on $H^2(B_n)$.

We now estimate the essential norm of C_φ by Proposition 4.1.

By (5.4), for $\frac{1}{2}\le r<1$, we get

$$\begin{aligned}
\|C_\varphi R_p f\|^2 &\le \frac{2b(n-1)!}{\pi^n}\int_{B_n} N_\varphi(w)|\mathrm{grad} R_p f|^2 \mathrm{d}\tau_w + |R_p f(\varphi(0))|^2\\
&\le \frac{2b(n-1)!}{\pi^n}\int_{B_n\setminus rB_n} N_\varphi(w)|\mathrm{grad} R_p f|^2 \mathrm{d}\tau_w\\
&\quad + \frac{2b(n-1)!}{\pi^n}\int_{rB_n} N_\varphi(w)|\mathrm{grad} R_p f|^2 \mathrm{d}\tau_w\\
&\quad + \frac{(2+p)^{n-1}|\varphi(0)|^{2(p+1)}}{(1-|\varphi(0)|^2)^n}\|f\|^2\\
&\le \frac{2b(n-1)!}{\pi^n}\int_{B_n\setminus rB_n} N_\varphi(w)|\mathrm{grad} R_p f|^2 \mathrm{d}\tau_w\\
&\quad + \frac{4b(n+1)!(2+p)^{(n+1)}|r|^{2p}}{\pi^n(1-r^2)^{n+2}}\|f\|^2\int_{rB_n} N_\varphi(w)\mathrm{d}\tau_w + \frac{(2+p)^{n-1}|\varphi(0)|^{2(p+1)}}{(1-|\varphi(0)|^2)^n}\|f\|^2\\
&\le \frac{2b(n-1)!}{\pi^n}\int_{B_n\setminus rB_n} N_\varphi(w)|\mathrm{grad} R_p f|^2 \mathrm{d}\tau_w + \frac{2bn(n+1)(2+p)^{(n+1)}|r|^{2p}}{a(1-r^2)^{n+2}}\|f\|^2\\
&\quad + \frac{(2+p)^{n-1}|\varphi(0)|^{2(p+1)}}{(1-|\varphi(0)|^2)^n}\|f\|^2\\
&\le \frac{2b(n-1)!}{\pi^n}\int_{B_n\setminus rB_n} \frac{N_\varphi(w)}{1-|w|}|\mathrm{grad}\, f|^2(1-|w|) \mathrm{d}\tau_w\\
&\quad + \frac{2bn(n+1)(2+p)^{(n+1)}|r|^{2p}}{a(1-r^2)^{n+2}}\|f\|^2 + \frac{(2+p)^{n-1}|\varphi(0)|^{2(p+1)}}{(1-|\varphi(0)|^2)^n}\|f\|^2\\
&\le \frac{2b(n-1)!}{\pi^n}\sup_{r\le|w|<1}\frac{N_\varphi(w)}{1-|w|}\int_{B_n\setminus rB_n} |\mathrm{grad}\, f|^2(1-|w|) \mathrm{d}\tau_w\\
&\quad + \frac{2bn(n+1)(2+p)^{(n+1)}|r|^{2p}}{a(1-r^2)^{n+2}}\|f\|^2 + \frac{(2+p)^{n-1}|\varphi(0)|^{2(p+1)}}{(1-|\varphi(0)|^2)^n}\|f\|^2\\
&\le 2b(n-1)\sup_{r\le|w|<1}\frac{N_\varphi(w)}{1-|w|}\|f\|^2 + \frac{2bn(n+1)(2+p)^{(n+1)}|r|^{2p}}{a(1-r^2)^{n+2}}\|f\|^2\\
&\quad + \frac{(2+p)^{n-1}|\varphi(0)|^{2(p+1)}}{(1-|\varphi(0)|^2)^n}\|f\|^2,
\end{aligned}$$

where the second, third and fourth inequalities are provided by (4.3)–(4.4) and (5.6), respectively, and the final inequality is provided by (5.5). Hence,

$$\frac{\|C_\varphi R_p f\|^2}{\|f\|^2}\le 2b(n-1)\sup_{r\le|w|<1}\frac{N_\varphi(w)}{1-|w|} + \frac{2bn(n+1)(2+p)^{(n+1)}|r|^{2p}}{a(1-r^2)^{n+2}} + \frac{(2+p)^{n-1}|\varphi(0)|^{2(p+1)}}{(1-|\varphi(0)|^2)^n},$$

i.e,

$$\|C_\varphi R_p\|^2 \le 2b(n-1)\sup_{r\le|w|<1}\frac{N_\varphi(w)}{1-|w|}+\frac{2bn(n+1)(2+p)^{(n+1)}|r|^{2p}}{a(1-r^2)^{n+2}}+\frac{(2+p)^{n-1}|\varphi(0)|^{2(p+1)}}{(1-|\varphi(0)|^2)^n}.$$

For each fixed r $\left(\frac{1}{2}\le r<1\right)$, letting $p\to+\infty$, we obtain

$$\|C_\varphi\|_e^2 \le 2b(n-1)\sup_{r\le|w|<1}\frac{N_\varphi(w)}{1-|w|}.$$

Hence,

$$\|C_\varphi\|_e^2 \le 2b(n-1)\limsup_{|w|\to1^-}\frac{N_\varphi(w)}{1-|w|},$$

when r tends to 1. If

$$\lim_{|w|\to1^-}\frac{N_\varphi(w)}{1-|w|}=0,$$

then $\|C_\varphi\|_e=0$, and C_φ is a compact operator.

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