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Null Controllability for Some Systems of Two Backward Stochastic Heat Equations with One Control Force*

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Abstract The authors establish the null controllability for some systems coupled by two backward stochastic heat equations. The desired controllability result is obtained by means of proving a suitable observability estimate for the dual system of the controlled system.

Keywords Backward stochastic heat equation, Null controllability, Observability estimate
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1 Introduction

Let $T > 0$, $G \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ be a given bounded domain with a C^4 boundary Γ, and G_0 be a nonempty open subset of G. Put

$$Q \triangleq (0,T) \times G, \quad \Sigma \triangleq (0,T) \times \Gamma.$$

Throughout this paper, we will use C to denote a generic positive constant depending only on G and G_0, which may change from line to line.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space on which a one dimensional standard Brownian motion $\{B(t)\}_{t\geq 0}$ is defined, such that $\{\mathcal{F}_{\sqcup}\}_{\sqcup\geq\prime}$ is the natural filtration generated by $\{B(t)\}_{t\geq 0}$. Let H be a Banach space. Denote by $L^2_{\mathcal{F}}(0,T;H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0,T;H)}) < \infty$, with the canonical norm; by $L^\infty_{\mathcal{F}}(0,T;H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted bounded processes; by $L^2_{\mathcal{F}}(\Omega; C([0,T];H))$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{C(0,T;H)}) < \infty$, with the canonical norm.

This paper is devoted to the study of the null controllability for the following coupled backward stochastic heat equations:

$$\begin{cases} \mathrm{d}y = -\Delta y \mathrm{d}t + (a_1 y + a_2 z + a_3 Y)\mathrm{d}t + Y \mathrm{d}B(t) & \text{in } Q, \\ \mathrm{d}z = -\Delta z \mathrm{d}t + (b_1 y + b_2 z + b_3 Z + \chi_{G_0} f)\mathrm{d}t + Z \mathrm{d}B(t) & \text{in } Q, \\ y = z = 0 & \text{on } \Sigma, \\ y(T) = y_T, \quad z(T) = z_T & \text{in } G, \end{cases} \tag{1.1}$$

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where

$$\begin{cases} a_i \in L^\infty_{\mathcal{F}}(0,T;L^\infty(G)), & i=1,2, \quad a_3 \in L^\infty_{\mathcal{F}}(0,T;W^{1,\infty}(G)), \\ b_i \in L^\infty_{\mathcal{F}}(0,T;L^\infty(G)), & i=1,2, \quad b_3 \in L^\infty_{\mathcal{F}}(0,T;W^{1,\infty}(G)), \end{cases} \tag{1.2}$$

and χ_{G_0} is the characteristic function of G_0. In the system (1.1), $(y_T, z_T) \in L^2(\Omega, \mathcal{F}_T, P; L^2(G)\times L^2(G))$ is a terminal state, (y,z) is a state variable and $f \in L^2_{\mathcal{F}}(0,T;L^2(G_0))$ is a control variable. By duality analysis as in [1], we can establish the existence and uniqueness for the solutions to the system (1.1) in the class of

$$\begin{aligned}(y,z;Y,Z) \in (&L^2_{\mathcal{F}}(\Omega; C([0,T]; L^2(G)\times L^2(G))) \cap L^2_{\mathcal{F}}(0,T;H^1_0(G)\times H^1_0(G))) \\ &\times L^2_{\mathcal{F}}(0,T;L^2(G)\times L^2(G)).\end{aligned}$$

The null controllability of the system (1.1) is formulated as follows.

Definition 1.1 *The system* (1.1) *is said to be null controllable at a time* $T>0$ *if for any given* $(y_T,z_T) \in L^2(\Omega,\mathcal{F}_T,P;L^2(G)\times L^2(G))$*, one can find a control* $f \in L^2_{\mathcal{F}}(0,T;L^2(G_0))$ *such that the solution* (y,z) *to the system* (1.1) *satisfies* $(y(0),z(0))=(0,0)$ *in* G*,* P*-a.s.*

There are a great many works on the controllability theory of deterministic heat equations and heat systems (see [2–7] and the references therein). However, things are quite different in the stochastic case. To our best knowledge, [8–11] are the only four published papers in which the null controllability for stochastic heat equations is studied. As far as we know, there is no published paper which is concerned with the null controllability of stochastic heat system.

Noting that we only act one control on the system (1.1), it is reasonable to expect that the action of z to y will be sufficiently effective. Hence we put the following condition on a_2.

Condition 1.1 There exists a nonempty subdomain $G_1 \subset G_0$ and a constant $\sigma>0$ such that $a_2(x,t)\ge\sigma$ or $a_2(x,t)\le-\sigma$, a.e. $(x,t)\in G_1\times(0,T)$, P-a.s.

In this paper, we prove the following result.

Theorem 1.1 *Let Condition* 1.1 *hold. For any terminal state* $(y_T,z_T)\in L^2(\Omega,\mathcal{F}_T,P;L^2(G)\times L^2(G))$*, we can find a control* $f\in L^2_{\mathcal{F}}(0,T;L^2(G))$ *such that the solution to the system* (1.1) *with this control satisfies that* $(y(0),z(0))=(0,0)$ *in* G*,* P*-a.s. Moreover, we have the following estimate for the control:*

$$|f|_{L^2_{\mathcal{F}}(0,T;L^2(G))} \le C\mathrm{e}^{C(T^{-4}+T)(1+p^2)}|(y_T,z_T)|_{L^2(\Omega,\mathcal{F}_0,P;L^2(G)\times L^2(G))} \tag{1.3}$$

with

$$p \triangleq \sum_{i=1}^{2}(|a_i|_{L^\infty(0,T;L^\infty(G))}+|b_i|_{L^\infty(0,T;L^\infty(G))})+|a_3|_{L^\infty(0,T,W^{1,\infty}(G))}+|b_3|_{L^\infty(0,T,W^{1,\infty}(G))}.$$

By means of the classical dual argument (see [11]), the null controllability of the system (1.1) can be reduced to the observability estimate for the following coupled forward stochastic heat equations:

$$\begin{cases} \mathrm{d}w = \Delta w\mathrm{d}t - (a_1 w + b_1 v)\mathrm{d}t - a_3 w \mathrm{d}B(t) & \text{in } Q, \\ \mathrm{d}v = \Delta v\mathrm{d}t - (a_2 w + b_2 v)\mathrm{d}t - b_3 v \mathrm{d}B(t) & \text{in } Q, \\ w = v = 0 & \text{on } \Sigma, \\ w(0)=w_0, \quad v(0)=v_0 & \text{in } G, \end{cases} \tag{1.4}$$

where $(w_0, z_0) \in L^2(\Omega, \mathcal{F}_0, P; L^2(G) \times L^2(G))$. We refer to [12] for the well-posedness of the system (1.4) under suitable assumptions in the class

$$(w, v) \in L^2_{\mathcal{F}}(\Omega; C([0,T]; L^2(G) \times L^2(G))) \cap L^2_{\mathcal{F}}(0,T; H^1_0(G) \times H^1_0(G)).$$

In order to prove Theorem 1.1, we only need to derive the following observability estimate for the system (1.4).

Theorem 1.2 *Let Condition* 1.1 *hold. Then any solution of the system* (1.4) *satisfies*

$$|(w,v)|_{L^2(\Omega,\mathcal{F}_T,P;L^2(G)\times L^2(G))} \le C\mathrm{e}^{C(T^{-4}+T)(1+p^2)}|v|_{L^2_{\mathcal{F}}(0,T;L^2(G_0))}. \tag{1.5}$$

The idea for the proof of Theorem 1.2 comes from the proof of an analogous result of Theorem 1.2 for deterministic heat systems (see [6]). We construct a functional $A(t)$ (see Section 3 for the details) to connect the suitable norms of w and v. The difference here is that we need to utilize Itô calculus for the computation. This will lead to some additional terms, compared with the deterministic case. How to treat these additional terms is the main difficulty which we need to overcome.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we prove Theorem 1.2. At last, in Section 4, we prove Theorem 1.1.

2 Some Preliminaries

This section is dedicated to giving some preliminaries. To begin with, we introduce the following function.

Let G_2 and G_3 be two nonempty open subsets of G such that $\overline{G}_2 \subset G_1$ and $\overline{G}_3 \subset G_2$. From Lemma 5.1 in [11], we know that there is a $\psi \in C^4(\overline{G})$ such that

$$\begin{cases} \psi > 0 & \text{in } G, \\ \psi = 0 & \text{on } \partial G, \\ |\nabla\psi| > 0 & \text{for all } x \in G \setminus G_3. \end{cases} \tag{2.1}$$

Put

$$\alpha(t,x) = \frac{\mathrm{e}^{\lambda\psi(x)} - \mathrm{e}^{2\lambda|\psi|_{L^\infty(G)}}}{t^2(T-t)^2}, \quad \varphi(t,x) = \frac{\mathrm{e}^{\lambda\psi(x)}}{t^2(T-t)^2}. \tag{2.2}$$

We have the following lemma for the observability estimate of backward stochastic heat equations.

Lemma 2.1 (see [11, Theorem 5.1]) *For any $T > 0$, there exists a constant $\lambda_0 = \lambda_0(G, G_2) > 0$ such that for all $\lambda \ge \lambda_0$, one can find two constants $C = C(\lambda) > 0$ and $s_0 = s_0(\lambda) > 0$ so that for all $p \in L^2_{\mathcal{F}}(\Omega; C([0,T]; L^2(G))) \cap L^2_{\mathcal{F}}(0,T; H^1_0(G))$, $f \in L^2_{\mathcal{F}}(0,T; L^2(G))$ and $g \in L^2_{\mathcal{F}}(0,T; H^1(G))$ satisfying*

$$\mathrm{d}p - \Delta p \mathrm{d}t = f\mathrm{d}t + g\mathrm{d}B(t), \tag{2.3}$$

and all $s \ge s_1 = s_1(\lambda, T) \triangleq s_0(\lambda)\max(1, T^2)$, it holds that

$$\begin{aligned} & s^3\lambda^4\mathbb{E}\int_Q \varphi^3\mathrm{e}^{2s\alpha}p^2\mathrm{d}x\mathrm{d}t + s\lambda^2\mathbb{E}\int_Q \varphi\mathrm{e}^{2s\alpha}|\nabla p|^2\mathrm{d}x\mathrm{d}t \\ & \le C\Big\{\mathbb{E}\int_Q \mathrm{e}^{2s\alpha}f^2\mathrm{d}x\mathrm{d}t + s^3\lambda^4\mathbb{E}\int_0^T\int_{G_2}\varphi^3\mathrm{e}^{2s\alpha}p^2\mathrm{d}x\mathrm{d}t + s\lambda^2\mathbb{E}\int_Q \varphi\mathrm{e}^{2s\alpha}g^2\mathrm{d}x\mathrm{d}t \end{aligned}$$

$$+\mathbb{E}\int_Q \varphi e^{2s\alpha}\sum_{i=1}^n[(g_{x_i}+s^2\alpha_{x_i}g)^2-(s\alpha_{x_i}^2+s\alpha_{x_ix_i})g^2]\mathrm{d}x\mathrm{d}t\Big\}. \tag{2.4}$$

Applying Lemma 2.1 to the first and second equations in the system (1.4), respectively, we obtain

$$\begin{aligned}&s^3\lambda^4\mathbb{E}\int_Q\varphi^3e^{2s\alpha}w^2\mathrm{d}x\mathrm{d}t+s\lambda^2\mathbb{E}\int_Q\varphi e^{2s\alpha}|\nabla w|^2\mathrm{d}x\mathrm{d}t\\&\le C\Big\{\mathbb{E}\int_Q e^{2s\alpha}(a_1w+b_1v)^2\mathrm{d}x\mathrm{d}t+s^3\lambda^4\mathbb{E}\int_0^T\int_{G_2}\varphi^3e^{2s\alpha}w^2\mathrm{d}x\mathrm{d}t+s\lambda^2\mathbb{E}\int_Q\varphi e^{2s\alpha}(a_3w)^2\mathrm{d}x\mathrm{d}t\\&\quad+\mathbb{E}\int_Q\varphi e^{2s\alpha}\sum_{i=1}^n[((a_3w)_{x_i}+s^2\alpha_{x_i}(a_3w))^2-(s\alpha_{x_i}^2+s\alpha_{x_ix_i})(a_3w)^2]\mathrm{d}x\mathrm{d}t\Big\}\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}&s^3\lambda^4\mathbb{E}\int_Q\varphi^3e^{2s\alpha}v^2\mathrm{d}x\mathrm{d}t+s\lambda^2\mathbb{E}\int_Q\varphi e^{2s\alpha}|\nabla v|^2\mathrm{d}x\mathrm{d}t\\&\le C\Big\{\mathbb{E}\int_Q e^{2s\alpha}(a_1w+b_1v)^2\mathrm{d}x\mathrm{d}t+s^3\lambda^4\mathbb{E}\int_0^T\int_{G_2}\varphi^3e^{2s\alpha}v^2\mathrm{d}x\mathrm{d}t+s\lambda^2\mathbb{E}\int_Q\varphi e^{2s\alpha}(a_3w)^2\mathrm{d}x\mathrm{d}t\\&\quad+\mathbb{E}\int_Q\varphi e^{2s\alpha}\sum_{i=1}^n[((b_3v)_{x_i}+s^2\alpha_{x_i}(b_3v))^2-(s\alpha_{x_i}^2+s\alpha_{x_ix_i})(b_3v)^2]\mathrm{d}x\mathrm{d}t\Big\}.\end{aligned} \tag{2.6}$$

From the inequality (2.5) and the inequality (2.6), and choosing

$$s\ge s_2\triangleq\max\{p^{\frac{2}{3}},s_1\},$$

we get

$$\mathbb{E}\int_Q\varphi^3e^{2s\alpha}(w^2+v^2)\mathrm{d}x\mathrm{d}t\le C\mathbb{E}\int_0^T\int_{G_2}\varphi^3e^{2s\alpha}(w^2+v^2)\mathrm{d}x\mathrm{d}t. \tag{2.7}$$

Hence we obtain the following proposition.

Proposition 2.1 *Let* (w,v) *be a solution to the system* (1.4)*. Then for each* $\lambda\ge\lambda_0$ *and all* $s\ge s_2$*, the inequality* (2.7) *holds.*

3 Proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2.

Proof of Theorem 1.2 From Condition 1.1, we know that $a_2(x,t)\ge\sigma$ or $a_2(x,t)\le-\sigma$, a.e. $(x,t)\in G_1\times(0,T)$, P-a.s. Without loss of generality, we assume that $a_2(x,t)\le-\sigma$, a.e. $(x,t)\in G_1\times(0,T)$, P-a.s.

By the definition of α, we know

$$s^3\lambda^4\mathbb{E}\int_Q\varphi^3(w^2+v^2)\mathrm{d}x\mathrm{d}t\le C\,\mathbb{E}\int_0^T\int_{G_2}e^{\frac{5}{3}s\alpha}(w^2+v^2)\mathrm{d}x\mathrm{d}t. \tag{3.1}$$

Let $\xi\in C^\infty(\mathbb{R}^n)$ be a cut-off function satisfying

$$\xi=1\text{ in }G_2,\quad \xi=0\text{ in }\mathbb{R}^n\setminus G_1,\quad 0\le\xi\le1\text{ in }G_1. \tag{3.2}$$

Put $\eta = \xi^6$. Let β_0, β_1, k, l be positive numbers, which will be specified later.

Let

$$A(t) \triangleq \mathbb{E}\int_G (\mathrm{e}^{k\tau\alpha}\eta^{\frac{4}{3}}w^2 + \beta_0\mathrm{e}^{2\tau\alpha}\eta wv + \beta_1\mathrm{e}^{l\tau\alpha}\eta^{\frac{2}{3}}v^2)\mathrm{d}x. \tag{3.3}$$

Then we have

$$\begin{aligned}\mathrm{d}A(t) = \mathbb{E}\int_G \{&k\tau\mathrm{e}^{k\tau\alpha}\alpha_t\eta^{\frac{4}{3}}w^2\mathrm{d}t + 2\mathrm{e}^{k\tau\alpha}\eta^{\frac{4}{3}}w\mathrm{d}w + \mathrm{e}^{k\tau\alpha}\eta^{\frac{4}{3}}(\mathrm{d}w)^2\\ &+ 2\tau\beta_0\mathrm{e}^{2\tau\alpha}\alpha_t\eta wv\mathrm{d}t + \beta_0\mathrm{e}^{2\tau\alpha}\eta(v\mathrm{d}w + w\mathrm{d}v + \mathrm{d}w\mathrm{d}v)\\ &+ \beta_1 l\tau\mathrm{e}^{l\tau\alpha}\alpha_t\eta^{\frac{2}{3}}v^2\mathrm{d}t + 2\beta_1\mathrm{e}^{l\tau}\eta^{\frac{2}{3}}v\mathrm{d}v + \beta_1\mathrm{e}^{l\tau\alpha}\eta^{\frac{2}{3}}(\mathrm{d}v)^2\}\mathrm{d}x.\end{aligned} \tag{3.4}$$

Noting that (w,v) is the solution to the system (1.4), from the equality (3.4), we obtain

$$\begin{aligned}\mathrm{d}A(t) = \mathbb{E}\int_G \{&k\tau\mathrm{e}^{k\tau\alpha}\alpha_t\eta^{\frac{4}{3}}w^2\mathrm{d}t + 2\mathrm{e}^{k\tau\alpha}\eta^{\frac{4}{3}}w(\Delta w - a_1w - b_1v)\mathrm{d}t + \mathrm{e}^{k\tau\alpha}\eta^{\frac{4}{3}}(a_3w)^2\mathrm{d}t\\ &+ \beta_0\mathrm{e}^{2\tau\alpha}\eta[2\tau\alpha_t wv + v(\Delta w - a_1w - b_1v) + w(\Delta v - a_2w - b_2v) + a_3wb_3v]\mathrm{d}t\\ &+ \beta_1 l\tau\mathrm{e}^{l\tau\alpha}\alpha_t\eta^{\frac{2}{3}}v^2\mathrm{d}t + \beta_1\mathrm{e}^{l\tau\alpha}\eta^{\frac{2}{3}}v(\Delta v - a_2w - b_2v)\mathrm{d}t + \beta_1\mathrm{e}^{l\tau\alpha}\eta^{\frac{2}{3}}(b_3v)^2\mathrm{d}t\}\mathrm{d}x.\end{aligned} \tag{3.5}$$

Integrating the equality (3.5) in $[0,T]$, we get

$$\begin{aligned}0 = &-\beta_0\mathbb{E}\int_Q \mathrm{e}^{2\tau\alpha}\eta a_2w^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q (k\tau\mathrm{e}^{k\tau\alpha}\alpha_t\eta^{\frac{4}{3}}w^2 - 2\mathrm{e}^{k\tau\alpha}\eta^{\frac{4}{3}}a_1w^2 + \mathrm{e}^{k\tau\alpha}\eta^{\frac{4}{3}}a_3^2w^2)\mathrm{d}x\mathrm{d}t\\ &- \mathbb{E}\int_Q (2\mathrm{e}^{k\tau\alpha}\eta^{\frac{4}{3}}b_1wv - 2\beta_0\tau\mathrm{e}^{2\tau\alpha}\alpha_t\eta wv + \beta_0\mathrm{e}^{2\tau\alpha}\eta(a_1 + b_2 - a_3b_3)wv + 2\beta_1\mathrm{e}^{l\tau\alpha}\eta^{\frac{2}{3}}a_2wv)\mathrm{d}x\mathrm{d}t\\ &+ \mathbb{E}\int_Q (\beta_1 l\tau\mathrm{e}^{l\tau\alpha}\alpha_t\eta^{\frac{2}{3}}v^2 - \beta_0\mathrm{e}^{2\tau\alpha}\eta b_1v^2 - 2\beta_1\mathrm{e}^{l\tau\alpha}\eta^{\frac{2}{3}}b_2v^2 + \beta_1\mathrm{e}^{l\tau\alpha}\eta^{\frac{2}{3}}b_3^2v^2)\mathrm{d}x\mathrm{d}t\\ &+ \mathbb{E}\int_Q (2\mathrm{e}^{k\tau\alpha}\eta^{\frac{4}{3}}w\Delta w + \beta_0\mathrm{e}^{2\tau\alpha}\eta(v\Delta w + w\Delta v) + 2\beta_1\mathrm{e}^{l\tau\alpha}\eta^{\frac{2}{3}}v\Delta v)\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.6}$$

Denoting by I_i $(i = 1,2,3,4)$ the last four terms on the right-hand side of the equality (3.6), we obtain

$$\beta_0\mathbb{E}\int_Q a_2\mathrm{e}^{2\tau\alpha}\eta w^2\mathrm{d}x\mathrm{d}t = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4. \tag{3.7}$$

Now we are going to estimate I_i $(i = 1,2,3,4)$.

Choosing $k > 2$, $r \in [\frac{3}{2}, 2)$, $l > 1 + \frac{r}{2}$, by the definition of α, we know that there exists an $s_3 > 0$ such that for all $s \ge s_3$, it holds that

$$\begin{cases}|k\tau\mathrm{e}^{(k-2)\tau\alpha}\alpha_t|_{L^\infty(Q)} \le 1, & |\mathrm{e}^{(k-2)\tau\alpha}|_{L^\infty(Q)} \le 1, \quad |l\tau\mathrm{e}^{(l-r)\tau\alpha}\alpha_t|_{L^\infty(Q)} \le 1,\\ |\mathrm{e}^{(2-r)\tau\alpha}|_{L^\infty(Q)} \le 1, & |\mathrm{e}^{(l-r)\tau\alpha}|_{L^\infty(Q)} \le 1, \quad |\mathrm{e}^{(k-1-\frac{r}{2})\tau\alpha}|_{L^\infty(Q)} \le 1,\\ |\mathrm{e}^{(1-\frac{r}{2})\tau\alpha}\alpha_t|_{L^\infty(Q)} \le 1, & |\mathrm{e}^{(1-\frac{r}{2})\tau\alpha}|_{L^\infty(Q)} \le 1, \quad |\mathrm{e}^{(l-1-\frac{r}{2})\tau\alpha}|_{L^\infty(Q)} \le 1,\\ |\tau|\nabla\alpha|\mathrm{e}^{\frac{k-2}{2}\tau\alpha}|_{L^\infty(Q)} \le 1, & |\tau\varphi\mathrm{e}^{(k-2)\tau\alpha}|_{L^\infty(Q)} \le 1.\end{cases} \tag{3.8}$$

By virtue of the first and second inequalities in (3.8), we know

$$\begin{aligned}\mathrm{I}_1 &= \mathbb{E}\int_Q (k\tau\mathrm{e}^{k\tau\alpha}\alpha_t\eta^{\frac{4}{3}}w^2 - 2\mathrm{e}^{k\tau\alpha}\eta^{\frac{4}{3}}a_1w^2 + \mathrm{e}^{k\tau\alpha}\eta^{\frac{4}{3}}a_3^2w^2)\mathrm{d}x\mathrm{d}t\\ &= \mathbb{E}\int_Q \mathrm{e}^{2\tau\alpha}(k\tau\mathrm{e}^{(k-2)\tau\alpha}\alpha_t\eta^{\frac{4}{3}}w^2 - 2\mathrm{e}^{(k-2)\tau\alpha}\eta^{\frac{4}{3}}a_1w^2 + \mathrm{e}^{(k-2)\tau\alpha}\eta^{\frac{4}{3}}a_3^2w^2)\mathrm{d}x\mathrm{d}t\end{aligned}$$

$$\leq C(p+p^2+1)\mathbb{E}\int_Q \mathrm{e}^{2\tau\alpha}\eta w^2\mathrm{d}x\mathrm{d}t. \tag{3.9}$$

As the estimate of I_1, by the third, fourth and fifth inequalities in (3.8), one can easily obtain

$$\begin{aligned}
\mathrm{I}_3 &= \mathbb{E}\int_Q (\beta_1 l\tau \mathrm{e}^{l\tau\alpha}\alpha_t\eta^{\frac{2}{3}}v^2 - \beta_0\mathrm{e}^{2\tau\alpha}\eta b_1 v^2 - 2\beta_1\mathrm{e}^{l\tau\alpha}\eta^{\frac{2}{3}}b_2v^2 + \beta_1\mathrm{e}^{l\tau\alpha}\eta^{\frac{2}{3}}b_3^2v^2)\mathrm{d}x\mathrm{d}t\\
&= \mathbb{E}\int_Q \mathrm{e}^{r\tau\alpha}\eta^{\frac{1}{3}}v^2(\beta_1 l\tau\mathrm{e}^{(l-r)\tau\alpha}\alpha_t\eta^{\frac{2}{3}} - \beta_0\mathrm{e}^{(2-r)\tau\alpha}\eta b_1 - 2\beta_1\mathrm{e}^{(l-r)\tau\alpha}\eta^{\frac{2}{3}}b_2 + \beta_1\mathrm{e}^{(l-r)\tau\alpha}\eta^{\frac{2}{3}}b_3^2)\mathrm{d}x\mathrm{d}t\\
&\leq C[(\beta_0+\beta_1)(p+p^2)+\beta_1]\mathbb{E}\int_Q \mathrm{e}^{r\tau\alpha}\eta^{\frac{1}{3}}v^2\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.10}$$

Now we estimate I_2. By Cauchy-Schwartz inequality, utilizing the sixth, seventh, eighth and ninth inequalities in (3.8), we have

$$\begin{aligned}
\mathrm{I}_2 &= -\mathbb{E}\int_Q [2\mathrm{e}^{k\tau\alpha}\eta^{\frac{4}{3}}b_1wv - 2\beta_0\tau\mathrm{e}^{2\tau\alpha}\alpha_t\eta wv + \beta_0\mathrm{e}^{2\tau\alpha}\eta(a_1+b_2-a_3b_3)wv\\
&\quad + 2\beta_1\mathrm{e}^{l\tau\alpha}\eta^{\frac{2}{3}}a_2wv]\mathrm{d}x\mathrm{d}t\\
&\leq \frac{1}{4}\mathbb{E}\int_Q \mathrm{e}^{2\tau\alpha}\eta w^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q \mathrm{e}^{r\tau\alpha}\eta^{\frac{1}{3}}v^2[2\mathrm{e}^{(k-1-\frac{r}{2})\tau\alpha}\eta^{\frac{2}{3}}b_1 - 2\tau\beta_0\mathrm{e}^{(1-\frac{r}{2})\tau\alpha}\alpha_t\eta^{\frac{1}{3}}\\
&\quad + \beta_0\mathrm{e}^{(1-\frac{r}{2})\tau\alpha}\eta^{\frac{1}{3}}(a_1+b_2-a_3b_3) + 2\beta_1\mathrm{e}^{(l-1-\frac{r}{2})\tau\alpha}a_2]^2\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.11}$$

Recalling $l > 1+\frac{r}{2}$ and noticing $1+\frac{r}{2} > r$, we obtain

$$\mathrm{I}_2 \leq C[(\beta_0+\beta_1)(p+p^2)]\mathbb{E}\int_Q \mathrm{e}^{r\tau\alpha}\eta^{\frac{1}{3}}v^2\mathrm{d}x\mathrm{d}t + \frac{1}{4}\mathbb{E}\int_Q \mathrm{e}^{2\tau\alpha}\eta w^2\mathrm{d}x\mathrm{d}t. \tag{3.12}$$

At last, we estimate I_4.

$$\begin{aligned}
\mathrm{I}_4 &= \mathbb{E}\int_Q (2\mathrm{e}^{k\tau\alpha}\eta^{\frac{4}{3}}w\Delta w + \beta_0\mathrm{e}^{2\tau\alpha}\eta(v\Delta w + w\Delta v) + 2\beta_1\mathrm{e}^{l\tau\alpha}\eta^{\frac{2}{3}}v\Delta v)\mathrm{d}x\mathrm{d}t\\
&= \mathbb{E}\int_Q \mathrm{e}^{k\tau\alpha}\eta^{\frac{4}{3}}\Delta w^2\mathrm{d}x\mathrm{d}t - 2\mathbb{E}\int_Q \mathrm{e}^{k\tau\alpha}\eta^{\frac{4}{3}}|\nabla w|^2\mathrm{d}x\mathrm{d}t\\
&\quad + \beta_0\mathbb{E}\int_Q \mathrm{e}^{2\tau\alpha}\eta\Delta(wv)\mathrm{d}x\mathrm{d}t - 2\beta_0\mathbb{E}\int_Q \mathrm{e}^{2\tau\alpha}\eta\nabla w\cdot\nabla v\mathrm{d}x\mathrm{d}t\\
&\quad + \beta_1\mathbb{E}\int_Q \mathrm{e}^{l\tau\alpha}\eta^{\frac{2}{3}}\Delta v^2\mathrm{d}x\mathrm{d}t - 2\beta_1\mathbb{E}\int_Q \mathrm{e}^{l\tau\alpha}\eta^{\frac{2}{3}}|\nabla v|^2\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.13}$$

By virtue of integration by parts, we get

$$\begin{aligned}
&\mathbb{E}\int_Q \mathrm{e}^{k\tau\alpha}\eta^{\frac{4}{3}}\Delta w^2\mathrm{d}x\mathrm{d}t\\
&= \mathbb{E}\int_Q \Delta(\mathrm{e}^{k\tau\alpha}\eta^{\frac{4}{3}})w^2\mathrm{d}x\mathrm{d}t\\
&= \mathbb{E}\int_Q \mathrm{e}^{k\tau\alpha}w^2\Big(k^2\tau^2|\nabla\alpha|^2\eta^{\frac{4}{3}} + k\tau\Delta\alpha\eta^{\frac{4}{3}} + \frac{4}{3}|\nabla\eta|^2\eta^{-\frac{2}{3}}\\
&\quad + \frac{4}{3}\eta^{\frac{1}{3}}\Delta\eta + \frac{8}{3}k\tau\eta^{\frac{1}{3}}\nabla\alpha\cdot\nabla\eta\Big).
\end{aligned} \tag{3.14}$$

It is easy to check that

$$\eta^{-\frac{5}{6}}\nabla\eta = 6\nabla\xi \in L^\infty(Q), \quad \eta^{-\frac{2}{3}}\Delta\eta = 30|\nabla\xi|^2 + 6\xi\Delta\xi \in L^\infty(Q). \tag{3.15}$$

Recalling $k > 2$, by means of the last two inequalities in (3.8), we can obtain

$$\mathbb{E}\int_Q \mathrm{e}^{k\tau\alpha}\eta^{\frac{4}{3}}\Delta w^2 \mathrm{d}x\mathrm{d}t \le C\mathbb{E}\int_Q \mathrm{e}^{2\tau\alpha}\eta w^2 \mathrm{d}x\mathrm{d}t. \tag{3.16}$$

By the similar argument to obtain the inequality (3.12) and the inequality (3.16), we can show

$$\beta_0\mathbb{E}\int_Q \mathrm{e}^{2\tau\alpha}\eta\Delta(wv)\mathrm{d}x\mathrm{d}t \le \frac{1}{4}\mathbb{E}\int_Q \mathrm{e}^{2\tau\alpha}\eta w^2 \mathrm{d}x\mathrm{d}t + C\beta_0^2\mathbb{E}\int_Q \mathrm{e}^{r\tau\alpha}\eta^{\frac{1}{3}}v^2\mathrm{d}x\mathrm{d}t \tag{3.17}$$

and

$$\beta_1\mathbb{E}\int_Q \mathrm{e}^{l\tau\alpha}\eta^{\frac{2}{3}}\Delta v^2\mathrm{d}x\mathrm{d}t \le C\beta_1\mathbb{E}\int_Q \mathrm{e}^{r\tau\alpha}\eta^{\frac{1}{3}}v^2\mathrm{d}x\mathrm{d}t. \tag{3.18}$$

Then, it follows from (3.13)–(3.18) that

$$\begin{aligned} \mathrm{I}_4 \le{}& C\mathbb{E}\int_Q \mathrm{e}^{2\tau\alpha}\eta w^2\mathrm{d}x\mathrm{d}t + C(\beta_0^2+\beta_1)\mathbb{E}\int_Q \mathrm{e}^{r\tau\alpha}\eta^{\frac{1}{3}}v^2\mathrm{d}x\mathrm{d}t \\ &- 2\mathbb{E}\int_Q \mathrm{e}^{k\tau\alpha}\eta^{\frac{4}{3}}|\nabla w|^2\mathrm{d}x\mathrm{d}t - 2\beta_0\mathbb{E}\int_Q \mathrm{e}^{2\tau\alpha}\eta\nabla w\cdot\nabla v\mathrm{d}x\mathrm{d}t \\ &- 2\beta_1\mathbb{E}\int_Q \mathrm{e}^{l\tau\alpha}\eta^{\frac{2}{3}}|\nabla v|^2\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.19}$$

Let $k+l<4$ and $\beta_1 > \frac{\beta_0^2}{4}$. Then we know

$$\begin{aligned} &- 2\mathbb{E}\int_Q \mathrm{e}^{k\tau\alpha}\eta^{\frac{4}{3}}|\nabla w|^2\mathrm{d}x\mathrm{d}t - 2\beta_0\mathbb{E}\int_Q \mathrm{e}^{2\tau\alpha}\eta\nabla w\cdot\nabla v\mathrm{d}x\mathrm{d}t \\ &- 2\beta_1\mathbb{E}\int_Q \mathrm{e}^{l\tau\alpha}\eta^{\frac{2}{3}}|\nabla v|^2\mathrm{d}x\mathrm{d}t \le 0. \end{aligned} \tag{3.20}$$

Therefore, we find

$$\mathrm{I}_4 \le C\mathbb{E}\int_Q \mathrm{e}^{2\tau\alpha}\eta w^2\mathrm{d}x\mathrm{d}t + C(\beta_0^2+\beta_1)\mathbb{E}\int_Q \mathrm{e}^{r\tau\alpha}\eta^{\frac{1}{3}}v^2\mathrm{d}x\mathrm{d}t. \tag{3.21}$$

From (3.9)–(3.10), (3.12) and (3.21), we see

$$\begin{aligned} \beta_0\mathbb{E}\int_Q a_2\mathrm{e}^{2\tau\alpha}\eta w^2\mathrm{d}x\mathrm{d}t \le{}& C(1+p^2)(\beta_0^2+\beta_1^2)\mathbb{E}\int_Q \mathrm{e}^{r\tau\alpha}\eta^{\frac{1}{3}}v^2\mathrm{d}x\mathrm{d}t \\ &+ C(p^2+1)\int_Q \mathrm{e}^{2\tau\alpha}\eta w^2\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.22}$$

Hence, by setting $\beta_0 = 2C(1+p^2)$, we obtain

$$\mathbb{E}\int_Q \mathrm{e}^{2\tau\alpha}\eta w^2\mathrm{d}x\mathrm{d}t \le C(1+p^{10})\mathbb{E}\int_Q \mathrm{e}^{r\tau\alpha}\eta^{\frac{1}{3}}v^2\mathrm{d}x\mathrm{d}t. \tag{3.23}$$

Taking into account Proposition 2.1, the inequality (3.1) and the inequality (3.23), for $\lambda \geq \lambda_0$ and $s \geq \max\{s_2, s_3\}$, we deduce

$$\mathbb{E}\int_Q \varphi^3 \mathrm{e}^{2s\alpha}(w^2+v^2)\mathrm{d}x\mathrm{d}t \leq C(1+p^{10})\mathbb{E}\int_0^T\int_{G_0} \mathrm{e}^{\frac{3}{2}s\alpha}v^2\mathrm{d}x\mathrm{d}t. \tag{3.24}$$

Recalling the definitions of α and ϕ (see (2.2)), we have

$$\mathbb{E}\int_Q \varphi^3 \mathrm{e}^{2s\alpha}(w^2+v^2)\mathrm{d}x\mathrm{d}t \geq \min_{x\in G}\Big[\varphi^3\Big(x,\frac{T}{2}\Big)\mathrm{e}^{2s\alpha(x,\frac{T}{2})}\Big]\mathbb{E}\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_G (w^2+v^2)\mathrm{d}x\mathrm{d}t \tag{3.25}$$

and

$$\mathbb{E}\int_0^T\int_{G_0} \mathrm{e}^{\frac{3}{2}s\alpha}v^2\mathrm{d}x\mathrm{d}t \leq \max_{(x,t)\in\overline{Q}}(\mathrm{e}^{\frac{3}{2}s\alpha(x,t)})\mathbb{E}\int_0^T\int_{G_0} v^2\mathrm{d}x\mathrm{d}t. \tag{3.26}$$

From (3.24)–(3.26), we obtain

$$\mathbb{E}\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_G (w^2+v^2)\mathrm{d}x\mathrm{d}t \leq C\mathrm{e}^{CT^{-4}(1+p^2)}\mathbb{E}\int_0^T\int_{G_0} v^2\mathrm{d}x\mathrm{d}t. \tag{3.27}$$

Noting $\mathrm{d}(w^2+v^2) = 2w\mathrm{d}w + (\mathrm{d}w)^2 + 2v\mathrm{d}v + (\mathrm{d}v)^2$, and applying the usual energy estimate to the system (1.4), it is easy to see that, for any $0 \leq t_1 \leq t_2 \leq T$, it holds

$$\begin{aligned}
&\mathbb{E}\int_G [w^2(t_2)+v^2(t_2)]\mathrm{d}x - \mathbb{E}\int_G [w^2(t_1)+v^2(t_1)]\mathrm{d}x\\
&= \mathbb{E}\int_{t_1}^{t_2}\int_G [2w\mathrm{d}w + (\mathrm{d}w)^2 + 2v\mathrm{d}v + (\mathrm{d}v)^2]\mathrm{d}x\mathrm{d}t\\
&= \mathbb{E}\int_{t_1}^{t_2}\int_G [2w(\Delta w - a_1 w - b_1 v) + (a_3 w)^2 + 2v(\Delta v - a_2 w - b_2 v) + (b_3 v)^2]\mathrm{d}x\mathrm{d}t\\
&\leq C(1+p^2)\mathbb{E}\int_{t_1}^{t_2}\int_G (w^2+v^2)\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.28}$$

Hence, in terms of Gronwall inequality, it follows

$$\mathbb{E}\int_G [w^2(t_2)+v^2(t_2)]\mathrm{d}x \leq \mathrm{e}^{CT(1+p^2)}\mathbb{E}\int_G [w^2(t_1)+v^2(t_1)]\mathrm{d}x. \tag{3.29}$$

By the inequality (3.27) and the inequality (3.29), we conclude that the solution (w, v) to the system (1.4) satisfies the inequality (1.5).

4 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1 which is our controllability result. The proof is almost the standard dual argument. However, for the sake of completeness, we still give it here.

Proof of Theorem 1.1 For any $(y_T, z_T) \in L^2(\Omega, \mathcal{F}_T, \mathcal{P}; \mathcal{L}^{\in}(\mathcal{G}) \times \mathcal{L}^{\in}(\mathcal{G}))$, we need to find a control $f \in L^2_{\mathcal{F}}(0,T;L^2(G_0))$ such that the solution to the system (1.1) satisfies $(y(0), z(0)) = (0,0)$ in G, P-a.s. We use the duality argument.

We introduce the following linear subspace of $L^2_{\mathcal{F}}(0,T;L^2(G_0)) \times L^2_{\mathcal{F}}(0,T;L^2(G))$:

$$X \triangleq \{v|_{[0,T]\times G_0\times\Omega} \mid (w,v) \text{ solves the system (1.4) with some}$$

$$(w_0, v_0) \in L^2(\Omega, \mathcal{F}_\prime, \mathcal{P}; \mathcal{L}^\in(\mathcal{G}) \times \mathcal{L}^\in(\mathcal{G}))\},$$

and define a linear functional on X as follows:

$$L(v|_{[0,T]\times G_0\times\Omega}) = \mathbb{E}\int_G (y_T w(T) + z_T v(T))\mathrm{d}x.$$

By means of the observability estimate (see Theorem 1.2), we know

$$\begin{aligned}|L(v|_{[0,T]\times G_0\times\Omega})| &\le \Big(\mathbb{E}\int_G (|y_T|^2 + |z_T|^2)\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\mathbb{E}\int_G (|w(T)|^2 + |v(T)|^2)\mathrm{d}x\Big)^{\frac{1}{2}} \\ &\le C\mathrm{e}^{C[T^{-4}(1+p^2)+T(1+p^2)]}\Big(\mathbb{E}\int_G (|y_T|^2 + |z_T|^2)\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\mathbb{E}\int_0^T\int_{G_0} |v|^2\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{2}}.\end{aligned}$$

Thus, L is a bounded linear functional on X such that the norm of l is bounded by

$$C\mathrm{e}^{C[T^{-4}(1+p^2)+T(1+p^2)]}\Big(\mathbb{E}\int_G (|y_T|^2 + |z_T|^2)\mathrm{d}x\Big)^{\frac{1}{2}}.$$

By Hahn-Banach theorem, L can be extended to be a bounded linear functional with the same norm on $L^2_{\mathcal{F}}(0, T; L^2(G_0))$. For simplicity, we use the same notation for this extension. Now, the Riesz representation theorem allows us to find a random field $f \in L^2_{\mathcal{F}}(0, T; L^2(G_0))$ such that

$$\mathbb{E}\int_G [y_T w(T) + z_T v(T)]\mathrm{d}x = \mathbb{E}\int_0^T\int_{G_0} fv\mathrm{d}x\mathrm{d}t \tag{4.1}$$

and

$$|f|_{L^2_{\mathcal{F}}(0,T;L^2(G))} \le C\mathrm{e}^{C[T^{-4}(1+p^2)+T(1+p^2)]}|(y_T, z_T)|_{L^2(\Omega,\mathcal{F}_0,P;L^2(G)\times L^2(G))}. \tag{4.2}$$

We claim that this random field f is exactly the control we need. In fact, by means of Itô formula, we know that

$$\mathrm{d}(yw) = y\mathrm{d}w + w\mathrm{d}y + \mathrm{d}y\mathrm{d}w \tag{4.3}$$

and

$$\mathrm{d}(zv) = z\mathrm{d}v + v\mathrm{d}z + \mathrm{d}z\mathrm{d}v, \tag{4.4}$$

where (y, z) is the solution to the system (1.1) and (w, v) is the solution to the system (1.4). From (4.3), we obtain

$$\begin{aligned}&\mathbb{E}\int_G y_T w(T)\mathrm{d}x - \mathbb{E}\int_G y(0)w_0\mathrm{d}x \\ &= \mathbb{E}\int_Q (y\mathrm{d}w + w\mathrm{d}y + \mathrm{d}y\mathrm{d}w)\mathrm{d}x \\ &= \mathbb{E}\int_Q y(\Delta w - a_1 w - b_1 v)\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q w(-\Delta y + a_1 y + a_2 z + a_3 Y)\mathrm{d}x\mathrm{d}t \\ &\quad + \mathbb{E}\int_Q Y(-a_3 w)\mathrm{d}x\mathrm{d}t \\ &= \mathbb{E}\int_Q (a_2 wz - b_1 vy)\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{4.5}$$

From (4.4), we know

$$\begin{aligned}
&\mathbb{E}\int_G z_T v(T)\mathrm{d}x - \mathbb{E}\int_G z(0)v_0\mathrm{d}x \\
=&\,\mathbb{E}\int_Q (z\mathrm{d}v + v\mathrm{d}z + \mathrm{d}z\mathrm{d}v)\mathrm{d}x \\
=&\,\mathbb{E}\int_Q z(\Delta v - a_2 w - b_2 v)\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q v(-\Delta z + b_1 y + b_2 z + b_3 Z + \chi_{G_0} f)\mathrm{d}x\mathrm{d}t \\
&+ \mathbb{E}\int_Q Z(-b_3 v)\mathrm{d}x\mathrm{d}t \\
=&\,\mathbb{E}\int_Q (b_1 v y - a_2 w z + \chi_{G_0} f v)\mathrm{d}x\mathrm{d}t. \qquad (4.6)
\end{aligned}$$

Combining the equalities (4.1) and (4.5)–(4.6), we find

$$\mathbb{E}\int_G y(0)w_0\mathrm{d}x + \mathbb{E}\int_G z(0)v_0\mathrm{d}x = 0.$$

Since (w_0, v_0) can be chosen arbitrarily, this implies that $(y(0), z(0)) = 0$ in G, P-a.s.

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