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Smoothing Effects for the Classical Solutions to the Landau-Fermi-Dirac Equation*

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Abstract The smoothness of the solutions to the full Landau equation for Fermi-Dirac particles is investigated. It is shown that the classical solutions near equilibrium to the Landau-Fermi-Dirac equation have a regularizing effects in all variables (time, space and velocity), that is, they become immediately smooth with respect to all variables.

Keywords Landau-Fermi-Dirac equation, Classical solutions, Smoothing effect
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1 Introduction and the Statement of Our Main Results

In this paper, we study the regularity of the solutions to the Landau-Fermi-Dirac (LFD) equation for the Pauli exclusion principle which reads

$$\begin{aligned} &\partial_t f + v \cdot \nabla_x f = Q(f, f), \quad t > 0, \\ &f(0, x, v) = f_0(x, v), \end{aligned} \tag{1.1}$$

where $f(t, x, v) \geq 0$ is the spatially periodic distribution function for the particles at time $t \geq 0$, with spatial coordinates $x = (x_1, x_2, x_3) \in [-\pi, \pi]^3 = \mathbb{T}^3$ and velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. $Q(f, f)$ is the nonlinear collision operator defined by

$$Q(f, f) = \sum_{i,j=1}^{3} \partial_{v_i} \int_{\mathbb{R}^3} \psi^{ij}(v - u)[(1 - f(u))f(u)\partial_{v_j} f(v) - (1 - f(v))f(v)\partial_{u_j} f(u)]\mathrm{d}u.$$

The non-negative matrix ψ is

$$\psi^{ij}(v) = \Big\{\delta^{ij} - \frac{v_i v_j}{|v|^2}\Big\}|v|^{\gamma+2}.$$

Here, γ is a parameter leading to the standard classification of the hard potential ($\gamma > 0$), Maxwellian molecule ($\gamma = 0$) or soft potential ($\gamma < 0$) (cf. [11]). In particular, $\gamma = -3$ corresponds to the Coulomb interaction in plasma physics. We recall that the Coulomb potential is, however, the only one to have a physical relevance. In this paper, we restrict our discussion to the case $-3 \leq \gamma < -2$.

The Landau equation, which was proposed by Landau in 1936, was formally obtained in a singular limit of the Boltzmann equation (cf. [2, 6, 12]). Sometimes, the quantum effects such as

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the Pauli exclusion principle would be taken into account and both the Boltzmann and Landau equations have to be modified (cf. [6, 10, 14, 19]). Among them, the Boltzmann-Fermi-Dirac (BFD) equation and the LFD equation are two typical models. We mention that using a new sequence of cross-sections in the BFD operator and taking a limit (grazing collision limit) lead to the quantum LFD operator (cf. [10, 17]). While the classical Landau equations are the subject of several papers [6, 9, 11, 13, 16, 20] and the references therein, few studies were devoted to the LFD equation. For the LFD equation, a formal derivation from the BFD equation in the grazing collisions limit and a spectral analysis of its linearization near an equilibrium were studied in [10] and [17] respectively. In the spatially homogenous setting, the well-posedness of the Cauchy problem was considered in [3] and the equilibrium states were given in [4]. For the spatially inhomogeneous case, very recently, the global-in-time classical solutions near equilibrium have been constructed in [18].

We are now concerned with the regularity issues. In the spatially homogeneous setting, the regularity of the solutions to the Landau equation was investigated by Arsen'ev-Buryak [2] in the Coulomb case. The instantaneous smoothing effect was shown by Desvillettes-Villani [13], El Safadi [15] and Chen-Li-Xu [7] for not necessarily smooth initial data in the case of hard potentials. In the spatially inhomogeneous setting, recently, Chen-Desvillettes-He [9] and also Alexandre et al. [1] have developed independent machinery to study these general smoothing effects for kinetic equations. We note that the well-posedness results of the above mentioned equations can be found in the references of the corresponding papers.

As far as we know, for the regularity properties of the LFD equation, very few results are available. We would like to mention that Chen [8] got the smoothing effect of Bagland's weak solutions (cf. [3]) to the spatially homogeneous LFD equation for hard potentials. Our goal in this paper is to obtain the regularity effect of the spatially inhomogeneous LFD equation. We obtain the smoothness in the velocity variable by using energy methods and the dissipative property of the LFD collision operator, where smoothness in the velocity variable was obtained by the elliptic property of the diffusive matrix to the LFD collision operator. Since the LFD nonlinear operator Q can be written as the diffusive operator and some error terms, smoothness in the position variable can be shown by using the classical averaging lemma (cf. [5]). Lastly, we prove the smoothness in the time variable as [9] and deduce the smoothness in all variables by the iterative methods in [1, 8–9]. Although our main results are proved by using the novel idea of [9], there are two main difficulties in this paper. The first new difficulty is due to the complexity of the nonlinear term $f(1-f)$. The L^∞-norm of f is repeatedly used to overcome this difficulty. The second one is to obtain the lower bound of $\sum\limits_{ij}\overline{a}_{ij}\xi_i\xi_j$, and unlike the classical Landau equation, the smallness of the solutions in the norm $H^s_{x,v}$ and the Sobolev embedding as well as the velocity splitting method introduced in [13] guarantee the elliptic property of $\overline{a}_{ij}$. We have to detailedly use the property of the quadratic term $f(1-f)$ and the Pauli exclusion principle to get the corresponding estimates.

Now we introduce some notations and definitions. For simplicity, we omit the integrating domains $\mathbb{T}^3$ and $\mathbb{R}^3$, which correspond to variables x and v, respectively. For example, we write $L^2_{x,v}$ instead of $L^2_x(\mathbb{T}^3; L^2_v(\mathbb{R}^3))$. For $s \in \mathbb{R}$, we use the standard notation H^s to denote the usual Sobolev space, and use $\dot{H}^s$ to denote the homogeneous Sobolev space. For any integer

$N \geq 0$, letting $l \geq 0$, we define the weighted Sobolev space

$$H^{N,l}_{x,v} = \Big\{ f(x,v) : \sum_{|\alpha|+|\beta|\leq N} \|(\partial^\alpha_\beta f)(1+|v|^2)^{\frac{l}{2}}\|_{L^2_{x,v}} < +\infty \Big\},$$

where $\alpha = [\alpha_1, \alpha_2, \alpha_3]$, $\beta = [\beta_1, \beta_2, \beta_3]$ denote multi-index with length $|\alpha|$ and $|\beta|$, respectively, and

$$\partial^\alpha_\beta = \partial^{\alpha_1}_{x_1} \partial^{\alpha_2}_{x_2} \partial^{\alpha_3}_{x_3} \partial^{\beta_1}_{v_1} \partial^{\beta_2}_{v_2} \partial^{\beta_3}_{v_3}.$$

Furthermore, define $\beta' \leq \beta$ if no component of β' is greater than the component of β, and $\beta' < \beta$, if $\beta' \leq \beta$ and $|\beta'| < |\beta|$. It is obvious that $H^{N,0}_{x,v} = H^N_{x,v}$. We also define $H^\infty_{x,v}$ and $H^{\infty,l}_{x,v}$ as

$$H^\infty_{x,v} = \bigcup_{N\geq 0} H^N_{x,v}, \quad H^{\infty,l}_{x,v} = \bigcap_{N\geq 0} H^{N,l}_{x,v}.$$

For $k \in \mathbb{Z}^+$, we also use the following notations in this paper:

$$\|f(1+|v|^2)^{\frac{l}{2}}\|_{H^k_{x,v}} = \sum_{|\alpha|+|\beta|\leq k} \|(\partial^\alpha_\beta f)(1+|v|^2)^{\frac{l}{2}}\|_{L^2_{x,v}}.$$

For the fractional order Sobolev space $H^s(\mathbb{T}^3)$ $(0 < s < 1)$, more direct characterizations come from considering the L^2-modulus of continuity. Given a point $k \in \mathbb{T}^3$, $f \in H^s$, we define

$$\triangle_k f = f(x+k) - f(x).$$

From now on, we use C or c to denote a generic positive constant that may be different from line to line. $A \sim B$, means $cA \leq B \leq \frac{A}{c}$ for a generic constant $0 < c < 1$.

For $1 \leq i, j \leq 3$, we define

$$\overline{a}_{ij}(t,x,v) = [\psi^{ij} * (f(1-f))], \quad \overline{b}_i(t,x,v) = \sum_j (\partial_{v_j} \psi^{ij}) * f.$$

Then equation (1.1) can be rewritten as

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\overline{a} \nabla_v f - \overline{b} f(1-f)), \tag{1.2}$$

where $\overline{a} = (\overline{a}_{ij})$, $\overline{b} = (\overline{b}_i)$.

We denote a normalized global Maxwellian by $M(v) = (2\pi)^{-\frac{3}{2}} \exp\big(-\frac{|v|^2}{2}\big)$. It is easy to check that (1.1) has a stationary solution $M_q(v) = \frac{M(v)}{1+M(v)}$. We introduce the standard perturbation $f(t,x,v)$ with respect to M_q as $f = M_q + \sqrt{\widetilde{M}} F$, where a suitable choice of $\widetilde{M}$ is $\widetilde{M}(v) = \frac{M(v)}{(1+M(v))^2} = M_q(v)(1 - M_q(v))$.

The total conversation laws often play an important role in the study of the existence of the solution to the kinetic equation (such as Boltzmann equation, Landau equation, etc.) over the bounded domain, because the Poincaré inequality is able to be applied (cf. [16, 18]). By assuming that $f_0(x,v)$ has the same mass, moment and energy as the Fermi-Dirac function M_q, we can get the conservation laws as

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} F(t,x,v) \sqrt{M_q} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \mathrm{d}x\mathrm{d}v \equiv 0. \tag{1.3}$$

Throughout this paper, we shall assume $N \geq 8$. The results of [18] can be summarized as follows.

Theorem 1.1 *Let $1 \geq f(0,x,v) = M_q + \sqrt{\widetilde{M}}F_0(x,v) \geq 0$ and $-3 \leq \gamma < -2$. Assuming that $F_0(x,v)$ satisfies (1.3), then there exists a suitably small constant $\epsilon_0 > 0$, such that if $\sqrt{\mathcal{E}_N(F_0)} \leq \epsilon_0$, equation (1.1) has a unique global classical solution $f(t,x,v)$ satisfying $1 \geq f(t,x,v) = M_q + \sqrt{\widetilde{M}}F(t,x,v) \geq 0$. Moreover, there exists a $C_0 > 0$ (depending on γ, N, ϵ_0), such that*

$$\mathcal{E}_N(F)(t) \leq C_0 \mathcal{E}_N(F_0)$$

for any $t \geq 0$, where $\mathcal{E}_N(F)(t)$ is defined as

$$\mathcal{E}_N(F)(t) \sim \sum_{|\alpha|+|\beta| \leq N} \|(\partial^\alpha_\beta F)(1+|v|^2)^{\frac{(\gamma+2)|\beta|}{2}}\|^2_{L^2_{x,v}}.$$

Our main result shows that the classical solution to equation (1.1) belonging to $H^8_{x,v}$ lies in fact in $C^\infty_{x,v}$ for any time $t > 0$.

Theorem 1.2 *Let $-3 \leq \gamma < -2$, there exists a small constant $\epsilon_0 > 0$, such that if $\sqrt{\mathcal{E}_8(F_0)} \leq \epsilon_0$, the unique classical nonnegative solution to equation (1.1) given by Theorem 1.1 satisfies (for any $0 < \tau_1 < \tau < T < \infty$ and $l \geq 0$) :*

$$f \in C^\infty_t\Big([\tau,T]; \bigcap_{l \geq 0} H^{\infty,l}_{x,v}(\mathbb{T}^3 \times \mathbb{R}^3)\Big).$$

Remark 1.1 Our main results in Theorems 1.1 and 1.2 are concerned with the soft potential case, i.e., $-3 \leq \gamma < -2$, and in this case, as in [16], a special weight was designed to derive the global existence. And to obtain the regularity properties listed in Theorem 1.2, one has to kill the singularity caused by the collision kernel ψ. In this paper, we mainly use Lemma 2.2 to overcome this difficulty.

The rest of this paper is arranged as follows. In Section 2, we give some basic estimates for later use. Section 3 is devoted to the regularity of x and v in the case $\gamma \in [-3,-2)$. Theorem 1.2 is proved in Section 4.

2 Preliminaries

In this section, we give some preliminary lemmas which will be used in the proof of Theorem 1.2. Firstly, we introduce the following lemma about the fractional order Sobolev space.

Lemma 2.1 *Let $0 < \delta < 1$. Then $f(x) \in H^\delta(\mathbb{T}^n)$, if and only if $f(x) \in L^2(\mathbb{T}^n)$ and $\|f_{\delta,k}\|_{L^2_{x,k}} < +\infty$, where $f_{\delta,k} = (\triangle_k f)|k|^{-\frac{n}{2}-\delta}$, i.e., there exists a $c(n) > 0$, such that*

$$\|f_{\delta,k}\|^2_{L^2_{x,k}} = c(n) \sum_{m \in \mathbb{Z}^n} |m|^{2\delta} |\widehat{f}(m)|^2,$$

where $\widehat{f}(m) = \int_{\mathbb{T}^n} f(x) \mathrm{e}^{-\mathrm{i}m\cdot x} \mathrm{d}x$ is the m-th Fourier coefficient of f with respect to the x variable.

Moreover, it holds that

$$\int_{\mathbb{T}^n} |\widehat{f}_{\delta,k}|^2 \mathrm{d}k = c(n)|m|^{2\delta}|\widehat{f}(m)|^2. \tag{2.1}$$

The following lemma is devoted to a basic estimate of the convolution of two functions.

Lemma 2.2 *Let $0 < l_1 < n$, $l_2 > n$, $v \in \mathbb{R}^n$. Assume that there is a constant $C > 0$, such that*

$$|f(v)| \le C|v|^{-l_1}, \quad |g(v)| \le C(1+|v|)^{-l_2}.$$

Then there is a $C_1 > 0$, depending on l_1, l_2, C and n, such that

$$|[f * g](v)| \le C_1(1+|v|)^{-l_1}.$$

Proof Noticing that

$$f * g = \int_{\mathbb{R}^n} f(v-u)g(u)\mathrm{d}u = \int_{|v-u|>\frac{1}{2}[1+|v|]} + \int_{|v-u|\le\frac{1}{2}[1+|v|]}.$$

The first integral is bounded by

$$C\Big(\frac{1}{2}[1+|v|]\Big)^{-l_1}\int_{\mathbb{R}^n}(1+|u|)^{-l_2}\mathrm{d}u \le C_1(1+|v|)^{-l_1}.$$

On the other hand, since $1+|u| \ge 1+|v|-|v-u| \ge \frac{1}{2}[1+|v|]$ in the second part, the second integral is bounded by

$$C\Big(\frac{1}{2}[1+|v|]\Big)^{-l_2}\int_{|v-u|\le\frac{1}{2}[1+|v|]}|v-u|^{-l_1}\mathrm{d}u \le C\Big(\frac{1}{2}[1+|v|]\Big)^{-l_2-l_1+n}.$$

Therefore Lemma 2.2 is valid.

The next lemma shows the positivity of the operator $\overline{a}_{ij}$.

Lemma 2.3 *Let $-3 \le \gamma < -2$, $N \ge 8$ and f be a nonnegative classical solution to equation* (1.1) *given by Theorem* 1.1. *If $\sqrt{\mathcal{E}_N(F_0)} \le \epsilon_0$ for ϵ_0 small enough, there exists a constant $K > 0$, depending on N, ϵ_0 and γ, such that for any $t \in \mathbb{R}_+$, $x \in \mathbb{T}^3$, $v \in \mathbb{R}^3$ and $\xi \in \mathbb{R}^3$,*

$$\sum_{ij}\overline{a}_{ij}(t,x,v)\xi_i\xi_j \ge K(1+|v|^2)^{\frac{\gamma}{2}}|\xi|^2. \tag{2.2}$$

Proof Our proof is carried out by borrowing the method of the proof of Proposition 4 in [13]. In view of Theorem 1.1, we get from Sobolev's embedding theorem that there is a constant S (depending on γ, N and ϵ_0), such that

$$\|\sqrt{\widetilde{M}}F\|_{L^\infty_t([0,+\infty);L^\infty_{x,v})} \le S\epsilon_0,$$

which implies that for $|v| \le R$ (R will be chosen later),

$$\frac{1}{2}(2\pi)^{-\frac{3}{2}}\mathrm{e}^{-\frac{R^2}{2}} - S\epsilon_0 \le f \le (2\pi)^{-\frac{3}{2}} + S\epsilon_0.$$

Choosing $0 < \epsilon_0 < \frac{1}{2}(2\pi)^{-\frac{3}{2}}(1+S)^{-1} < \frac{1}{2}$ and $R = R_0 = \sqrt{-2\ln(2(2\pi)^{\frac{3}{2}}\epsilon_0(1+S))}$, we obtain

$$\epsilon_0 \le f \le 1-\epsilon_0 \quad \text{for } |v| \le R_0.$$

By setting $\epsilon_1 = \epsilon_0(1-\epsilon_0)$, we further get that

$$f(1-f) \geq \epsilon_1 \quad \text{for } |v| \leq R_0.$$

After getting the lower bound of $f(1-f)$ in the case $|v| \leq R_0$, and then performing the similar calculations as [13, Proposition 4], we can obtain (2.2), which completes the proof of Lemma 2.3.

Now we present estimates for coefficients $\overline{a}_{ij}$ and $\overline{b}_i$, which will be used repeatedly in the proof of Theorem 1.2.

Lemma 2.4 *Let $-3 \leq \gamma < -2$. Then there exists a positive constant C which depends only on γ, such that for all nonnegative $f = f(t,x,v)$ for which the derivatives are defined, for any multi-index α, β and Ω interval included in $[0, +\infty)$, we have*

$$\begin{aligned}&\|\partial_\beta^\alpha \overline{a}_{ij}(t,x,\cdot)\|_{L_t^\infty(\Omega;L_x^\infty)}(v)\\ &\leq C \sum_{|\alpha_1|+|\beta_1|\leq[\frac{1}{2}(|\alpha|+|\beta|)]} (1+\|\partial_{\beta_1}^{\alpha_1} f\|_{L_t^\infty(\Omega;H_{x,v}^4)})\|(\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f)(1+|v|^2)\|_{L_t^\infty(\Omega;H_{x,v}^2)}\end{aligned} \tag{2.3}$$

and

$$\|\partial_\beta^\alpha \overline{b}_i(t,x,\cdot)\|_{L_t^\infty(\Omega;L_x^\infty)}(v) \leq C \sum_{|\sigma|=1} \|(\partial_{\beta+\sigma}^\alpha f)(1+|v|^2)\|_{L_t^\infty(\Omega;H_{x,v}^2)}. \tag{2.4}$$

Proof We only prove (2.3). Letting $|\alpha_1|+|\beta_1| \leq \frac{1}{2}(|\alpha|+|\beta|)$, we write

$$\partial_\beta^\alpha \overline{a}_{ij}(v) = \psi^{ij} * (\partial_\beta^\alpha f)(v) - \sum_{\substack{\alpha_1\leq\alpha\\ \beta_1\leq\beta}} C_\alpha^{\alpha_1} C_\beta^{\beta_1} \psi^{ij} * [\partial_{\beta_1}^{\alpha_1} f \partial_{\beta-\beta_1}^{\alpha-\alpha_1} f](v).$$

For brevity, we compute only the complicate term involving $\psi^{ij} * [\partial_\beta^\alpha (f^2)](v)$. By Sobolev's imbedding and Minkowski's inequality, we get

$$\begin{aligned}&\|\psi^{ij} * [\partial_{\beta_1}^{\alpha_1} f \partial_{\beta-\beta_1}^{\alpha-\alpha_1} f]\|_{L_t^\infty(\Omega;L_x^\infty)}(v)\\ &\leq C\|\partial_{\beta_1}^{\alpha_1} f\|_{L_t^\infty(\Omega;H_{x,v}^4)} \sup_{t\in\Omega} \int_{\mathbb{R}^3} |v-u|^{\gamma+2} \Big(\sum_{|\sigma|\leq 2} \int_{\mathbb{T}^3} |\partial_{\beta-\beta_1}^{\alpha-\alpha_1+\sigma} f|^2 \mathrm{d}x\Big)^{\frac{1}{2}} \mathrm{d}u\\ &\leq C\|\partial_{\beta_1}^{\alpha_1} f\|_{L_t^\infty(\Omega;H_{x,v}^4)} \Big(\int_{\mathbb{R}^3} |v-u|^{2(\gamma+2)}(1+|u|^2)^{-2} \mathrm{d}u\Big)^{\frac{1}{2}} \|(\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f)(1+|v|^2)\|_{L_t^\infty(\Omega;H_{x,v}^2)}\\ &\leq C\|\partial_{\beta_1}^{\alpha_1} f\|_{L_t^\infty(\Omega;H_{x,v}^4)} \|(\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f)(1+|v|^2)\|_{L_t^\infty(\Omega;H_{x,v}^2)}.\end{aligned} \tag{2.5}$$

Thus (2.3) holds.

Observing that $\partial_\beta^\alpha \overline{b}_i(v) = \sum\limits_j \psi^{ij} * \partial_\beta^\alpha \partial_{v_j} f$, as a consequence, we can easily get (2.4) by following the proof of (2.3).

3 Gain of Regularity in v and x

3.1 Gain of regularity in v

We prove the following lemma in this subsection.

Lemma 3.1 *Let $-3 \le \gamma < -2$, $N \ge 8$ be a given integer, and f be a smooth nonnegative solution to equation* (1.1) *given by Theorem* 1.1. *We suppose that for any $T > 0$, $l \ge 0$, there exists $0 \le \tau_1 < T$ $(N = 8, \tau_1 = 0)$ such that*

$$\|f(\tau_1, x, v)\|_{H^{N,l}_{x,v}} \le K_0, \quad \|f\|_{L^\infty_t([\tau_1,T];H^{N-1,l}_{x,v})} \le K_0, \quad \|f\|_{L^2_t([\tau_1,T];H^{N,l}_{x,v})} \le K_0,$$

where $K_0 = K_0(l, T, \gamma, N)$ is a suitably small constant.

Then there exists a constant $\widetilde{C}_1 > 0$, which depends on N, l, γ, T, K_0, such that

$$\sup_{t\in[\tau_1,T]} \int_{\mathbb{T}^3\times\mathbb{R}^3} |\partial^\alpha_\beta f|^2(1 + |v|^2)^l \mathrm{d}x\mathrm{d}v + \int_{[\tau_1,T]\times\mathbb{T}^3\times\mathbb{R}^3} |\nabla_v \partial^\alpha_\beta f|^2(1 + |v|^2)^{l+\frac{\gamma}{2}} \mathrm{d}x\mathrm{d}v\mathrm{d}t \le \widetilde{C}_1$$

with $|\alpha| + |\beta| \le N$.

Proof Denote

$$h = (\partial^\alpha_\beta f)(1 + |v|^2)^{\frac{l}{2}}. \tag{3.1}$$

Since equation (1.1) is equivalent to (1.2), we know from Leibniz's formula that h satisfies the following equation (denoting $g = \partial^\alpha_\beta f$ and $\widetilde{f} = f(1 - f)$):

$$\partial_t h + v \cdot \nabla_x h = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}), \tag{3.2}$$

where

$$\begin{aligned}
(\mathrm{I}) &= -\rho_{|\beta|} C^{\beta_1}_\beta (\partial_{\beta_1} v \cdot \nabla_x \partial^\alpha_{\beta-\beta_1} f)(1 + |v|^2)^{\frac{l}{2}}, \\
(\mathrm{II}) &= \partial_{v_i}(\overline{a}_{ij}\partial_{v_j} h) - l\overline{a}_{ij}(\partial_{v_i} g)(1 + |v|^2)^{\frac{l}{2}-1} v_i - \partial_{v_i}[l\overline{a}_{ij} g(1 + |v|^2)^{\frac{l}{2}-1} v_j] \\
&\quad - \partial_{v_i}[\overline{b}_i(\partial^\alpha_\beta \widetilde{f})(1 + |v|^2)^{\frac{l}{2}}] + l\overline{b}_i(\partial^\alpha_\beta \widetilde{f})(1 + |v|^2)^{\frac{l}{2}-1} v_i \\
&= \sum_{k=1}^5 (\mathrm{II})_k, \\
(\mathrm{III}) &= \rho_{|\alpha|+|\beta|} \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta \\ |\alpha_1|+|\beta_1|\ge 1}} C^{\alpha_1}_\alpha C^{\beta_1}_\beta \{\partial_{v_i}[(\partial^{\alpha_1}_{\beta_1}\overline{a}_{ij})(\partial_{v_j}\partial^{\alpha_2}_{\beta_2} f)(1 + |v|^2)^{\frac{l}{2}}] \\
&\quad - l(\partial^{\alpha_1}_{\beta_1}\overline{a}_{ij})(\partial_{v_i}\partial^{\alpha_2}_{\beta_2} f)(1 + |v|^2)^{\frac{l}{2}-1} v_i \\
&\quad - \partial_{v_i}[(\partial^{\alpha_1}_{\beta_1}\overline{b}_i)(\partial^{\alpha_2}_{\beta_2}\widetilde{f})(1 + |v|^2)^{\frac{l}{2}}] + l(\partial^{\alpha_1}_{\beta_1}\overline{b}_i)(\partial^{\alpha_2}_{\beta_2}\widetilde{f})(1 + |v|^2)^{\frac{l}{2}-1} v_i\} \\
&= \sum_{k=1}^4 (\mathrm{III})_k.
\end{aligned}$$

Here $(\mathrm{II})_k$ and $(\mathrm{III})_k$ $(k = 1, 2, \cdots)$ denote the corresponding terms in (II) and (III), and ρ_m is defined as

$$\rho_m = \begin{cases} 1, & m > 0, \\ 0, & m = 0. \end{cases}$$

In the following, we perform the standard energy estimates for equation (3.2). Multiplying equation (3.2) by h, and then integrating on t, x, v over $[\tau_1, T] \times \mathbb{T}^3 \times \mathbb{R}^3$, we shall estimate the resulting equation term by term.

We see that

$$\begin{aligned}
&\int_{[\tau_1,T]\times\mathbb{T}^3\times\mathbb{R}^3}(\partial_t h+v\cdot\nabla_x h)h\mathrm{d}t\mathrm{d}x\mathrm{d}v\\
&=\frac{1}{2}(\|h(T)\|^2_{L^2_{x,v}}-\|h(\tau_1)\|^2_{L^2_{x,v}})\\
&\geq\frac{1}{2}\|h(T)\|^2_{L^2_{x,v}}-\frac{1}{2}K_0^2.
\end{aligned}\tag{3.3}$$

For the term containing (I), we get from Hölder's inequality that

$$\Big|\int_{[\tau_1,T]\times\mathbb{T}^3\times\mathbb{R}^3}(\mathrm{I})h\mathrm{d}t\mathrm{d}x\mathrm{d}v\Big|\leq C\|h\|_{L^2_t([\tau_1,T];L^2_{x,v})}\|f\|_{L^2_t([\tau_1,T];H^{N,l}_{x,v})}\leq CK_0^2.$$

For the term containing $(\mathrm{II})_1$, utilizing integration by parts, we obtain

$$\begin{aligned}
\int_{[\tau_1,T]\times\mathbb{T}^3\times\mathbb{R}^3}(\mathrm{II})_1h\mathrm{d}t\mathrm{d}x\mathrm{d}v&=-\int_{[\tau_1,T]\times\mathbb{T}^3\times\mathbb{R}^3}\overline{a}_{ij}\partial_{v_i}g\partial_{v_j}g(1+|v|^2)^l\mathrm{d}t\mathrm{d}x\mathrm{d}v\\
&\quad-2l\int_{[\tau_1,T]\times\mathbb{T}^3\times\mathbb{R}^3}\overline{a}_{ij}(\partial_{v_i}g)g(1+|v|^2)^{l-1}v_j\mathrm{d}t\mathrm{d}x\mathrm{d}v\\
&\quad-l^2\int_{[\tau_1,T]\times\mathbb{T}^3\times\mathbb{R}^3}\overline{a}_{ij}g^2(1+|v|^2)^{l-2}v_jv_i\mathrm{d}t\mathrm{d}x\mathrm{d}v\\
&=\sum_{i=1}^3A_i,
\end{aligned}\tag{3.4}$$

where A_i $(i=1,2,3)$ denote the three terms on the right-hand side of (3.4). By Lemma 2.3, we get

$$A_1\leq-K\int_{[\tau_1,T]\times\mathbb{T}^3\times\mathbb{R}^3}(1+|v|^2)^{\frac{\gamma}{2}}|(\nabla_v g)(1+|v|^2)^{\frac{l}{2}}|^2\mathrm{d}t\mathrm{d}x\mathrm{d}v.$$

Thanks to (2.3) in Lemma 2.4 and Cauchy-Schwarz's inequality with ε, we discover that

$$\begin{aligned}
A_2&\leq C\int_{[\tau_1,T]\times\mathbb{T}^3\times\mathbb{R}^3}(1+|v|^2)^l|\nabla_v g||g|\mathrm{d}t\mathrm{d}x\mathrm{d}v(1+\|f\|_{L^\infty_t([\tau_1,T];H^4_{x,v})})\|f\|_{L^\infty_t([\tau_1,T];H^{2,2}_{x,v})}\\
&\leq C(1+K_0)K_0\{\varepsilon\|(\nabla_v g)(1+|v|^2)^{\frac{l}{2}+\frac{\gamma}{4}}\|^2_{L^2_t([\tau_1,T];L^2_{x,v})}+C(\varepsilon)K_0^2\}.
\end{aligned}$$

For A_3, we get from (2.3) that

$$|A_3|\leq C(1+K_0)K_0\|f\|^2_{L^2_t([\tau_1,T];H^{N,l}_{x,v})}\leq C(1+K_0)K_0^3.$$

The estimates for the terms containing $(\mathrm{II})_2,(\mathrm{II})_3,(\mathrm{II})_4$ and $(\mathrm{II})_5$ can be obtained similarly, and we omit the details for brevity. For $(\mathrm{III})_1$, by integration by parts,

$$\begin{aligned}
&\int_{[\tau_1,T]\times\mathbb{T}^3\times\mathbb{R}^3}\partial_{v_i}[(\partial^{\alpha_1}_{\beta_1}\overline{a}_{ij})(\partial_{v_j}\partial^{\alpha_2}_{\beta_2}f)(1+|v|^2)^{\frac{l}{2}}]h\mathrm{d}t\mathrm{d}x\mathrm{d}v\\
&=-\int_{[\tau_1,T]\times\mathbb{T}^3\times\mathbb{R}^3}(\partial^{\alpha_1}_{\beta_1}\overline{a}_{ij})(\partial_{v_j}\partial^{\alpha_2}_{\beta_2}f)(\partial_{v_i}g)(1+|v|^2)^l\mathrm{d}t\mathrm{d}x\mathrm{d}v\\
&\quad-l\int_{[\tau_1,T]\times\mathbb{T}^3\times\mathbb{R}^3}(\partial^{\alpha_1}_{\beta_1}\overline{a}_{ij})(\partial_{v_j}\partial^{\alpha_2}_{\beta_2}f)g(1+|v|^2)^{l-1}v_i\mathrm{d}t\mathrm{d}x\mathrm{d}v.
\end{aligned}\tag{3.5}$$

For the first term on the right-hand side of the above equality, we divide our estimation into the following two parts. The first part is devoted to the case that $1 \le |\alpha_1| + |\beta_1| \le [\frac{N}{2}] + 1$. Since $|\alpha_1| + |\beta_1| + 2 \le N - 1$, Lemma 2.4 implies

$$\begin{aligned}
\|\partial^{\alpha_1}_{\beta_1}\overline{a}_{ij}\|_{L^\infty_t([\tau_1,T];L^\infty_x)} &\le C\|\partial^{\alpha_1}_{\beta_1} f\|_{L^\infty_t([\tau_1,T];H^{2;2}_{x,v})} \\
&\quad + C\sum_{|\alpha_1'|+|\beta_1'|\le[\frac{1}{2}(|\alpha_1|+|\beta_1|)]} \|\partial^{\alpha_1'}_{\beta_1'} f\|_{L^\infty_t([\tau_1,T];H^4_{x,v})}\|\partial^{\alpha_1-\alpha_1'}_{\beta_1-\beta_1'} f\|_{L^\infty_t([\tau_1,T];H^{2;2}_{x,v})} \\
&\le CK_0 + CK_0^2,
\end{aligned}$$

so that

$$\begin{aligned}
&\Big|\int_{[\tau_1,T]\times\mathbb{T}^3\times\mathbb{R}^3}(\partial^{\alpha_1}_{\beta_1}\overline{a}_{ij})(\partial_{v_j}\partial^{\alpha_2}_{\beta_2} f)(\partial_{v_i} g)(1+|v|^2)^l \mathrm{d}t\mathrm{d}x\mathrm{d}v\Big| \\
&\le C(K_0+K_0^2)\{\varepsilon\|(\nabla_v g)(1+|v|^2)^{\frac{l}{2}+\frac{\gamma}{4}}\|^2_{L^2_t([\tau_1,T];L^2_{x,v})} + C(\varepsilon)\|f\|^2_{L^2_t([\tau_1,T];H^{N,l-\frac{\gamma}{2}}_{x,v})}\}.
\end{aligned}$$

Now we turn to the second part when $|\alpha_1| + |\beta_1| \ge [\frac{N}{2}] + 2$. Then $|\alpha_2| + |\beta_2| + 5 \le N - 1$. Using Sobolev's inequality, we see that

$$\|(1+|v|^2)^{\frac{l}{2}-\frac{\gamma}{4}+1}\partial_{v_j}\partial^{\alpha_2}_{\beta_2} f\|_{L^\infty_t([\tau_1,T];L^\infty_{x,v})} \le C\|(1+|v|^2)^{\frac{l}{2}-\frac{\gamma}{4}+1}\partial_{v_j}\partial^{\alpha_2}_{\beta_2} f\|_{L^\infty_t([\tau_1,T];H^4_{x,v})} \le CK_0.$$

Noticing that $|\psi^{ij}(v-u)| \le |v-u|^{\gamma+2}$, we have

$$\begin{aligned}
&\Big|\int_{[\tau_1,T]\times\mathbb{T}^3\times\mathbb{R}^3}(\partial^{\alpha_1}_{\beta_1}\overline{a}_{ij})(\partial_{v_j}\partial^{\alpha_2}_{\beta_2} f)(\partial_{v_i} g)(1+|v|^2)^l \mathrm{d}t\mathrm{d}x\mathrm{d}v\Big| \\
&\le CK_0\int_{[\tau_1,T]\times\mathbb{T}^3\times\mathbb{R}^3}\Big(\int_{\mathbb{R}^3}|v-u|^{\gamma+2}|\partial^{\alpha_1}_{\beta_1} f|\mathrm{d}u\Big)|\partial_{v_i} g|(1+|v|^2)^{\frac{l}{2}+\frac{\gamma}{4}-1}\mathrm{d}t\mathrm{d}x\mathrm{d}v \\
&\quad + CK_0\int_{[\tau_1,T]\times\mathbb{T}^3\times\mathbb{R}^3}\Big(\int_{\mathbb{R}^3}|v-u|^{\gamma+2}|\partial^{\alpha_1}_{\beta_1}(f^2)|\mathrm{d}u\Big)|\partial_{v_i} g|(1+|v|^2)^{\frac{l}{2}+\frac{\gamma}{4}-1}\mathrm{d}t\mathrm{d}x\mathrm{d}v. \qquad (3.6)
\end{aligned}$$

For the second term on the right-hand side of (3.6), letting $|\alpha_1'| + |\beta_1'| \le \frac{1}{2}(|\alpha_1| + |\beta_1|)$, by Hölder's inequality and Sovolev's embedding theorem, we have that

$$\begin{aligned}
&\int_{[\tau_1,T]\times\mathbb{T}^3\times\mathbb{R}^3}\Big(\int_{\mathbb{R}^3}|v-u|^{\gamma+2}|\partial^{\alpha_1}_{\beta_1}(f^2)|\mathrm{d}u\Big)|\partial_{v_i} g|(1+|v|^2)^{\frac{l}{2}+\frac{\gamma}{4}-1}\mathrm{d}t\mathrm{d}x\mathrm{d}v \\
&\le C\|\partial^{\alpha_1'}_{\beta_1'} f\|_{L^\infty_t([\tau_1,T];H^4_{x,v})}\Big(\int_{\mathbb{R}^3}|v-u|^{2(\gamma+2)}(1+|u|^2)^{-2}\mathrm{d}u\Big)^{\frac{1}{2}} \\
&\quad\times\int_{[\tau_1,T]\times\mathbb{T}^3}\|(1+|v|^2)\partial^{\alpha_1-\alpha_1'}_{\beta_1-\beta_1'} f\|_{L^2_v}\|(\nabla_v g)(1+|v|^2)^{\frac{l}{2}+\frac{\gamma}{4}}\|_{L^2_v}\mathrm{d}t\mathrm{d}x \\
&\le CK_0\{\varepsilon\|(\nabla_v g)(1+|v|^2)^{\frac{l}{2}+\frac{\gamma}{4}}\|^2_{L^2_t([\tau_1,T];L^2_{x,v})} + C(\varepsilon)\|f\|^2_{L^2_t([\tau_1,T];H^{N;2}_{x,v})}\}. \qquad (3.7)
\end{aligned}$$

Similarly, the first term on the right-hand side of (3.6) is bounded by

$$CK_0\{\varepsilon\|(\nabla_v g)(1+|v|^2)^{\frac{l}{2}+\frac{\gamma}{4}}\|^2_{L^2_t([\tau_1,T];L^2_{x,v})} + C(\varepsilon)\|f\|^2_{L^2_t([\tau_1,T];H^{N;2}_{x,v})}\}. \qquad (3.8)$$

Substituting (3.8) and (3.7) into (3.6) yields

$$\begin{aligned}
&\Big|\int_{[\tau_1,T]\times\mathbb{T}^3\times\mathbb{R}^3}(\partial^{\alpha_1}_{\beta_1}\overline{a}_{ij})(\partial_{v_j}\partial^{\alpha_2}_{\beta_2} f)(\partial_{v_i} g)(1+|v|^2)^l \mathrm{d}t\mathrm{d}x\mathrm{d}v\Big| \\
&\le C(K_0+K_0^2)\{\varepsilon\|(\nabla_v g)(1+|v|^2)^{\frac{l}{2}+\frac{\gamma}{4}}\|^2_{L^2_t([\tau_1,T];L^2_{x,v})} + CK_0^2\}.
\end{aligned}$$

Likewise, we can bound the second term on the right-hand side of (3.5) by $C(K_0^3+K_0^4)$. Then

$$\begin{aligned}&\Big|\int_{[\tau_1,T]\times\mathbb{T}^3\times\mathbb{R}^3}(\mathrm{III})_1 h\mathrm{d}t\mathrm{d}x\mathrm{d}v\Big|\\&\leq C(K_0+K_0^2)\{\varepsilon\|(\nabla_v g)(1+|v|^2)^{\frac{l}{2}+\frac{\gamma}{4}}\|^2_{L^2_t([\tau_1,T];L^2_{x,v})}+CK_0^2\}.\end{aligned}$$

The term containing $(\mathrm{III})_3$ can be estimated in the same way. We omit the details for brevity, but we simply notice that because of Lemma 2.4, the cases $1\leq|\alpha_1|+|\beta_1|\leq[\frac{N}{2}]$ and $|\alpha_1|+|\beta_1|\geq[\frac{N}{2}]+1$ are considered respectively. As a consequence,

$$\begin{aligned}&\Big|\int_{[\tau_1,T]\times\mathbb{T}^3\times\mathbb{R}^3}(\mathrm{III})_3 h\mathrm{d}t\mathrm{d}x\mathrm{d}v\Big|\\&\leq C(K_0+K_0^2)\{\varepsilon\|(\nabla_v g)(1+|v|^2)^{\frac{l}{2}+\frac{\gamma}{4}}\|^2_{L^2_t([\tau_1,T];L^2_{x,v})}+CK_0^2\}.\end{aligned}$$

Finally, one can also prove that

$$\Big|\int_{[\tau_1,T]\times\mathbb{T}^3\times\mathbb{R}^3}[(\mathrm{III})_2+(\mathrm{III})_4]h\mathrm{d}t\mathrm{d}x\mathrm{d}v\Big|\leq C(K_0^3+K_0^4). \tag{3.9}$$

Putting (3.3) through (3.9) together, we discover

$$\frac{1}{2}\|h(T,x,v)\|^2_{L^2_{x,v}}+[K-C(K_0+K_0^2)\varepsilon]\|(\nabla_v g)(1+|v|^2)^{\frac{l}{2}+\frac{\gamma}{4}}\|^2_{L^2_t([\tau_1,T];L^2_{x,v})}\leq C(K_0^3+K_0^4).$$

The proof of Lemma 3.1 is concluded by taking $\varepsilon>0$ small enough.

3.2 Gain of regularity in x

Lemma 3.2 *Let $-3\leq\gamma<-2$, $N\geq 8$ be a given integer, and f be a smooth nonnegative solution to equation* (1.1) *given by Theorem* 1.1. *We suppose that for any $T>0$, $l\geq 0$,*

$$\|f\|_{L^\infty_t([\tau_1,T];H^{N,\theta}_{x,v})}\leq K_1,$$

where $K_1=K_1(l,T,\gamma,N)$ is a suitably small constant.

Then there exists a constant $\widetilde{C}_2>0$, which depends on N,l,γ,T,K_1, such that

$$\int_{(\tau_1,T]\times\mathbb{R}^3}\|h\|^2_{\dot{H}_x^{\frac{1}{20}}}\mathrm{d}v\mathrm{d}t\leq\widetilde{C}_2. \tag{3.10}$$

Proof Let $p(t,x,v)=h(t,x,v)(1+|v|^2)^2$. To verify (3.10), we only need to prove

$$\int_{(\tau_1,T]\times\mathbb{R}^3}(1+|v|^2)^{-4}\Big(\sum_{m\in\mathbb{Z}^3}|m|^{\frac{1}{10}}|\widehat{p}(t,m,v)|^2\Big)\mathrm{d}v\mathrm{d}t\leq\widetilde{C}_2.$$

Let $\chi:=\chi(v)\in C_c^\infty(\mathbb{R}^3)$ be a test function which satisfies $\chi(v)\geq 0$ and $\int_{\mathbb{R}^3}\chi(v)\mathrm{d}v=1$. we introduce the regularizing sequence $\chi_\epsilon=\epsilon^{-3}\chi(\frac{v}{\epsilon})$ and write

$$\widehat{p}(t,m,v)=[\widehat{p}(t,m,v)-(\widehat{p}(t,m,\cdot)*\chi_\epsilon)(v)]+(\widehat{p}(t,m,\cdot)*\chi_\epsilon)(v).$$

Here, ϵ will be chosen later (and will depend on m).

For the first term of the right-hand side of the above equality, we use Minkowski's inequality and get

$$\begin{aligned}&\int_{(\tau_1,T]\times\mathbb{R}^3}(1+|v|^2)^{-4}\Big(\sum_{m\in\mathbb{Z}^3}|m|^{\frac{1}{10}}|\widehat{p}(t,m,v)-(\widehat{p}(t,m,\cdot\,)*\chi_\epsilon)(v)|^2\Big)\mathrm{d}v\mathrm{d}t\\&\le C\epsilon^2\int_{(\tau_1,T]\times\mathbb{R}^3}\sum_{m\in\mathbb{Z}^3}|m|^{\frac{1}{10}}|\nabla_v\widehat{p}(t,m,v)|^2\mathrm{d}v\mathrm{d}t.\end{aligned}\tag{3.11}$$

Remembering that $p(t,x,v)=h(t,x,v)(1+|v|^2)^2$, we see that p is the solution to equation (3.2) with l replaced by $l+4$. Then, we can write the equation satisfied by p in the form

$$\partial_t p+v\cdot\nabla_x p=p_1+\nabla_v\cdot p_2.$$

Here, $\nabla_v\cdot p_2$ is the sum of the terms $(\mathrm{II})_1$, $(\mathrm{II})_3$, $(\mathrm{II})_4$, $(\mathrm{III})_1$ and $(\mathrm{III})_3$, while p_1 is the sum of the remaining terms.

We claim that $p_1,p_2\in L_t^2((\tau_1,T];L^2_{x,v})$. We only present here the estimates for the term $(\mathrm{III})_1$, while the other terms can be shown similarly. For this, we divide our discussion into the following two cases.

Case 1 $1\le|\alpha_1|+|\beta_1|\le\big[\frac{N}{2}\big]+1$, and in this case, $\frac12\big(\big[\frac{N}{2}\big]+1\big)+4\le N$. We apply (2.3) as follows:

$$\begin{aligned}\|\partial^{\alpha_1}_{\beta_1}\overline{a}_{ij}\|_{L_t^\infty((\tau_1,T];L^\infty_{x,v})}&\le C\sum(1+\|\partial^{\alpha_1'}_{\beta_1'}f\|_{L_t^\infty(\Omega;H^4_{x,v})})\|(\partial^{\alpha_1-\alpha_1'}_{\beta_1-\beta_1'}f)(1+|v|^2)\|_{L_t^\infty(\Omega;H^2_{x,v})}\\&\le CK_1(1+K_1).\end{aligned}\tag{3.12}$$

This implies

$$\begin{aligned}&\|(\partial^{\alpha_1}_{\beta_1}\overline{a}_{ij})(\partial^{\alpha_2}_{\beta_2}\partial_{v_j}f)(1+|v|^2)^{\frac{l+4}{2}}\|_{L_t^2((\tau_1,T];L^2_{x,v})}\\&\le CK_1(1+K_1)\|(\partial^{\alpha_2}_{\beta_2}\partial_{v_j}f)(1+|v|^2)^{\frac{l+4}{2}}\|_{L_t^2((\tau_1,T];L^2_{x,v})}\\&\le C(1+K_1)K_1^2T^{\frac12}.\end{aligned}$$

Case 2 $|\alpha_1|+|\beta_1|\ge\big[\frac{N}{2}\big]+2$, and in this case, $|\alpha_2|+|\beta_2|+5\le N$. We know that

$$\|(1+|v|^2)^{\frac{l+6}{2}}\partial_{v_j}\partial^{\alpha_2}_{\beta_2}f\|_{L_t^\infty([\tau_1,T];L^\infty_{x,v})}\le C\|(1+|v|^2)^{\frac{l+6}{2}}\partial_{v_j}\partial^{\alpha_2}_{\beta_2}f\|_{L_t^\infty([\tau_1,T];H^4_{x,v})}\le CK_1.$$

Since $|\psi^{ij}(v-u)|\le|v-u|^{\gamma+2}$, we have

$$\begin{aligned}&\|(\partial^{\alpha_1}_{\beta_1}\overline{a}_{ij})(\partial_{v_j}\partial^{\alpha_2}_{\beta_2}f)(1+|v|^2)^{\frac{l+4}{2}}\|_{L_t^2((\tau_1,T];L^2_{x,v})}\\&\le CK_1\Big\|\Big(\int_{\mathbb{R}^3}|v-u|^{\gamma+2}|\partial^{\alpha_1}_{\beta_1}f|\mathrm{d}u\Big)(1+|v|^2)^{-1}\Big\|_{L_t^2((\tau_1,T];L^2_{x,v})}\\&\quad+CK_1\Big\|\Big(\int_{\mathbb{R}^3}|v-u|^{\gamma+2}|\partial^{\alpha_1}_{\beta_1}(f^2)|\mathrm{d}u\Big)(1+|v|^2)^{-1}\Big\|_{L_t^2((\tau_1,T];L^2_{x,v})}\\&=B_1+B_2,\end{aligned}\tag{3.13}$$

where B_1 and B_2 denote the corresponding two terms on the right-hand side of (3.13). Now we turn to estimate the delicate term B_2. Firstly, one can see that

$$B_2\le\sum C^{\alpha_1}_{\alpha}C^{\beta_1}_{\beta}K_1\Big\|\Big(\int_{\mathbb{R}^3}|v-u|^{\gamma+2}|(\partial^{\alpha_1'}_{\beta_1'}f)(\partial^{\alpha_1-\alpha_1'}_{\beta_1-\beta_1'}f)|\mathrm{d}u\Big)(1+|v|^2)^{-1}\Big\|_{L_t^2((\tau_1,T];L^2_{x,v})}.$$

Next, without loss of generality, we assume that $|\alpha_1'|+|\beta_1'|\le \frac{1}{2}(|\alpha_1|+|\beta_1|)$, and then by applying Hölder's inequality, Sovolev's embedding theorem and Lemma 2.2, we have that

$$\begin{aligned}B_2 &\le CK_1\|\partial^{\alpha_1'}_{\beta_1'}f\|_{L^\infty_t([\tau_1,T];H^4_{x,v})}\|(1+|v|^2)\partial^{\alpha_1-\alpha_1'}_{\beta_1-\beta_1'}f\|_{L^2_t((\tau_1,T];L^2_{x,v})}\\ &\quad\times\Big\|\Big(\int_{\mathbb{R}^3}|v-u|^{2(\gamma+2)}(1+|u|^2)^{-2}\mathrm{d}u\Big)^{\frac{1}{2}}(1+|v|^2)^{-1}\Big\|_{L^2_v}\\ &\le CK_1^3T^{\frac{1}{2}}.\end{aligned}$$

As to B_1, one can get that

$$B_1\le CK_1^2T^{\frac{1}{2}}. \tag{3.14}$$

Combining the estimates (3.12)–(3.14) together, we see that

$$\|(\partial^{\alpha_1}_{\beta_1}\overline{a}_{ij})(\partial_{v_j}\partial^{\alpha_2}_{\beta_2}f)(1+|v|^2)^{\frac{l+4}{2}}\|_{L^2_t((\tau_1,T];L^2_{x,v})}\le C(1+K_1)K_1^2T^{\frac{1}{2}}.$$

Now, employing the average lemma (cf. [5]), we can prove that

$$\begin{aligned}&|m|^{\frac{1}{2}}\int_{\tau_1}^T|\widehat{p}(t,m,\cdot\,)*\chi_\epsilon(v)|^2\mathrm{d}t\\ &\le C(\|\chi_\epsilon(v-u)(1+|u|^2)\|_{L^\infty_u}+\|\nabla\chi_\epsilon(v-u)(1+|u|^2)\|_{L^\infty_u})^2\\ &\quad\times(\|\widehat{p}(\tau_1,m,\cdot\,)\|^2_{L^2_v}+\|\widehat{p}(\,\cdot\,,m,\cdot\,)\|^2_{L^2_t((\tau_1,T];L^2_v)}\\ &\quad+\|\widehat{p}_1(\,\cdot\,,m,\cdot\,)\|^2_{L^2_t((\tau_1,T];L^2_v)}+\|\widehat{p}_2(\,\cdot\,,m,\cdot\,)\|^2_{L^2_t((\tau_1,T];L^2_v)}).\end{aligned}$$

Since $\|\chi_\epsilon(v-u)(1+|u|^2)\|_{L^\infty_u}\le C\epsilon^{-3}(1+|v|^2)$, and $\|\nabla\chi_\epsilon(v-u)(1+|u|^2)\|_{L^\infty_u}\le C\epsilon^{-4}(1+|v|^2)$, we see that

$$\begin{aligned}&\int_{(\tau_1,T]\times\mathbb{R}^3}(1+|v|^2)^{-4}\sum_{m\in\mathbb{Z}^3}|m|^{\frac{1}{10}}|\widehat{p}(t,m,\cdot\,)*\chi_\epsilon(v)|^2\mathrm{d}v\mathrm{d}t\\ &\le C\sum_{m\in\mathbb{Z}^3}|m|^{\frac{1}{10}-\frac{1}{2}}(\epsilon^{-6}+\epsilon^{-8})(\|\widehat{p}(\tau_1,m,\cdot\,)\|^2_{L^2_v}+\|\widehat{p}(\,\cdot\,,m,\cdot\,)\|^2_{L^2_t((\tau_1,T];L^2_v)}\\ &\quad+\|\widehat{p}_1(\,\cdot\,,m,\cdot\,)\|^2_{L^2_t((\tau_1,T];L^2_v)}+\|\widehat{p}_2(\,\cdot\,,m,\cdot\,)\|^2_{L^2_t((\tau_1,T];L^2_v)}).\end{aligned}\tag{3.15}$$

If we choose $\epsilon=\epsilon(|m|)=|m|^{-\frac{1}{20}}$, we can bound (3.15) remembering that $p(\tau_1,x,v)\in L^2_{x,v}$ and $p,p_1,p_2\in L^2_t((\tau_1,T];L^2_{x,v})$. Thus the proof of Lemma 3.2 is completed.

Roughly speaking, Lemma 3.2 together with Lemma 3.1 shows that when f is a solution to equation (1.1) constructed in Theorem 1.1, then $f\in L^2((\tau_1,T];H^{N+\frac{1}{20},l})$. To improve the regularity about the position variable x, we will first show a preliminary lemma.

Lemma 3.3 *Let $-3\le\gamma<-2$, $N\ge 8$ be a given integer, $\delta\in(0,\frac{19}{20}]$, and f be a smooth nonnegative solution to equation* (1.1) *given by Theorem* 1.1. *We suppose that for any $T>\tau_2>\tau_1$, $l\ge 0$,*

$$\|f(\tau_2,x,v)\|_{H^{N+\delta,l}_{x,v}}\le K_2,\quad \|f\|_{L^\infty_t([\tau_1,T];H^{N,l}_{x,v})}\le K_2,\quad \|f\|_{L^2_t((\tau_1,T];H^{N+\delta,l}_{x,v})}\le K_2,$$

where $K_2=K_2(l,T,\gamma,N,\delta)$ is a suitably small constant.

Then there exists a constant $\widetilde{C}_3 > 0$, which depends on N, l, γ, T, K_2 and δ, such that

$$\sup_{\tau_2 \le t \le T} \int_{\mathbb{T}^3\times\mathbb{T}^3\times\mathbb{R}^3} |g_{\delta,k}|^2 \mathrm{d}v\mathrm{d}x\mathrm{d}k + \int_{[\tau_2,T]\times\mathbb{T}^3\times\mathbb{T}^3\times\mathbb{R}^3} |\nabla_v g_{\delta,k}|^2 (1+|v|^2)^{\frac{\gamma}{4}} \mathrm{d}v\mathrm{d}x\mathrm{d}k\mathrm{d}t \le \widetilde{C}_3,$$

where $g_{\delta,k} = \triangle_k h(t,x,v)|k|^{-\delta-\frac{3}{2}}$.

Proof Note that $g_{\delta,k}$ satisfies the following equation:

$$\partial_t g_{\delta,k} + v\cdot\nabla_x g_{\delta,k} = (\mathrm{IV}) + (\mathrm{V}) + (\mathrm{VI}), \quad t > \tau_2, \tag{3.16}$$

where

$$\begin{aligned}
(\mathrm{IV}) &= -\rho_{|\beta|} C_\beta^{\beta_1} (\partial_{\beta_1} v\cdot\nabla_x \partial^\alpha_{\beta-\beta_1} \triangle_k f)|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{\frac{l}{2}},\\
(\mathrm{V}) &= \partial_{v_i}[\triangle_k(\overline{a}_{ij}\partial_{v_j} h)|k|^{-\delta-\frac{3}{2}}] - l\triangle_k[\overline{a}_{ij}(\partial_{v_i} g)]|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{\frac{l}{2}-1} v_i\\
&\quad - \partial_{v_i}[l\triangle_k(\overline{a}_{ij} g)|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{\frac{l}{2}-1} v_j] - \partial_{v_i}\{\triangle_k[\overline{b}_i(\partial^\alpha_\beta \widetilde{f})]|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{\frac{l}{2}}\}\\
&\quad + l\triangle_k[\overline{b}_i(\partial^\alpha_\beta \widetilde{f})]|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{\frac{l}{2}-1} v_i\\
&= \sum_{k=1}^{5} (\mathrm{V})_k,\\
(\mathrm{VI}) &= \rho_{|\alpha|+|\beta|} \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2=\beta\\ |\alpha_1|+|\beta_1|\ge 1}} C_\alpha^{\alpha_1} C_\beta^{\beta_1} \{\partial_{v_i}[\triangle_k((\partial^{\alpha_1}_{\beta_1}\overline{a}_{ij})(\partial_{v_j}\partial^{\alpha_2}_{\beta_2} f))|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{\frac{l}{2}}]\\
&\quad - l\triangle_k[(\partial^{\alpha_1}_{\beta_1}\overline{a}_{ij})(\partial_{v_i}\partial^{\alpha_2}_{\beta_2} f)]|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{\frac{l}{2}-1} v_i\\
&\quad - \partial_{v_i}[\triangle_k((\partial^{\alpha_1}_{\beta_1}\overline{b}_i)(\partial^{\alpha_2}_{\beta_2}\widetilde{f}))|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{\frac{l}{2}}]\\
&\quad + l\triangle_k((\partial^{\alpha_1}_{\beta_1}\overline{b}_i)(\partial^{\alpha_2}_{\beta_2}\widetilde{f}))|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{\frac{l}{2}-1} v_i\}\\
&= \sum_{k=1}^{4} (\mathrm{VI})_k,
\end{aligned}$$

where $(\mathrm{V})_k$ and $(\mathrm{VI})_k$ $(k = 1, 2, \cdots)$ denote the corresponding terms in (V) and (VI).

We multiply (IV), (V), (VI) by $g_{\delta,k}$, and then integrate them on (t, x, k, v) in the domain $[\tau_2, T]\times\mathbb{T}^3\times\mathbb{T}^3\times\mathbb{R}^3$. We only estimate the terms containing $(\mathrm{V})_1$ and $(\mathrm{VI})_1$, since the estimates for other terms are similar.

For any functions f and g,

$$\triangle_k[f(x)g(x)] = \triangle_k f(x) g(x+k) + f(x)\triangle_k g(x),$$

and therefore, we can rewrite

$$(\mathrm{V})_1 = \partial_{v_i}[\overline{a}_{ij}(x+k)(\partial_{v_j} g_{\delta,k})] + \partial_{v_i}[(\triangle_k \overline{a}_{ij})(\partial_{v_j} h)|k|^{-\delta-\frac{3}{2}}] = \mathrm{D}_1 + \mathrm{D}_2,$$

where D_1 and D_2 denote the corresponding two terms on the right-hand side of the above equality. After integration by parts,

$$\begin{aligned}
&\int_{[\tau_2,T]\times\mathbb{T}^3\times\mathbb{T}^3\times\mathbb{R}^3} \mathrm{D}_1 g_{\delta,k}\mathrm{d}t\mathrm{d}x\mathrm{d}v\mathrm{d}k\\
&= -\int_{[\tau_2,T]\times\mathbb{T}^3\times\mathbb{T}^3\times\mathbb{R}^3} \overline{a}_{ij}(x+k)[\partial_{v_i}(\triangle_k g)][\partial_{v_j}(\triangle_k g)](1+|v|^2)^l|k|^{-2\delta-3}\mathrm{d}t\mathrm{d}x\mathrm{d}v\mathrm{d}k\\
&\quad -2l\int_{[\tau_2,T]\times\mathbb{T}^3\times\mathbb{T}^3\times\mathbb{R}^3} \overline{a}_{ij}(x+k)[\partial_{v_i}(\triangle_k g)](\triangle_k g)(1+|v|^2)^{l-1}v_j|k|^{-2\delta-3}\mathrm{d}t\mathrm{d}x\mathrm{d}v\mathrm{d}k\\
&\quad -l^2\int_{[\tau_2,T]\times\mathbb{T}^3\times\mathbb{T}^3\times\mathbb{R}^3} \overline{a}_{ij}(x+k)(\triangle_k g)^2(1+|v|^2)^{l-2}v_jv_i|k|^{-2\delta-3}\mathrm{d}t\mathrm{d}x\mathrm{d}v\mathrm{d}k\\
&= \sum_{i=1}^3 \mathrm{D}_{1,i},
\end{aligned}\tag{3.17}$$

where $\mathrm{D}_{1,i}$ $(i=1,2,3)$ denote the three terms on the right-hand side of (3.17). By Lemma 2.3, we get

$$\mathrm{D}_{1,1} \le -K\int_{[\tau_2,T]\times\mathbb{T}^3\times\mathbb{T}^3\times\mathbb{R}^3} |[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{l}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}|^2\mathrm{d}t\mathrm{d}x\mathrm{d}v\mathrm{d}k.$$

For $\mathrm{D}_{1,2}$, by Lemma 2.4, we have

$$\|\overline{a}_{ij}(x+k)\|_{L^\infty_t([\tau_2,T],L^\infty_{x,k})} \le C(1+K_2)K_2.$$

Employing Cauchy-Schwarz's inequality with ε and the hypothesis of our lemma, we deduce

$$\begin{aligned}
|\mathrm{D}_{1,2}| &\le C(1+K_2)K_2\int_{[\tau_2,T]\times\mathbb{T}^3\times\mathbb{T}^3\times\mathbb{R}^3} ([\partial_{v_i}(\triangle_k g)](1+|v|^2)^{\frac{l}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}})\\
&\quad\times((\triangle_k g)(1+|v|^2)^{\frac{l}{2}-\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}})\mathrm{d}t\mathrm{d}x\mathrm{d}v\mathrm{d}k\\
&\le C(1+K_2)K_2\{\varepsilon\|[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{l}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}\|^2_{L^2_t([\tau_2,T],L^2_{x,v,k})}+CK_2^2\}.
\end{aligned}$$

Likewise, one can show that

$$|\mathrm{D}_{1,3}| \le C(1+K_2)K_2^3.$$

As to the term containing D_2, applying once more integration by parts, we find

$$\begin{aligned}
&\int_{[\tau_2,T]\times\mathbb{T}^3\times\mathbb{T}^3\times\mathbb{R}^3} \mathrm{D}_2 g_{\delta,k}\mathrm{d}t\mathrm{d}x\mathrm{d}v\mathrm{d}k\\
&= -\int_{[\tau_2,T]\times\mathbb{T}^3\times\mathbb{T}^3\times\mathbb{R}^3} (\triangle_k\overline{a}_{ij})(\partial_{v_i}(\triangle_k g))(\partial_{v_j}g)(1+|v|^2)^l|k|^{-2\delta-3}\mathrm{d}t\mathrm{d}x\mathrm{d}v\mathrm{d}k\\
&\quad -l\int_{[\tau_2,T]\times\mathbb{T}^3\times\mathbb{T}^3\times\mathbb{R}^3} (\triangle_k\overline{a}_{ij})[\partial_{v_i}(\triangle_k g)]g(1+|v|^2)^{l-1}v_j|k|^{-2\delta-3}\mathrm{d}t\mathrm{d}x\mathrm{d}v\mathrm{d}k\\
&\quad -l\int_{[\tau_2,T]\times\mathbb{T}^3\times\mathbb{T}^3\times\mathbb{R}^3} (\triangle_k\overline{a}_{ij})(\triangle_k g)(\partial_{v_j}g)(1+|v|^2)^{l-1}v_i|k|^{-2\delta-3}\mathrm{d}t\mathrm{d}x\mathrm{d}v\mathrm{d}k\\
&\quad -l^2\int_{[\tau_2,T]\times\mathbb{T}^3\times\mathbb{T}^3\times\mathbb{R}^3} (\triangle_k\overline{a}_{ij})(\triangle_k g)g(1+|v|^2)^{l-2}v_jv_i|k|^{-2\delta-3}\mathrm{d}t\mathrm{d}x\mathrm{d}v\mathrm{d}k\\
&= \sum_{i=1}^4 \mathrm{D}_{2,i}.
\end{aligned}$$

Noticing that

$$\triangle_k\overline{a}_{ij} = \psi^{ij} * [\triangle_k f] - \psi^{ij} * [\triangle_k (f^2)] = \psi^{ij} * [\triangle_k f] - \psi^{ij} * [(\triangle_k f) f] - \psi^{ij} * [f(x+k)\triangle_k f]$$

and $\|f(x+k)\|_{L^\infty_{x,k}(\mathbb{T}^3\times\mathbb{T}^3)} = \|f(x)\|_{L^\infty_x(\mathbb{T}^3)}$, performing the similar computations as deducing Lemma 2.4, we know that

$$\|\triangle_k\overline{a}_{ij}\|_{L^\infty_t([\tau_2,T],L^\infty_x)} \leq C(1+K_2)\|\triangle_k f(1+|v|^2)\|_{L^\infty_t([\tau_2,T],H^2_{x,v})}.$$

From this and the hypothesis of our lemma, we see that

$$\begin{aligned}
|\mathrm{D}_{2,1}| &\leq C(1+K_2)\int_{\mathbb{T}^3} \|\triangle_k f(x)(1+|v|^2)\|_{L^\infty_t([\tau_2,T],H^2_{x,v})} \\
&\quad \times \|\nabla_v \triangle_k g(1+|v|^2)^{\frac{l}{2}+\frac{\gamma}{4}}\|_{L^2_t([\tau_2,T],L^2_{x,v})}|k|^{-2\delta-3}\mathrm{d}k\|\nabla_v g(1+|v|^2)^{\frac{l}{2}-\frac{\gamma}{4}}\|_{L^2_t([\tau_2,T],L^2_{x,v})} \\
&\leq C(1+K_2)K_2\{\varepsilon\|[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{l}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}\|^2_{L^2_t([\tau_2,T],L^2_{x,v,k})} + C(\varepsilon)\widetilde{C}_1\},
\end{aligned}$$

where we also used Lemma 3.1 to get the last inequality. Likewise, we can obtain

$$\begin{aligned}
\sum_{i=2}^{4}|\mathrm{D}_{2,i}| &\leq C(1+K_2)K_2\{\varepsilon\|[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{l}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}\|^2_{L^2_t([\tau_2,T],L^2_{x,v,k})} \\
&\quad + C(\varepsilon)(K_2+\widetilde{C}_1)\}.
\end{aligned} \tag{3.18}$$

Then the estimate for $\int_{[\tau_2,T]\times\mathbb{T}^3\times\mathbb{T}^3\times\mathbb{R}^3}(\mathrm{V})_1 g_{\delta,k}\mathrm{d}t\mathrm{d}x\mathrm{d}v\mathrm{d}k$ follows the above discussion from (3.17) through (3.18).

As to the term $(\mathrm{VI})_1$, we note carefully that

$$\begin{aligned}
&\partial_{v_i}\{\triangle_k[(\partial^{\alpha_1}_{\beta_1}\overline{a}_{ij})(\partial_{v_j}\partial^{\alpha_2}_{\beta_2}f)]|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{\frac{l}{2}}\} \\
&= \partial_{v_i}\{(\partial^{\alpha_1}_{\beta_1}\overline{a}_{ij}(x+k))(\partial_{v_j}\partial^{\alpha_2}_{\beta_2}\triangle_k f)|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{\frac{l}{2}}\} \\
&\quad + \sum C^{\alpha_1'}_{\alpha_1}C^{\beta_1'}_{\beta_1}\partial_{v_i}\{[\psi^{ij} * (\partial^{\alpha_1'}_{\beta_1'}\triangle_k f\partial^{\alpha_1-\alpha_1'}_{\beta_1-\beta_1'}f(x+k))](\partial_{v_j}\partial^{\alpha_2}_{\beta_2}f)|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{\frac{l}{2}}\} \\
&\quad + \sum C^{\alpha_1'}_{\alpha_1}C^{\beta_1'}_{\beta_1}\partial_{v_i}\{[\psi^{ij} * (\partial^{\alpha_1'}_{\beta_1'}f\partial^{\alpha_1-\alpha_1'}_{\beta_1-\beta_1'}\triangle_k f)](\partial_{v_j}\partial^{\alpha_2}_{\beta_2}f)|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{\frac{l}{2}}\} \\
&= \sum_{i=1}^{3}\mathrm{E}_i.
\end{aligned}$$

We only estimate the terms containing E_3 in the following for brevity. By integration by parts, we have (omitting the coefficients of E_3)

$$\begin{aligned}
&\int_{[\tau_2,T]\times\mathbb{T}^3\times\mathbb{T}^3\times\mathbb{R}^3}\mathrm{E}_3 g_{\delta,k}\mathrm{d}t\mathrm{d}x\mathrm{d}v\mathrm{d}k \\
&= -\int_{[\tau_2,T]\times\mathbb{T}^3\times\mathbb{T}^3\times\mathbb{R}^3}\psi^{ij} * (\partial^{\alpha_1'}_{\beta_1'}f\partial^{\alpha_1-\alpha_1'}_{\beta_1-\beta_1'}\triangle_k f)(\partial_{v_j}\partial^{\alpha_2}_{\beta_2}f)|k|^{-2\delta-3}(1+|v|^2)^{l}\partial_{v_i}\triangle_k g\mathrm{d}t\mathrm{d}x\mathrm{d}v\mathrm{d}k \\
&\quad - l\int_{[\tau_2,T]\times\mathbb{T}^3\times\mathbb{T}^3\times\mathbb{R}^3}\psi^{ij} * (\partial^{\alpha_1'}_{\beta_1'}f\partial^{\alpha_1-\alpha_1'}_{\beta_1-\beta_1'}\triangle_k f)\partial_{v_j}\partial^{\alpha_2}_{\beta_2}f|k|^{-2\delta-3}(1+|v|^2)^{l-1}v_i\triangle_k g\mathrm{d}t\mathrm{d}x\mathrm{d}v\mathrm{d}k \\
&= \mathrm{E}_{3,1} + \mathrm{E}_{3,2}.
\end{aligned}$$

When $|\alpha_1|+|\beta_1|\le[\frac{N}{2}]+1$, without loss of generality, we also assume that $|\alpha_1'|+|\beta_1'|\ge\frac{1}{2}(|\alpha_1|+|\beta_1|)$, and recalling the estimate (2.5), we deduce that

$$\|\psi^{ij}*(\partial^{\alpha_1'}_{\beta_1'}f\partial^{\alpha_1-\alpha_1'}_{\beta_1-\beta_1'}\triangle_k f)\|_{L^\infty_t([\tau_2,T],L^\infty_x)}\le C\|\partial^{\alpha_1'}_{\beta_1'}f\|_{L^\infty_t([\tau_2,T],H^{2,2}_{x,v})}\|\partial^{\alpha_1-\alpha_1'}_{\beta_1-\beta_1'}\triangle_k f\|_{L^\infty_t([\tau_2,T],H^4_{x,v})}.$$

Proceeding as the estimate for $\mathrm{D}_{2,1}$, we obtain

$$|\mathrm{E}_{3,1}|\le CK_2^2\{\varepsilon\|[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{l}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}\|^2_{L^2_t([\tau_2,T],L^2_{x,v,k})}+C(\varepsilon)K_2^2\}.$$

When $|\alpha_1|+|\beta_1|\ge\left[\frac{N}{2}\right]+2$ and $|\alpha_1'|+|\beta_1'|\ge\frac{1}{2}(|\alpha_1|+|\beta_1|)$, in light of (2.5) and by direct computations, one can see that

$$|\psi^{ij}*(\partial^{\alpha_1'}_{\beta_1'}f\partial^{\alpha_1-\alpha_1'}_{\beta_1-\beta_1'}\triangle_k f)|\le C\|\partial^{\alpha_1'}_{\beta_1'}f\|_{L^{2,2}_v}\sup_{t,x,v}|\partial^{\alpha_1-\alpha_1'}_{\beta_1-\beta_1'}\triangle_k f|.$$

Then by taking the $L^\infty_{t,x,v}$-norm of $(\partial_{v_j}\partial^{\alpha_2}_{\beta_2}f)(1+|v|^2)^{\frac{l}{2}}$ and applying Sobolev's inequality, we discover

$$\begin{aligned}|\mathrm{E}_{3,1}|\le{}& CK_2\int_{\mathbb{T}^3}|k|^{-2\delta-3}\|\partial^{\alpha_1-\alpha_1'}_{\beta_1-\beta_1'}\triangle_k f\|_{L^\infty_t([\tau_2,T],H^4_{x,v})}\\&\times\|[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{l}{2}+\frac{\gamma}{4}}\|_{L^2_t([\tau_2,T],L^2_{x,v})}\mathrm{d}k\|\partial^{\alpha_1'}_{\beta_1'}f\|_{L^\infty_t([\tau_2,T],L^{2,2}_{x,v})}\\\le{}& CK_2^2\{\varepsilon\|[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{l}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}\|^2_{L^2_t([\tau_2,T],L^2_{x,v,k})}+C(\varepsilon)K_2^2\}.\end{aligned}$$

Likewise, one can obtain

$$|\mathrm{E}_{3,2}|\le CK_2^4.$$

Now we end up with

$$\begin{aligned}&\Big|\int_{[\tau_2,T]\times\mathbb{T}^3\times\mathbb{T}^3\times\mathbb{R}^3}(\mathrm{VI})g_{\delta,k}\mathrm{d}t\mathrm{d}x\mathrm{d}v\mathrm{d}k\Big|\\&\le C(1+K_2)K_2\{\varepsilon\|[\nabla_v(\triangle_k g)](1+|v|^2)^{\frac{l}{2}+\frac{\gamma}{4}}|k|^{-\delta-\frac{3}{2}}\|^2_{L^2_t([\tau_2,T],L^2_{x,v,k})}+C(\varepsilon)K_2^2\}.\end{aligned}$$

As in the end of the proof of Lemma 3.1, we can conclude the proof of Lemma 3.3 by choosing ε small enough.

Next we will lift the regularity of the position variable x and obtain the following lemma.

Lemma 3.4 *Let $-3\le\gamma<-2$, $N\ge 8$ be a given integer, $\delta\in(0,\frac{19}{20}]$, and f be a smooth nonnegative solution to equation* (1.1) *given by Theorem* 1.1. *Suppose that for any $T>\tau_2>\tau_1$, $l\ge0$,*

$$\|f(\tau_2,x,v)\|_{H^{N+\delta,l}_{x,v}}\le K_3,\quad\|f\|_{L^\infty_t([\tau_1,T];H^{N,l}_{x,v})}\le K_3,\quad\|f\|_{L^2_t((\tau_1,T];H^{N+\delta,l}_{x,v})}\le K_3,$$

where $K_3=K_3(l,T,\gamma,N)$ is a constant.

Then there exists a constant $\widetilde{C}_4>0$, which depends on N,l,γ,T,K_3, such that

$$\int_{(\tau_2,T]\times\mathbb{R}^3}\|h\|^2_{\dot H^{\delta+\frac{1}{20}}_x}\mathrm{d}v\mathrm{d}t\le\widetilde{C}_4,\tag{3.19}$$

where h is defined by (3.1).

Proof In view of (2.1) in Lemma 2.1, we know that

$$
\begin{aligned}
(3.19) \Leftrightarrow & \int_{(\tau_2,T]\times\mathbb{R}^3} \sum_{m\in\mathbb{Z}^3} |m|^{2\delta+\frac{1}{10}} |\widehat{h}(m)|^2 \mathrm{d}v\mathrm{d}t \le \widetilde{C}_4 \\
\Leftrightarrow & \int_{(\tau_2,T]\times\mathbb{R}^3} \sum_{m\in\mathbb{Z}^3} |m|^{\frac{1}{10}} |\widehat{g}_{\delta,k}|^2 \mathrm{d}v\mathrm{d}k\mathrm{d}t \le \widetilde{C}_4.
\end{aligned} \tag{3.20}
$$

In order to prove (3.20), we use the method of the proof of Lemma 3.2. We introduce $p_{\delta,k} = g_{\delta,k}(1+|v|^2)^2$, and write

$$
\widehat{p}_{\delta,k}(t,m,v) = [\widehat{p}_{\delta,k}(t,m,v) - (\widehat{p}_{\delta,k}(t,m,\cdot\,) * \chi_\epsilon)(v)] + (\widehat{p}_{\delta,k}(t,m,\cdot\,) * \chi_\epsilon)(v),
$$

the parameter ϵ being chosen later. Following the proof of estimate (3.11), we get

$$
\begin{aligned}
& \int_{(\tau_2,T]\times\mathbb{R}^3\times\mathbb{T}^3} (1+|v|^2)^{-4} \sum_{m\in\mathbb{Z}^3} |m|^{\frac{1}{10}} |\widehat{p}_{\delta,k}(t,m,v) - (\widehat{p}_{\delta,k}(t,m,\cdot\,) * \chi_\epsilon)(v)|^2 \mathrm{d}v\mathrm{d}k\mathrm{d}t \\
\le\, & C \int_{(\tau_2,T]\times\mathbb{R}^3\times\mathbb{T}^3} \epsilon^2 \sum_{m\in\mathbb{Z}^3} |m|^{\frac{1}{10}} |\nabla_v \widehat{p}_{\delta,k}(t,m,v)|^2 \mathrm{d}v\mathrm{d}k\mathrm{d}t.
\end{aligned}
$$

Noticing that $p_{\delta,k}$ is the solution to (3.16) with l replaced by $l+4$, we can write the equation with the solution $p_{\delta,k}$ in the form

$$
\partial_t p_{\delta,k} + v\cdot\nabla_x p_{\delta,k} = p^{(1)}_{\delta,k} + \nabla_v \cdot p^{(2)}_{\delta,k},
$$

where $p^{(1)}_{\delta,k}$ consists in the sum of terms $(\mathrm{IV}), (\mathrm{V})_2, (\mathrm{V})_5, (\mathrm{VI})_2$ and $(\mathrm{VI})_4$.

We assert that $p^{(1)}_{\delta,k}, p^{(2)}_{\delta,k} \in L^2_t((\tau_2,T];L^2_{x,v})$. We only present here the estimates for the term $(\mathrm{VI})_1$. The other terms are similar. We write

$$
\begin{aligned}
& \partial_{v_i}\{\triangle_k[(\partial^{\alpha_1}_{\beta_1}\overline{a}_{ij})(\partial_{v_j}\partial^{\alpha_2}_{\beta_2} f)]|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{\frac{l+4}{2}}\} \\
=\, & \partial_{v_i}\{(\partial^{\alpha_1}_{\beta_1}\overline{a}_{ij}(x+k))(\partial_{v_j}\partial^{\alpha_2}_{\beta_2}\triangle_k f)|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{\frac{l+4}{2}}\} \\
& + \partial_{v_i}\{(\partial^{\alpha_1}_{\beta_1}\triangle_k\overline{a}_{ij})(\partial_{v_j}\partial^{\alpha_2}_{\beta_2} f)|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{\frac{l+4}{2}}\},
\end{aligned}
$$

and only estimate the first term of the right-hand side of the above equality, since the estimate for the second term is similar.

When $|\alpha_1|+|\beta_1| \le \left[\frac{N}{2}\right]$, then $|\alpha_1|+|\beta_1|+4 \le N$, and therefore, we employ Lemma 2.4 to deduce

$$
\|\partial^{\alpha_1}_{\beta_1}\overline{a}_{ij}(x+k)\|_{L^\infty_t((\tau_2,T];L^\infty_{x,k})} \le CK_3(1+K_3).
$$

From this and the hypothesis of our lemma, we see that

$$
\begin{aligned}
& \|(\partial^{\alpha_1}_{\beta_1}\overline{a}_{ij}(x+k))(\partial_{v_j}\partial^{\alpha_2}_{\beta_2}\triangle_k f)|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{\frac{l+4}{2}}\|_{L^2_t((\tau_2,T];L^2_{x,v,k})} \\
\le\, & CK_3(1+K_3)\|(\partial_{v_j}\partial^{\alpha_2}_{\beta_2}\triangle_k f)|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{\frac{l+4}{2}}\|_{L^2_t((\tau_2,T];L^2_{x,v,k})} \\
\le\, & C(1+K_3)K_3^2.
\end{aligned}
$$

When $|\alpha_1|+|\beta_1| \ge \left[\frac{N}{2}\right]+1$, and in this case, $|\alpha_2|+|\beta_2|+5 \le N$, we know that

$$
\begin{aligned}
& \|(1+|v|^2)^{\frac{l+6}{2}}(\partial_{v_j}\partial^{\alpha_2}_{\beta_2}\triangle_k f)|k|^{-\delta-\frac{3}{2}}\|_{L^2_t((\tau_2,T];L^\infty_{x,v})} \\
\le\, & C\|(1+|v|^2)^{\frac{l+6}{2}}(\partial_{v_j}\partial^{\alpha_2}_{\beta_2}\triangle_k f)|k|^{-\delta-\frac{3}{2}}\|_{L^2_t((\tau_2,T];H^4_{x,v})}.
\end{aligned}
$$

Since $|\psi^{ij}| \le |v-u|^{\gamma+2}$, we have

$$\begin{aligned}
&\|(\partial_{\beta_1}^{\alpha_1}\overline{a}_{ij}(x+k))(\partial_{v_j}\partial_{\beta_2}^{\alpha_2}\triangle_k f)|k|^{-\delta-\frac{3}{2}}(1+|v|^2)^{\frac{l+4}{2}}\|_{L_t^2((\tau_2,T];L^2_{x,v,k})}\\
\le{}& CK_3\Big\|\Big(\int_{\mathbb{R}^3}|v-u|^{\gamma+2}|\partial_{\beta_1}^{\alpha_1}f(x+k)|\mathrm{d}u\Big)(1+|v|^2)^{-1}\Big\|_{L_t^\infty((\tau_2,T];L^2_{x,v})}\\
&+CK_3\Big\|\Big(\int_{\mathbb{R}^3}|v-u|^{\gamma+2}|\partial_{\beta_1}^{\alpha_1}(f^2(x+k))|\mathrm{d}u\Big)(1+|v|^2)^{-1}\Big\|_{L_t^\infty((\tau_2,T];L^2_{x,v})}.
\end{aligned}\tag{3.21}$$

Then proceeding as the study of the term $(\mathrm{III})_1$ in the proof of Lemma 3.2, we get that (3.21) is bounded by $C(K_3^2+K_3^3)$.

Performing the similar calculations as (3.15), one can see that

$$\begin{aligned}
&\int_{(\tau_1,T]\times\mathbb{T}^3\times\mathbb{R}^3}(1+|v|^2)^{-4}\sum_{m\in\mathbb{Z}^3}|m|^{\frac{1}{10}}|\widehat{p}_{\delta,k}(t,m,\cdot)*\chi_\epsilon(v)|^2\mathrm{d}v\mathrm{d}k\mathrm{d}t\\
\le{}& C\sum_{m\in\mathbb{Z}^3}|m|^{\frac{1}{10}-\frac{1}{2}}(\epsilon^{-6}+\epsilon^{-8})(\|\widehat{p}_{\delta,k}(\tau_1,m,\cdot)\|^2_{L^2_{v,k}}+\|\widehat{p}_{\delta,k}(\cdot,m,\cdot)\|^2_{L_t^2((\tau_1,T];L^2_{v,k})}\\
&+\|\widehat{p}^{(1)}_{\delta,k}(\cdot,m,\cdot)\|^2_{L_t^2((\tau_1,T];L^2_{v,k})}+\|\widehat{p}^{(2)}_{\delta,k}(\cdot,m,\cdot)\|^2_{L_t^2((\tau_1,T];L^2_{v,k})}).
\end{aligned}\tag{3.22}$$

Given $\epsilon=\epsilon(|m|)=|m|^{-\frac{1}{20}}$, then (3.22) is bounded. Thus the proof of Lemma 3.4 is completed.

By using Lemmas 3.1 and 3.2 and iterating Lemmas 3.3 and 3.4 nineteen times, we have the following proposition.

Proposition 3.1 *Letting* $-3\le\gamma<-2$, $N\ge 8$ *be a given integer, and* f *be a smooth nonnegative solution to equation* (1.1) *given by Theorem* 1.1, *we suppose that for any* $T>0$, $l\ge 0$, *there exists* $0\le\tau_1<T$, *such that*

$$\|f\|_{L_t^\infty([\tau_1,T];H^{N,l}_{x,v})}\le\overline{K},$$

where $\overline{K}=\overline{K}(l,T,\gamma,N)$ *is a constant.*

Then for any $T>t_*>\tau_1$, *there exists a constant* $\widetilde{C}_0>0$, *which depends on* $N,l,\gamma,T,\overline{K}$, *such that*

$$\|f\|_{L_t^\infty([t_*,T];H^{N+1,l}_{x,v})}\le\widetilde{C}_0.$$

4 Proof of Theorem 1.2

We now complete the proof of Theorem 1.2. Employing Proposition 3.1 repeatedly, we get by induction on N that for any $0<\tau<T<+\infty$ and $l\ge 0$ large enough,

$$\|f\|_{L_t^\infty([\tau,T];H^{\infty,l}_{x,v})}\le\widetilde{C},\tag{4.1}$$

where $\widetilde{C}$ is a positive constant. We now prove by induction on m that $\partial_t^m f\in L_t^\infty([\tau,T];H^{\infty,l}_{x,v})$ in the sense of distribution. In light of (4.1), this is true for $m=0$. Let us assume that the induction hypothesis holds for any integer $k \leq m$. Then for all multi-index α, β and any $l \geq 0$,

$$\begin{aligned}[\partial_\beta^\alpha \partial_t^{m+1} f](1+|v|^2)^{\frac{l}{2}} = &-(1+|v|^2)^{\frac{l}{2}} \partial_\beta^\alpha [v \cdot \nabla_x \partial_t^m f] \\ &+ (1+|v|^2)^{\frac{l}{2}} \partial_\beta^\alpha \Big\{ \sum_{k=0}^m C_k^m \nabla_v \cdot \{[\psi * (\partial_t^k f - \partial_t^k (f^2))] \nabla_v \partial_t^{m-k} f \\ &- (\psi * \nabla_v \partial_t^k f)(\partial_t^{m-k} f - \partial_t^{m-k}(f^2))\} \Big\}.\end{aligned}$$

It is clear that

$$(1+|v|^2)^{\frac{l}{2}} \partial_\beta^\alpha [v \cdot \nabla_x \partial_t^m f] \in L^2_{x,v}$$

according to the induction hypothesis.

For the term $(1+|v|^2)^{\frac{l}{2}} \partial_\beta^\alpha \{[\psi * \partial_{v_i} \partial_t^k (f^2)] \partial_{v_i} \partial_t^{m-k} f\}$, recalling (2.5), we see that

$$\|\psi^{ij} * (\partial_{\beta_1}^{\alpha_1} \partial_{v_i} \partial_t^k (f^2))\|_{L^\infty_x} \leq C \|\partial_{\beta_1'}^{\alpha_1'} \partial_t^{k_1} f\|_{H^4_{x,v}} \|(\partial_{\beta_1 - \beta_1'}^{\alpha_1 - \alpha_1'} \partial_{v_i} \partial_t^{k-k_1} f)(1+|v|^2)\|_{H^2_{x,v}},$$

which implies

$$\begin{aligned}&\|\psi^{ij} * (\partial_{\beta_1}^{\alpha_1} \partial_{v_i} \partial_t^k (f^2))(\partial_{v_i} \partial_{\beta_2}^{\alpha_2} \partial_t^{m-k} f)(1+|v|^2)^{\frac{l}{2}}\|_{L^\infty_t([\tau,T];L^2_{x,v})} \\ \leq\ & C \|\partial_{\beta_1'}^{\alpha_1'} \partial_t^{k_1} f\|_{L^\infty_t([\tau,T];H^4_{x,v})} \|(\partial_{\beta_1-\beta_1'}^{\alpha_1-\alpha_1'} \partial_{v_i} \partial_t^{k-k_1} f)(1+|v|^2)\|_{L^\infty_t([\tau,T];H^2_{x,v})} \\ &\times \|(\partial_{v_i} \partial_{\beta_2}^{\alpha_2} \partial_t^{m-k} f)(1+|v|^2)^{\frac{l}{2}}\|_{L^\infty_t([\tau,T];L^2_{x,v})}.\end{aligned}$$

From the induction hypothesis, we conclude that $(1+|v|^2)^{\frac{l}{2}} \partial_\beta^\alpha \nabla_v \cdot \{[\psi * \partial_t^k (f^2)] \nabla_v (\partial_t^{m-k} f)\} \in L^2_{x,v}$. The other terms can be treated in the same way. Therefore, we know that

$$\|\partial_t^{m+1} f\|_{L^\infty_t([\tau,T];H^{\infty;l}_{x,v})} \leq \widetilde{C}.$$

This completes the proof of Theorem 1.2.

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