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Parseval Frame Wavelet Multipliers in $L^2(\mathbb{R}^d)$*

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Abstract Let A be a $d \times d$ real expansive matrix. An A-dilation Parseval frame wavelet is a function $\psi \in L^2(\mathbb{R}^d)$, such that the set $\{|\det A|^{\frac{n}{2}}\psi(A^n\mathbf{t}-\ell) : n \in \mathbb{Z}, \ell \in \mathbb{Z}^d\}$ forms a Parseval frame for $L^2(\mathbb{R}^d)$. A measurable function f is called an A-dilation Parseval frame wavelet multiplier if the inverse Fourier transform of $f\widehat{\psi}$ is an A-dilation Parseval frame wavelet whenever ψ is an A-dilation Parseval frame wavelet, where $\widehat{\psi}$ denotes the Fourier transform of ψ. In this paper, the authors completely characterize all A-dilation Parseval frame wavelet multipliers for any integral expansive matrix A with $|\det(A)| = 2$. As an application, the path-connectivity of the set of all A-dilation Parseval frame wavelets with a frame MRA in $L^2(\mathbb{R}^d)$ is discussed.

Keywords Parseval frame wavelet, Wavelet multiplier, Frame multiresolution analysis
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1 Introduction

A countable family $(f_j), j \in J$, in a separable Hilbert space H, is a Parseval frame for H if the equality $\|f\|^2 = \sum\limits_{j\in J} |\langle f, f_j\rangle|^2$ is satisfied for all $f \in H$.

Let A be a $d \times d$ real expansive matrix, i.e., a matrix with real entries whose eigenvalues are all of modules greater than one.

An A-dilation Parseval frame wavelet is a function $\psi \in L^2(\mathbb{R}^d)$, such that the set

$$\{|\det A|^{\frac{n}{2}}\psi(A^n\mathbf{t}-\ell) : n \in \mathbb{Z}, \ell \in \mathbb{Z}^d\}$$

forms a Parseval frame for $L^2(\mathbb{R}^d)$. For any function $f(\mathbf{t}) \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, its Fourier transform is defined by

$$\mathcal{F}(f(\mathbf{t})) = \widehat{f}(\mathbf{s}) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(\mathbf{t})\mathrm{e}^{-\mathrm{i}\mathbf{t}\circ\mathbf{s}}\mathrm{d}\mathbf{t}, \tag{1.1}$$

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where the Fourier transform defined is normalized to be a unitary operator, and $\mathbf{t} \circ \mathbf{s}$ is the standard inner product of the vectors $\mathbf{s}, \mathbf{t} \in \mathbb{R}^d$. The inverse Fourier transform will be denoted by $\mathcal{F}^{-1}$.

We denote by $\mathbb{T}^d$ the d-dimensional torus. For $f, g \in L^2(\mathbb{R}^d)$, we denote by $[f, g]$ the $2\pi\mathbb{Z}^d$ periodic function $[f, g] = \sum_{k \in \mathbb{Z}^d} f(\mathbf{s} + 2\pi k)\overline{g(\mathbf{s} + 2\pi k)}$ a.e. It is easily seen that $[\cdot, \cdot]$ is a sesquilinear form, $[f, g] \in L^1(\mathbb{T}^d)$ for all $f, g \in L^2(\mathbb{R}^d)$, and

$$\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle = \int_{\mathbb{T}^d} [\widehat{f}, \widehat{g}](\mathbf{s})\mathrm{d}\mathbf{s}. \tag{1.2}$$

For $\sigma_f = [\widehat{f}, \widehat{f}]$, let Ω_f be the $2\pi\mathbb{Z}^d$-translation invariant subset of $\mathbb{R}^d$ defined, modulo a null set, by

$$\Omega_f = \operatorname{supp} \sigma_f = \{\mathbf{s} \in \mathbb{R}^d : \widehat{f}(\mathbf{s} + 2\pi k) \neq 0 \text{ for some } k \in \mathbb{Z}^d\}. \tag{1.3}$$

One of the many problems in wavelet theory concerns the construction of wavelets. Naturally, one may attempt to construct new wavelets from an existing one. This approach leads to the concept of wavelet multipliers (see [7]). A measurable function f is called an A-dilation wavelet multiplier if the inverse Fourier transform of $(f\widehat{\psi})$ is an A-dilation wavelet for any A-dilation wavelet ψ. The wavelet multipliers have been studied extensively and completely characterized for the one dimensional case (see [10, 18, 22]). For high dimensional cases, they were studied for any dilation matrices which are expansive matrices with integer entries and the determinant ± 2 (see [15–17]). It is well-known that the frame wavelet is a generalized concept of the (orthonormal) wavelet. So it should have a counterpart theory. Frame wavelet multiplier in the one dimensional case was introduced in [20]. In this paper, we will study frame wavelet multipliers in high dimensional case by using some results of [1]. We will generalize wavelet multiplier results to frame wavelet multipliers in high dimensional cases.

Our study in this paper concerns the case where the dilation matrix A is an expansive matrix with integer entries, such that $|\det(A)| = 2$.

The rest of the paper is organized as follows. In the next section, we introduce the notations and terms needed in this paper, with some preliminary results needed in the later sections. In Section 3, we state and prove our results of frame wavelet multipliers on $L^2(\mathbb{R}^d)$. In Section 4, we discuss the path-connectivity of a subclass of all A-dilation Parseval frame wavelets with FMRA.

2 Notations, Definitions and Preliminary Results

Let $M_d^{(2)}(\mathbb{Z})$ be the set of all $d \times d$ expansive integral matrices (i.e., matrices with integer entries) whose determinants are ± 2. Throughout this paper, we will limit our discussion to matrices $A \in M_d^{(2)}(\mathbb{Z})$. We will use T and D_A as the translation and dilation unitary operators on $L^2(\mathbb{R}^d)$, respectively, which are defined by $(T^\ell f)(\mathbf{t}) = f(\mathbf{t} - \ell)$, $(D_A f)(\mathbf{t}) = |\det(A)|^{\frac{1}{2}} f(A\mathbf{t})$, $\forall f \in L^2(\mathbb{R}^d)$, $\mathbf{t} \in \mathbb{R}^d$ and $\ell \in \mathbb{Z}^d$. We denote $\psi_{j,\mathbf{n}}(\mathbf{s}) = |\det A|^{\frac{j}{2}} \psi(A^j\mathbf{s} - \mathbf{n}), \forall j \in \mathbb{Z}, \mathbf{n} \in \mathbb{Z}^2$, $\psi(\mathbf{s}) \in L^2(\mathbb{R}^d)$. Whenever we state that two functions f, $g \in L^2(\mathbb{R}^d)$ are equal, it is understood that $f(\mathbf{s}) = g(\mathbf{s})$ for almost all $\mathbf{s} \in \mathbb{R}^d$. Furthermore, we say that $E = F$ for two measurable sets E and F in $\mathbb{R}^d$ if $(E \setminus F) \cup (F \setminus E)$ is a measure zero set. A function f with the property $|f| = 1$ is called a unimodular function.

The following definition of the frame MRA was introduced in [2].

Definition 2.1 *A sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}^d)$ is called an A-dilation frame multiresolution analysis (or A-dilation FMRA) if the following conditions hold:*

(i) $V_j \subset V_{j+1},\ \forall j \in \mathbb{Z}$.

(ii) $\bigcap\limits_{j\in\mathbb{Z}} V_j = \{0\},\ \overline{\bigcup\limits_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^d)$.

(iii) $DV_j = V_{j+1}$.

(iv) *There exists $\varphi(\mathbf{t})$ in V_0, such that $\{\phi(\mathbf{t}-\ell) : \ell \in \mathbb{Z}^d\}$ is a Parseval frame for V_0. We say that φ is a frame scaling function for $\{V_j : j \in \mathbb{Z}\}$.*

If $\{V_j : j \in \mathbb{Z}\}$ is an FMRA, we denote by W_j $(j \in Z)$, the orthogonal complement of V_j in V_{j+1}, i.e., $W_j = V_{j+1} \ominus V_j$. Notice that $L^2(\mathbb{R}^d) = \bigoplus\limits_{j\in\mathbb{Z}} W_j$. Suppose that ψ is a function in W_0, such that the family $\{\psi(\mathbf{t}-\ell) :\ \ell \in \mathbb{Z}^d\}$ is a Parseval frame for W_0. Then it is clear that the system $\Psi = \{\psi_{j,k}, j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ is a Parseval frame for $L^2(\mathbb{R}^d)$. Thus ψ is a Parseval frame wavelet. In particular, if $\{\psi(\mathbf{t}-\ell) : \ell \in \mathbb{Z}^d\}$ is an orthonormal basis for W_0, then ψ is an orthonormal wavelet.

The following Lemmas 2.1–2.9 are well-known results and come from [1].

Lemma 2.1 *ψ is an A-dilation Parseval frame wavelet iff the following conditions hold:*

(i) $\sum\limits_{j\in\mathbb{Z}} |\widehat{\psi}((A^\tau)^j\mathbf{s})|^2 = \frac{1}{(2\pi)^d}$.

(ii) $\sum\limits_{j=0}^{\infty} \widehat{\psi}((A^\tau)^j\mathbf{s})\overline{\widehat{\psi}((A^\tau)^j(\mathbf{s}+2\pi\ell))} = 0,\ \forall \ell \in \mathbb{Z}^d \setminus A^\tau\mathbb{Z}^d$.

The following Lemma 2.2 describes some special properties of a matrix in the set $M_d^{(2)}(\mathbb{Z})$. Its proof is elementary, and the readers can also refer to [1].

Lemma 2.2 *Let $A \in M_d^{(2)}(\mathbb{Z})$. Then the group $\mathbb{Z}^d/A^\tau\mathbb{Z}^d$ is isomorphic to $(A^\tau)^{-1}\mathbb{Z}^d/\mathbb{Z}^d$, and the order of $\mathbb{Z}^d/A^\tau\mathbb{Z}^d$ is equal to 2. In particular, for any $\mathbf{h}_1 \in \mathbb{Z}^d \setminus A^\tau\mathbb{Z}^d$, we have that $\mathbb{Z}^d = A^\tau\mathbb{Z}^d \cup (A^\tau\mathbb{Z}^d + \mathbf{h}_1)$ and $(A^\tau)^{-1}\mathbb{Z}^d = \mathbb{Z}^d \cup (\mathbb{Z}^d + \mathbf{h}_2)$, where $\mathbf{h}_2 = (A^\tau)^{-1}\mathbf{h}_1$.*

Remark 2.1 Since $A \in M_d^{(2)}(\mathbb{Z})$, any non-integer entry in $(A^\tau)^{-1}$ is a rational number with the denominator 2 (namely a number of the form $\frac{1}{2}(2r+1)$ with $r \in \mathbb{Z}$). It follows that $\mathbf{h}_2 = (A^\tau)^{-1}\mathbf{h}_1 \notin \mathbb{Z}^d$ has at least one non-integer entry and all non-integer entries are rational numbers with the denominator 2. We will use $\rho(\mathbf{h}_2)$ to denote the index of the first of such non-integer entries in $\mathbf{h}_2$.

Remark 2.2 Notice that for $\mathbf{h}_1, \mathbf{h}_1' \in \mathbb{Z}^d \setminus A^\tau(\mathbb{Z}^d)$ and $\mathbf{h}_2 = (A^\tau)^{-1}\mathbf{h}_1,\ \mathbf{h}_2' = (A^\tau)^{-1}\mathbf{h}_1'$, we have $\rho(\mathbf{h}_2) = \rho(\mathbf{h}_2^p)$ since $\mathbf{h}_2 - \mathbf{h}_2' \in \mathbb{Z}^d$. Thus the index $\rho(\mathbf{h}_2)$ only depends on A. Hence, it is appropriate to denote such an index by $\rho(A)$. Let $\mathbf{u} \in \mathbb{R}^d$ be the vector with all of its entries being zero except its entry at the $\rho(A)$-th coordinate, which is 1. Then $\mathrm{e}^{\pm \mathrm{i}2\pi\mathbf{h}_2\circ\mathbf{u}} = -1$. It is easy to verify that there is a unique element $\mathbf{h}_1 \in \mathbb{Z}^d \setminus A^\tau\mathbb{Z}^d$, such that $(A^\tau)^{-1}\mathbf{h}_1 = \mathbf{h}_2$ is a non-zero vector whose entries are either $\frac{1}{2}$ or 0. In this case, $\mathbf{h}_2 \circ \mathbf{u} = \frac{1}{2}$. From now on, $\mathbf{h}_1$ and $\mathbf{h}_2$ will be understood as two uniquely determined vectors to avoid any confusion.

Lemma 2.3 *Let $B \in M_d^{(2)}(\mathbb{Z})$, $\mathbf{h}_1 \in \mathbb{Z}^d \setminus B\mathbb{Z}^d$ and $\mathbf{h}_2 = B^{-1}\mathbf{h}_1$. Suppose that $\mathbf{s}$ is a $2\pi\mathbb{Z}^d$ periodic, unimodular function on $\mathbb{R}^d$. Then there exists a $2\pi\mathbb{Z}^d$ periodic, unimodular function*

t on $\mathbb{R}^d$, such that

$$t(\mathbf{s}) = t(B\mathbf{s})\overline{t(\mathbf{s}+2\pi\mathbf{h}_2)}\,\mathbf{s}(B\mathbf{s}). \tag{2.1}$$

Lemma 2.4 *A function $\varphi \in L^2(\mathbb{R}^d)$ is a frame scaling function for an FMRA $\{V_j : j \in \mathbb{Z}\}$ iff the following three conditions are satisfied:*

(i) $\widehat{\varphi}(A^\tau \mathbf{s}) = m(\mathbf{s})\widehat{\varphi}(\mathbf{s})$ *for some* $m \in L^2(\mathbb{T}^d)$.

(ii) $\lim\limits_{j\to\infty} |\widehat{\varphi}((A^\tau)^{-j}\mathbf{s})| = \frac{1}{(2\pi)^{\frac{d}{2}}}$.

(iii) $\sum\limits_{k\in\mathbb{Z}^d} |\widehat{\varphi}(\mathbf{s}+2\pi k|^2 = \frac{1}{(2\pi)^d}\chi_{\Omega_\varphi}(\mathbf{s})$.

Let $\{V_j : j \in \mathbb{Z}\}$ be an FMRA with a frame scaling function φ. Define

$$N = \Omega_\varphi \cap (\Omega_\varphi - 2\pi\mathbf{h}_2) \cap ((A^\tau)^{-1}\Omega_\varphi)^C, \tag{2.2}$$

where A^C denotes the complementary set of A.

Lemma 2.5 *Let $\{V_j : j \in \mathbb{Z}\}$ be an FMRA. Then $\{V_j : j \in \mathbb{Z}\}$ admits associated Parseval frame wavelets iff $|N| = 0$, where $|S|$ denotes the Lebesgure measure of S.*

Lemma 2.6 *Let $\{V_j : j \in \mathbb{Z}\}$ be an FMRA with a frame scaling function φ, such that $|N| = 0$. Then there exists a function $m_0 \in L^2(\mathbb{T}^d)$, such that*

$$\widehat{\varphi}(A^\tau \mathbf{s}) = m_0(\mathbf{s})\widehat{\varphi}(\mathbf{s}), \tag{2.3}$$

$$|m_0(\mathbf{s})|^2 + |m_0(\mathbf{s}+2\pi\mathbf{h}_2)|^2 = 1, \tag{2.4}$$

where m_0 is called a canonical low pass filter.

Lemma 2.7 *Let φ be a frame scaling function for an FMRA $\{V_j : j \in \mathbb{Z}\}$. Then*

(i) *There exists a low pass filter m_0 (the canonical low pass filter) with $|m_0(\mathbf{s})|^2 + |m_0(\mathbf{s}+2\pi\mathbf{h}_2)|^2 = 1$.*

(ii) *$\psi \in L^2(\mathbb{R}^d)$ is an FMRA Parseval frame wavelet associated with $\{V_j : j \in \mathbb{Z}\}$ iff*

$$\widehat{\psi}(A^\tau \mathbf{s}) = \mathrm{e}^{\mathrm{i}\mathbf{s}\circ\mathbf{u}} s(A^\tau \mathbf{s})\overline{m_0(\mathbf{s}+2\pi\mathbf{h}_2)}\widehat{\varphi}(\mathbf{s}) \tag{2.5}$$

for some unimodular function $s \in L^2(\mathbb{T}^d)$.

Lemma 2.8 *Let ψ be an FMRA Parseval frame wavelet associated with an FMRA $\{V_j : j \in \mathbb{Z}\}$ whose frame scaling function is φ. Then there exists a frame scaling function φ_ψ for $\{V_j : j \in \mathbb{Z}\}$ and a function $m_0^\psi \in L^2(\mathbb{T}^d)$, such that*

$$\widehat{\psi}(A^\tau \mathbf{s}) = \mathrm{e}^{\mathrm{i}\mathbf{s}\circ\mathbf{u}}\overline{m_0^\psi(\mathbf{s}+2\pi\mathbf{h}_2)}\widehat{\varphi}_\psi(\mathbf{s}). \tag{2.6}$$

Lemma 2.9 *Let $\varphi \in L^2(\mathbb{R}^d)$ be a scaling function of an FMRA $\{V_j : j \in \mathbb{Z}\}$. Suppose that functions $m_0, m_1 \in L^2(\mathbb{T}^d)$ and $\psi \in L^2(\mathbb{R}^d)$, such that*

(i) $\widehat{\varphi}(A^\tau \mathbf{s}) = m_0(\mathbf{s})\widehat{\varphi}(\mathbf{s})$.

(ii) $\widehat{\psi}(A^\tau \mathbf{s}) = m_1(\mathbf{s})\varphi(\mathbf{s})$.

(iii) *The filter matrix $M(\mathbf{s})$ defined by*

$$\begin{pmatrix} m_0(\mathbf{s}) & m_0(\mathbf{s}+2\pi\mathbf{h}_2) \\ m_1(\mathbf{s}) & m_1(s+2\pi\mathbf{h}_2) \end{pmatrix}$$

is unitary.

Then ψ is an FMRA Parseval frame wavelet (or PFW for short) associated with $\{V_j : j \in \mathbb{Z}\}$.

Lemma 2.10 (see [10, Corollary 3.28]) *Let ψ be an MRA PFW. If φ is a corresponding scaling function, then*

$$|\widehat{\varphi}(\mathbf{s})|^2 = \sum_{j=1}^{\infty} |\widehat{\psi}((A^\tau)^j \mathbf{s})|^2, \tag{2.7}$$

$$\|\varphi\| = |\psi\|. \tag{2.8}$$

Proposition 2.1 (see [16, Proposition 2.1]) *Let $\phi \in L^2(\mathbb{R}^d)$ be an orthonormal scaling function for an A-dilation orthonormal MRA $\{V_j\}$, and let m be its associated low pass filter. Let $\psi \in W_0 = V_1 \cap V_0^\perp$. Then $\{\psi(\mathbf{t}-\ell) : \ \ell \in \mathbb{Z}^d\}$ is an orthonormal basis for W_0 iff there exists a $2\pi\mathbb{Z}^d$ periodic, measurable and unimodular function $v : \ \mathbb{R}^d \to \mathbb{C}$, such that*

$$\widehat{\psi}(A^\tau \mathbf{s}) = \mathrm{e}^{\mathrm{i}(\mathbf{s}\circ\mathbf{u})} v(A^\tau \mathbf{s}) \overline{m(\mathbf{s} + 2\pi \mathbf{h}_2)} \widehat{\phi}(\mathbf{s}), \tag{2.9}$$

where $\mathbf{u}$ *is the vector defined in Remark* 2.2.

Proposition 2.2 (see [16, Proposition 2.2]) *Let ψ be an A-dilation orthonormal MRA wavelet. Then $\mathrm{e}^{\mathrm{i}(\mathbf{s}\circ A^{-1}\mathbf{u})}|\widehat{\psi}(\mathbf{s})|$ is the Fourier transform of an A-dilation orthonormal MRA wavelet.*

3 *A*-Dilation Frame Wavelet Multipliers

A measurable function f is called an A-dilation Parseval frame wavelet multiplier (or PFW multiplier for short) if the inverse Fourier transform of $f\widehat{\psi}$ is an A-dilation Parseval frame wavelet whenever ψ is an A-dilation Parseval frame wavelet. In this section, we characterize the A-dilation Parseval frame wavelet multipliers.

Proposition 3.1 *Let $v \in L^\infty(\mathbb{R}^d)$ be an A-dilation PFW multiplier. Then v is unimodular.*

Proof Let ψ_0 be the same as that in [1, Example 5.14] with $\widehat{\psi_0}(\mathbf{s}) \neq 0$. We first show that $|v(\mathbf{s})| \leq 1$. Since

$$\{\mathbf{s} \in \mathbb{R}^d : |v(\mathbf{s})| > 1\} = \bigcup_{n=1}^{\infty} F_n, \quad F_n = \Big\{\mathbf{s} \in \mathbb{R}^d : |v(\mathbf{s})| \geq 1 + \frac{1}{n}\Big\},$$

it suffices to show that $|F_n| = 0$. By the assumption, for any fixed $n \in \mathbb{N}$, there exists an $\varepsilon > 0$, such that

$$|\{\mathbf{s} \in \mathbb{R}^d, |\widehat{\psi_0}(\mathbf{s})| > \varepsilon\} \cap F_n| \geq \frac{1}{2}|F_n|. \tag{3.1}$$

Take $N \in \mathbb{N}$, such that $\varepsilon(1 + \frac{1}{n})^N > 1$. Then

$$|(v(\mathbf{s}))^N \cdot \widehat{\psi_0}(\mathbf{s})| > 1$$

for $\mathbf{s} \in \{\mathbf{s} \in \mathbb{R}^d : |\widehat{\psi_0}(\mathbf{s})| > \varepsilon\} \cap F_n$. Since $(v(\mathbf{s}))^N \cdot \widehat{\psi_0}(\mathbf{s})$ is the Fourier translation of an A-dilation PFW, $|(v(\mathbf{s}))^N \cdot \widehat{\psi_0}(\mathbf{s})| \leq 1$ a.e. on $\mathbb{R}^d$ (since $\|(v(\mathbf{s}))^N \cdot \widehat{\psi_0}(\mathbf{s})\| \leq 1$). Thus $|\{\mathbf{s} \in \mathbb{R}^d : \widehat{\psi_0}(\mathbf{s})| > \varepsilon\} \cap F_n| = 0$. This together with (3.1) yields $|F_n| = 0$. Therefore, $|v(\mathbf{s})| \leq 1$ a.e. on $\mathbb{R}^d$. Now we show that $|v(\mathbf{s}) = 1|$ a.e.

Since $\psi_0(\mathbf{s})$ and $v(\mathbf{s})\psi_0(\mathbf{s})$ are Parseval frame wavelets, by Lemma 2.1,

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi_0}((A^\tau)^j\mathbf{s})|^2(1-|v((A^\tau)^j)\mathbf{s})|=0.$$

So we have $|v(\mathbf{s})|=1$.

Theorem 3.1 *A unimodular function $v\in L^\infty(\mathbb{R}^d)$ is an A-dilation Parseval frame wavelet multiplier iff the function $k(\mathbf{s})=v(A^\tau\mathbf{s})/v(\mathbf{s})$ is $2\pi\mathbb{Z}^d$ periodic.*

Proof "$\Longleftarrow$" Assume that $v\in L^\infty(\mathbb{R}^d)$ is a unimodular function, and $k(\mathbf{s})=v(A^\tau\mathbf{s})/v(\mathbf{s})$ is $2\pi\mathbb{Z}^d$ periodic. To show that v is a wavelet multiplier, we need to show that for any A-dilation Parseval frame wavelet ψ, $\eta=\mathcal{F}^{-1}(v\widehat{\psi})$ is also a Parseval frame wavelet. It suffices to verify that $\widehat{\eta}$ satisfies conditions (i)–(ii) in Lemma 2.1. It is obvious that Lemma 2.1(i) holds since v is unimodular. We show that Lemma 2.1(ii) holds. By the assumption, $k(\mathbf{s})$ is $2\pi\mathbb{Z}^d$ periodic and unimodular.

Applying the relation $v(A^\tau\mathbf{s})=k(\mathbf{s})v(\mathbf{s})$ repeatedly, for any $j\geq 1$ and $\ell\in\mathbb{Z}^d$, we obtain

$$v((A^\tau)^j\mathbf{s})=k((A^\tau)^{j-1}\mathbf{s})\cdots k(A^\tau\mathbf{s})k(\mathbf{s})v(\mathbf{s}) \tag{3.2}$$

and

$$\begin{aligned}v((A^\tau)^j(\mathbf{s}+2\pi\ell))&=k((A^\tau)^{j-1}(\mathbf{s}+2\pi\ell))k((A^\tau)^{j-2}(\mathbf{s}+2\pi\ell))\\&\quad\cdots k(A^\tau(\mathbf{s}+2\pi\ell))k(\mathbf{s}+2\pi\ell)v(\mathbf{s}+2\pi\ell)\\&=k((A^\tau)^{j-1}\mathbf{s})\cdots k(A^\tau\mathbf{s})k(\mathbf{s})v(\mathbf{s}+2\pi\ell).\end{aligned}$$

Since $k(\mathbf{s})$ is unimodular, this leads to

$$\begin{aligned}&v((A^\tau)^j\mathbf{s})\cdot\overline{v((A^\tau)^j(\mathbf{s}+2\pi\ell))}\\&=k((A^\tau)^{j-1}\mathbf{s})\cdots k(A^\tau\mathbf{s})k(\mathbf{s})v(\mathbf{s})\cdot\overline{k((A^\tau)^{j-1}\mathbf{s})\cdots k(A^\tau\mathbf{s})k(\mathbf{s})v(\mathbf{s}+2\pi\ell)}\\&=v(\mathbf{s})\overline{v(\mathbf{s}+2\pi\ell)}\end{aligned}$$

for any $j\geq 0$ and $\ell\in\mathbb{Z}^d$. Thus

$$\begin{aligned}&\sum_{j=0}^{\infty}\widehat{\eta}((A^\tau)^j\mathbf{s})\overline{\widehat{\eta}((A^\tau)^j(\mathbf{s}+2\pi\ell))}\\&=v(\mathbf{s})\overline{v(\mathbf{s}+2\pi\ell)}\sum_{j=0}^{\infty}\widehat{\psi}((A^\tau)^j\mathbf{s})\overline{\widehat{\psi}((A^\tau)^j(\mathbf{s}+2\pi\ell))}\\&=0\end{aligned}$$

for any $\ell\in\mathbb{Z}^d\backslash A^\tau\mathbb{Z}^d$. So the condition (ii) of Lemma 2.1 holds for $\widehat{\eta}$ as well.

"$\Longrightarrow$" We need to show that $k(\mathbf{s})=f(A^\tau\mathbf{s})/f(\mathbf{s})$ is $2\pi\mathbb{Z}^d$ periodic. Let ψ be any A-dilation orthonormal MRA wavelet, such that $\mathrm{supp}(\widehat{\psi})=\mathbb{R}^d$ (the existence of such ψ is proved in [1, Example 5.14]). By Proposition 2.2, the function $\psi_1(t)$ defined by

$$\widehat{\psi}_1=\mathrm{e}^{\mathrm{i}(A^\tau)^{-1}\mathbf{s}\circ\mathbf{u}}|\widehat{\psi}(\mathbf{s})|=\mathrm{e}^{\mathbf{i}\mathbf{s}\circ A^{-1}\mathbf{u}}|\widehat{\psi}_1(\mathbf{s})| \tag{3.3}$$

is an A-dilation wavelet, which is also an A-dilation Parseval frame wavelet. Since $\mathcal{F}^{-1}(v\widehat{\psi}_1)$ is also an A-dilation Parseval frame wavelet, both $\widehat{\psi}_1$ and $v\widehat{\psi}_1$ satisfy the condition (ii) of Lemma 2.1, i.e.,

$$\sum_{j=0}^{\infty} \widehat{\psi}_1((A^\tau)^j\mathbf{s}) \cdot \overline{\widehat{\psi}_1((A^\tau)^j(\mathbf{s}+2\pi\ell))} = 0, \tag{3.4}$$

$$\sum_{j=0}^{\infty} v((A^\tau)^j\mathbf{s})\widehat{\psi}_1((A^\tau)^j\mathbf{s}) \cdot \overline{v((A^\tau)^j(\mathbf{s}+2\pi\ell))}\,\overline{\widehat{\psi}_1((A^\tau)^j(\mathbf{s}+2\pi\ell))} = 0 \tag{3.5}$$

for any $\ell \in \mathbb{Z}^d \setminus A^\tau\mathbb{Z}^d$. Since $\ell \in \mathbb{Z}^d \setminus A^\tau\mathbb{Z}^d$, there exists an $\ell_1 \in \mathbb{Z}^d$, such that $\ell = A^\tau\ell_1 + \mathbf{h}_1 = A^\tau(\ell_1 + \mathbf{h}_2)$ by Lemma 2.2. Thus

$$\begin{aligned}
\widehat{\psi}_1(\mathbf{s})\overline{\widehat{\psi}_1(\mathbf{s}+2\pi\ell)} &= \mathrm{e}^{\mathrm{i}\mathbf{s}\circ A^{-1}\mathbf{u}}|\widehat{\psi}_1(\mathbf{s})| \cdot \mathrm{e}^{-\mathrm{i}(\mathbf{s}+2\pi\ell)\circ A^{-1}\mathbf{u}}|\widehat{\psi}_1(\mathbf{s}+2\pi\ell)| \\
&= \mathrm{e}^{-\mathrm{i}2\pi(\ell_1+\mathbf{h}_2)\circ\mathbf{u}}|\widehat{\psi}_1(\mathbf{s})| \cdot |\widehat{\psi}_1(\mathbf{s}+2\pi\ell)| \\
&= \mathrm{e}^{-\mathrm{i}2\pi\mathbf{h}_2\circ\mathbf{u}}|\widehat{\psi}_1(\mathbf{s})| \cdot |\widehat{\psi}_1(\mathbf{s}+2\pi\ell)| \\
&= -|\widehat{\psi}_1(\mathbf{s})| \cdot |\widehat{\psi}_1(\mathbf{s}+2\pi\ell)|,
\end{aligned}$$

since $\ell_1 \circ \mathbf{u}$ is an integer and $\mathrm{e}^{-\mathrm{i}2\pi\mathbf{h}_2\circ\mathbf{u}} = -1$ (see Remark 2.2). On the other hand, for any $j > 0$,

$$\begin{aligned}
&\widehat{\psi}_1((A^\tau)^j\mathbf{s})\overline{\widehat{\psi}_1((A^\tau)^j(\mathbf{s}+2\pi\ell))} \\
&= \mathrm{e}^{\mathrm{i}(A^\tau)^j\mathbf{s}\circ A^{-1}\mathbf{u}}|\widehat{\psi}_1((A^\tau)^j\mathbf{s})| \cdot \mathrm{e}^{-\mathrm{i}(A^\tau)^j(\mathbf{s}+2\pi\ell)\circ A^{-1}\mathbf{u}}|\widehat{\psi}_1((A^\tau)^j(\mathbf{s}+2\pi\ell))| \\
&= |\widehat{\psi}_1((A^\tau)^j\mathbf{s})| \cdot |\widehat{\psi}_1((A^\tau)^j(\mathbf{s}+2\pi\ell))|.
\end{aligned}$$

Thus, (3.4)–(3.5) can be rewritten as

$$\begin{aligned}
&|\widehat{\psi}_1(\mathbf{s})| \cdot |\widehat{\psi}_1(\mathbf{s}+2\pi\ell)| \\
&= \sum_{j=1}^{\infty} |\widehat{\psi}_1((A^\tau)^j\mathbf{s})| \cdot |\widehat{\psi}_1((A^\tau)^j(\mathbf{s}+2\pi\ell))|,
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
&v(\mathbf{s})\overline{v(\mathbf{s}+2\pi\ell)} \cdot |\widehat{\psi}_1(\mathbf{s})| \cdot |\widehat{\psi}_1(\mathbf{s}+2\pi\ell)| \\
&= \sum_{j=1}^{\infty} v((A^\tau)^j\mathbf{s})\overline{v((A^\tau)^j(\mathbf{s}+2\pi\ell))}|\widehat{\psi}_1((A^\tau)^j\mathbf{s})| \cdot |\widehat{\psi}_1((A^\tau)^j(\mathbf{s}+2\pi\ell))|.
\end{aligned} \tag{3.7}$$

Since v is unimodular, $\overline{v} = \frac{1}{v}$. Hence, (3.7) can be rewritten as

$$\begin{aligned}
&\frac{v(\mathbf{s})}{v(\mathbf{s}+2\pi\ell)}|\widehat{\psi}_1(\mathbf{s})| \cdot |\widehat{\psi}_1(\mathbf{s}+2\pi\ell)| \\
&= \sum_{j=1}^{\infty} \frac{v((A^\tau)^j\mathbf{s})}{v((A^\tau)^j(\mathbf{s}+2\pi\ell))}|\widehat{\psi}_1((A^\tau)^j\mathbf{s})| \cdot |\widehat{\psi}_1((A^\tau)^j(\mathbf{s}+2\pi\ell))|.
\end{aligned} \tag{3.8}$$

Combining this with (3.6), we have

$$\begin{aligned}
&\sum_{j=1}^{\infty} |\widehat{\psi}_1((A^\tau)^j\mathbf{s})| \cdot |\widehat{\psi}_1((A^\tau)^j(\mathbf{s}+2\pi\ell))| \\
&= \sum_{j=1}^{\infty} \frac{v(\mathbf{s}+2\pi\ell)}{v(\mathbf{s})}\frac{v((A^\tau)^j\mathbf{s})}{v((A^\tau)^j(\mathbf{s}+2\pi\ell))}|\widehat{\psi}_1((A^\tau)^j\mathbf{s})| \cdot |\widehat{\psi}_1((A^\tau)^j(\mathbf{s}+2\pi\ell))|.
\end{aligned} \tag{3.9}$$

Finally, since $|\widehat{\psi}_1((A^\tau)^j\mathbf{s})|\cdot|\widehat{\psi}_1((A^\tau)^j(\mathbf{s}+2\pi\ell))|>0$, by the choice of ψ_1 and

$$\Big|\frac{v(\mathbf{s}+2\pi\ell)}{v(\mathbf{s})}\frac{v((A^\tau)^j\mathbf{s})}{v((A^\tau)^j(\mathbf{s}+2\pi\ell))}\Big|=1,$$

it follows that

$$\frac{v(\mathbf{s}+2\pi\ell)}{v(\mathbf{s})}\frac{v((A^\tau)^j\mathbf{s})}{v((A^\tau)^j(\mathbf{s}+2\pi\ell))}=1.$$

In particular,

$$k(\mathbf{s})=\frac{v(A^\tau\mathbf{s})}{v(\mathbf{s})}=\frac{v(A^\tau(\mathbf{s}+2\pi\ell))}{v(\mathbf{s}+2\pi\ell)}=k(\mathbf{s}+2\pi\ell),\quad\forall\ell\in\mathbb{Z}^d\setminus A^\tau\mathbb{Z}^d.$$

If $\ell\in A^\tau\mathbb{Z}^d$, then $\ell-\mathbf{h}_1\notin A^\tau\mathbb{Z}^d$, and

$$k(\mathbf{s}+2\pi\ell)=k(\mathbf{s}+2\pi\mathbf{h}_1+2\pi(\ell-\mathbf{h}_1))=k(\mathbf{s}+2\pi\mathbf{h}_1)=k(\mathbf{s})$$

as well. Therefore, $k(\mathbf{s})=\frac{v(A^\tau\mathbf{s})}{v(\mathbf{s})}$ is $2\pi\mathbb{Z}^d$ periodic.

Theorem 3.2 *A unimodular function v is an A-dilation FMRA PFW multiplier iff $k(\mathbf{s})=\frac{v(A^\tau\mathbf{s})}{v(\mathbf{s})}$ is $2\pi\mathbb{Z}^d$ periodic.*

Proof By Theorem 3.1, we only prove the sufficiency. By Lemma 2.3, there exists a unimodular $2\pi\mathbb{Z}^d$ periodic function t, such that

$$k(\mathbf{s})=\overline{t(\mathbf{s})}t((A^\tau)^{-1}\mathbf{s}+2\pi\mathbf{h}_2)t((A^\tau)^{-1}\mathbf{s}).\tag{3.10}$$

Let $\mu(\mathbf{s})=v(\mathbf{s})t((A^\tau)^{-1}\mathbf{s})t((A^\tau)^{-1}\mathbf{s}+2\pi\mathbf{h}_2)$. Then μ is unimodular, and

$$\begin{aligned}\mu(A^\tau\mathbf{s})\overline{\mu(\mathbf{s})}&=v(A^\tau\mathbf{s})\cdot t(\mathbf{s})t(\mathbf{s}+2\pi\mathbf{h}_2)\cdot\overline{v(\mathbf{s})}\cdot\overline{t((A^\tau)^{-1}\mathbf{s})}\cdot\overline{t((A^\tau)^{-1}\mathbf{s}+2\pi\mathbf{h}_2)}\\&=k(\mathbf{s})t(\mathbf{s})t(\mathbf{s}+2\pi\mathbf{h}_2)\overline{t((A^\tau)^{-1}\mathbf{s})}\cdot\overline{t((A^\tau)^{-1}\mathbf{s}+2\pi\mathbf{h}_2)}\\&=t(\mathbf{s}+2\pi\mathbf{h}_2)\end{aligned}$$

is a $2\pi\mathbb{Z}^d$ periodic function.

Let ψ be an MRA PFW, φ be its scaling function, and m_0 be its canonical low pass filter. Then by Lemma 2.7, we have

$$\widehat{\psi}(A^\tau\mathbf{s})=\mathrm{e}^{\mathrm{i}\mathbf{s}\circ\mathbf{u}}s(A^\tau\mathbf{s})\overline{m_0(\mathbf{s}+2\pi\mathbf{h}_2)}\widehat{\varphi}(\mathbf{s})\tag{3.11}$$

for some unimodular function $s\in L^2(\mathbb{T}^d)$.

Let $\widetilde{m}(\mathbf{s})=m_0(\mathbf{s})t(\mathbf{s}+2\pi\mathbf{h}_2)$ and $\widehat{\widetilde{\varphi}}(\mathbf{s})=\widehat{\varphi}(\mathbf{s})\mu(\mathbf{s})$. Then

$$\widehat{\widetilde{\varphi}}(A^\tau\mathbf{s})=\widehat{\widetilde{\varphi}}(\mathbf{s})\widetilde{m}(\mathbf{s}).$$

By Lemma 2.4, $\widehat{\widetilde{\varphi}}(\mathbf{s})$ is a scaling function.

Let $\widehat{\widetilde{\psi}}(\mathbf{s})=v(\mathbf{s})\widehat{\psi}(\mathbf{s})$. Since ψ is a PFW, we can use Theorem 3.1 to deduce that $\widetilde{\psi}$ is a PFW. Since

$$\begin{aligned}&\mathrm{e}^{\mathrm{i}\mathbf{s}\circ\mathbf{u}}\cdot\overline{\widetilde{m}(\mathbf{s}+2\pi\mathbf{h}_2)}\cdot\widehat{\widetilde{\varphi}}(\mathbf{s})s(A^\tau\mathbf{s})\\&=\mathrm{e}^{\mathrm{i}\mathbf{s}\circ\mathbf{u}}\cdot\overline{m_0(\mathbf{s}+2\pi\mathbf{h}_2)}\cdot\overline{t(\mathbf{s})}\cdot\widehat{\varphi}(\mathbf{s})\mu(\mathbf{s})\cdot s(A^\tau\mathbf{s})\\&=\widehat{\psi}(A^\tau\mathbf{s})\cdot\overline{t(\mathbf{s})}\mu(A^\tau\mathbf{s})\overline{t(\mathbf{s}+2\pi\mathbf{h}_2)}\\&=\widehat{\psi}(A^\tau\mathbf{s})\cdot v(A^\tau\mathbf{s})\\&=\widehat{\widetilde{\psi}}(A^\tau\mathbf{s}),\end{aligned}$$

that is, $\widehat{\widetilde{\psi}}(A^\tau \mathbf{s}) = m_1(\mathbf{s})\widehat{\widetilde{\varphi}}(\mathbf{s})$, where $m_1(\mathbf{s}) = \mathrm{e}^{\mathrm{i}\mathbf{s}\circ\mathbf{u}} \cdot \overline{\widetilde{m}(\mathbf{s}+2\pi\mathbf{h}_2)}\, \mathbf{s}(A^\tau \mathbf{s})$, by computing, matrix

$$\begin{pmatrix} \widetilde{m}(\mathbf{s}) & \widetilde{m}(\mathbf{s}+2\pi\mathbf{h}_2) \\ m_1(\mathbf{s}) & m_1(\mathbf{s}+2\pi\mathbf{h}_2) \end{pmatrix}$$

is unitary. By Lemma 2.9, $\widetilde{\psi}$ is an FMRA PFW, so v is an MRA PFW multiplier.

4 Path-Connectivity of the Set of A-Dilation FMRA Parseval Frame Wavelets

In this section, we discuss the path connectivity of the set of all A-dilation FMRA Parseval frame wavelets. For more discussions and related results on this topic, interested readers may refer to [18, 21–22].

Suppose that ψ_0 is an FMRA PFW and φ_0 is a scaling function associated with ψ_0 (in the sense that φ_0 and ψ_0 satisfy (2.6)–(2.7)). Although such φ_0 is not uniquely determined by ψ_0, we know by (2.7) that $|\widehat{\varphi}_0|$ is unique.

In the following definitions, let φ denote a scaling function associated with an FMRA PFW ψ by (2.6), and v denote an FMRA PFW multiplier. Define

$$W_{\psi_0}^{\mathrm{PF}} = \{\psi : |\widehat{\psi}(\mathbf{s})| = |\widehat{\psi}_0(\mathbf{s})|\}, \tag{4.1}$$

$$S_{\psi_0}^{\mathrm{PF}} = \{\psi : |\widehat{\varphi}(\mathbf{s})| = |\widehat{\varphi}_0(\mathbf{s})|\}, \tag{4.2}$$

$$M_{\psi_0}^{\mathrm{PF}} = \{\psi : \text{There exists } a\ v, \text{ such that } \widehat{\psi}(\mathbf{s}) = v(\mathbf{s})\widehat{\psi}_0(\mathbf{s})\}. \tag{4.3}$$

Theorem 4.1 *If ψ_0 is an FMRA PFW, then*

$$W_{\psi_0}^{\mathrm{PF}} = S_{\psi_0}^{\mathrm{PF}} = M_{\psi_0}^{\mathrm{PF}}.$$

Proof Notice that (2.7) immediately implies $W_{\psi_0}^{\mathrm{PF}} \subseteq S_{\psi_0}^{\mathrm{PF}}$ and

$$|\widehat{\psi}(\mathbf{s})|^2 = |\widehat{\varphi}((A^\tau)^{-1}\mathbf{s})|^2 - |\widehat{\varphi}(\mathbf{s})|^2. \tag{4.4}$$

Obviously, (4.4) implies $S_{\psi_0}^{\mathrm{PF}} \subseteq W_{\psi_0}^{\mathrm{PF}}$.

By Theorem 3.3, we know an MRA PFW multiplier is unimodular. Thus $M_{\psi_0}^{\mathrm{PF}} \subseteq W_{\psi_0}^{\mathrm{PF}}$. It remains to prove $S_{\psi_0}^{\mathrm{PF}} \subseteq M_{\psi_0}^{\mathrm{PF}}$.

Suppose that $\psi_1 \in S_{\psi_0}^{\mathrm{PF}}$. By (2.6), there exists a scaling function φ_1 and a low pass filter m_1, such that $|\widehat{\varphi}_0(\mathbf{s})| = |\widehat{\varphi}_1(\mathbf{s})|$ a.e. and

$$\widehat{\psi}_0(A^\tau \mathbf{s}) = \mathrm{e}^{\mathrm{i}\mathbf{s}\circ\mathbf{u}}\overline{m_0(\mathbf{s}+2\pi\mathbf{h}_2)}\widehat{\varphi}_0(\mathbf{s}),$$
$$\widehat{\psi}_1(A^\tau \mathbf{s}) = \mathrm{e}^{\mathrm{i}\mathbf{s}\circ\mathbf{u}}\overline{m_1(\mathbf{s}+2\pi\mathbf{h}_2)}\widehat{\varphi}_1(\mathbf{s}).$$

In particular, since $W_{\psi_0}^{\mathrm{PF}} = S_{\psi_0}^{\mathrm{PF}}$, $|\widehat{\psi}_0(\mathbf{s})| = |\widehat{\psi}_1(\mathbf{s})|$ a.e. Therefore, it makes sense to define $\widetilde{\psi} \in L^2(\mathbb{R}^d)$ by

$$|\widehat{\widetilde{\psi}}(\mathbf{s})| = \mathrm{e}^{\mathrm{i}((A^\tau)^{-1}\mathbf{s})\circ\mathbf{u}}|\widehat{\psi}_j(\mathbf{s})|, \quad j = 0, 1. \tag{4.5}$$

$|m_j|$ and $\mathcal{F}^{-1}(|\widehat{\varphi}_i|)$ are also low pass filters and scaling functions. By Lemma 2.9, we know that $\widetilde{\psi}$ is an FMRA PFW, and it is enough to show that there exist MRA PFW multipliers v_j, such that $\widehat{\psi}_j = v_j\widehat{\widetilde{\psi}}$ $(j = 0, 1)$. Without loss of generality, we shall consider the case $j = 1$.

Toward this end, let

$$F = \{\mathbf{s} \in \mathbb{R}^d : \widehat{\varphi}_1((A^\tau)^{\ell+1}\mathbf{s}) = m_1((A^\tau)^\ell \mathbf{s})\widehat{\varphi}_1((A^\tau)^\ell \mathbf{s}) \text{ for all } \ell \in \mathbb{Z}\}. \tag{4.6}$$

It is clear that

$$|\mathbb{R}^d \setminus F| = 0. \tag{4.7}$$

Also let $E = \{\mathbf{s} \in F : \widehat{\varphi}_1(\mathbf{s}) \neq 0\}$. Then we have $E \subset A^\tau E$. Consequently, for $n = 0, 1, \cdots,$

$$(A^\tau)^n E \subset (A^\tau)^{n+1} E. \tag{4.8}$$

It follows that if we define $\triangle_0 = E,\ \triangle_n = (A^\tau)^n E \backslash (A^\tau)^{n-1} E$ for $n \geq 1$, then $\triangle_m \cap \triangle_n = \varnothing$ for $m \neq n$. We claim that

$$\Big| F \Big\backslash \bigcup_{n\geq 0} (A^\tau)^n E \Big| = 0. \tag{4.9}$$

If we accept (4.9), the rest of the proof follows the proof in [18]. Indeed, since

$$\mathbb{R}^d \Big\backslash \bigcup_{n\geq 0} (A^\tau)^n E = \Big\{ (\mathbb{R}^d \setminus F) \Big\backslash \bigcup_{n\geq 0} (A^\tau)^n E \Big\} \bigcup \Big\{ F \Big\backslash \bigcup_{n\geq 0} (A^\tau)^n E \Big\},$$

(4.7) and (4.9) imply

$$\Big| \mathbb{R}^d \Big\backslash \bigcup_{n\geq 0} (A^\tau)^n E \Big| = 0.$$

Thus, it suffices to define v_1 on the disjoint union $\bigcup\limits_{n\geq 0} \triangle_n = \bigcup\limits_{n\geq 0} (A^\tau)^n E$, which has full measure. For this purpose, we first define the function μ, such that $\mu(\mathbf{s})|m_1(\mathbf{s})| = m_1(\mathbf{s})$ if $m_1(\mathbf{s}) \neq 0$, and $\mu(\mathbf{s}) = 1$ if $m_1(\mathbf{s}) = 0$ (notice that μ is unimodular and $2\pi\mathbb{Z}^d$ periodic). Then we define the function $t(\mathbf{s})$ inductively on $\bigcup\limits_{n\geq 0} \triangle_n$ such that

$$t(\mathbf{s}) = \frac{|\widehat{\varphi}_1(\mathbf{s})|}{\widehat{\varphi}_1(\mathbf{s})}, \quad \mathbf{s} \in \triangle_0$$

and

$$t(\mathbf{s}) = \overline{\mu((A^\tau)^{-1}\mathbf{s})}\, t((A^\tau)^{-1}\mathbf{s}), \quad \mathbf{s} \in \triangle_n.$$

We define v_1 by

$$v_1(\mathbf{s}) = \overline{\mu((A^\tau)^{-1}\mathbf{s} + 2\pi\mathbf{h}_2)} \cdot \overline{t((A^\tau)^{-1}\mathbf{s})}. \tag{4.10}$$

It follows that v_1 is unimodular and $v_1(A^\tau\mathbf{s})\overline{v_1(\mathbf{s})} = \overline{\mu(\mathbf{s} + 2\pi\mathbf{h}_2)}\mu((A^\tau)^{-1}\mathbf{s} + 2\pi\mathbf{h}_2) \cdot \mu((A^\tau)^{-1}\mathbf{s})$ is $2\pi\mathbb{Z}^d$-periodic, i.e., v_1 is an MRA PFW multiplier and $\widehat{\psi}_1 = v_1\widehat{\widehat{\psi}}$ a.e.

Therefore, it remains to prove (4.9). Suppose that K is a measurable subset of $F \Big\backslash \bigcup\limits_{n\geq 0} (A^\tau)^n E$ $= \bigcap\limits_{n\geq 0} (F \setminus (A^\tau)^n E)$. If $\mathbf{s} \in K$, then $\mathbf{s} \in F = (A^\tau)^n F$ for all $n \in \mathbb{Z}$. Hence $((A^\tau)^{-n}\mathbf{s}) \in F$ for all $n \in \mathbb{Z}$; moreover, $((A^\tau)^{-n}\mathbf{s}) \notin E$ for all $n \geq 0$. It follows that $\widehat{\varphi}_1((A^\tau)^{-n}\mathbf{s}) = 0$ for all $n \geq 0$. We conclude that for all $n \geq 0$,

$$\chi_K(\mathbf{s})\widehat{\varphi}_1((A^\tau)^{-n}\mathbf{s}) = 0. \tag{4.11}$$

Using (4.4) for $\psi_1(\mathbf{s})$ and φ_1, we conclude that for all $j \leq 0,\ j \in \mathbb{Z}$,

$$\chi_K(\mathbf{s}) \mid \widehat{\psi}_1((A^\tau)^j \mathbf{s}) = 0. \tag{4.12}$$

Since ψ_1 is a PFW, it satisfies Lemma 2.1(i). Thus by (2.7), we obtain

$$\frac{1}{(2\pi)^d} = \sum_{j\in\mathbb{Z}} |\widehat{\psi}(A^\tau)^j \mathbf{s})|^2 = |\widehat{\varphi_1}(\mathbf{s})|^2 + \sum_{j\leq 0} |\widehat{\psi}_1((A^\tau)^j \mathbf{s})|^2 \quad \text{a.e.} \tag{4.13}$$

Observe that (4.11) for $n = 0$, and (4.12)–(4.13) imply that $\chi_K(\mathbf{s}) \cdot \frac{1}{(2\pi)^d} = 0$ a.e., i.e., $|K| = 0$. This proves (4.9) and completes the proof of Theorem 4.1.

Theorem 4.2 *If ψ_0 is a PFW, then $M_{\psi_0}^{\mathrm{PF}}$ is path-connected in $L^2(\mathbb{R}^d)$.*

Proof Observe that the class $M_{\psi_0}^{\mathrm{PF}}$ defined by (4.3) is well-defined for an arbitrary PFW ψ_0. Since the properties of wavelet multipliers are exactly the same as the properties of PFW multipliers in [16, Theorem 3.1], using similar arguments in the proof of Theorem 1.3 in [15], we complete the proof.

Using Theorem 4.1, we obtain the following proposition in the FMRA case.

Proposition 4.1 *If ψ_0 is an FMRA PFW, then $W_{\psi_0}^{\mathrm{PF}}$ is path-connected in $L^2(\mathbb{R}^d)$.*

Remark 4.1 One natural question is whether all A-dilation Parseval frame wavelets with FMRA is path-connected. By Proposition 4.1, the question is reduced to answer the following question.

Suppose that ψ_0 and ψ_1 are FMRA PFWs with scaling functions φ_0 and φ_1, whose $\widehat{\varphi}_0$ and $\widehat{\varphi}_1$ are nonnegative, respectively. Can we connect ψ_0 and ψ_1 with a continuous path in $L^2(\mathbb{R}^d)$ with the class of FMRA PFWS? This question was answered positively for orthonormal MRA wavelets ψ_0 and ψ_1 in [16]. But the argument does not work for the frame MRA PFWs case. Therefore, the above question remains open and would require a different method.

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