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On the Fourth Moment of Coefficients of Symmetric Square *L*-Function*

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Abstract Let $f(z)$ be a holomorphic Hecke eigencuspform of weight k for the full modular group. Let $\lambda_f(n)$ be the nth normalized Fourier coefficient of $f(z)$. Suppose that $L(\mathrm{sym}^2 f, s)$ is the symmetric square L-function associated with $f(z)$, and $\lambda_{\mathrm{sym}^2 f}(n)$ denotes the nth coefficient $L(\mathrm{sym}^2 f, s)$. In this paper, it is proved that

$$\sum_{n\le x}\lambda^4_{\mathrm{sym}^2 f}(n)=xP_2(\log x)+O(x^{\frac{79}{81}+\varepsilon}),$$

where $P_2(t)$ is a polynomial in t of degree 2. Similarly, it is obtained that

$$\sum_{n\le x}\lambda^4_f(n^2)=x\widetilde{P}_2(\log x)+O(x^{\frac{79}{81}+\varepsilon}),$$

where $\widetilde{P}_2(t)$ is a polynomial in t of degree 2.

Keywords Fourier coefficient of cusp form, Symmetric power L-function, Rankin-Selberg L-function
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1 Introduction

According to the Langlands program, the "most general" L-function should be a product of L-functions of automorphic cuspidal representations of $\mathrm{GL}_m/\mathbb{Q}$. Therefore, these automorphic L-functions deserve further investigation. The symmetric power L-functions are important automorphic L-functions.

Let k be a positive even integer, and H_k^* be the set of all normalized Hecke primitive eigencuspforms of weight k for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. The Fourier expansion of $f\in H_k^*$ at the cusp ∞ is

$$f(z)=\sum_{n=1}^{\infty}\lambda_f(n)n^{\frac{k-1}{2}}\mathrm{e}^{2\pi\mathrm{i}nz},$$

where $\lambda_f(n)$ is the eigenvalue of the (normalized) Hecke operator T_n. Then $\lambda_f(n)$ is real and satisfies the multiplicative property

$$\lambda_f(m)\lambda_f(n)=\sum_{d|(m,n)}\lambda_f\Big(\frac{mn}{d^2}\Big), \tag{1.1}$$

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where $m \geq 1$ and $n \geq 1$ are any integers. In 1974, Deligne [2] proved the Ramanujan-Petersson conjecture

$$|\lambda_f(n)| \leq d(n), \tag{1.2}$$

where $d(n)$ is the divisor function.

The Hecke L-function attached to $f \in H_k^*$ is defined for $\mathrm{Re}(s) > 1$ by

$$L(f,s) = \sum_{n=1}^{\infty} \lambda_f(n) n^{-s}.$$

By the theory of Hecke operators and the famous work of Deligne [2] on the Ramanujan conjecture, we have

$$L(f,s) = \prod_p \{(1-\alpha_f(p)p^{-s})(1-\beta_f(p)p^{-s})\}^{-1},$$

where $\alpha(p)$ and $\beta(p)$ satisfy

$$\lambda_f(p) = \alpha_f(p) + \beta_f(p), \quad \overline{\alpha_f(p)} = \beta_f(p), \quad |\alpha_f(p)| = 1. \tag{1.3}$$

The jth symmetric power L-function attached to $f \in H_k^*$ is defined as

$$L(\mathrm{sym}^j f, s) := \prod_p \prod_{m=0}^{j} (1 - \alpha_f(p)^{j-m} \beta_f(p)^m p^{-s})^{-1} \tag{1.4}$$

for $\mathrm{Re}(s) > 1$. In particular, $L(\mathrm{sym}^0 f, s)$ is the Riemann zeta-function $\zeta(s)$, and $L(\mathrm{sym}^1 f, s) = L(f,s)$. In the half-plane $\mathrm{Re}(s) > 1$, we can denote $L(\mathrm{sym}^j f, s)$ as a Dirichlet series

$$L(\mathrm{sym}^j f, s) = \sum_{n=1}^{\infty} \lambda_{\mathrm{sym}^j f}(n) n^{-s}, \tag{1.5}$$

where $\lambda_{\mathrm{sym}^j f}(n)$ is a multiplicative function.

Recently, Fomenko [4] has studied the mean square estimate for the coefficients of the symmetric square L-function attached to $f \in H_k^*$, and has shown that

$$\sum_{n \leq x} \lambda^2_{\mathrm{sym}^2 f}(n) = c_1 x + O(x^{\gamma}),$$

where $\gamma < 1$ is a constant, and the notation $f(x) = O(g(x))$ (or $f(x) \ll g(x)$) means that there exists a constant C such that $|f(x)| < Cg(x)$.

In [15], Lao established the following results: for any $\varepsilon > 0$,

$$\sum_{n \leq x} \lambda^2_{\mathrm{sym}^3 f}(n) = c_2 x + O(x^{\frac{8}{9}+\varepsilon}),$$

$$\sum_{n \leq x} \lambda^2_{\mathrm{sym}^4 f}(n) = c_3 x + O(x^{\frac{25}{27}+\varepsilon}).$$

In this paper, we are interested in the fourth moment of coefficients of the symmetric square L-function, and are able to show the following result.

Theorem 1.1 *Let $f \in H_k^*$, and $\lambda_{\mathrm{sym}^2 f}(n)$ denote the nth coefficient of the symmetric square L-function associated with f. Then we have that for any $\varepsilon > 0$,*

$$\sum_{n \le x} \lambda_{\mathrm{sym}^2 f}^4(n) = xP_2(\log x) + O(x^{\frac{79}{81}+\varepsilon}),$$

where $P_2(t)$ is a polynomial in t of degree 2.

The Fourier coefficient of cusp forms is an interesting object. For the sum of normalized Fourier coefficients over natural numbers, Rankin [23] proved that

$$S(x) = \sum_{n \le x} \lambda_f(n) \ll x^{\frac{1}{3}}(\log x)^{-\delta},$$

where $0 < \delta < 0.06$. Subsequently, many mathematicians studied the asymptotic behavior of the sum

$$\sum_{n \le x} \lambda_f(n^2)$$

(see [3, 6, 26]). In 1983, Moreno and Shahidi [20] proved

$$\sum_{n \le x} \tau_0^4(n) \sim cx \log x, \quad x \to \infty,$$

where $\tau_0(n) = \frac{\tau(n)}{n^{\frac{11}{2}}}$ is the normalized Ramanujan τ-function. Obviously, the result of Moreno and Shahidi also holds, if we replace $\tau_0(n)$ by the normalized Fourier coefficient $\lambda_f(n)$. Later, some authors of [16, 18–19] generalized this result to kth moment of $\lambda_f(n)$ ($k \le 8$).

In this paper, we also want to investigate the sum

$$\sum_{n \le x} \lambda_f^4(n^2).$$

Since $\lambda_f(n^j)$ is closely related to $\lambda_{\mathrm{sym}^j f}(n)$ (see Lemma 2.2), we are able to prove the following result.

Theorem 1.2 *Let $f \in H_k^*$, and $\lambda_f(n)$ denote the nth normalized Fourier coefficients associated with f. Then, we have that for any $\varepsilon > 0$,*

$$\sum_{n \le x} \lambda_f^4(n^2) = x\widetilde{P}_2(\log x) + O(x^{\frac{79}{81}+\varepsilon}),$$

where $\widetilde{P}_2(t)$ is a polynomial in t of degree 2.

2 Some Lemmas

Lemma 2.1 *Let $f \in H_k^*$, and $\lambda_{\mathrm{sym}^2 f}(n)$ denote the nth coefficient of the symmetric square L-function associated with f. We introduce*

$$L_1(s) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^2 f}^4(n)}{n^s}$$

for $\mathrm{Re}(s)>1$. *For* $j=2,3,4$, *let* $L(\mathrm{sym}^j f,s)$ *be the* j*th symmetric power* L*-function associated with* f, *and* $L(\mathrm{sym}^i f\times\mathrm{sym}^j f,s)$ *be the Rankin-Selberg* L*-function of* $\mathrm{sym}^i f$ *and* $\mathrm{sym}^j f$. *Then, we have that for* $\mathrm{Re}(s)>1$,

$$\begin{aligned}L_1(s)=\zeta(s)&L^2(\mathrm{sym}^2 f,s)L^2(\mathrm{sym}^4 f,s)L^2(\mathrm{sym}^2 f\times\mathrm{sym}^4 f,s)\\&\times L(\mathrm{sym}^2 f\times\mathrm{sym}^2 f,s)L(\mathrm{sym}^4 f\times\mathrm{sym}^4 f,s)U_1(s),\end{aligned}$$

where $U_1(s)$ *converges uniformly and absolutely in the half plane* $\mathrm{Re}(s)\geq\frac{1}{2}+\varepsilon$ *for any* $\varepsilon>0$.

Proof The Riemann zeta-function is defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_p(1-p^{-s})^{-1}\tag{2.1}$$

for $\mathrm{Re}(s)>1$. The jth symmetric power L-function attached to $f\in H_k^*$ is defined by

$$L(\mathrm{sym}^j f,s)=\prod_p\prod_{m=0}^{j}(1-\alpha_f(p)^{j-m}\beta_f(p)^m p^{-s})^{-1}=:\prod_p L_p(\mathrm{sym}^j f,s)\tag{2.2}$$

for $\mathrm{Re}(s)>1$. The product over primes gives a Dirichlet series representation for $L(\mathrm{sym}^j f,s)$, i.e., for $\mathrm{Re}(s)>1$,

$$L(\mathrm{sym}^j f,s)=\sum_{n=1}^{\infty}\frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s},$$

where $\lambda_{\mathrm{sym}^j f}(n)$ is a multiplicative function. Then we have that for $\mathrm{Re}(s)>1$,

$$L(\mathrm{sym}^j f,s)=\prod_p\Big(1+\frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s}+\cdots+\frac{\lambda_{\mathrm{sym}^j f}(p^k)}{p^{ks}}+\cdots\Big).\tag{2.3}$$

From (2.2)–(2.3), we have

$$\lambda_{\mathrm{sym}^j}(p)=\sum_{m=0}^{j}\alpha_f(p)^{j-m}\beta_f(p)^m.\tag{2.4}$$

From (1.3), we have

$$|\lambda_{\mathrm{sym}^j f}(n)|\leq d_{j+1}(n),\tag{2.5}$$

where $d_k(n)$ is the nth coefficient of the Dirichlet series $\zeta^k(s)$. This shows that $L(\mathrm{sym}^j f,s)$ converges absolutely in the half plane $\mathrm{Re}(s)>1$.

The Rankin-Selberg L-function attached to $\mathrm{sym}^i f$ and $\mathrm{sym}^j f$ is defined by

$$L(\mathrm{sym}^i f\times\mathrm{sym}^j f,s):=\prod_p\prod_{m=0}^{i}\prod_{u=0}^{j}(1-\alpha_f(p)^{i-m}\beta_f(p)^m\alpha_f(p)^{j-u}\beta_f(p)^u p^{-s})^{-1}\tag{2.6}$$

for $\mathrm{Re}(s)>1$. The product over primes also gives a Dirichlet series representation for $L(\mathrm{sym}^i f\times\mathrm{sym}^j f,s)$, i.e., for $\mathrm{Re}(s)>1$,

$$L(\mathrm{sym}^i f\times\mathrm{sym}^j f,s)=\sum_{n=1}^{\infty}\frac{\lambda_{\mathrm{sym}^i f\times\mathrm{sym}^j f}(n)}{n^s},$$

where $\lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(n)$ is a multiplicative function. Then we have that for $\mathrm{Re}(s)>1$,

$$L(\mathrm{sym}^i f\times \mathrm{sym}^j f, s)=\prod_p\Big(1+\frac{\lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(p)}{p^s}+\cdots+\frac{\lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(p^k)}{p^{ks}}+\cdots\Big). \tag{2.7}$$

From (2.6)–(2.7), we have

$$\begin{aligned}\lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(p)&=\sum_{m=0}^{i}\sum_{u=0}^{j}\alpha_f(p)^{i-m}\beta_f(p)^m\alpha_f(p)^{j-u}\beta_f(p)^u\\&=\lambda_{\mathrm{sym}^i f}(p)\lambda_{\mathrm{sym}^j f}(p).\end{aligned} \tag{2.8}$$

From (1.3), we have

$$|\lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j f}(n)|\le d_{(i+1)(j+1)}(n). \tag{2.9}$$

This shows that $L(\mathrm{sym}^i f\times \mathrm{sym}^j f, s)$ converges absolutely in the half plane $\mathrm{Re}(s)>1$.

For $\mathrm{Re}(s)>1$, we can write $\zeta(s)L^2(\mathrm{sym}^2 f,s)L^2(\mathrm{sym}^4 f,s)L^2(\mathrm{sym}^2 f\times\mathrm{sym}^4 f,s)L(\mathrm{sym}^2 f\times \mathrm{sym}^2 f,s)L(\mathrm{sym}^4 f\times \mathrm{sym}^4 f,s)$ as an Euler product,

$$\begin{aligned}&\zeta(s)L^2(\mathrm{sym}^2 f,s)L^2(\mathrm{sym}^4 f,s)L^2(\mathrm{sym}^2 f\times\mathrm{sym}^4 f,s)\\&\times L(\mathrm{sym}^2 f\times \mathrm{sym}^2 f,s)L(\mathrm{sym}^4 f\times \mathrm{sym}^4 f,s)\\&=:\prod_p\Big(1+\frac{b(p)}{p^s}+\cdots+\frac{b(p^k)}{p^{ks}}+\cdots\Big).\end{aligned} \tag{2.10}$$

From (2.1), (2.3) and (2.7), we have

$$\begin{aligned}b(p)=1&+2\lambda_{\mathrm{sym}^2 f}(p)+2\lambda_{\mathrm{sym}^4 f}(p)+2\lambda_{\mathrm{sym}^2 f\times\mathrm{sym}^4 f}(p)\\&+\lambda_{\mathrm{sym}^2 f\times\mathrm{sym}^2 f}(p)+\lambda_{\mathrm{sym}^4 f\times\mathrm{sym}^4 f}(p).\end{aligned} \tag{2.11}$$

From (1.3), (2.4) and (2.8), it is easy to check that

$$b(p)=\lambda^4_{\mathrm{sym}^2 f}(p). \tag{2.12}$$

On the other hand, from (2.5), we learn that

$$L_1(s)=\sum_{n=1}^{\infty}\frac{\lambda^4_{\mathrm{sym}^2 f}(n)}{n^s}$$

is absolutely convergent in the half plane $\mathrm{Re}(s)>1$. Noting that $\lambda^4_{\mathrm{sym}^2 f}(n)$ is a multiplicative function, we have that for $\mathrm{Re}(s)>1$,

$$L_1(s)=\prod_p\Big(1+\frac{\lambda^4_{\mathrm{sym}^2 f}(p)}{p^s}+\frac{\lambda^4_{\mathrm{sym}^2 f}(p^2)}{p^{2s}}+\cdots+\frac{\lambda^4_{\mathrm{sym}^2 f}(p^k)}{p^{ks}}+\cdots\Big). \tag{2.13}$$

Therefore, from (2.10) and (2.12)–(2.13), we have that for $\mathrm{Re}(s)>1$,

$$\begin{aligned}L_1(s)&=\zeta(s)L^2(\mathrm{sym}^2 f,s)L^2(\mathrm{sym}^4 f,s)L^2(\mathrm{sym}^2 f\times\mathrm{sym}^4 f,s)\\&\quad\times L(\mathrm{sym}^2 f\times \mathrm{sym}^2 f,s)L(\mathrm{sym}^4 f\times \mathrm{sym}^4 f,s)\\&\quad\times\prod_p\Big(1+\frac{\lambda^4_{\mathrm{sym}^2 f}(p^2)-b(p^2)}{p^{2s}}+\cdots\Big)\\&=:\zeta(s)L^2(\mathrm{sym}^2 f,s)L^2(\mathrm{sym}^4 f,s)L^2(\mathrm{sym}^2 f\times\mathrm{sym}^4 f,s)\\&\quad\times L(\mathrm{sym}^2 f\times \mathrm{sym}^2 f,s)L(\mathrm{sym}^4 f\times \mathrm{sym}^4 f,s)U_1(s).\end{aligned}$$

From (1.2), (2.5) and (2.9), it is obvious that $U_1(s)$ converges uniformly and absolutely in the half plane $\mathrm{Re}(s) \geq \frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$. This completes the proof of Lemma 2.1.

Lemma 2.2 *Let $f \in H_k^*$, and $\lambda_f(n)$ denote the nth normalized Fourier coefficients associated with f. We introduce*

$$L_2(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^4(n^2)}{n^s} \tag{2.14}$$

for $\mathrm{Re}(s) > 1$. For $j = 2, 3, 4$, let $L(\mathrm{sym}^j f, s)$ be the jth symmetric power L-function associated with f, and $L(\mathrm{sym}^i f \times \mathrm{sym}^j f, s)$ be the Rankin-Selberg L-function of $\mathrm{sym}^i f$ and $\mathrm{sym}^j f$. Then, we have that for $\mathrm{Re}(s) > 1$,

$$\begin{aligned} L_2(s) = {} & \zeta(s)L^2(\mathrm{sym}^2 f, s)L^2(\mathrm{sym}^4 f, s)L^2(\mathrm{sym}^2 f \times \mathrm{sym}^4 f, s) \\ & \times L(\mathrm{sym}^2 f \times \mathrm{sym}^2 f, s)L(\mathrm{sym}^4 f \times \mathrm{sym}^4 f, s)U_2(s), \end{aligned} \tag{2.15}$$

where $U_2(s)$ converges uniformly and absolutely in the half plane $\mathrm{Re}(s) \geq \frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$.

Proof By (1.3) and the theory of Hecke operators, it is easy to show that for $j \geq 1$,

$$\lambda_f(p^j) = \frac{\alpha_f(p)^{j+1} - \beta_f(p)^{j+1}}{\alpha_f(p) - \beta_f(p)} = \sum_{m=0}^{j} \alpha_f(p)^{j-m}\beta_f(p)^m. \tag{2.16}$$

From (2.4) and (2.16), we have

$$\lambda_f(p^j) = \lambda_{\mathrm{sym}^j f}(p).$$

In particular, we have

$$\lambda_f(p^2) = \lambda_{\mathrm{sym}^2 f}(p).$$

Therefore, from (2.12), we have

$$b(p) = \lambda_f^4(p^2).$$

The rest of the proof is similar to that of Lemma 2.1.

Lemma 2.3 (see [1]) *Let $f \in H_k^*$, and the jth symmetric power L-function associated with $\mathrm{sym}^j f$ be defined in (1.4). For $j = 1, 2, 3, 4$, the archimedean local factor of $L(\mathrm{sym}^j f, s)$ is*

$$L_\infty(\mathrm{sym}^j f, s) = \begin{cases} \displaystyle\prod_{v=0}^{n} \Gamma_{\mathbb{C}}\Big(s + \Big(v + \frac{1}{2}\Big)\Big(k-1\Big)\Big), & \text{if } \ j = 2n+1, \\ \displaystyle\Gamma_{\mathbb{R}}(s + \delta_{2\nmid n}) \prod_{v=1}^{n} \Gamma_{\mathbb{C}}(s + v(k-1)), & \text{if } \ j = 2n, \end{cases}$$

where $\Gamma_{\mathbb{R}} = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$, $\Gamma_{\mathbb{C}} = 2(2\pi)^{-s}\Gamma(s)$, and

$$\delta_{2\nmid n} = \begin{cases} 1, & \text{if } \ 2 \nmid n, \\ 0, & \text{otherwise.} \end{cases}$$

For $1 \le j \le 4$, it is known that the complete L-function

$$\Lambda(\mathrm{sym}^j f, s) := L_\infty(\mathrm{sym}^j f, s) L(\mathrm{sym}^j f, s)$$

is an entire function on the whole complex plane $\mathbb{C}$ and satisfies the functional equation

$$\Lambda(\mathrm{sym}^j f, s) = \epsilon_{\mathrm{sym}^j f} \Lambda(\mathrm{sym}^j f, 1-s),$$

where $\epsilon_{\mathrm{sym}^j f} = \pm 1$.

Based on the work of Cogdell and Michel [1], Lau and Wu [17] showed that for $j = 2, 3, 4$, $L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s)$ has a meromorphic continuation to the whole complex plane and satisfies a functional equation.

Lemma 2.4 (see [17]) *Let $f \in H_k^*$, and the Rankin-Selberg L-function associated with $\mathrm{sym}^j f$ and $\mathrm{sym}^j f$ be defined in (2.6). For $j = 1, 2, 3, 4$, the archimedean local factor of $L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s)$ is*

$$L_\infty(\mathrm{sym}^j f \times \mathrm{sym}^j f, s) = \Gamma_{\mathbb{R}}(s)^{\delta_{2|j}} \Gamma_{\mathbb{C}}(s)^{[\frac{j}{2}] + \delta_{2\nmid j}} \prod_{v=1}^{j} \Gamma_{\mathbb{C}}(s + v(k-1))^{j-v+1},$$

where $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})$, $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$, $\delta_{2|j} = 1 - \delta_{2\nmid j}$ and

$$\delta_{2\nmid j} = \begin{cases} 1, & \text{if } 2 \nmid j, \\ 0, & \text{otherwise.} \end{cases}$$

Then the complete L-function

$$\Lambda(\mathrm{sym}^j f \times \mathrm{sym}^j f, s) =: L_\infty(\mathrm{sym}^j f \times \mathrm{sym}^j f, s) L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s)$$

is entire except possibly for simple poles at $s = 0, 1$, and satisfies the functional equation

$$\Lambda(\mathrm{sym}^j f \times \mathrm{sym}^j f, s) = \epsilon_{\mathrm{sym}^j f \times \mathrm{sym}^j f} \Lambda(\mathrm{sym}^j f \times \mathrm{sym}^j f, 1-s),$$

where $\epsilon_{\mathrm{sym}^j f \times \mathrm{sym}^j f} = \pm 1$.

Remark 2.1 In addition, from the famous works of Gelbart and Jacquet [5], Kim and Shahidi [12–13], and Kim [11], we learn that for $1 \le j \le 4$, the symmetric power L-function $L(\mathrm{sym}^j f, s)$ agrees with the L-function associated with an automorphic cuspidal selfdual representation $\mathrm{sym}^j \pi_f$ of $\mathrm{GL}_{j+1}(\mathbb{A}_{\mathbb{Q}})$. Then from the works of Jacquet and Shalika [9–10], Shahidi [28–29], and the reformulation of Rudnick and Sarnak [24], the Rankin-Selberg L-function $L(\mathrm{sym}^i f \times \mathrm{sym}^j f, s)$ can be extended to be an entire function on the whole complex plane $(1 \le i, j \le 4,\ i \ne j)$, which satisfies a functional equation of Riemann-type.

Lemma 2.5 *Let $i, j = 2, 3, 4$. Then for any $\varepsilon > 0$, $0 \le \sigma \le 2$ and $|t| \ge 2$, we have*

$$L(\mathrm{sym}^j f, \sigma + it) \ll_{f,\varepsilon} (1 + |t|)^{\max\left\{\frac{j+1}{2}(1-\sigma), 0\right\} + \varepsilon},$$
$$L(\mathrm{sym}^i f \times \mathrm{sym}^j f, \sigma + it) \ll_{f,\varepsilon} (1 + |t|)^{\max\left\{\frac{(i+1)(j+1)}{2}(1-\sigma), 0\right\} + \varepsilon}.$$

Proof Using the absolute convergences of $L(\mathrm{sym}^j f, \sigma+\mathrm{i}t)$ and $L(\mathrm{sym}^i f \times \mathrm{sym}^j f, \sigma+\mathrm{i}t)$ for $\sigma > 1$, the corresponding functional equation, Stirling's estimate for the gamma function and the Phragmen-Lindelöf convexity principle, we can follow standard arguments to establish the convexity bound for $L(\mathrm{sym}^j f, \sigma+\mathrm{i}t)$ and $L(\mathrm{sym}^i f \times \mathrm{sym}^j f, \sigma+\mathrm{i}t)$ in the critical strip $0 \leq \sigma \leq 1$ (see, e.g., [8, Chapter 5]), namely,

$$L(\mathrm{sym}^j f, \sigma+\mathrm{i}t) \ll_{f,\varepsilon} (1+|t|)^{\frac{j+1}{2}(1-\sigma)+\varepsilon},$$
$$L(\mathrm{sym}^i f \times \mathrm{sym}^j f, \sigma+\mathrm{i}t) \ll_{f,\varepsilon} (1+|t|)^{\frac{(i+1)(j+1)}{2}(1-\sigma)+\varepsilon}.$$

In addition, since $L(\mathrm{sym}^j f, \sigma+\mathrm{i}t)$ and $L(\mathrm{sym}^i f \times \mathrm{sym}^j f, \sigma+\mathrm{i}t)$ are both absolutely convergent when $\sigma > 1$, according to [21, Theorem 6.6.3], we have

$$L(\mathrm{sym}^j f, \sigma+\mathrm{i}t) \ll_{f,\varepsilon} (1+|t|)^{\varepsilon},$$
$$L(\mathrm{sym}^i f \times \mathrm{sym}^j f, \sigma+\mathrm{i}t) \ll_{f,\varepsilon} (1+|t|)^{\varepsilon}$$

for $\sigma > 1$. This completes the proof of Lemma 2.5.

Lemma 2.6 (see [14]) *Let $j = 2, 3, 4$. Then for $T \geq T_0$ (where T_0 is sufficiently large), we have the estimate*

$$\int_T^{2T} \Big|L\Big(\mathrm{sym}^j f \times \mathrm{sym}^j f, \frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\Big|^2 \mathrm{d}t \ll_{f,\varepsilon} T^{\frac{(j+1)^2}{2}+\varepsilon},$$

where ε is any positive constant.

In general, we have the following result.

Lemma 2.7 *Let $L(f,s)$ be a Dirichlet series with Euler product of degree $m \geq 2$, which means*

$$L(f,s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_{p<\infty} \prod_{j=1}^{m} \Big(1 - \frac{\alpha_f(p,j)}{p^s}\Big)^{-1},$$

where $\alpha_f(p,j)$, $j = 1, \cdots, m$ are the local parameters of $L(f,s)$ at the prime p, and $\lambda_f(n) \ll n^{\varepsilon}$. Assume that this series and its Euler product are absolutely convergent for $\mathrm{Re}(s) > 1$. *Let the gamma factor*

$$L_\infty(f,s) = \prod_{j=1}^{m} \pi^{-\frac{s+\mu_f(j)}{2}} \Gamma\Big(\frac{s+\mu_f(j)}{2}\Big),$$

where $\mu_f(j)$, $j = 1, \cdots, m$ are the local parameters of $L(f,s)$ at ∞. We also define the completed L-function $\Lambda(f,s)$ by

$$\Lambda(f,s) = L_\infty(f,s) L(f,s).$$

We assume that $\Lambda(f,s)$ admits an analytic continuation to the whole complex plane $\mathbb{C}$ and is entire except possibly for simple poles at $s = 0, 1$. It also satisfies a functional equation

$$\Lambda(f,s) = \epsilon_f \Lambda(\widetilde{f}, 1-s),$$

where ϵ_f is the root number with $|\epsilon_f| = 1$, and $\widetilde{f}$ is the dual of f such that $\lambda_{\widetilde{f}}(n) = \overline{\lambda_f(n)}$, $\mu_{\widetilde{f}}(j) = \overline{\mu_f(j)}$.

Then we have that for $T \geq T_0$ (where T_0 is sufficiently large),

$$\int_T^{2T} \Big|L\Big(f, \frac{1}{2} + \varepsilon + \mathrm{i}t\Big)\Big|^2 \mathrm{d}t \ll T^{\frac{m}{2}+\varepsilon},$$

where ε is any positive constant.

Proof The proof of this lemma is similar to that of Lemma 2.6. From the functional equation

$$\Lambda(f, s) = \epsilon_f \Lambda(\widetilde{f}, 1 - s),$$

we have

$$L\,(f, s) = \chi(s) L(\widetilde{f}, 1 - s),$$

where

$$|\chi(s)| \asymp |t|^{\frac{m}{2}(1-2\sigma)}, \quad \text{as} \quad |t| \to \infty,$$

in any fixed strip $a \leq \sigma \leq b$. Then we can also follow the arguments of [25, Theorem 4.1(i)] to show

$$\int_T^{2T} \Big|L\Big(f, \frac{1}{2} + \varepsilon + \mathrm{i}t\Big)\Big|^2 \mathrm{d}t \ll T^{\frac{m}{2}+\varepsilon}.$$

Here we choose free parameters Y and Y_1 such that $Y = Y_1 = cT^{\frac{m}{2}}$, where c is a suitable positive constant.

3 Proof of Theorems 1.1–1.2

In this section, we give the proof of Theorem 1.1. The proof of Theorem 1.2 is similar to that of Theorem 1.1. In order to avoid repetition, we omit the proof of Theorem 1.2.

Recall that we defined

$$L_1(s) = \sum_{n=1}^{\infty} \frac{\lambda^4_{\mathrm{sym}^2 f}(n)}{n^s} \tag{3.1}$$

for $\mathrm{Re}(s) > 1$. From Lemmas 2.1 and 2.3–2.4, we learn that

$$\begin{aligned} L_1(s) = \zeta(s) &L^2(\mathrm{sym}^2 f, s) L^2(\mathrm{sym}^4 f, s) L^2(\mathrm{sym}^2 f \times \mathrm{sym}^4 f, s) \\ &\times L(\mathrm{sym}^2 f \times \mathrm{sym}^2 f, s) L(\mathrm{sym}^4 f \times \mathrm{sym}^4 f, s) U_1(s) \end{aligned}$$

can be analytically continued on the half plane $\mathrm{Re}(s) > \frac{1}{2}$. In this region, $L_1(s)$ only has a pole $s = 1$ of order 3.

Now we begin to prove our main results. By (3.1) and Perron's formula (see [8, Proposition 5.54]), we have

$$\sum_{n \leq x} \lambda^4_{\mathrm{sym}^2 f}(n) = \frac{1}{2\pi \mathrm{i}} \int_{b-\mathrm{i}T}^{b+\mathrm{i}T} L_1(s) \frac{x^s}{s} \mathrm{d}s + O\Big(\frac{x^{1+\varepsilon}}{T}\Big), \tag{3.2}$$

where $b = 1 + \varepsilon$ and $1 \le T \le x$ is a parameter to be chosen later. Here we have used (1.2).

Next we move the integration to the parallel segment with $\mathrm{Re}(s) = \frac{1}{2} + \varepsilon$. By Cauchy's residue theorem, we have

$$\begin{aligned}\sum_{n\le x}\lambda^4_{\mathrm{sym}^2 f}(n) &= \operatorname*{Res}_{s=1} L_1(s)\frac{x^s}{s} + \frac{1}{2\pi \mathrm{i}}\Big\{\int_{\frac{1}{2}+\varepsilon-\mathrm{i}T}^{\frac{1}{2}+\varepsilon+\mathrm{i}T} + \int_{\frac{1}{2}+\varepsilon+\mathrm{i}T}^{b+\mathrm{i}T} + \int_{b-\mathrm{i}T}^{\frac{1}{2}+\varepsilon-\mathrm{i}T}\Big\} L_1(s)\frac{x^s}{s}\mathrm{d}s \\ &\quad + O\Big(\frac{x^{1+\varepsilon}}{T}\Big) \\ &=: xP_2(\log x) + \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + O\Big(\frac{x^{1+\varepsilon}}{T}\Big), \end{aligned} \tag{3.3}$$

where $P_2(t)$ is a polynomial in t of degree 2.

For convenience, we write

$$\begin{aligned} L_{01}(s) &= \zeta(s)L^2(\mathrm{sym}^2 f, s)L^2(\mathrm{sym}^4 f, s), \\ L_{02}(s) &= L^2(\mathrm{sym}^2 f\times \mathrm{sym}^4 f, s)L(\mathrm{sym}^2 f\times\mathrm{sym}^2 f, s)L(\mathrm{sym}^4 f\times \mathrm{sym}^4 f, s). \end{aligned}$$

Further, we recall that $L_0(s) = L_{01}(s)L_{02}(s)$ is a Riemann-type nice L-function with Euler product of degree $m = 81$.

For I_1, by Lemma 2.1, we have

$$\mathrm{I}_1 \ll x^{\frac{1}{2}+\varepsilon}\int_1^T \Big|L_0\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)U_1\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\Big| t^{-1}\mathrm{d}t + x^{\frac{1}{2}+\varepsilon}.$$

Then by Cauchy-Schwarz inequality, we have

$$\begin{aligned}\mathrm{I}_1 &\ll x^{\frac{1}{2}+\varepsilon}\log T\max_{T_1\le T}\Big\{\frac{1}{T_1}\Big(\int_{\frac{T_1}{2}}^{T_1}\Big|L_{01}\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\Big|^2\mathrm{d}t\Big)^{\frac{1}{2}} \\ &\quad\times\Big(\int_{\frac{T_1}{2}}^{T_1}\Big|L_{02}\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\Big|^2\mathrm{d}t\Big)^{\frac{1}{2}}\Big\} + x^{\frac{1}{2}+\varepsilon} \\ &\ll x^{\frac{1}{2}+\varepsilon}T^{\frac{77}{4}+\varepsilon}, \end{aligned}\tag{3.4}$$

where we have used Lemma 2.7 in the following forms:

$$\begin{aligned}\int_{\frac{T_1}{2}}^{T_1}\Big|L_{01}\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\Big|^2\mathrm{d}t &\ll T^{\frac{17}{2}+\varepsilon}, \\ \int_{\frac{T_1}{2}}^{T_1}\Big|L_{02}\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\Big|^2\mathrm{d}t &\ll T^{32+\varepsilon}. \end{aligned}$$

For the integrals over the horizontal segments, we use Lemma 2.5 (note that $L_0(\sigma+\mathrm{i}t)$ converges absolutely when $\sigma > 1$) to get

$$\begin{aligned}\mathrm{I}_2 + \mathrm{I}_3 &\ll \int_{\frac{1}{2}+\varepsilon}^{b} x^\sigma\,|L_0(\sigma+\mathrm{i}T)|\,T^{-1}\mathrm{d}\sigma \\ &= \int_{\frac{1}{2}+\varepsilon}^{1} x^\sigma\,|L_0(\sigma+\mathrm{i}T)|\,T^{-1}\mathrm{d}\sigma + \int_1^{1+\varepsilon} x^\sigma\,|L_0(\sigma+\mathrm{i}T)|\,T^{-1}\mathrm{d}\sigma \\ &\ll \max_{\frac{1}{2}+\varepsilon\le\sigma\le 1} x^\sigma T^{\frac{81}{2}(1-\sigma)+\varepsilon}T^{-1} + x^{1+\varepsilon}T^\varepsilon T^{-1} \end{aligned}$$

$$= \max_{\frac{1}{2}+\varepsilon\le\sigma\le 1}\left(\frac{x}{T^{\frac{81}{2}}}\right)^{\sigma}T^{\frac{79}{2}+\varepsilon}+x^{1+\varepsilon}T^{-1+\varepsilon}$$
$$\ll x^{1+\varepsilon}T^{-1+\varepsilon}+x^{\frac{1}{2}+\varepsilon}T^{\frac{77}{4}+\varepsilon}. \tag{3.5}$$

From (3.3)–(3.5), we have

$$\sum_{n\le x}\lambda^4_{\mathrm{sym}^2 f}(n)=xP_2(\log x)+O(x^{1+\varepsilon}T^{-1+\varepsilon})+O(x^{\frac{1}{2}+\varepsilon}T^{\frac{77}{4}+\varepsilon}). \tag{3.6}$$

Taking $T=x^{\frac{2}{81}}$ in (3.6), we have

$$\sum_{n\le x}\lambda^4_{\mathrm{sym}^2 f}(n)=xP_2(\log x)+O(x^{\frac{79}{81}+\varepsilon}).$$

This completes the proof of Theorem 1.1.

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