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Gradient Flow for the Helfrich Functional*

Yannan LIU[1]

Abstract The author studies the L^2 gradient flow of the Helfrich functional, which is a functional describing the shapes of human red blood cells. For any $\lambda_i \geq 0$ and c_0, the author obtains a lower bound on the lifespan of the smooth solution, which depends only on the concentration of curvature for the initial surface.

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1 Introduction

Let $F : \Sigma \to \mathbb{R}^3$ be a smooth compact embedded surface, and H be the mean curvature of the surface with respect to the outer normal vector ν. Consider the functional

$$W_{c_0}(F) = \int_\Sigma (H - c_0)^2 \mathrm{d}\mu,$$

where c_0 is a constant and is called the spontaneous curvature. In the special case of $c_0 = 0$, $W_{c_0}(F)$ is the well-known Willmore functional. Generally, $W_{c_0}(F)$ is a model describing the shapes of human red blood cells, and the problem is to look for the critical points of $W_{c_0}(F)$ subject to the fixed area as well as the fixed volume enclosed by the surface (see [1] for details).

Let $A(F)$ and $V(F)$ denote the area of the surface $F : \Sigma \to \mathbb{R}^3$ and the volume enclosed by the surface, respectively. For $\lambda_i \in \mathbb{R}^1$ $(i = 1, 2)$, consider the functional

$$H_{c_0}(F) = \frac{1}{2} W_{c_0}(F) + \lambda_1 A(F) + \lambda_2 V(F). \tag{1.1}$$

The Euler-Lagrange equation of $H_{c_0}(F)$ is

$$\Xi := \triangle H + H\Big(|A|^2 - \frac{1}{2}|H|^2\Big) + c_0(|H|^2 - |A|^2) - H\Big(\frac{1}{2}c_0^2 + \lambda_1\Big) - \lambda_2 = 0. \tag{1.2}$$

In this paper, we study the flow

$$\begin{aligned}\frac{\mathrm{d}F(t)}{\mathrm{d}t} &= \Xi\nu \\ &= \Big(\triangle H + H\Big(|A|^2 - \frac{1}{2}|H|^2\Big) + c_0(|H|^2 - |A|^2) - H\Big(\frac{1}{2}c_0^2 + \lambda_1\Big) - \lambda_2\Big)\nu, \end{aligned}\tag{1.3}$$

which is a fourth order parabolic equation. The short-time existence of (1.3) was obtained in [2].

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[1]Department of Mathematics, Beijing Technology and Business University, Beijing 100048, China.
E-mail: liuyn@th.btbu.edu.cn
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Theorem 1.1 (see [2]) *For any* λ_i $(i = 1, 2)$, c_0 *and any smooth immersion* $F_0 : \Sigma \to \mathbb{R}^3$, *there exists a unique, nonextendable smooth solution* $F_t : \Sigma \times [0, T) \to \mathbb{R}^3$ *to* (1.3) *with* $F(\cdot, 0) = F_0$, *where* $0 < T \leq \infty$.

The long-time existence of the flow (1.3) was proved in [2] for some special λ_1, λ_2, where the center manifold method is used, so the existence is in the sense of weak solutions and there is no subconvergence available. Note that when $c_0 = \lambda_1 = \lambda_2 = 0$, the flow is the well-known Willmore flow, which has been studied extensively in [3–5]. In [4], the authors gave a lower bound on the lifespan to the smooth solution of the Willmore flow, which is the key step in the proof of the long-time existence. For the curve case of the Helfrich flow, see [6] for reference.

The main result of this paper is that we obtain a lower bound on the lifespan which depends only on the concentration of curvature for the initial surface.

Theorem 1.2 *Let* $F_0 : \Sigma \to \mathbb{R}^3$ *be a smooth immersion. There exist constants* $\epsilon_0 > 0$, $c < \infty$ *depending only on* c_0, λ_i $(i = 1, 2)$, *such that if* $\rho > 0$ *is chosen with*

$$\int_{B_\rho(x)} |A_0|^2 \mathrm{d}\mu_0 \leq \epsilon_0 \leq \epsilon \quad \text{for any } x \in \mathbb{R}^3, \tag{1.4}$$

then, the maximal time T *of the smooth existence of the flow* (1.3) *with the initial datum* F_0 *satisfies*

$$T \geq \frac{1}{c}\rho^4, \tag{1.5}$$

and one has the estimate

$$\int_{B_\rho(x)} |A|^2 \mathrm{d}\mu \leq c\epsilon \quad \text{for } 0 \leq t \leq \frac{1}{c}\rho^4. \tag{1.6}$$

The difficulty to prove Theorem 1.2 comes from the fact that (1.3) is a fourth order parabolic equation, so the method based on the maximum principle to study the usual curvature flow (a second order parabolic equation) cannot be applied in this case. Instead, we use an energy estimate method in [4] to prove Theorem 1.2 in Section 4. For this purpose, we want to use ideas in [7–9] to derive the evolution equations for the curvature and its derivatives, which are exploited in Section 2. In Section 3, we give some energy type inequalities.

2 Evolution Equations for the Curvature and Its Derivatives

Throughout this paper, we use the following notations. $\langle\, ,\, \rangle$ denotes the usual inner product in $\mathbb{R}^3$. If Σ is given as in Section 1 and F denotes its parametrization in $\mathbb{R}^3$, the metrics $\{g_{ij}\}$ are given by

$$g_{ij}(x) = \Big\langle \frac{\partial F(x)}{\partial x_i}, \frac{\partial F(x)}{\partial x_j} \Big\rangle, \quad x \in M.$$

Let ∇ denote the Levi-Civita connection on Σ, and D denote the standard metric of $\mathbb{R}^3$. Indices are raised and lowered w.r.t g^{ij} and g_{ij}. Also, we use $\langle\, ,\, \rangle$ to denote the scalar product on M if there is no confusion.

The second fundamental form in direction ν is denoted by

$$h_{ij}(x) = -\langle \nu, \nabla_i \nabla_j F \rangle.$$

The norm of the second fundamental form is denoted by

$$|A|^2 = g^{ij} g^{pl} h_{ip} h_{lj}.$$

The mean curvature on Σ is given by

$$H = g^{ij} h_{ij}.$$

$\mathrm{d}\mu$ denotes the area element of $F(\Sigma)$.

Let φ and ϕ be forms having normal values along F. We denote by $\varphi * \phi$ a normal-valued multilinear form depending on φ and ϕ in a universal bilinear way. In particular, we have the properties $|\varphi * \phi| \le c|\varphi||\phi|$ and $\nabla(\varphi * \phi) = \nabla\varphi * \phi + \varphi * \nabla\phi$.

Lemma 2.1 *Assume that the flow* (1.3) *smoothly exists on* $[0, T)$ *with* $0 < T \le \infty$. *The following equations hold:*

$$\frac{\partial g_{ij}}{\partial t} = 2\Xi h_{ij}, \tag{2.1}$$

$$\frac{\partial \mathrm{d}\mu_t}{\partial t} = H\Xi \mathrm{d}\mu_t, \tag{2.2}$$

$$\frac{\partial h_{ij}}{\partial t} = -\nabla_i \nabla_j \Xi + \Xi h_i^l h_{lj}, \tag{2.3}$$

$$\frac{\partial H}{\partial t} = -\Delta \Xi - \Xi |A|^2. \tag{2.4}$$

Proof Since $\frac{\mathrm{d}F(t)}{\mathrm{d}t} = \Xi\nu$, we have that

$$\frac{\partial g_{ij}}{\partial t} = \Big\langle \nabla_i \frac{\mathrm{d}F(t)}{\mathrm{d}t}, \nabla_j F \Big\rangle + \Big\langle \nabla_j \frac{\mathrm{d}F(t)}{\mathrm{d}t}, \nabla_i F \Big\rangle = 2\Xi h_{ij},$$

and (2.2) follows from (2.1). For (2.3), we have

$$\frac{\partial h_{ij}}{\partial t} = -\Big\langle \nabla_i \nabla_j \frac{\mathrm{d}F(t)}{\mathrm{d}t}, \nu \Big\rangle = -\nabla_i \nabla_j \Xi + \Xi h_i^l h_{lj}.$$

(2.4) follows from (2.1)–(2.2),

$$\frac{\partial H}{\partial t} = \frac{\partial g^{ij}}{\partial t} h_{ij} + \frac{\partial h_{ij}}{\partial t} g^{ij} = -\Delta \Xi - \Xi |A|^2.$$

Next, we use $P_m^n(A)$ to denote any linear combinations of terms of the type $\nabla^{i_1} A * \cdots * \nabla^{i_m} A$ with universal constant coefficients, where $n = i_1 + \cdots + i_m$ is the total number of derivatives. Using this notation, we can write Ξ as

$$\Xi = \triangle H + \sum_{0 \le i \le 3} P_i^0(A) = P_1^2(A) + \sum_{0 \le i \le 3} P_i^0(A). \tag{2.5}$$

Lemma 2.2 *The evolution equation of* h_{ij} *is*

$$\frac{\partial h_{ij}}{\partial t} + \triangle^2 h_{ij} = \sum_{1 \le i \le 3} P_i^2(A) + \sum_{2 \le i \le 5} P_i^0(A).$$

Proof By Lemma 2.1, we have

$$\begin{aligned}\frac{\partial h_{ij}}{\partial t} &= -\nabla_i\nabla_j\triangle H - \nabla_i\nabla_j\Big(H\Big(|A|^2 - \frac{1}{2}|H|^2\Big)\Big)\\ &\quad - c_0\nabla_i\nabla_j(|H|^2 - |A|^2) + \Big(\frac{1}{2}c_0^2 + \lambda_1\Big)\nabla_i\nabla_j H + h_i^l h_{lj}\Xi.\end{aligned}$$

By Mainardi-Codazzi, Gauss and Ricci equations, we have

$$\triangle^2 h_{ij} - \nabla_i\nabla_j(\triangle H) = A * A * \nabla^2 H + A * \nabla A * \nabla H + \nabla(A * A * \nabla H + A * \nabla A * H).$$

Substituting this equation into the above equality and using the expression of $P_m^n(A)$ and (2.5), we can easily obtain the result.

Next, we recall a result of [4].

Lemma 2.3 (see [4]) *Let ϕ be a form with normal values along a variation $F : \Sigma\times[0,T) \to \mathbb{R}^n$ with the normal velocity $\partial_t F = N$. If $\partial_t^\perp\phi + \triangle^2\phi = Y$, then $\psi = \nabla\phi$ satiesfies an equation*

$$\begin{aligned}\partial_t^\perp\psi + \triangle^2\psi &= \nabla Y + \sum_{i+j+k=3}\nabla^i A * \nabla^j A * \nabla^k A\\ &\quad + A * \nabla N * \phi + \nabla A * V * \phi. \qquad (2.6)\end{aligned}$$

Using Lemmas 2.2–2.3, we can easily get the following result by induction.

Lemma 2.4 *The following equation holds:*

$$\partial_t(\nabla^m A) + \triangle^2(\nabla^m A) = \sum_{1\le i\le 3} P_i^{m+2}(A) + \sum_{2\le i\le 5} P_i^m(A). \qquad (2.7)$$

3 Energy Type Inequalities

In this section, we obtain some energy type inequalities. First, we introduce the following lemma coming from [4].

Lemma 3.1 (see [4]) *Let $F : \Sigma\times[0,T) \to \mathbb{R}^n$ be a variation with the normal $\partial_t F = N$, and ϕ be an l-linear form along F which satisfies $\partial_t^\perp\phi + \triangle^2\phi = Y$. Then, for any $\gamma \in C^2(\Sigma\times I)$, $s \ge 4$ and $c = c(n,s)$, we have*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma |\phi|^2\gamma^s\mathrm{d}\mu + \int_\Sigma |\nabla^2\phi|^2\gamma^s\mathrm{d}\mu - \int_\Sigma 2\langle Y,\phi\rangle\gamma^s\mathrm{d}\mu\\ &\le \int_\Sigma \langle A * \phi * \phi, N\rangle\gamma^s\mathrm{d}\mu + \int_\Sigma |\phi|^2 s\gamma^{s-1}\partial_t\gamma\mathrm{d}\mu\\ &\quad + c\int_\Sigma |\phi|^2\gamma^{s-4}(|\nabla\gamma|^4 + \gamma^2|\nabla^2\gamma|^2)\mathrm{d}\mu + c\int_\Sigma |\phi|^2(|A|^4 + |\nabla A|^2)\gamma^s\mathrm{d}\mu. \qquad (3.1)\end{aligned}$$

Similar to [4], we assume that $\gamma = \widetilde{\gamma}\circ F$, where $0 \le \widetilde{\gamma} \le 1$ and $\|\widetilde{\gamma}\|_{C^2(\mathbb{R}^3)} \le c < \infty$. This implies that $\nabla\gamma = (D\widetilde{\gamma}\circ F)DF$ and $\nabla^2\gamma = (D^2\widetilde{\gamma}\circ F)(DF, DF) + (D\widetilde{\gamma}\circ F)A(\,\cdot\,,\,\cdot\,)$. Therefore, we have

$$|\nabla\gamma| \le c, \quad |\nabla^2\gamma| \le c(1 + |A|). \qquad (3.2)$$

Lemma 3.2 *Assume that the flow* (1.3) *smoothly exists on* $[0,T]$ *with* $0<T\le\infty$. *Then for* $\gamma=\widetilde{\gamma}\circ F$ *satisfying* (3.2), $\phi=\nabla^m A$ *with a positive integer* m *and* $s\ge 2m+4$, *we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma|\phi|^2\gamma^s\mathrm{d}\mu+\frac{3}{4}\int_\Sigma|\nabla^2\phi|^2\gamma^s\mathrm{d}\mu \le\int_\Sigma\Big(\sum_{1\le i\le 3}P_i^{m+2}(A)+\sum_{2\le i\le 5}P_i^m(A)\Big)*\phi\gamma^s\mathrm{d}\mu+c\int_{[\gamma>0]}|A|^2\gamma^{s-4-2m}\mathrm{d}\mu. \tag{3.3}$$

Proof We will estimate the terms in (3.1). Since $Y=\sum\limits_{1\le i\le 3}P_i^{m+2}(A)+\sum\limits_{2\le i\le 5}P_i^m(A)$, by (2.5), we have

$$2\int_\Sigma\langle Y,\phi\rangle\gamma^s\mathrm{d}\mu+\int_\Sigma\langle A*\phi*\phi,V\rangle\gamma^s\mathrm{d}\mu+c\int_\Sigma|\phi|^2(|A|^4+|\nabla A|^2)\gamma^s\mathrm{d}\mu \le\int_\Sigma\Big(\sum_{1\le i\le 3}P_i^{m+2}(A)+\sum_{2\le i\le 5}P_i^m(A)\Big)*\phi\gamma^s\mathrm{d}\mu. \tag{3.4}$$

Under (1.3), we have

$$\partial_t\gamma=(D\widetilde{\gamma}\circ F)\cdot\Xi=(D\widetilde{\gamma}\circ F)\cdot\Big(\triangle H+\sum_{0\le i\le 3}P_i^0(A)\Big). \tag{3.5}$$

Hence

$$\int_\Sigma|\phi|^2s\gamma^{s-1}\partial_t\gamma\mathrm{d}\mu=\int_\Sigma|\phi|^2\gamma^{s-1}(D\widetilde{\gamma}\circ F)\cdot\Big(\triangle H+\sum_{0\le i\le 3}P_i^0(A)\Big)\mathrm{d}\mu. \tag{3.6}$$

Because $\|\widetilde{\gamma}\|_{C^2(R^3)}\le c<\infty$, we have the following estimates by Young's inequality:

$$\int_\Sigma|\phi|^2\gamma^{s-1}(D\widetilde{\gamma}\circ F)\cdot P_3^0(A)\mathrm{d}\mu\le c\int_\Sigma|\phi|^2\gamma^s|A|^4\mathrm{d}\mu+c\int_\Sigma|\phi|^2\gamma^{s-4}\mathrm{d}\mu,$$
$$\int_\Sigma|\phi|^2\gamma^{s-1}(D\widetilde{\gamma}\circ F)\cdot P_2^0(A)\mathrm{d}\mu\le c\int_\Sigma|\phi|^2\gamma^s|A|^4\mathrm{d}\mu+c\int_\Sigma|\phi|^2\gamma^{s-2}\mathrm{d}\mu,$$
$$\int_\Sigma|\phi|^2\gamma^{s-1}(D\widetilde{\gamma}\circ F)\cdot P_1^0(A)\mathrm{d}\mu\le c\int_\Sigma|\phi|^2\gamma^s|A|^4\mathrm{d}\mu+c\int_\Sigma|\phi|^2\gamma^{s-\frac{4}{3}}\mathrm{d}\mu,$$
$$\int_\Sigma|\phi|^2\gamma^{s-1}(D\widetilde{\gamma}\circ F)\cdot P_0^0(A)\mathrm{d}\mu\le c\int_\Sigma|\phi|^2\gamma^{s-1}\mathrm{d}\mu.$$

Since $0\le\gamma\le 1$ and $s\ge 2m+4$, by the above four inequalities, we have

$$\begin{aligned}&\int_\Sigma|\phi|^2\gamma^{s-1}\Big((D\widetilde{\gamma}\circ F)\cdot\sum_{0\le i\le 3}P_i^0(A)\Big)\mathrm{d}\mu\\ &\le c\int_\Sigma|\phi|^2\gamma^s|A|^4\mathrm{d}\mu+c\int_\Sigma|\phi|^2\gamma^{s-4}\mathrm{d}\mu\\ &\le\int_\Sigma P_5^m(A)*\phi\gamma^s\mathrm{d}\mu+c\int_\Sigma|\phi|^2\gamma^{s-4}\mathrm{d}\mu.\end{aligned} \tag{3.7}$$

For the other term on the right-hand side of (3.6), by using the result of [4], we have

$$\begin{aligned}\int_\Sigma|\phi|^2\gamma^{s-1}(D\widetilde{\gamma}\circ F)\cdot\triangle H\mathrm{d}\mu\le{}&\epsilon\int_\Sigma|\nabla^2\phi|^2\gamma^s\mathrm{d}\mu+c(\epsilon)\int_{[\gamma>0]}|A|^2\gamma^{s-4-2m}\mathrm{d}\mu\\ &+\int_\Sigma(P_3^{m+2}(A)+P_5^m(A))*\phi\gamma^s\mathrm{d}\mu,\end{aligned} \tag{3.8}$$

where $\epsilon > 0$ is arbitrary.

An interpolation inequality in [4] implies that

$$\int_\Sigma |\phi|^2\gamma^{s-4}\mathrm{d}\mu + \int_\Sigma |\nabla\phi|^2\gamma^{s-2}\mathrm{d}\mu \\ \le \epsilon\int_\Sigma |\nabla^2\phi|^2\gamma^s\mathrm{d}\mu + c(\epsilon)\int_{[\gamma>0]} |A|^2\gamma^{s-4-2m}\mathrm{d}\mu. \tag{3.9}$$

Now, from (3.7)–(3.9), we obtain

$$\int_\Sigma |\phi|^2 s\gamma^{s-1}\partial_t\gamma\mathrm{d}\mu \le \epsilon\int_\Sigma |\nabla^2\phi|^2\gamma^s\mathrm{d}\mu + c(\epsilon)\int_{[\gamma>0]} |A|^2\gamma^{s-4-2m}\mathrm{d}\mu \\ + \int_\Sigma (P_3^{m+2}(A) + P_5^m(A)) * \phi\gamma^s\mathrm{d}\mu. \tag{3.10}$$

The remaining term in (3.1) can be estimated as (see [4])

$$\int_\Sigma |\phi|^2\gamma^{s-4}(|\nabla\gamma|^4 + \gamma^2|\nabla^2\gamma|^2)\mathrm{d}\mu \le c\int_\Sigma |\phi|^2\gamma^{s-4}\mathrm{d}\mu + c\int_\Sigma |\phi|^2|A|^4\gamma^s\mathrm{d}\mu. \tag{3.11}$$

Now from (3.4) and (3.9)–(3.11), we can get the result.

4 Proof of Theorem 1.2

In this section, we will complete the proof of Theorem 1.2. First, we give two important lemmas in the proof of Theorem 1.2.

Lemma 4.1 *Assume that the flow* (1.3) *smoothly exists on* $[0,T]$, γ *is as in* (3.2) *and*

$$\epsilon = \sup_{0\le t\le T} \|A\|^2_{2,[\gamma>0]} \le \epsilon_0 \tag{4.1}$$

for some ϵ_0 *small enough depending on the constants in* (3.2). *Then for any* $t\in[0,T]$, *we have*

$$\int_{[\gamma=1]} |A|^2\mathrm{d}\mu + \frac{1}{8}\int_0^t\int_{[\gamma=1]} (|\nabla^2A|^2 + |A|^2|\nabla A|^2 + |A|^6)\mathrm{d}\mu\mathrm{d}\tau \\ \le \int_{[\gamma_0>0]} |A_0|^2\mathrm{d}\mu_0 + c\epsilon t. \tag{4.2}$$

Proof Let $m=0$ and $s=4$ in (3.3). We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma |A|^2\gamma^4\mathrm{d}\mu + \frac{3}{4}\int_\Sigma (|\nabla^2A|^2\gamma^4 + |A|^2|\nabla A|^2\gamma^4 + |A|^6\gamma^4)\mathrm{d}\mu \\ \le c\int_\Sigma \Big(\sum_{1\le i\le 3} P_i^2(A) + \sum_{2\le i\le 5} P_i^0(A)\Big) * A\gamma^4\mathrm{d}\mu + c\int_{[\gamma>0]} |A|^2\mathrm{d}\mu. \tag{4.3}$$

Since

$$\int_\Sigma \Big(\sum_{1\le i\le 3} P_i^2(A) + \sum_{2\le i\le 5} P_i^0(A)\Big) * A\gamma^4\mathrm{d}\mu \\ \le c\int_\Sigma \Big(|A|^3|\nabla^2A| + |A|^2|\nabla A|^2 + |A||\nabla A|^2 + |A||\nabla^2A| + \sum_{3\le i\le 6}|A|^i\Big)\gamma^4\mathrm{d}\mu, \tag{4.4}$$

by Young's inequality, we have

$$\begin{aligned} c\int_\Sigma |A|^3|\nabla^2 A|\gamma^4 \mathrm{d}\mu &\le \frac{1}{4}\int_\Sigma |\nabla^2 A|^2\gamma^4\mathrm{d}\mu + c\int_\Sigma |A|^6\gamma^4\mathrm{d}\mu,\\ c\int_\Sigma |A||\nabla^2 A|\gamma^4 \mathrm{d}\mu &\le \frac{1}{4}\int_\Sigma |\nabla^2 A|^2\gamma^4\mathrm{d}\mu + c\int_\Sigma |A|^2\gamma^4\mathrm{d}\mu,\\ c\int_\Sigma \sum_{3\le i\le 6}|A|^i\gamma^4\mathrm{d}\mu &\le c\int_\Sigma |A|^6\gamma^4\mathrm{d}\mu + c\int_\Sigma |A|^2\gamma^4\mathrm{d}\mu,\\ c\int_\Sigma |A||\nabla A|^2\mathrm{d}\mu &\le c\int_\Sigma |A|^2|\nabla A|^2\gamma^4\mathrm{d}\mu + \frac{1}{4}\int_\Sigma |\nabla A|^2\gamma^4\mathrm{d}\mu. \end{aligned} \tag{4.5}$$

Integrating by parts, we can estimate the last term in (4.5)

$$\begin{aligned} \int_\Sigma |\nabla A|^2\gamma^4\mathrm{d}\mu &\le \int_\Sigma |\nabla^2 A||A|\gamma^4\mathrm{d}\mu + \int_\Sigma |\nabla A||A|4\gamma^3|\nabla\gamma|\mathrm{d}\mu\\ &\le \frac{1}{4}\int_\Sigma |\nabla^2 A|\gamma^4\mathrm{d}\mu + \frac{1}{2}\int_\Sigma |\nabla A|^2\gamma^4\mathrm{d}\mu + c\int_{[\gamma>0]}|A|^2\mathrm{d}\mu. \end{aligned}$$

So,

$$\frac{1}{2}\int_\Sigma |\nabla A|^2\gamma^4\mathrm{d}\mu \le \frac{1}{4}\int_\Sigma |\nabla^2 A|\gamma^4\mathrm{d}\mu + c\int_{[\gamma>0]}|A|^2\mathrm{d}\mu,$$

which implies

$$c\int_\Sigma |A||\nabla A|^2\mathrm{d}\mu \le c\int_\Sigma |A|^2|\nabla A|^2\gamma^4\mathrm{d}\mu + \frac{1}{8}\int_\Sigma |\nabla^2 A|\gamma^4\mathrm{d}\mu + c\int_{[\gamma>0]}|A|^2\mathrm{d}\mu. \tag{4.6}$$

For the other estimates, by [4, Lemma 4.2], we have

$$\begin{aligned} &\int_\Sigma |A|^6\gamma^4\mathrm{d}\mu + \int_\Sigma |A|^2|\nabla A|^2\gamma^4\mathrm{d}\mu\\ &\le c\int_{[\gamma>0]}|A|^2\mathrm{d}\mu\int_\Sigma (|A|^6\gamma^4 + |\nabla^2 A|^2\gamma^4)\mathrm{d}\mu + c\Big(\int_{[\gamma>0]}|A|^2\mathrm{d}\mu\Big)^2. \end{aligned} \tag{4.7}$$

Combining (4.4)–(4.7), we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma |A|^2\gamma^4\mathrm{d}\mu + \frac{3}{4}\int_\Sigma (|\nabla^2 A|^2\gamma^4 + |A|^2|\nabla A|^2\gamma^4 + |A|^6\gamma^4)\mathrm{d}\mu\\ &\le \frac{5}{8}\int_\Sigma |\nabla^2 A|^2\gamma^4\mathrm{d}\mu + c\int_{[\gamma>0]}|A|^2\mathrm{d}\mu\\ &\quad + c\int_{[\gamma>0]}|A|^2\mathrm{d}\mu\int_\Sigma (|A|^6\gamma^4 + |\nabla^2 A|^2\gamma^4)\mathrm{d}\mu + c\Big(\int_{[\gamma>0]}|A|^2\mathrm{d}\mu\Big)^2. \end{aligned} \tag{4.8}$$

Noting that $\int_{[\gamma>0]}|A|^2\mathrm{d}\mu \le \epsilon_0$ in (4.1), by (4.8), we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma |A|^2\gamma^4\mathrm{d}\mu + \frac{1}{8}\int_\Sigma (|\nabla^2 A|^2\gamma^4 + |A|^2|\nabla A|^2\gamma^4 + |A|^6\gamma^4)\mathrm{d}\mu \le c\epsilon,$$

and the result follows from the integration over $[0, t]$.

Lemma 4.2 *Assume that the flow* (1.3) *smoothly exists on* $[0,T]$ *with* $0<T\le\infty$. *Then for* $\gamma=\widetilde{\gamma}\circ F$ *as in* (3.2), $\phi=\nabla^m A$ *with a positive integer* m *and* $s\ge 2m+4$, *we have*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma|\phi|^2\gamma^s\mathrm{d}\mu+\frac{1}{4}\int_\Sigma|\nabla^2\phi|^2\gamma^s\mathrm{d}\mu\\ &\le c(1+\|A\|^4_{\infty,[\gamma>0]})\int_\Sigma|\phi|^2\gamma^s\mathrm{d}\mu+c(1+\|A\|^4_{\infty,[\gamma>0]})\|A\|^2_{2,[\gamma>0]}.\end{aligned}\tag{4.9}$$

Proof By Lemma 3.2, we only need to prove that

$$\begin{aligned}&\int_\Sigma\Big(\sum_{1\le i\le 3}P_i^{m+2}(A)+\sum_{2\le i\le 5}P_i^m(A)\Big)*\phi\gamma^s\mathrm{d}\mu\\ &\le\frac{1}{2}\int_\Sigma|\nabla^2\phi|\gamma^s\mathrm{d}\mu+c(1+\|A\|^4_{\infty,[\gamma>0]})\int_\Sigma|\phi|^2\gamma^s\mathrm{d}\mu\\ &\quad+c(1+\|A\|^4_{\infty,[\gamma>0]})\|A\|^2_{2,[\gamma>0]}.\end{aligned}\tag{4.10}$$

We recall an inequality in [4] as follows:

$$\Big|\int_\Sigma\nabla^{i_1}\phi*\cdots*\nabla^{i_r}\phi\gamma^s\mathrm{d}\mu\Big|\le c\|\phi\|_\infty^{r-2}\Big(\int_\Sigma|\nabla^k\phi|^2\gamma^s\mathrm{d}\mu+\|\phi\|^2_{2,[\gamma>0]}\Big),\tag{4.11}$$

where $0\le i_1,\cdots,i_r\le k$, $i_1+\cdots+i_r=2k$ and $s\ge 2k$. Using (4.11), we can obtain

$$\begin{aligned}\int_\Sigma\sum_{2\le i\le 5}P_i^m(A)*\phi\gamma^s\mathrm{d}\mu&\le c(\|A\|_{\infty,[\gamma>0]}+\|A\|^2_{\infty,[\gamma>0]}+\|A\|^3_{\infty,[\gamma>0]}+\|A\|^4_{\infty,[\gamma>0]})\\ &\quad\cdot\Big(\int_\Sigma|\phi|^2\gamma^s\mathrm{d}\mu+\|A\|^2_{2,[\gamma>0]}\Big).\end{aligned}$$

By Young's inequality, we have

$$\int_\Sigma\sum_{2\le i\le 5}P_i^m(A)*\phi\gamma^s\mathrm{d}\mu\le c(1+\|A\|^4_{\infty,[\gamma>0]})\cdot\Big(\int_\Sigma|\phi|^2\gamma^s\mathrm{d}\mu+\|A\|^2_{2,[\gamma>0]}\Big).\tag{4.12}$$

Using (4.11) again with $r=4$, $k=m+1$, we have

$$\int_\Sigma P_1^{m+2}(A)*\phi\gamma^s\mathrm{d}\mu\le c\Big(\int_\Sigma|\nabla\phi|^2\gamma^s\mathrm{d}\mu+\|A\|^2_{2,[\gamma>0]}\Big),\tag{4.13}$$

$$\int_\Sigma P_2^{m+2}(A)*\phi\gamma^s\mathrm{d}\mu\le c\|A\|_{\infty,[\gamma>0]}\Big(\int_\Sigma|\nabla\phi|^2\gamma^s\mathrm{d}\mu+\|A\|^2_{2,[\gamma>0]}\Big),\tag{4.14}$$

$$\int_\Sigma P_3^{m+2}(A)*\phi\gamma^s\mathrm{d}\mu\le c\|A\|^2_{\infty,[\gamma>0]}\Big(\int_\Sigma|\nabla\phi|^2\gamma^s\mathrm{d}\mu+\|A\|^2_{2,[\gamma>0]}\Big).\tag{4.15}$$

The above three inequalities imply that

$$\begin{aligned}\int_\Sigma\sum_{1\le i\le 3}P_i^{m+2}(A)\gamma^s\mathrm{d}\mu&\le c(1+\|A\|_{\infty,[\gamma>0]}+\|A\|^2_{\infty,[\gamma>0]})\\ &\quad\cdot\Big(\int_\Sigma|\nabla\phi|^2\gamma^s\mathrm{d}\mu+\|A\|^2_{2,[\gamma>0]}\Big).\end{aligned}\tag{4.16}$$

The following estimate was obtained in [4] with a slight difference:

$$\begin{aligned}
&c(1+\|A\|_{\infty,[\gamma>0]}+\|A\|^2_{\infty,[\gamma>0]})\int_\Sigma |\nabla\phi|^2\gamma^s\mathrm{d}\mu \\
\le{}& c(1+\|A\|^2_{\infty,[\gamma>0]})\Big(\int_\Sigma |\phi|^2\gamma^s\mathrm{d}\mu\Big)^{\frac12}\Big(\int_\Sigma |\nabla^2\phi|^2\gamma^s\mathrm{d}\mu\Big)^{\frac12} \\
&+c(1+\|A\|^2_{\infty,[\gamma>0]})\Big(\int_\Sigma |\phi|^2\gamma^s\mathrm{d}\mu\Big)^{\frac12}\Big(\int_\Sigma |\nabla\phi|^2\gamma^{s-2}\mathrm{d}\mu\Big)^{\frac12} \\
\le{}& \frac14\int_\Sigma |\nabla^2\phi|^2\gamma^s\mathrm{d}\mu + c(1+\|A\|^4_{\infty,[\gamma>0]})\int_\Sigma |\phi|^2\gamma^s\mathrm{d}\mu + c\int_\Sigma |\nabla\phi|^2\gamma^{s-2}\mathrm{d}\mu \\
\le{}& \frac12\int_\Sigma |\nabla^2\phi|^2\gamma^s\mathrm{d}\mu + \int_{[\gamma>0]}|A|^2\mathrm{d}\mu + c(1+\|A\|^4_{\infty,[\gamma>0]})\int_\Sigma |\phi|^2\gamma^s\mathrm{d}\mu.
\end{aligned}\tag{4.17}$$

Now (4.12) and (4.16)–(4.17) imply (4.10).

Using the same method as in [4], by Lemmas 4.1–4.2, we can obtain the following regularity of curvature.

Lemma 4.3 *Assume that the flow* (1.3) *smoothly exists on* $[0,T]$, γ *is as in* (3.2)*. If*

$$\epsilon = \sup_{0\le t\le T}\int_{[\gamma>0]}|A|^2\mathrm{d}\mu \le \epsilon_0$$

for some ϵ_0 *small enough depending on the constants in* (3.2)*, then*

$$\|\nabla^m A\|_{\infty,[\gamma=1]} \le c(m,T,\alpha_0(m+2)),\tag{4.18}$$

where $\alpha_0(m)=\sum\limits_{j=0}^m \|\nabla^j A_0\|_{2,[\gamma_0>0]}$.

The following proof of Theorem 1.2 is almost the same as the result in [4], and we write it here for completion.

Proof of Theorem 1.2 By rescaling, we may assume that $\rho=1$,

$$\epsilon(t)=\sup_{x\in\mathbb{R}^3}\int_{B_1(x)}|A|^2\mathrm{d}\mu.$$

By a trivial covering agument, for some constant $\Gamma>1$, we get

$$\epsilon(t)\le \Gamma\sup_{x\in\mathbb{R}^3}\int_{B_{\frac12}(x)}|A|^2\mathrm{d}\mu.\tag{4.19}$$

Now let $\lambda>0$ be a parameter, and define

$$t_0 := \sup\{0\le t\le \min(T,\Gamma) : \epsilon(\tau)\le 3\Gamma\epsilon \text{ for } 0\le\tau\le t\}.\tag{4.20}$$

By the continuity of $\epsilon(t)$, we have that $t_0>0$ and

$$\epsilon(t_0)=3\Gamma\epsilon,\quad \text{if } t_0<\min(T,\lambda).\tag{4.21}$$

Let $\widetilde{\gamma}\in C^2(\mathbb{R}^3)$ be a cutoff function with $\|\widetilde{\gamma}\|_{C^2(\mathbb{R}^3)}$ and $\chi_{B_{\frac12}}(x)\le\widetilde{\gamma}\le\chi_{B_1}(x)$. Then $\gamma=\widetilde{\gamma}\circ F$ satisfies (3.2). We note that (4.20) implies the condition of Lemma 4.1 on $[0,t_0)$. Therefore,

$$\int_{B_{\frac12}(x)}|A|^2\mathrm{d}\mu \le \int_{B_1(x)}|A_0|^2\mathrm{d}\mu_0 + c\Gamma\epsilon t \le 2\epsilon \quad \text{for } 0\le t\le t_0,$$

if we take $\lambda = (c\Gamma)^{-1}$. From (4.19), we conclude that

$$\epsilon(t) \leq 2\Gamma\epsilon \quad \text{for } 0 \leq t \leq t_0. \tag{4.22}$$

Thus (4.21) implies that $t_0 = \min(T, (c\Gamma)^{-1})$. Now if $t_0 = (c\Gamma)^{-1}$, then (1.5) holds, and (4.22) implies (1.6). Theorem 1.2 is proved. If $t_0 = T$, we will get a contradiction. By (4.22), $T = t_0 \leq (c\Gamma)^{-1}$ and Lemma 4.3, we obtain

$$\|\nabla^m A\|_\infty \leq c(m, F_0).$$

This implies that we can extend the flow to an interval $[0, T + \delta)$ for some $\delta > 0$, which contradicts the maximality of T. Then the theorem is proved.

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