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An Improved Result in Almost Sure Central Limit Theory for Products of Partial Sums with Stable Distribution*

Qunying WU[1]

Abstract Consider a sequence of i.i.d. positive random variables with the underlying distribution in the domain of attraction of a stable distribution with an exponent in $(1, 2]$. A universal result in the almost sure limit theorem for products of partial sums is established. Our results significantly generalize and improve those on the almost sure central limit theory previously obtained by Gonchigdanzan and Rempale and by Gonchigdanzan. In a sense, our results reach the optimal form.

Keywords Almost sure central limit theorem, Product of partial sums, Stable distribution
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1 Introduction

Let $(X, X_n)_{n\geq 1}$ be a sequence of identically distributed random variables with a non-degenerate distribution function F. For each $n \geq 1$, set $S_n = \sum\limits_{j=1}^{n} X_j$. We say that the r.v. X (d.f. F of X) belongs to the domain of attraction of a characteristic stable distribution G_α with an exponent $\alpha \in (0, 2]$ if there exist constants $a_n > 0$ and $b_n \in \mathrm{R}$ such that

$$\frac{S_n - b_n}{a_n} \xrightarrow{d} G_\alpha. \tag{1.1}$$

We write $X \in DA(\alpha)$ ($F \in DA(\alpha)$) and say that (X_n) satisfies the central limit theorem (CLT) with the limit G_α.

It is well-known that (1.1) holds with $\alpha \in (0, 2)$, if and only if

$$F(-x) = \frac{c_1(x)}{x^\alpha} L(x), \quad 1 - F(x) = \frac{c_2(x)}{x^\alpha} L(x), \quad x \to \infty, \tag{1.2}$$

where

$$c_i(x) \geq 0, \quad \lim_{x\to\infty} c_i(x) = c_i, \quad i = 1, 2,\ c_1 + c_2 > 0,$$

and $L(x) \geq 0$ is a slowly varying function at infinity.

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[1]College of Science, Guilin University of Technology, Guilin 541004, Guangxi, China.
E-mail: wqy666@glut.edu.cn
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It is also known that (1.1) holds with $\alpha = 2$, i.e., F is in the domain of attraction of the normal distribution if and only if

$$\lim_{x\to\infty} \frac{x^2 P(|X| > x)}{\mathrm{E}X^2 I(|X| \le x)} = 0. \tag{1.3}$$

From (1.2)–(1.3), we obtain that if $X \in DA(\alpha)$, then

$$\mathrm{E}|X|^p < \infty \text{ for } p < \alpha. \tag{1.4}$$

In particular, $\mathrm{E}|X| < \infty$ for $\alpha > 1$.

From [11], the normalising constants a_n and the centring constants b_n in (1.1) can be chosen as

$$a_n = n^{\frac{1}{\alpha}} L_1(n) \tag{1.5}$$

for an appropriate slowly varying function L_1 and

$$b_n = \begin{cases} n\mathrm{E}X, & \text{if } \alpha \in (1,2], \\ 0, & \text{if } \alpha \in (0,1), \\ 0, & \text{if } \alpha = 1 \text{ and } F \text{ is symmetric.} \end{cases} \tag{1.6}$$

In particular, if $\mathrm{Var}X = \sigma^2 < \infty$ and $\mathrm{E}X = 0$, then $a_n \sim n^{\frac{1}{2}}\sigma$, $n \to \infty$.

A series of limit results for distributions in the domain of attraction of a stable law has been established. One can refer to Chover [3], Mikosch [7], Vasudeva [12] for the laws of the iterated logarithm and the weak convergence for partial sums of independent sequences, Wu and Jiang [13–15] for the law of the iterated logarithm for partial sums of NA and $\widetilde{\rho}$-mixing sequences of random variables, and Wu [16] for the almost sure limit theorems for partial sums. Recently there have been several studies the products of partial sums with stable distributions. Qi [9] obtained the central limit theorem (CLT), and Gonchigdanzan [6] obtained an almost sure limit central theorem (ASLCT) for the products of the partial sums of the domain of attraction of an $\alpha \in (1,2]$ stable distribution.

Under mild moment conditions ASLCT follows from the ordinary CLT, but in general the validity of ASLCT is a delicate question of a totally different character. The difference between CLT and ASLCT lies in the summation method.

Let $\mathbf{D} = (D_n)$ be a positive non-decreasing sequence with $\lim_{n\to\infty} D_n = \infty$ and set $d_k = D_k - D_{k-1}$. We say that $(x_k)_{k\ge1}$ is $\mathbf{D}$-summable to x if

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k x_k = x. \tag{1.7}$$

By a result of Harly (see [2]), if $\mathbf{D}$ and $\mathbf{D}^*$ are summation procedures with $D_n^* = O(D_n)$, then under minor technical assumptions, the summation $\mathbf{D}^*$ is stronger than $\mathbf{D}$, i.e. if a sequence (x_n) is $\mathbf{D}$-summable to x, then it is also $\mathbf{D}^*$-summable to x. Also, if (d_k) grows exponentially (or fastly), then (1.7) is equivalent to the convergence of the sequence (x_k), and hence this is the weakest summation method. By a result of Zygmund (see [2]), if $D_n^\alpha \le D_n^* \le D_n^\beta$ $(n \ge n_0)$ for some $\alpha > 0$, $\beta > 0$, then $\mathbf{D}$ and $\mathbf{D}^*$ are equivalent; and if $D_n^* = O(D_n^\varepsilon)$ for any $\varepsilon > 0$, then $\mathbf{D}^*$ is strictly stronger than $\mathbf{D}$. These results show that the larger the norming sequence D_n in (1.7) is, the stronger the relation (1.7) becomes. From this, the larger the weight sequence in

ASCLT is, the stronger the ASCLT becomes. By this argument, one should also expect to get stronger results if we use larger weights.

The purpose of this paper is to make substantial improvements of weight sequences on the result in [6]. We will show that the ASCLT holds under a fairly general growth condition on $d_k = k^{-1}\mathrm{e}^{\ln^\alpha k}$, $0 \le \alpha < 1$. In some sense, our results reach the optimal form.

2 Results and Proofs

In the following, we assume that $(X, X_n)_{n\ge1}$ is a sequence of i.i.d. positive random variables in the domain of attraction of an α-stable law of G_α with $1 < \alpha \le 2$ and $\mathrm{E}X = \mu$. Let $b_{k,n} = \sum_{j=k}^{n} \frac{1}{j}$, $S_k = \sum_{i=1}^{k} X_i$, $\widetilde{S}_k = \sum_{i=1}^{k} (X_i - \mu)$, $S_{k,n} = \sum_{i=1}^{k} b_{i,n}(X_i - \mu)$ for $1 \le k \le n$, and $d_k = \frac{\mathrm{e}^{\ln^\beta k}}{k}$, $0 \le \beta < 1$, $D_n = \sum_{k=1}^{n} d_k$. $H(x)$ is the distribution function of the G_α. $\mathcal{C}_H$ denotes the set of continuity points of $H(x)$. I denotes the indicator function. $A \sim B$ denotes $\frac{A}{B} \to 1$. $a_n \ll b_n$ denotes that there exists a constant $c > 0$ such that $a_n \le cb_n$ for the sufficiently large n. The symbol c stands for a generic positive constant which may differ from one place to another.

Our theorems are formulated in a more general setting.

Theorem 2.1 *Let* $(X, X_n)_{n\ge1}$ *be a sequence of* i.i.d. *positive random variables in the domain of attraction of an* α*-stable law of* G_α *with* $1 < \alpha \le 2$ *and* $\mathrm{E}X = \mu$. *Then*

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\Big\{\frac{S_{n,n}}{a_n} \le x\Big\} = H(x) \quad \text{a.s. } \textit{for any } x \in \mathcal{C}_H$$

if and only if

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k P\Big(\frac{S_{n,n}}{a_n} \le x\Big) = H(x) \quad \textit{for any } x \in \mathcal{C}_H,$$

where a_n *is defined by* (1.1).

Theorem 2.2 *Suppose that the assumptions of Theorem* 2.1 *hold. Then for any continuity point* x *of* $F(x)$,

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\bigg(\bigg(\frac{\prod_{i=1}^{k} S_i}{k!\mu^k}\bigg)^{\frac{\mu}{a_k}} \le x\bigg) = F(x) \quad \text{a.s.}, \tag{2.1}$$

where a_k *is defined by* (1.1), *and* F *is the distribution function of the random variable* $\exp(\Gamma^{\frac{1}{\alpha}}(1+\alpha)G_\alpha)$.

Remark 2.1 By the property of **D**-summable, the Theorems 2.1–2.2 remain valid if we replace the weight sequence (d_k) by any (d_k^*) such that $0 \le d_k^* \le d_k$, and $\sum_{k=1}^{\infty} d_k^* = \infty$.

Remark 2.2 From [11], if $\mathrm{E}X^2 < \infty$, then $X \sim DA(2)$, $\Gamma(\alpha+1) = \Gamma(3) = 2$, $G_2 = \mathcal{N}$, $\sigma = \mathrm{Var}X$, and $a_k = \frac{1}{\sigma\sqrt{k}}$. Let $\beta = 0$. We have $d_k \sim \frac{\mathrm{e}}{k}$, $D_n \sim \mathrm{e}\ln n$. Hence, Theorem 2 in [5] is a particular case of our Theorem 2.2.

Remark 2.3 Theorem 1.1 in [6] is a particular case of Theorem 2.2 for $1 < \alpha < 2$ and $\beta = 0$. Our Theorem 2.2 substantially improves the weight sequence in [6].

Remark 2.4 By the Theorem 1 of [10], for $\beta = 1$, i.e., $d_k = 1$, ASLCT does not hold. Therefore, in a sense, our Theorem 2.2 reach the optimal form.

Denote by $\mathcal{A}$ the class of bounded functions with the Lipchitz condition. For any $f \in \mathcal{A}$, let

$$\begin{aligned}\xi_k &= f\Big(\frac{S_{k,k}}{a_k}\Big) - \mathrm{E}f\Big(\frac{S_{k,k}}{a_k}\Big),\\ \xi_{k,l} &= f\Big(\frac{S_{l,l} - S_{k,k} - b_{k+1,l}\widetilde{S}_k}{a_l}\Big) - \mathrm{E}f\Big(\frac{S_{l,l} - S_{k,k} - b_{k+1,l}\widetilde{S}_k}{a_l}\Big).\end{aligned}$$

Clearly, there exists a constant $c > 0$ such that

$$|f(x)| \le c, \quad |f(x) - f(y)| \le 2c\min(|x-y|, 1), \quad \forall x, y \in \mathrm{R}, \quad |\xi_n| \le 2c,\ \forall n. \tag{2.2}$$

To extend the weights from $d_k = \frac{1}{k}$ to $d_k = k^{-1}\mathrm{e}^{\ln^\beta k}$, $0 \le \beta < 1$, we encountered great difficulties and challenges. To overcome the difficulties and challenges, the following lemma plays an important role.

Lemma 2.1 *Suppose that the assumptions of Theorem* 2.1 *hold. Then there exist positive constants* $\delta < 1$ *and* c *such that*

$$|\mathrm{E}\xi_k\xi_l| \le c\min\Big(\mathrm{E}\Big|\frac{S_{k,k} + b_{k+1,l}\widetilde{S}_k}{a_l}\Big|, 1\Big), \quad \mathrm{E}\Big|\frac{S_{k,k} + b_{k+1,l}\widetilde{S}_k}{a_l}\Big| \le c\Big(\frac{k}{l}\Big)^\delta \ln k, \quad l \ge k. \tag{2.3}$$

For every $p \in \mathbf{N}$*, there exist positive constants* A_p *and* c_p *such that*

$$\mathrm{E}\Big|\sum_{l=m}^{n} d_l(\xi_l - \xi_{k,l})\Big|^p \le A_p\Big(\sum_{l=m}^{n} l d_l^2\Big)^{\frac{p}{2}} \ln n, \quad k \le m \le n \tag{2.4}$$

and

$$\mathrm{E}\Big|\sum_{k=1}^{n} d_k\xi_k\Big|^p \le c_p\Big(\sum_{1\le k\le l\le n} d_k d_l \min\Big(\Big(\frac{k}{l}\Big)^\delta \ln k, 1\Big)\Big)^{\frac{p}{2}} \ln n, \tag{2.5}$$

where δ *is the constant in* (2.3)*.*

Proof Noting that

$$S_{l,l} - S_{k,k} = b_{k+1,l}\widetilde{S}_k + (b_{k+1,l}(X_{k+1} - \mu) + \cdots + b_{l,l}(X_l - \mu)), \quad l \ge k,$$

we see that $S_{l,l} - S_{k,k} - b_{k+1,l}\widetilde{S}_k$ and $S_{k,k}$ are independent. Hence for $l \ge k$,

$$\begin{aligned}|\mathrm{E}\xi_k\xi_l| &= \Big|\mathrm{cov}\Big(f\Big(\frac{S_{k,k}}{a_k}\Big), f\Big(\frac{S_{l,l}}{a_l}\Big)\Big)\Big|\\ &= \Big|\mathrm{cov}\Big(f\Big(\frac{S_{k,k}}{a_k}\Big), f\Big(\frac{S_{l,l}}{a_l}\Big) - f\Big(\frac{S_{l,l} - S_{k,k} - b_{k+1,l}\widetilde{S}_k}{a_l}\Big)\Big)\Big|.\end{aligned}$$

By (2.2) and the Jensen inequality, we have

$$|\mathrm{E}\xi_k\xi_l| \le c\min\Big(\mathrm{E}\Big|\frac{S_{k,k} + b_{k+1,l}\widetilde{S}_k}{a_l}\Big|, 1\Big).$$

On the other hand,

$$\mathrm{E}\Big|\frac{S_{k,k}+b_{k+1,l}\widetilde{S}_k}{a_l}\Big| \le \frac{a_k}{a_l}\mathrm{E}\frac{|S_{k,k}|}{a_k}+\frac{b_{k+1,l}a_k}{a_l}\mathrm{E}\frac{|\widetilde{S}_k|}{a_k}. \tag{2.6}$$

By (1.1), applying Theorem 6.1 of [4] and (1.4),

$$\lim_{n\to\infty}\mathrm{E}\Big|\frac{S_n-b_n}{a_n}\Big|^p=\mathrm{E}|G_\alpha|^p<\infty, \quad 0<p<\alpha.$$

Thus, letting $p=1<\alpha$ and $b_n=n\mu$ (see (1.6)), there exists a constant $c>0$ such that

$$\mathrm{E}\Big|\frac{\widetilde{S}_n}{a_n}\Big|\le c \quad \text{for any } n.$$

By (1.5) and the properties of a slowly varying function at infinity, a_k is quasi-decreasing. Hence

$$\begin{aligned}
\frac{\mathrm{E}|S_{k,k}|}{a_k} &= \mathrm{E}\Big|\frac{1}{a_k}\sum_{i=1}^k b_{i,k}(X_i-\mu)\Big| \sim \mathrm{E}\Big|\frac{1}{a_k}\sum_{i=1}^k(\ln k-\ln i)(X_i-\mu)\Big| \\
&= \mathrm{E}\Big|\frac{1}{a_k}\sum_{i=1}^k\sum_{j=i+1}^k(\ln j-\ln(j-1))(X_i-\mu)\Big| \\
&= \mathrm{E}\Big|\frac{1}{a_k}\sum_{j=2}^k\Big(\sum_{i=1}^{j-1}(X_i-\mu)\Big)(\ln j-\ln(j-1))\Big| \\
&\ll \sum_{j=2}^k\mathrm{E}\Big|\frac{\widetilde{S}_{j-1}}{a_{j-1}}\Big|(\ln j-\ln(j-1)) \\
&\ll \sum_{j=2}^k(\ln j-\ln(j-1)) \\
&= \ln k.
\end{aligned} \tag{2.7}$$

Letting $0<\varepsilon<\frac{1}{2\alpha}$, by (1.5), (2.6)–(2.7), and the properties of a slowly varying function at infinity, for sufficiently large $k\le l$, we get

$$\mathrm{E}\Big|\frac{S_{k,k}+b_{k+1,l}\widetilde{S}_k}{a_l}\Big| \ll \frac{a_k}{a_l}\ln k+\frac{a_k}{a_l}\ln\frac{l}{k} \ll \Big(\frac{k}{l}\Big)^{\frac{1}{\alpha}-\varepsilon}\ln k+\Big(\frac{k}{l}\Big)^{\frac{1}{\alpha}-2\varepsilon} \ll \Big(\frac{k}{l}\Big)^{\delta}\ln k,$$

where $0<\delta=\frac{1}{\alpha}-2\varepsilon<\frac{1}{\alpha}<1$. Hence, (2.3) holds.

By $|\xi_k|\le 2c$, $|\xi_{k,l}|\le 2c$, and (2.3), we have

$$\begin{aligned}
\mathrm{E}|\xi_l-\xi_{k,l}|^p &\le (4c)^{p-1}\mathrm{E}|\xi_l-\xi_{k,l}| \\
&\ll \mathrm{E}\Big|f\Big(\frac{S_{l,l}}{a_l}\Big)-f\Big(\frac{S_{l,l}-S_{k,k}-b_{k+1,l}S_k}{a_l}\Big)\Big| \\
&\ll \min\Big(\mathrm{E}\Big|\frac{S_{k,k}+b_{k+1,l}\widetilde{S}_k}{a_l}\Big|,1\Big) \\
&\ll \min\Big(\Big(\frac{k}{l}\Big)^{\delta}\ln k,1\Big).
\end{aligned}$$

Thus, using the Hölder inequality, for $k \le m \le n$, we obtain

$$\begin{aligned}\mathrm{E}\Big|\sum_{l=m}^{n} d_l(\xi_l-\xi_{k,l})\Big|^p &\le \sum_{l_1=m}^{n}\cdots\sum_{l_p=m}^{n} d_{l_1}\cdots d_{l_p}\Big(\mathrm{E}|\xi_{l_1}-\xi_{k,l_1}|^p\cdots\mathrm{E}|\xi_{l_p}-\xi_{k,l_p}|^p\Big)^{\frac{1}{p}}\\ &\ll k^{\delta}\sum_{l_1=m}^{n}\cdots\sum_{l_p=m}^{n} d_{l_1}\cdots d_{l_p} l_1^{-\frac{\delta}{p}}\cdots l_p^{-\frac{\delta}{p}}\ln k\\ &= k^{\delta}\ln k\Big(\sum_{l=m}^{n} d_l l^{-\frac{\delta}{p}}\Big)^p\\ &\le \ln k\Big(\sum_{l=m}^{n} d_l^2 l\Big)^{\frac{p}{2}} m^{\delta}\Big(\sum_{l=m}^{n} l^{-\frac{2\delta}{p}-1}\Big)^{\frac{p}{2}}\\ &\ll \ln k\Big(\sum_{l=m}^{n} d_l^2 l\Big)^{\frac{p}{2}}\\ &\le \ln n\Big(\sum_{l=m}^{n} d_l^2 l\Big)^{\frac{p}{2}}.\end{aligned}$$

This completes the proof of (2.4).

Set

$$\begin{aligned}U_{m,n} &= \sum_{m\le k\le l\le n} d_k d_l \min\Big(\Big(\frac{k}{l}\Big)^{\delta}\ln k, 1\Big)\\ &= \sum_{l=m}^{n} d_l l^{-\delta}\Big(\sum_{k=m}^{l} d_k \min(k^{\delta}\ln k, \quad l^{\delta})\Big), \quad 1\le m\le n.\end{aligned}$$

Put further

$$c_p = (4\gamma)^{p^2}. \tag{2.8}$$

We show that if the number γ chosen is large enough, then

$$\mathrm{E}\Big|\sum_{k=m}^{n} d_k\xi_k\Big|^p \le c_p U_{m,n}^{\frac{p}{2}}\ln n, \quad \text{for all } 1\le m\le n. \tag{2.9}$$

Since if let $m=1$ in (2.9), we have that (2.9) becomes (2.5), this will prove (2.5). We use induction on p. By (2.3), we get

$$\begin{aligned}\mathrm{E}\Big(\sum_{k=m}^{n} d_k\xi_k\Big)^2 &\le 2\sum_{m\le k\le l\le n} d_k d_l|\mathrm{E}\xi_k\xi_l|\\ &\le 2c\sum_{m\le k\le l\le n} d_k d_l \min\Big(\Big(\frac{k}{l}\Big)^{\delta}\ln k, 1\Big)\\ &\le 2cU_{m,n}.\end{aligned}$$

Hence if we choose γ so large that $(4\gamma)^4 \ge 2c$, then (2.9) holds for $p=2$.

Assume now that (2.9) is true for $p-1\ge 2$. From $kd_k\ln k = \exp(\ln^{\beta} k)\ln k \gg 1$, it follows that there exists an $A>0$ such that

$$\sum_{k=m}^{l} d_k \min(k^{\delta}\ln k, l^{\delta}) \ge Al^{\delta}.$$

Now, we choose γ so large that the c_p defined in (2.8) satisfies $c_p > \left(\frac{2c}{A}\right)^p \gamma^{\frac{p}{2}}$. Then, using $|\xi_l| \le 2c$, we get that for $U_{m,n} \le \gamma$,

$$\Big|\sum_{l=m}^{n} d_l \xi_l\Big| \le \frac{2c}{A}\sum_{l=m}^{n} d_l l^{-\delta}\Big(\sum_{k=m}^{l} d_k \min(k^\delta \ln k, l^\delta)\Big)$$
$$= \frac{2c}{A} U_{m,n} \le \frac{2c}{A}\gamma^{\frac12} U_{m,n}^{\frac12},$$

which implies

$$\mathrm{E}\Big|\sum_{l=m}^{n} d_l \xi_l\Big|^p \le \Big(\frac{2c}{A}\Big)^p \gamma^{\frac{p}{2}} U_{m,n}^{\frac{p}{2}} \le c_p U_{m,n}^{\frac{p}{2}}.$$

Hence, in the case $U_{m,n} \le \gamma$, the relation (2.9) is valid. We now show that if $B \ge \gamma$ is arbitrary and (2.9) holds for $U_{m,n} \le B$, then (2.9) will also hold for $U_{m,n} \le \frac{3B}{2}$. As the validity of (2.9) is already verified for $U_{m,n} \le \gamma$, we will show that (2.9) holds for any value of $U_{m,n}$, and will complete the induction step.

Assume $U_{m,n} \le \frac{3B}{2}$. Set

$$\mathrm{W}_1 + \mathrm{W}_2 = \sum_{k=m}^{q} d_k \xi_k + \sum_{k=q+1}^{n} d_k \xi_k, \quad m \le q \le n$$

and

$$\mathrm{T}_2 = \sum_{k=q+1}^{n} d_k \xi_{j,k}.$$

By the Stolz Theorem, and $\ln(1+x) \sim x$, $\mathrm{e}^x - 1 \sim x$, for $x \to 0$, we get

$$D_n \sim \frac{1}{\beta}\ln^{1-\beta} n \exp(\ln^\beta n) \quad \text{for } \beta > 0, \quad D_n \sim \ln n \quad \text{for } \beta = 0. \tag{2.10}$$

For fixed m and n, we choose q such that

$$U_{m,q} \le B, \quad U_{q+1,n} \le B, \quad \frac{U_{q+1,n}}{U_{m,q}} = \lambda \in \Big[\frac12, 1\Big]. \tag{2.11}$$

This is possible for some sufficiently large γ, since $U_{m,n} \ge \gamma$, by (2.10), and $\mathrm{e}^{2\ln^\beta n}$ $(0 \le \beta < 1)$ is a slowly varying function at infinity, so we have

$$U_{m,n} - U_{m,n-1} = d_n n^{-\delta}\sum_{k=m}^{n} d_k \min(k^\delta \ln k, 1)$$
$$\le d_n \ln n D_n$$
$$\ll \frac{\mathrm{e}^{2\ln^\beta n}\ln^{2-\beta} n}{n} \to 0, \quad n \to \infty.$$

Now we prove that

$$\mathrm{E}|\mathrm{W}_1 + \mathrm{W}_2|^p \le c_p U_{m,n}^{\frac{p}{2}} \ln n.$$

By the properties of a slowly varying function at infinity, we have that d_k and $d_k k^\delta \ln k$ $(\delta < 1)$ are quasi-decreasing, which implies that there exists a $c > 0$ such that

$$c\sum_{k=m}^{l} d_k \min(k^\delta \ln k, l^\delta) \ge l^{1+\delta} d_l \quad \text{for all } l \ge 1.$$

This shows that

$$\sum_{l=m}^{n} l d_l^2 = \sum_{l=m}^{n} l^{-\alpha} d_l d_l l^{\alpha+1}$$
$$\le c \sum_{l=m}^{n} l^{-\alpha} d_l \Big(\sum_{k=m}^{l} d_k \min(k^{\delta} \ln k, l^{\delta}) \Big) = c U_{m,n}, \quad 1 \le m \le n. \tag{2.12}$$

Set $F_j = c^{\frac{j}{2}} A_j$, where A_j is the constant in (2.4). Using (2.4), (2.11)–(2.12), we have that for all $j \ge 1$,

$$\begin{aligned} \mathrm{E}|\mathrm{W}_2 - \mathrm{T}_2|^j &= \mathrm{E}\Big| \sum_{l=q+1}^{n} d_l (\xi_l - \xi_{q,l}) \Big|^j \\ &\le A_j \Big(\sum_{l=q+1}^{n} l d_l^2 \Big)^{\frac{j}{2}} \ln n \\ &\le A_j c^{\frac{j}{2}} U_{q+1,n}^{\frac{j}{2}} \ln n \\ &= F_j \lambda^{\frac{j}{2}} U_{m,q}^{\frac{j}{2}} \ln n. \end{aligned} \tag{2.13}$$

We also have

$$\mathrm{E}|\mathrm{W}_1|^j \le c_j U_{m,q}^{\frac{j}{2}}, \quad 1 \le j \le p \tag{2.14}$$

and

$$\mathrm{E}|\mathrm{W}_2|^j \le c_j U_{q+1,n}^{\frac{j}{2}} = c_j \lambda^{\frac{j}{2}} U_{m,q}^{\frac{j}{2}}, \quad 1 \le j \le p. \tag{2.15}$$

For $1 \le j \le p-1$ the last two inequalities are valid by the induction hypothesis, and for $j = p$ they follow from the validity of (2.9) for $U_{m,n} \le B$. Hence the c_r inequality yields

$$\begin{aligned} \mathrm{E}|\mathrm{T}_2|^j &\le \mathrm{E}(|\mathrm{W}_2| + |\mathrm{W}_2 - \mathrm{T}_2|)^j \\ &\le 2^{j-1} (\mathrm{E}|\mathrm{W}_2|^j + \mathrm{E}|\mathrm{W}_2 - \mathrm{T}_2|^j) \\ &\le 2^j c_j \lambda^{\frac{j}{2}} U_{m,q}^{\frac{j}{2}}, \quad 1 \le j \le p. \end{aligned} \tag{2.16}$$

The Hölder inequality with the latter results implies that for $j = 1, 2, \cdots, p-1$,

$$\begin{aligned} \mathrm{E}|\mathrm{W}_1|^j |\mathrm{W}_2 - \mathrm{T}_2| |\mathrm{W}_2|^{p-j-1} &\le (\mathrm{E}|\mathrm{W}_1|^p)^{\frac{1}{p}} (\mathrm{E}|\mathrm{W}_2 - \mathrm{T}_2|^p)^{\frac{1}{p}} (\mathrm{E}|\mathrm{W}_2|^p)^{\frac{p-j-1}{p}} \\ &\le c_p^{\frac{p-1}{p}} F_p^{\frac{1}{p}} \ln^{\frac{1}{p}} n \lambda^{\frac{p-j}{2}} U_{m,q}^{\frac{p}{2}}, \end{aligned} \tag{2.17}$$

and

$$\mathrm{E}|\mathrm{W}_1|^j |\mathrm{W}_2 - \mathrm{T}_2| |\mathrm{T}_2|^{p-j-1} \le 2^{p-j-1} c_p^{\frac{p-1}{p}} F_p^{\frac{1}{p}} \lambda^{\frac{p-j}{2}} \ln n U_{m,q}^{\frac{p}{2}}. \tag{2.18}$$

From the mean value theorem, we obtain

$$|\mathrm{W}_2^j - \mathrm{T}_2^j| \le j |\mathrm{W}_2 - \mathrm{T}_2| (|\mathrm{W}_2|^{j-1} + |\mathrm{T}_2|^{j-1}), \quad j \ge 1. \tag{2.19}$$

By (2.8),

$$c_p^{-1} F_p^{\frac{1}{p}} \le c(4\gamma)^{-p}, \quad \frac{c_j c_{p-j}}{c_p} \le (4\gamma)^{-p}, \quad \lambda \le 1. \tag{2.20}$$

Since W_1 and T_2 are independent, by the binomial formula and the triangle inequality, we get

$$\mathrm{E}|\mathrm{W}_1+\mathrm{W}_2|^p \le \mathrm{E}|\mathrm{W}_1|^p + \mathrm{E}|\mathrm{W}_2|^p + \sum_{j=1}^{p-1}\binom{p}{j}(\mathrm{E}|\mathrm{W}_1|^j|\mathrm{W}_2^{p-j}-\mathrm{T}_2^{p-j}| + \mathrm{E}|\mathrm{W}_1|^j\mathrm{E}|\mathrm{T}_2|^{p-j}).$$

We substitute (2.11) and (2.13)–(2.20) in the above inequality. Then, since $\lambda \ge \frac{1}{2}$, we get that for a large enough γ,

$$\begin{aligned}
&\mathrm{E}|\mathrm{W}_1+\mathrm{W}_2|^p \\
&\le c_p U_{m,q}^{\frac{p}{2}}\Big(1+\lambda^{\frac{p}{2}} + c_p^{-\frac{1}{p}}F_p^{\frac{1}{p}}\sum_{j=1}^{p-1}2^{p-j}\tbinom{p}{j}(p-j)\lambda^{\frac{p-j}{2}} + c_p^{-1}\sum_{j=1}^{p-1}2^{p-j}\lambda^{\frac{p-j}{2}}\tbinom{p}{j}c_jc_{p-j}\Big)\ln n \\
&\le c_p U_{m,q}^{\frac{p}{2}}(1+\lambda^{\frac{p}{2}} + c(4\gamma)^{-p}p(2\lambda^{\frac{1}{2}}+1)^p + (4\gamma)^{-p}(1+2\lambda^{\frac{1}{2}})^p)\ln n \\
&\le c_p U_{m,q}^{\frac{p}{2}}(1+\lambda^{\frac{p}{2}} + c\gamma^{-p})\ln n \\
&\le c_p(1+\lambda)^{\frac{p}{2}}U_{m,q}^{\frac{p}{2}}\ln n \\
&= c_p U_{m,n}^{\frac{p}{2}}\ln n.
\end{aligned}$$

Thus, we proved the validity of (2.9) for $U_{m,n} \le \frac{3B}{2}$, and the proof of (2.9) is completed.

Proof of Theorem 2.1 By Remark 2.1, without loss of generality, we can suppose $\beta > 0$. By Theorem 7.1 of [1] and Section 2 of [8], to prove Theorem 2.1, it suffices to show that

$$T_n \widehat{=} \frac{1}{D_n}\sum_{k=1}^n d_k\xi_k \to 0 \quad \text{a.s.,} \quad n\to\infty. \tag{2.21}$$

Since

$$\begin{aligned}
&\sum_{1\le k\le l\le n} d_kd_l \min\Big(\Big(\frac{k}{l}\Big)^\delta \ln k, 1\Big) \\
&\le \sum_{\substack{1\le k\le l\le n\\ \frac{k}{l}\le(\ln^2 D_n\ln n)^{-\frac{1}{\delta}}}} d_kd_l\Big(\frac{k}{l}\Big)^\delta \ln k + \sum_{\substack{1\le k\le l\le n\\ \frac{k}{l}>(\ln^2 D_n\ln n)^{-\frac{1}{\delta}}}} d_kd_l \\
&\triangleq \mathrm{T}_{n_1} + \mathrm{T}_{n_2},
\end{aligned}\tag{2.22}$$

we have

$$\begin{aligned}
\mathrm{T}_{n_1} &\ll \sum_{\substack{1\le k\le l\le n\\ \frac{k}{l}\le(\ln^2 D_n\ln n)^{-\frac{1}{\delta}}}} d_kd_l\Big(\frac{k}{l}\Big)^\delta \ln n \\
&\le \sum_{\substack{1\le k\le l\le n\\ \frac{k}{l}\le(\ln^2 D_n\ln n)^{-\frac{1}{\delta}}}} d_kd_l \ln n\frac{1}{\ln^2 D_n \ln n} \\
&\le \frac{D_n^2}{\ln^2 D_n}.
\end{aligned}\tag{2.23}$$

By (2.10),

$$\mathrm{e}^{\ln^\beta n} \sim \frac{\beta D_n}{(\ln D_n)^{\frac{1-\beta}{\beta}}}, \quad \ln D_n \sim \ln^\beta n, \quad \ln\ln D_n \sim \beta\ln\ln n.$$

Hence,

$$\begin{aligned} \mathrm{T}_{n_2} &\le \sum_{k=1}^{n} d_k \sum_{k\le l<k(\ln^2 D_n \ln n)^{\frac{1}{\delta}}} \frac{1}{l}\mathrm{e}^{\ln^\beta n} \\ &\ll \frac{D_n}{(\ln D_n)^{\frac{1-\beta}{\beta}}} \sum_{k=1}^{n} d_k \ln\ln D_n \\ &\ll \frac{D_n^2 \ln\ln D_n}{(\ln D_n)^{\frac{1-\beta}{\beta}}}. \end{aligned} \tag{2.24}$$

Thus, letting $\beta_1 = \min(2, \frac{1-\beta}{\beta}) > 0$, and by (2.22)–(2.24), we get

$$\sum_{1\le k\le l\le n} d_k d_l \min\Big(\Big(\frac{k}{l}\Big)^\delta \ln k, 1\Big) \ll \frac{D_n^2 \ln\ln D_n}{(\ln D_n)^{\beta_1}}. \tag{2.25}$$

Hence by $\ln n \sim \ln^{\frac{1}{\beta}} D_n$, $(\ln\ln D_n)^{\frac{p}{2}} = o(\ln D_n)$, $\forall p$. Let $p > \frac{2(3\beta+1)}{\beta_1\beta}$, i.e., $\frac{\beta_1 p}{2} - \frac{1}{\beta} - 1 > 2$, and by the Markov inequality, (2.5) and (2.25), for sufficiently large n, we have

$$\begin{aligned} P\Big(\Big|\frac{1}{D_n}\sum_{k=1}^{n} d_k\xi_k\Big| > \varepsilon\Big) &\ll \frac{1}{D_n^p}\mathrm{E}\Big|\sum_{k=1}^{n} d_k\xi_k\Big|^p \\ &\ll \frac{1}{D_n^p}\Big(\sum_{1\le k\le l\le n} d_k d_l \min\Big(\Big(\frac{k}{l}\Big)^\delta \ln k, 1\Big)\Big)^{\frac{p}{2}} \ln n \\ &\ll \frac{1}{D_n^p}\Big(\frac{D_n^2 \ln\ln D_n}{(\ln D_n)^{\beta_1}}\Big)^{\frac{p}{2}} \ln n \\ &\ll \frac{1}{(\ln D_n)^{\frac{\beta_1 p}{2} - \frac{1}{\beta} - 1}} \\ &\le \frac{1}{\ln^2 D_n}. \end{aligned}$$

By (2.10), we have $D_{n+1} \sim D_n$. Let $n_k = \inf\{n; D_n \ge \exp(k^{\frac{2}{3}})\}$. Then $D_{n_k} \ge \exp(k^{\frac{2}{3}})$, $D_{n_k-1} < \exp(k^{\frac{2}{3}})$. Therefore,

$$1 \le \frac{D_{n_k}}{\exp(k^{\frac{2}{3}})} \sim \frac{D_{n_k-1}}{\exp(k^{\frac{2}{3}})} < 1 \to 1,$$

that is

$$D_{n_k} \sim \exp(k^{\frac{2}{3}}).$$

Therefore,

$$\sum_{k=1}^{\infty} P\Big(\Big|\frac{1}{D_{n_k}}\sum_{i=1}^{n_k} d_i\xi_i\Big| > \varepsilon\Big) \ll \sum_{k=1}^{\infty}\frac{1}{k^{\frac{4}{3}}} < \infty,$$

that is

$$T_{n_k} \to 0 \quad \text{a.s.}$$

For $n_k \le n < n_{k+1}$,

$$|T_n| \le |T_{n_k}| + \frac{2c}{D_{n_k}}(D_{n_{k+1}} - D_{n_k}) \to 0 \quad \text{a.s.}$$

from $\frac{D_{n_{k+1}}}{D_{n_k}} = \exp((k+1)^{\frac{2}{3}} - k^{\frac{2}{3}}) = \exp\left(k^{\frac{2}{3}}\left[\left(1+\frac{1}{k}\right)^{\frac{2}{3}} - 1\right]\right) \sim \exp\left(\frac{2k^{-\frac{1}{3}}}{3}\right) \to 1$. Therefore, (2.21) holds.

Proof of Theorem 2.2 (2.1) is equivalent to

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\Big(\frac{\mu}{a_k} \sum_{i=1}^{k} \ln\left(\frac{S_i}{i\mu}\right) \le x\Big) = G(x) \quad \text{a.s.} \quad \text{for any } x, \tag{2.26}$$

where $G(x)$ is the distribution function of the random variable $\Gamma^{\frac{1}{\alpha}}(1+\alpha)G_\alpha$.

By Lemma 2.3 in [9],

$$\frac{1}{a_n} \sum_{k=1}^{n} \ln\left(\frac{n+1}{k}\right)(X_k - \mu) \xrightarrow{d} \Gamma^{\frac{1}{\alpha}}(1+\alpha)G_\alpha, \quad n \to \infty.$$

Hence, using the fact that $\ln \frac{n+1}{k} \sim b_{k,n}$, we get

$$\frac{1}{a_n} S_{n,n} = \frac{1}{a_n} \sum_{k=1}^{n} \frac{b_{k,n}}{\ln \frac{n+1}{k}} \ln\left(\frac{n+1}{k}\right)(X_k - \mu) \xrightarrow{d} \Gamma^{\frac{1}{\alpha}}(1+\alpha)G_\alpha.$$

This implies

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k P\Big(\frac{S_{k,k}}{a_k} \le x\Big) = G(x). \tag{2.27}$$

Note that

$$\frac{S_{n,n}}{a_n} = \frac{1}{a_n} \sum_{k=1}^{n} b_{k,n}(X_k - \mu) = \frac{\mu}{a_n} \sum_{k=1}^{n} \Big(\frac{S_k}{\mu k} - 1\Big).$$

By Theorem 2.1, (2.27) is equivalent to

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\Big(\frac{S_{k,k}}{a_k} \le x\Big) = \lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\Big(\frac{\mu}{a_k} \sum_{i=1}^{k} \Big(\frac{S_i}{\mu i} - 1\Big) \le x\Big) = G(x) \quad \text{a.s.} \tag{2.28}$$

Let $p \in \left(\frac{2\alpha}{1+\alpha}, \alpha\right)$. Then $\mathrm{E}|X| < \infty$ and $\mathrm{E}|X|^p < \infty$. Using the Marcinkiewicz-Zygmund strong large number law, we have

$$\frac{S_k}{\mu k} - 1 = \frac{1}{\mu}\Big(\frac{S_k - \mu}{k}\Big) \to 0 \quad \text{a.s.},$$
$$S_k - \mu k = o(k^{\frac{1}{p}}) \quad \text{a.s.}$$

Hence, by $|\ln(1+x) - x| = O(x^2)$ for $|x| < \frac{1}{2}$ and (1.10),

$$\begin{aligned}
&\Big|\frac{1}{a_k} \sum_{i=1}^{k} \ln\left(\frac{S_i}{\mu i}\right) - \frac{1}{a_k} \sum_{i=1}^{k} \left(\frac{S_i}{\mu i} - 1\right)\Big| \\
&\ll \frac{1}{a_k} \sum_{i=1}^{k} \left(\frac{S_i - \mu i}{\mu i}\right)^2 \\
&\le \frac{1}{a_k} \sum_{i=1}^{k} i^{2(\frac{1}{p}-1)} \\
&\ll \frac{k^{\frac{2}{p}-1}}{k^{\frac{1}{\alpha}} L_1(k)} \to 0 \quad \text{a.s.}, \quad k \to \infty
\end{aligned}$$

from $\frac{2}{p} - 1 < \frac{1}{\alpha}$.

Hence, for almost every event ω and any $\varepsilon > 0$ there exists $k_0 = k_0(\omega, \varepsilon, x)$ such that for $k > k_0$,

$$I\Big(\frac{\mu}{a_k}\sum_{i=1}^{k}\ln\frac{S_i}{i\mu} \le x-\varepsilon\Big) \le I\Big(\frac{\mu}{a_k}\sum_{i=1}^{k}\Big(\frac{S_i}{i\mu}-1\Big) \le x\Big) \le I\Big(\frac{\mu}{a_k}\sum_{i=1}^{k}\ln\frac{S_i}{i\mu} \le x+\varepsilon\Big),$$

and thus (2.28) implies (2.26).

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