Chin. Ann. Math.
33B(6), 2012, 903–908
DOI: 10.1007/s11401-012-0744-x

Chinese Annals of Mathematics, Series B

On the Cellular Indecomposable Property of Semi-Fredholm Operators*

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Abstract The authors prove that an operator with the cellular indecomposable property has no singular points in the semi-Fredholm domain, by applying the 4×4 matrix model of semi-Fredholm operators due to Fang in 2004. This result fills a gap in the result of Olin and Thomson in 1984.

Keywords Cellular indecomposable property, Semi-Fredholm operator, Singular point
2000 MR Subject Classification 47A53

1 Introduction

In [3–5], Olin and Thomson introduced and studied the cellular indecomposable property (CIP) which is a basic notion in the operator theory. An operator $T \in B(H)$ has (CIP) if any two nontrivial invariant subspaces $M_1, M_2 \subset H$ of T have a nontrivial intersection $M_1 \cap M_2 \neq \{0\}$. Note that if T has (CIP), then so does $T - \lambda$ for any $\lambda \in \mathbb{C}$, since T and $T - \lambda$ have the same invariant subspace lattice.

The principal question underlying Olin and Thomson's research is what the spectral picture (see [6]) of a CIP operator can look like. For instance, one can show that the Fredholm index of a CIP operator cannot be positive, and hence the adjoint is quasi-triangular (see [1, 6]). It is easy to achieve the index 0 or -1, but it is still not known whether the index can be -2 or smaller.

Motivated by the spectral picture problem, Olin and Thomson made a thorough analysis of subnormal operators with (CIP). For general operators, they proved a result on semi-Fredholm operators (see [3, Lemma 4]) which is needed in the proof of the main result in [3]. The proof of Lemma 4 in [3], however, contains a gap in handling singular points in the semi-Fredholm domain as explained below.

On the other hand, their result is almost certainly useful for further study of the spectral theory of a general CIP operator. This prompts us to find a complete proof. In this paper, we prove a result (see Theorem 2.2) which suffices to fill the gap and is of independent interests, since we show that a CIP operator has no singularity at all.

Manuscript received June 23, 2011. Revised July 23, 2012.
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*Project supported by the National Natural Science Foundation of China (No. 11101312) and the National Science Foundation (No. 0801174).

Our main technical tool is the 4×4 matrix model of semi-Fredholm operators developed in [2].

2 Main Results

Recall that a singular point $\lambda_0 \in \rho_F(T)$ in the Fredholm domain. $\rho_F(T)$ of an operator $T \in B(H)$ acting on a Hilbert space H is a point λ_0, such that the dimension function of the kernel

$$\lambda \to \dim(\ker(T - \lambda))$$

is not continuous at λ_0. When $\lambda_0 \in \rho_{sF}(T)$ (the semi-Fredholm domain), λ_0 is singular if the projection $P_{\ker(T-\lambda)}$ does not converge to $P_{\ker(T-\lambda_0)}$ as $\lambda \to \lambda_0$ in the strong operator topology. In this paper, we mainly consider those singular points in the semi-Fredholm domain.

To overcome the complexity caused by a singular point, Olin and Thomson [3] used a translation argument: For a semi-Fredholm T, possibly singular at 0, they replaced T by $T - \lambda$ for some small λ, so they assume that T is regular at 0. However, they implicitly used the following argument: If T is analytic, then so is $T - \lambda$. Here, an operator T is analytic if

$$\bigcap_{k \geq 0} T^k H = \{0\}$$

(see the first line and the last line in [3, p. 402]). This is not true as illustrated by the following one dimensional extension of a pure isometry $S \in B(H)$:

$$T = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \in B(\mathbb{C} \oplus H). \tag{2.1}$$

The statement of the following Theorem 2.1 is the same as Lemma 4 in [3].

Theorem 2.1 *If T is a semi-Fredholm operator, such that*
(1) *the Fredholm index satisfies* $\text{index}(T) \notin \{0, -1\}$,
(2) *T is analytic, i.e.,* $\bigcap_{k \geq 0} T^k H = \{0\}$,
then T is cellular decomposable, that is, it has no (CIP).

The arguments of the proof in [3] do not work for the above T in (2.1). The obstacle at the end of [3, p. 402] is as follows: After a translation of $T - \lambda$, the second analytic condition (2) in Theorem 2.1 is no longer satisfied. Moreover, Olin and Thomson [3] actually proved Theorem 2.1 under an extra condition, that is,

$(*)$ T has no singularity at 0.

The main result of this paper is as follows.

Theorem 2.2 *If the Hilbert space H is infinite dimensional,* $\dim(H) = \infty$, *and $T \in B(H)$ is cellular indecomposable, then T has no singular points in its semi-Fredholm domain.*

So Theorem 2.1 follows from Theorem 2.2 and the proof of Olin and Thomson in [3]. Note that Theorem 2.2 does not hold on a finite dimensional Hilbert space, as illustrated by a single nilpotent Jordan block, which indeed has (CIP) and is singular at the origin.

Corollary 2.1 *If $T \in B(H)$ is a semi-Fredholm operator with the cellular indecomposable property, then T has the following matrix decomposition*:

$$T = \begin{pmatrix} T_1 & A \\ 0 & T_2 \end{pmatrix}, \tag{2.2}$$

where the decomposition is with respect to $H_1 \oplus H_1^{\perp}$ with $H_1 = \bigcap_{k \geq 1} T^k H$, $T_1 \in B(H_1)$ is invertible, and T_2 is a pure shift.

Recall that a pure shift is a left-invertible operator which is also analytic (see [2]). The proof of Corollary 2.1 is essentially contained in the proof of Theorem 2.2.

It is an interesting question to observe when the entry T_1 in (2.2) is indeed void. If $\mathrm{index}(T) \leq -2$, then Theorem 2.1 implies that T_1 cannot be void. Again, we do not know whether $\mathrm{index}(T) \leq -2$ can happen for a CIP operator.

The rest of this paper is devoted to the proof of Theorem 2.2.

Proof of Theorem 2.2 We first recall the 4×4 upper-triangular matrix model of semi-Fredholm operators developed in [2] which we rely on heavily.

For any semi-Fredholm $T \in B(H)$, we can decompose $H = H_1 \oplus H_2 \oplus H_3 \oplus H_4$ into the direct sum of four closed subspaces, with some components possibly void, such that the associated matrix of T has the form

$$T = \begin{pmatrix} T_1 & * & * & * \\ 0 & T_2 & * & * \\ 0 & 0 & T_3 & * \\ 0 & 0 & 0 & T_4 \end{pmatrix}. \tag{2.3}$$

The properties of T_1, T_2, T_3, T_4 which we will need are listed below.

(i) T_4 is a pure shift semi-Fredholm operator. See the definition after Corollary 2.1. Or, to be more speficic, recall that a semi-Fredholm operator $S \in B(K)$ is a pure shift if

(a) $\ker(S) = \{0\}$,

(b) S is analytic, i.e., $\bigcap_{k \geq 0} S^k K = \{0\}$.

In particular, if S is a pure shift, then $\ker(S^*) \neq \{0\}$ and $\dim(\ker(S^* - \lambda))$ is a constant in a small open neighborhood of the origin by the general Fredholm theory.

(ii) T_1^* is a pure shift.

(iii) T_2 is invertible.

(iv) T_3 is a finite nilpotent matrix. In particular,

$$\dim(H_3) = N < \infty. \tag{2.4}$$

It follows that

$$T_3^N = 0. \tag{2.5}$$

These two conditions will play important roles in the proof.

(v) The origin 0 is a singular point in the semi-Fredholm domain of T if and only if $H_3 \neq \{0\}$. So our goal is to show $H_3 = \{0\}$.

First, we show that $H_1 = \{0\}$. Otherwise, $H' = \ker(T_1) \neq \{0\}$ is a nontrivial invariant subspace of T. Since T_1^* is a pure shift, and

$$\dim(\ker(T_1)) = \dim(\ker(T_1 - \lambda))$$

when λ is small enough, but nonzero, we have

$$H'' = \ker(T_1 - \lambda) \neq \{0\}$$

to be another nontrivial invariant subspace of T_1, and hence of T. Clearly, $H' \cap H'' = \{0\}$, since they consist of eigenvectors of different eigenvalues. This is a contradiction, since T has (CIP).

Next, we show that at most one of H_2 and H_3 can be nonzero. Otherwise, H_2 is a nontrivial invariant subspace. Since H_3 is nonzero, by (v) above, 0 is a singular point of T. Hence

$$\ker(T) \neq \{0\},$$

which is another nontrivial invariant subspace. Since $T_2 = T|_{H_2}$ is invertible, T is bounded below on H_2. It follows that $H_2 \cap \ker(T) = \{0\}$. It is again in contradiction with (CIP).

If $H_3 = \{0\}$, then the proof is completed.

Next, we assume that $H_2 = \{0\}$, and H_3 is a nontrivial invariant subspace. In this case, $H = H_3 \oplus H_4$.

Since $\dim(H) = \infty$ and $\dim(H_3) = N < \infty$, we know that H_4 is nontrivial. Since T_4 is a pure shift, we can choose a unit vector

$$k \in \ker(T_4^*),$$

and let $H_k \subset H$ denote the invariant subspace generated by $\binom{0}{k}$ under the action of T.

Claim 2.1 $H_k \cap H_3 = \{0\}$.

This will be in contradiction with (CIP). So it follows $H_3 = \{0\}$, and we are done then. The rest of the proof is devoted to proving this claim.

Next, we assume that there is a sequence of polynomials $p_t(z) \in \mathbb{C}[z]$, such that

$$\lim_{t\to\infty} p_t(T) \begin{pmatrix} 0 \\ k \end{pmatrix} = \begin{pmatrix} e \\ 0 \end{pmatrix} \in H_k \cap H_3,$$

and we wish to show $e = 0$.

Let

$$T = \begin{pmatrix} T_3 & A \\ 0 & T_4 \end{pmatrix}$$

for some $A \in B(H_4, H_3)$, and for any polynomial

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n,$$

we write

$$p(T) \begin{pmatrix} 0 \\ k \end{pmatrix} = \begin{pmatrix} p(T_3) & B_p \\ 0 & p(T_4) \end{pmatrix} \begin{pmatrix} 0 \\ k \end{pmatrix} = \begin{pmatrix} B_p k \\ p(T_4) k \end{pmatrix},$$

where B_p is a noncommutative polynomial of T_3, A and T_4. If we can show that for any polynomial p,

$$\|B_p k\| \le C\|p(T_4)k\| \tag{2.6}$$

for some constant C independent of p, then we can conclude that $e = 0$.

Without loss of generality, we assume that

$$n \ge N = \dim(H_3),$$

since otherwise we can choose

$$a_{n+1} = \cdots = a_N = 0,$$

so that p is formally of degree N. This will make the bookkeeping in the proof of (2.8) easier. (2.8) is a key step toward the proof of (2.6).

Next, we calculate B_p directly. For any $i = 1, 2, \cdots, N$, let

$$B_i = a_i T_3^{i-1} A + a_{i+1} T_3^{i-1} A T_4 + \cdots + a_n T_3^{i-1} A T_4^{n-i}.$$

By using

$$T_3^N = 0 \tag{2.7}$$

and all terms in B_p, we have

$$B_p = B_1 + \cdots + B_N. \tag{2.8}$$

The proof of (2.8) involves some work on bookkeeping, but there is nothing challenging. To write out all terms of B_p, one just needs to keep (2.7) in mind.

Note that $N = \dim(H_3)$ is independent of $p = p(z)$. So it suffices to show that for each $i = 1, 2, \cdots, N$,

$$\|B_i k\| \le C\|p(T_4)k\|$$

for some constant C independent of p. Let

$$B_i' = a_i + a_{i+1} T_4 + \cdots + a_n T_4^{n-i}.$$

Then

$$B_i = T_3^{i-1} A B_i'.$$

Hence it suffices to show

$$\|B_i' k\| \le C\|p(T_4)k\| \tag{2.9}$$

for some constant C independent of p.

Next, we show (2.9) by induction. First for $i = 1$, since T_4 is a pure shift, it is bounded below. So we assume

$$\|T_4 x\| \ge c\|x\|$$

for some $c > 0$ and any $x \in H_4$.

Write

$$p(T_4)k = a_0k + T_4(a_1 + a_2T_4 + \cdots + a_nT_4^{n-1})k.$$

By our choice of k, $k \perp T_4H_4$, so we have

$$\begin{aligned}\|p(T_4)k\|^2 &= \|a_0k\|^2 + \|T_4(a_1 + a_2T_4 + \cdots + a_nT_4^{n-1})k\|^2 \\ &\geq c^2\|(a_1 + a_2T_4 + \cdots + a_nT_4^{n-1})k\|^2,\end{aligned}$$

which is the case of $i = 1$ for (2.9).

Now replacing $p(z)$ by $q(z) = a_1+a_2z+\cdots+a_nz^{n-1}$, and applying the case of $i = 1$ for (2.9) to $q(z)$, one obtains the case of $i = 2$ for (2.9) to $p(z)$ with a different constant C. Iterating this process and the proof of (2.9), the whole proof can be completed.

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