Chin. Ann. Math.
34B(1), 2013, 1–28
DOI: 10.1007/s11401-012-0760-x

Chinese Annals of Mathematics, Series B

Finite Volume Multilevel Approximation of the Shallow Water Equations*

Arthur BOUSQUET[1] Martine MARION[2] Roger TEMAM[1]

(*In honor of the scientific heritage of Jacques-Louis Lions*)

Abstract The authors consider a simple transport equation in one-dimensional space and the linearized shallow water equations in two-dimensional space, and describe and implement a multilevel finite-volume discretization in the context of the utilization of the incremental unknowns. The numerical stability of the method is proved in both cases.

Keywords Finite-volume methods, Multilevel methods, Shallow water equations, Stability analysis
2000 MR Subject Classification 65M60, 65N21, 65N99

1 Introduction

This article is closely related to and complements the article (see [1]), in which the authors implemented multilevel finite-volume discretizations of the shallow water equations in two-dimensional space, as a model for geophysical flows. The geophysical context is presented in [1] as well as practical issues concerning the implementation. In this article, we recall the motivation, present the algorithm, and discuss the numerical analysis of some variations of the algorithm, and in particular the stability in time.

The shallow water equations are a simplified model of the primitive equations (or PEs for short) of the atmosphere and the oceans. As shown in [20, 24], in rectangular geometry, the PEs can be expanded by using a certain vertical modal decomposition. With such a decomposition, we obtain an infinite system of coupled equations, which resemble the shallow water equations. See [6–7] for the actual numerical resolution of these coupled systems. However, it appears in these articles that the problems to be solved are very difficult (demanding), and performable numerical methods are needed to tackle more and more realistic problems. We turned to multilevel finite-volume methods in [1], finite-volume methods are desirable for the treatment of

Manuscript received July 31, 2012.
[1]The Institute for Scientific Computing and Applied Mathematics, Indiana University, Bloomington, IN 47405 USA. E-mail: arthbous@indiana.edu temam@indiana.edu
[2]The Institute for Scientific Computing and Applied Mathematics, Indiana University, Bloomington, IN 47405 USA. Département Mathématique Informatique, Université de Lyon, Ecole Centrale de Lyon, CNRS UMR 5208, 36 avenue Guy de Collongue, 69134 Ecully Cedex, France.
E-mail: Martine.Marion@ec-lyon.fr
*Project supported by the National Science Foundation (Nos. DMS 0906440, DMS 1206438) and the Research Fund of Indiana University.

complicated geometrical domains such as the oceans, and multilevel methods of the incremental unknown type are useful for the implementation of multilevel methods. Such methods have been introduced in the context of the nonlinear Galerkin method in [18] (see also [19]), finite differences in [23], and spectral methods and turbulence in [8]. As continuation of [1], this article explores the finite-volume implementation of the incremental unknowns.

Considering to simplify a rectangular geometry, we divide the domain into cells of size $\Delta x \times \Delta y$, which we regroup at the first level of increment, in cells of size $3\Delta x \times 3\Delta y$. The unknowns on the small cells being the original unknowns, we introduce for the coarse cells suitably averaged values of the unknowns. The dynamic strategy, which may take many different forms (see [1, 8]), consists in solving alternatively the system for a number of time steps on the fine mesh grid and then for a number of time steps, the system considered on the coarse mesh during which the increments as defined below, remain frozen. This coarsening can be repeated once more considering cells of size $9\Delta x \times 9\Delta y$, and possibly several times as the programming cost is repetitive and thus small, but we restrict ourselves in this article to one coarsening.

We have chosen to present the method for the shallow water (or SW for short) equations for the reasons mentioned above. We consider the SW equations without viscosity, linearized around a constant flow. The well-posedness of these linear hyperbolic equations has been established very recently (see [12]). We choose in this article one of many situations presented in [12], i.e., the fully supercritical case, since the boundary conditions depend on the nature of the flow (subcritical versus supercritical, subsonic versus supersonic). Other implementation of multilevel methods in geophysical fluid dynamics appear in [16]. See also [14–15] for more developments on the primitive equations. Further developments along the lines of this work will appear in an article in [4].

This article is organized as follows. We start in Section 2 with a simple model corresponding to a one-dimensional transport equation. We then proceed in Section 3 with the shallow water equation presenting first the equations (see Section 3.1), then the multilevel finite-volume discretization (see Section 3.2) and then the multilevel temporal discretization (see Section 3.3). In Section 4, we consider another related form of the algorithm. In Sections 2 and 3, the algorithm on the coarse grid is the same as the algorithm on the fine grid (in space) with just a different spatial mesh. In this section, we consider another algorithm on which we started, where the spatial scheme on the coarse grid is obtained by averaging, in each coarse cell the equations for the corresponding fine cells. The study of the stability of the scheme in this case has not been completed yet. We present the analysis in one-dimensional space, for the simple transport equation (see Section 4.1) and for the one-dimensional linearized equation (see Section 4.2). The boundary condition is space periodicity and the stability analysis is conducted by the classical von Neumann method.

2 The One-Dimensional Case

We start with the one-dimensional space and consider the problem

$$\frac{\delta u}{\delta t}(x,t) + \frac{\delta u}{\delta x}(x,t) = f(x,t) \tag{2.1}$$

for $(x,t) \in (0,L) \times (0,T)$, with the boundary condition

$$u(0,t) = 0 \tag{2.2}$$

and the initial condition

$$u(x,0) = u^0(x). \tag{2.3}$$

We set $\mathcal{M} = (0,L)$ and $H = L^2(\mathcal{M})$, and also introduce the operator $Au = u_x$ with domain $D(A) = \{v \in H^1(\mathcal{M}), v(0) = 0\}$. Then for $f, f' \in L^1(0,T;H)$, $u^0 \in D(A)$, problem (2.1)–(2.3) possesses a unique solution u, such that

$$u \in C([0,T];H) \cap L^\infty(0,T;D(A)), \quad \frac{\mathrm{d}u}{\mathrm{d}t} \in L^\infty(0,T;D(A)).$$

Our multilevel spatial discretization is presented in Section 2.1, while Section 2.2 deals with time and space discretization.

2.1 Multilevel spatial discretization

We consider, on the interval $(0,L)$, $3N$ cells $(k_i)_{1\le i\le 3N}$ of uniform length Δx with $3N\Delta x = L$. For $i = 0, \cdots, 3N$, we set

$$x_{i+\frac{1}{2}} = i\Delta x,$$

so that

$$k_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}).$$

We also introduce the center of each cell,

$$x_i = \frac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2} = (i-1)\Delta x + \frac{\Delta x}{2}, \quad 1 \le i \le 3N.$$

The discrete unknowns are denoted by u_i $(1 \le i \le 3N)$, and u_i is expected to be some approximation of the mean value of u over k_i. The equation (2.1) integrated over the cell k_i yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{k_i} u(x,t)\mathrm{d}x + u(x_{i+\frac{1}{2}},t) - u(x_{i-\frac{1}{2}},t) = \int_{k_i} f(x,t)\mathrm{d}x.$$

Here the term $u(x_{i+\frac{1}{2}},t)$ is approximated by $u_i(t)$ using an “upwind” scheme due to the direction of the characteristics for equation (2.1). Setting $f_i(t) = \frac{1}{\Delta x}\int_{k_i} f(x,t)\mathrm{d}x$, the upwind finite-volume discretization now reads

$$\frac{\mathrm{d}u_i}{\mathrm{d}t}(t) + \frac{u_i(t) - u_{i-1}(t)}{\Delta x} = f_i(t), \quad 1 \le i \le 3N, \tag{2.4}$$

where we have set

$$u_0(t) = 0. \tag{2.5}$$

These equations are supplemented with the initial condition

$$u_i(0) = \frac{1}{\Delta x}\int_{k_i} u^0(x)\mathrm{d}x, \quad 1 \le i \le 3N. \tag{2.6}$$

To rewrite the scheme in a more abstract form, we introduce the space V_h ($h = \Delta x$) of step functions u_h, which are constant on the intervals k_i, $i = 0, \cdots, 3N$ with $u_h|_{k_i} = u_i$ and $u_0 = 0$. Here to take into account the boundary condition, we have added the fictitious cell $k_0 = (-\Delta x, 0)$. The discrete space V_h is equipped with the norm induced by $L^2(\mathcal{M})$, that is,

$$|u_h|^2 = \Delta x \sum_{i=0}^{3N} |u_i|^2 = \Delta x \sum_{i=1}^{3N} |u_i|^2.$$

Next let us introduce the backward difference operator

$$\delta_h u_h = \frac{u_i - u_{i-1}}{\Delta x} \quad \text{on} \;\; k_i, \; 1 \le i \le 3N.$$

Then (2.4) can be rewritten as

$$\frac{\mathrm{d}u_h}{\mathrm{d}t} + \delta_h u_h = f_h$$

with $f_h|_{k_i} = f_i$.

We now introduce a coarser mesh consisting of the intervals K_l ($1 \le l \le N$), with length $3\Delta x$ obtained as

$$K_l = k_{3l-2} \cup k_{3l-1} \cup k_{3l}^* = (x_{3l-2-\frac{1}{2}}, x_{3l-\frac{1}{2}}). \tag{2.7}$$

Let $(u_i)_{1 \le i \le 3N}$ still denote the approximation of u on the fine mesh $(k_i)_{1 \le i \le 3N}$. Then an approximation of u on the coarse mesh is given by

$$U_l = \frac{1}{3}[u_{3l-2} + u_{3l-1} + u_{3l}], \quad 1 \le l \le N. \tag{2.8}$$

We introduce the incremental unknowns

$$Z_{3l-\alpha} = u_{3l-\alpha} - U_l \tag{2.9}$$

for $\alpha = 0, 1, 2, \; \ell = 1, \cdots, N$, so that

$$Z_{3\ell} + Z_{3\ell-1} + Z_{3\ell-2} = 0. \tag{2.10}$$

Remark 2.1 The definition of Z in (2.9) is at our disposal. In this case, Z are the order of Δx. For example, using Taylor's formula, we obtain

$$\begin{aligned} Z_{3l-2} &= u_{3l-2} - \frac{1}{3}[u_{3l-2} + u_{3l-1} + u_{3l}] \\ &= \frac{1}{3}[2u_{3l-2} - (u_{3l-2} + \mathcal{O}(\Delta x)) - (u_{3l-2} + \mathcal{O}(\Delta x))] \\ &= \mathcal{O}(\Delta x). \end{aligned}$$

* Including, strictly speaking, the separation points.

We will discuss elsewhere other definitions of the incremental unknown Z, and in particular those of order Δx^2 considered in [1].

The unknowns on the fine grid are thus written as the sum of the coarse grid unknowns $(U_l)_{1\le l\le N}$ and associated increments $(Z_i)_{1\le i\le 3N}$.

With this in mind, we consider a coarse grid discretization of the equation similar to (2.4), that is,

$$\frac{\mathrm{d}U_\ell(t)}{\mathrm{d}t} + \frac{1}{3\Delta x}(U_\ell(t) - U_{\ell-1}(t)) = F_\ell(t), \quad 1 \le \ell \le N \tag{2.11}$$

with

$$U_0(t) = 0, \tag{2.12}$$

$$F_\ell(t) = \frac{1}{3}\sum_{\alpha=0}^{2} f_{3\ell-\alpha}(t) \tag{2.13}$$

and

$$U_\ell(0) = \frac{1}{3}\sum_{\alpha=0}^{2} u_{3\ell-\alpha}(0). \tag{2.14}$$

Independent of the equation under consideration and the numerical scheme, let us make the following algebraic observation: for $u_h \in V_h,\ u_h = (u_i)_{1\le i\le 3N}$, we have

$$\begin{aligned}
|u_h|^2 &= h\sum_{i=1}^{3N} u_i^2 = h\sum_{\alpha=0}^{2}\sum_{\ell=1}^{N} |u_{3\ell-\alpha}|^2 \\
&= h\sum_{\alpha=0}^{2}\sum_{\ell=1}^{N} |U_\ell + Z_{3\ell-2}|^2 \\
&= 3h\sum_{\ell=1}^{N} |U_\ell|^2 + h\sum_{i=1}^{3N} |Z_i|^2 \quad \text{(because of (2.10))} \\
&= |U_h|^2 + |Z_h|^2.
\end{aligned} \tag{2.15}$$

In some sense, because of (2.10), the coarse component U and the increment Z are L^2-orthogonal.

2.2 Euler implicit time discretization and estimates

We define a time step Δt with $N_T\Delta t = T$, and set $t_n = n\Delta t$ for $0 \le n \le N_T$. We denote by $\{u_i^n, 1 \le i \le 3N, 0 \le n \le N_T\}$ the discrete unknowns. The value u_i^n is an expected approximation

$$u_i^n \simeq \frac{1}{\Delta x}\int_{k_i} u(x, t_n)\mathrm{d}x.$$

Our spatial discretization was presented in the previous section in (2.4)–(2.6), for the fine grid, and (2.11)–(2.14) for the coarse grid. We now discretize this equation in time by using the implicit Euler scheme with the time step $\frac{\Delta t}{p}$ on the fine mesh and time step Δt on the coarse mesh. More precisely, let $p > 1$ and $q > 1$ be two fixed integers. The multi-step discretization consists in alternating p steps on (2.4) with time step $\frac{\Delta t}{p}$, from t_n to t_{n+1} and then q steps on (2.11) with time step Δt, the incremental unknowns Z_i being frozen at t_{n+1} from t_{n+1} to t_{n+q+1}. Then, using equations (2.9), we can go back to the finer mesh for p steps from t_{n+q+1} to t_{n+q+2}. For simplicity, we suppose that N_T is a multiple of $q+1$, and set $N_q = \frac{N_T}{q+1}$.

Suppose that n is a multiple of $(q+1)$, and the $(u_i^n)_{1\le i\le 3N}$ are known. We introduce the discrete unknowns $u_i^{n+\frac{s}{p}}$ with $t_{n+\frac{s}{p}} = t_n + s\frac{\Delta t}{p}$ for $0 \le s \le p$ and $1 \le i \le 3N$. We successively determine the $u_i^{n+\frac{s}{p}}$ $(1 \le i \le 3N,\ 1 \le s \le p)$ with p iterations of the following scheme:

$$\begin{cases} \dfrac{p}{\Delta t}(u_i^{n+\frac{s+1}{p}} - u_i^{n+\frac{s}{p}}) + \dfrac{1}{\Delta x}(u_i^{n+\frac{s+1}{p}} - u_{i-1}^{n+\frac{s+1}{p}}) = f_i^{n+\frac{s+1}{p}}, \\ u_0^{n+\frac{s+1}{p}} = 0 \end{cases} \tag{2.16}$$

for $1 \le i \le 3N$, $0 \le s \le p-1$, where

$$f_i^{n+\frac{s+1}{p}} = \frac{1}{\frac{\Delta t}{p}} \frac{1}{\Delta x} \int_{(n+\frac{s}{p})\Delta t}^{(n+\frac{s+1}{p})\Delta t} \int_{k_i} f(x,t) \mathrm{d}x \mathrm{d}t. \tag{2.17}$$

It is convenient to introduce the step functions $u_h^{n+\frac{s}{p}},\ f_h^{n+\frac{s}{p}}$ defined for $0 \le s \le p$ by

$$u_h^{n+\frac{s}{p}}(x) = u_i^{n+\frac{s}{p}}, \quad f_h^{n+\frac{s}{p}}(x) = f_i^{n+\frac{s}{p}}, \quad x \in k_i,\ 1 \le i \le 3N.$$

We also introduce the backward difference operator δ_h defined by

$$\delta_h g_i^n = \frac{g_i^n - g_{i-1}^n}{\Delta x} \quad \text{or} \quad \delta_h g(x) = \frac{g(x) - g(x-h)}{\Delta x},$$

so that (2.16) can now be rewritten as

$$\frac{p}{\Delta t}(u_h^{n+\frac{s+1}{p}} - u_h^{n+\frac{s}{p}}) + \delta_h u_h^{n+\frac{s+1}{p}} = f_h^{n+\frac{s+1}{p}}. \tag{2.18}$$

Our goal now is to estimate $|u_h^{n+1}|$ in terms of $|u_h^n|$. We take the scalar product in $L^2(\mathcal{M})$ of (2.18) with $2\frac{\Delta t}{p} u_h^{n+\frac{s+1}{p}}$. Denoting by $(\,\cdot\,,\cdot\,)$ the L^2 scalar product and using the well-known relation

$$2(a-b,a) = |a|^2 - |b|^2 + |a-b|^2,$$

we find

$$\begin{aligned} &|u_h^{n+\frac{s+1}{p}}|^2 - |u_h^{n+\frac{s}{p}}|^2 + |u_h^{n+\frac{s+1}{p}} - u_h^{n+\frac{s}{p}}|^2 + \frac{2\Delta t}{p}(\delta_h u_h^{n+\frac{s+1}{p}}, u_h^{n+\frac{s+1}{p}}) \\ &= \frac{2\Delta t}{p}\big(f_h^{n+\frac{s+1}{p}}, u_h^{n+\frac{s+1}{p}}\big). \end{aligned} \tag{2.19}$$

We have, for every $u_h \in V_h$,

$$2(\delta_h u_h, u_h) = |u_{3N}|^2 + \sum_{i=1}^{3N} |u_i - u_{i-1}|^2. \tag{2.20}$$

Indeed

$$
\begin{aligned}
2(\delta_h u_h, u_h) &= 2\sum_{i=1}^{3N}(u_i - u_{i-1})u_i \\
&= \sum_{i=1}^{3N}\left(|u_i|^2 - |u_{i-1}|^2 + |u_i - u_{i-1}|^2\right),
\end{aligned}
$$

and (2.20) follows, since $u_0 = 0$.

Using (2.20) and Schwarz inequality, (2.19) yields

$$
\begin{aligned}
&|u_h^{n+\frac{s+1}{p}}|^2 - |u_h^{n+\frac{s}{p}}|^2 + |u_h^{n+\frac{s+1}{p}} - u_h^{n+\frac{s}{p}}|^2 \\
&+ \frac{\Delta t}{p}\Big[|u_{3N}^{n+\frac{s+1}{p}}|^2 + \sum_{i=1}^{3N}|u_i^{n+\frac{s+1}{p}} - u_{i-1}^{n+\frac{s+1}{p}}|^2\Big] \\
\le\ & \frac{\Delta t}{p}|f_h^{n+\frac{s+1}{p}}|^2 + \frac{\Delta t}{p}|u_h^{n+\frac{s+1}{p}}|^2,
\end{aligned}
\tag{2.21}
$$

so that

$$
\Big(1 - \frac{\Delta t}{p}\Big)|u_h^{n+\frac{s+1}{p}}|^2 \le \frac{\Delta t}{p}|f_h^{n+\frac{s+1}{p}}|^2 + |u_h^{n+\frac{s}{p}}|^2. \tag{2.22}
$$

This yields readily for $1 \le s \le p$,

$$
|u_h^{n+\frac{s}{p}}|^2 \le \frac{1}{\left(1 - \frac{\Delta t}{p}\right)^s}\Big[|u_h^n|^2 + \frac{\Delta t}{p}\sum_{d=0}^{s-1}|f_h^{n+\frac{d+1}{p}}|^2\Big]. \tag{2.23}
$$

Here, in view of definition (2.17), we observe that

$$
\begin{aligned}
\frac{\Delta t}{p}|f_h^{n+\frac{d}{p}}|^2 &= \frac{\Delta t}{p}\Delta x\sum_{i=1}^{3N}|f_i^{n+\frac{d}{p}}|^2 = \Big(\int_{(n+\frac{d}{p})\Delta t}^{(n+\frac{d+1}{p})\Delta t}\int_0^L f(x,t)\mathrm{d}x\mathrm{d}t\Big)^2 \\
&\le \int_{(n+\frac{d}{p})\Delta t}^{(n+\frac{d+1}{p})\Delta t}\int_0^L |f(x,t)|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

By adding these inequalities for $d = 0, \cdots, p-1$, we obtain

$$
\frac{\Delta t}{p}\sum_{d=0}^{p-1}|f_h^{n+\frac{d}{p}}|^2 \le \int_{n\Delta t}^{(n+1)\Delta t}\int_0^L |f(x,t)|^2\mathrm{d}x\mathrm{d}t.
$$

Combining this bound with (2.23) provides

$$
|u_h^{n+\frac{s}{p}}|^2 \le \frac{1}{\left(1 - \frac{\Delta t}{p}\right)^s}\Big[|u_h^n|^2 + \int_{n\Delta t}^{(n+1)\Delta t}\int_0^L |f(x,t)|^2\mathrm{d}x\mathrm{d}t\Big].
$$

Since $1 - x \ge 4^{-x}$ for $x \in \left[0, \frac{1}{2}\right]$, we see that, if $\frac{\Delta t}{p} \le \frac{1}{2}$,

$$
|u_h^{n+\frac{s}{p}}|^2 \le 4^{\frac{s}{p}\Delta t}\Big[|u_h^n|^2 + \int_{n\Delta t}^{(n+1)\Delta t}\int_0^L |f(x,t)|^2\mathrm{d}x\mathrm{d}t\Big]. \tag{2.24}
$$

Here s varies between 1 and p, and therefore the bound for $s = p$ reads

$$|u_h^{n+1}|^2 \leq 4^{\Delta t}\Big[|u_h^n|^2 + \int_{n\Delta t}^{(n+1)\Delta t}\int_0^L |f(x,t)|^2 \mathrm{d}x\mathrm{d}t\Big]. \tag{2.25}$$

We now define the u_h^{n+s} for $2 \leq s \leq q+1$, by applying q-times the implicit Euler scheme to equation (2.11) with step Δt, that is,

$$\begin{cases} \dfrac{U_l^{n+s+1} - U_l^{n+s}}{\Delta t} + \dfrac{U_\ell^{n+s+1} - U_{\ell-1}^{n+s+1}}{3\Delta x} = F_l^{n+s+1}, \\ U_0^{n+s+1} = u_0^{n+s+1} = 0, \end{cases} \tag{2.26}$$

where

$$\begin{aligned} F_l^{n+s+1} &= \frac{1}{3}[f_{3l-2}^{n+s+1} + f_{3l-1}^{n+s+1} + f_{3l}^{n+s+1}] \\ &= \frac{1}{3\Delta t\Delta x}\int_{(n+s)\Delta t}^{(n+s+1)\Delta t}\int_{K_l} f(x,t)\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{2.27}$$

As we said at the beginning of the section, the Z_i's are frozen between t_{n+1} and t_{n+q+1}, and therefore for $2 \leq s \leq q+1$, $1 \leq l \leq N$,

$$\begin{cases} U_l^{n+s} = \dfrac{1}{3}[u_{3l-2}^{n+s} + u_{3l-1}^{n+s} + u_{3l}^{n+s}], \\ Z_{3l-\alpha}^{n+s} = Z_{3l-\alpha}^{n+1} = u_{3l-\alpha}^{n+1} - U_l^{n+1}, \quad \alpha = 0,1,2. \end{cases} \tag{2.28}$$

We can invert this system (2.28) to obtain

$$u_{3l-\alpha}^{n+s} = U_l^{n+s} + Z_{3l-\alpha}^{n+1}, \quad \alpha = 0,1,2. \tag{2.29}$$

Classically these equations allow us to uniquely define the terms U_ℓ^{n+s+1}, when the terms U_ℓ^{n+1} are known. Then the equations (2.29) allow us to compute the u_i^{n+s+1} ($i = 1,\cdots,3N,\ s = 1,\cdots,q$).

To derive suitable a priori estimates, we multiply (2.26) by $6\Delta t\Delta x U_\ell^{n+s+1}$ and sum for $\ell = 1,\cdots,N$. Setting $\tau = n+s+1$, we find

$$\begin{aligned} &3\Delta x\sum_{\ell=1}^N(|U_\ell^\tau|^2 - |U_\ell^{\tau-1}|^2) + 3\Delta x\sum_{\ell=1}^N |U_l^\tau - U_l^{\tau-1}|^2 \\ &+ 2\Delta t|U_N^\tau|^2 + \Delta t\sum_{\ell=1}^N |U_\ell^\tau - U_{\ell-1}^\tau|^2 \\ = {}& 6\Delta t\Delta x\sum_{\ell=1}^N F_\ell^\tau U_\ell^\tau. \end{aligned} \tag{2.30}$$

Hence, as for equations (2.21)–(2.25),

$$|U_h^\tau|^2 \leq 4^{\Delta t}\Big[|U_h^{\tau-1}|^2 + \int_{(\tau-1)\Delta t}^{\tau\Delta t}\int_0^L |f(x,t)|^2\mathrm{d}x\mathrm{d}t\Big]. \tag{2.31}$$

We write equation (2.31) for $\tau = n+2, \cdots, n+q+1$, multiply the equation for $\tau = n+s$ by $4^{(q+1-s)\Delta t}$ and add for $s = 2, \cdots, q+1$. We obtain

$$|U_h^{n+q+1}|^2 \leq 4^{q\Delta t}\Big[|U_h^{n+1}|^2 + \int_{(n+1)\Delta t}^{(n+q+1)\Delta t} \int_0^L |f(x,t)|^2 \mathrm{d}x\mathrm{d}t\Big]. \tag{2.32}$$

We add $|Z_h^{n+1}|^2$ to both sides and, in view of (2.15) and the second formula of (2.28), we find

$$|u_h^{n+q+1}|^2 \leq 4^{q\Delta t}\Big[|u_h^{n+1}|^2 + \int_{(n+1)\Delta t}^{(n+q+1)\Delta t} \int_0^L |f(x,t)|^2 \mathrm{d}x\mathrm{d}t\Big]. \tag{2.33}$$

Taking into account (2.25), we find that

$$|u_h^{n+q+1}|^2 \leq 4^{(q+1)\Delta t}\Big[|u_h^{n}|^2 + \int_{n\Delta t}^{(n+q+1)\Delta t} |f(\cdot,t)|_2^2 \mathrm{d}t\Big]. \tag{2.34}$$

More generally, we have the stability result

$$\begin{aligned} |u_h^m|^2 &\leq 4^{m\Delta t}\Big[|u_h^0|^2 + \int_0^{m\Delta t} |f(\cdot,t)|_2^2 \mathrm{d}t\Big] \\ &\leq 4^T\Big[|u^0|^2 + \int_0^T |f(\cdot,t)|_{L^2}^2 \mathrm{d}t\Big]. \end{aligned} \tag{2.35}$$

To summarize, we show the following result.

Theorem 2.1 *The multilevel scheme defined by the equations* (2.16) *and* (2.26) *is stable in* $L^\infty(0,T;L^2(\mathcal{M}))$ *in the sense of* (2.35).

3 The Linear Shallow Water Equations

We now want to extend the previous results to the more complex case of the shallow water equations linearized around a constant flow $(\widetilde{u}_0, \widetilde{v}_0, \widetilde{\phi}_0)$ (see (3.2) below). As shown in [12] the boundary conditions, which can be associated with these equations, depend on the relative values of the velocities ($\widetilde{u}_0^2, \widetilde{v}_0^2 >$ (or $<$) $g\widetilde{\phi}_0$), that is, whether these velocities are sub- or supercritical (sub- or supersonic). We consider here the case, where

$$\widetilde{\phi}_0 > 0, \quad \widetilde{u}_0 > \sqrt{g\widetilde{\phi}_0}, \quad \widetilde{v}_0 > \sqrt{g\widetilde{\phi}_0}. \tag{3.1}$$

3.1 The equations

We consider, in the domain $\mathcal{M} = (0,L_1)\times(0,L_2)$, the equations

$$\begin{cases} \dfrac{\delta u}{\delta t} + \widetilde{u}_0\dfrac{\delta u}{\delta x} + \widetilde{v}_0\dfrac{\delta u}{\delta y} + g\dfrac{\delta \phi}{\delta x} = f_u, \\ \dfrac{\delta v}{\delta t} + \widetilde{u}_0\dfrac{\delta v}{\delta x} + \widetilde{v}_0\dfrac{\delta v}{\delta y} + g\dfrac{\delta \phi}{\delta y} = f_v, \\ \dfrac{\delta \phi}{\delta t} + \widetilde{u}_0\dfrac{\delta \phi}{\delta x} + \widetilde{v}_0\dfrac{\delta \phi}{\delta y} + \widetilde{\phi}_0\Big(\dfrac{\delta u}{\delta x} + \dfrac{\delta v}{\delta y}\Big) = f_\phi. \end{cases} \tag{3.2}$$

Here (u,v) is the velocity, and ϕ is the potential height. The advecting velocities $\widetilde{u}_0$, $\widetilde{v}_0$ and the mean geopotential height $\widetilde{\phi}_0$ are constants. $\mathbf{f} = (f_u, f_v, f_\phi)$ is the source term. For the subcritical flow under consideration, we supplement (3.2) with the boundary conditions,

$$\mathbf{u} = (u,v,\phi) = 0, \quad \text{at } \{x=0\} \cup \{y=0\}, \tag{3.3}$$

and the initial conditions

$$\mathbf{u} = (u,v,\phi) = \mathbf{u}^0 = (u^0, v^0, \phi^0), \quad \text{at } t=0. \tag{3.4}$$

The system becomes

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \mathbf{A}\mathbf{u} = \mathbf{f},$$

where $\mathbf{A}\mathbf{u} = (A_1\mathbf{u}, A_2\mathbf{u}, A_3\mathbf{u})$ is given by

$$\begin{cases} A_1\mathbf{u} = \widetilde{u}_0 \dfrac{\delta u}{\delta x} + \widetilde{v}_0 \dfrac{\delta u}{\delta y} + g\dfrac{\delta \phi}{\delta x}, \\ A_2\mathbf{u} = \widetilde{u}_0 \dfrac{\delta v}{\delta x} + \widetilde{v}_0 \dfrac{\delta v}{\delta y} + g\dfrac{\delta \phi}{\delta y}, \\ A_3\mathbf{u} = \widetilde{u}_0 \dfrac{\delta \phi}{\delta x} + \widetilde{v}_0 \dfrac{\delta \phi}{\delta y} + \widetilde{\phi}_0\Big(\dfrac{\delta u}{\delta x} + \dfrac{\delta v}{\delta y}\Big). \end{cases} \tag{3.5}$$

It may also be convenient to decompose $\mathbf{A}$ with respect to its x and y derivatives, that is,

$$\begin{aligned} &\mathbf{A} = \mathbf{A}^x + \mathbf{A}^y, \\ &\mathbf{A}^x\mathbf{u} = (A_1^x\mathbf{u}, A_2^x\mathbf{u}, A_3^x\mathbf{u}), \quad \mathbf{A}^y\mathbf{u} = (A_1^y\mathbf{u}, A_2^y\mathbf{u}, A_3^y\mathbf{u}) \end{aligned}$$

with

$$\mathbf{A}^x\mathbf{u} = \begin{cases} \widetilde{u}_0 \dfrac{\partial u}{\partial x} + g\dfrac{\partial \phi}{\partial x}, \\ \widetilde{u}_0 \dfrac{\partial v}{\partial x}, \\ \widetilde{u}_0 \dfrac{\partial \phi}{\partial x} + \widetilde{\phi}_0 \dfrac{\partial u}{\partial x}, \end{cases} \qquad \mathbf{A}^y\mathbf{u} = \begin{cases} \widetilde{v}_0 \dfrac{\partial u}{\partial y}, \\ \widetilde{v}_0 \dfrac{\partial v}{\partial y} + g\dfrac{\partial \phi}{\partial y}, \\ \widetilde{v}_0 \dfrac{\partial \phi}{\partial y} + \widetilde{\phi}_0 \dfrac{\partial v}{\partial y}. \end{cases}$$

We define the scalar product on $H = (L^2(\mathcal{M}))^3$ as follows: for $\mathbf{u} = (u,v,\phi)$, $\mathbf{u}' = (u',v',\phi')$, and we set

$$\langle \mathbf{u}, \mathbf{u}' \rangle = (u,u') + (v,v') + \frac{g}{\widetilde{\phi}_0}(\phi,\phi'), \tag{3.6}$$

where $(\,\cdot\,,\,\cdot\,)$ denotes the standard scalar product on $L^2(\mathcal{M})$. Then the following positivity result for $\mathbf{A}$ holds.

Lemma 3.1 *Under the assumption* (3.1)*, for all sufficiently smooth* $\mathbf{u}$ *satisfying* (3.3)*, we have* $\langle \mathbf{A}\mathbf{u}, \mathbf{u} \rangle \geq 0$.

Proof We write

$$\langle \mathbf{A}\mathbf{u}, \mathbf{u} \rangle = \langle \mathbf{A}^x\mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{A}^y\mathbf{u}, \mathbf{u} \rangle \tag{3.7}$$

with

$$\langle \mathbf{A}^x \mathbf{u}, \mathbf{u} \rangle = \iint_{\mathcal{M}} \Big[\widetilde{u}_0 u_x u + g\phi_x u + \widetilde{u}_0 v_x v + \frac{g}{\widetilde{\phi}_0} \widetilde{u}_0 \phi_x \phi + g u_x \phi \Big] \mathrm{d}x\mathrm{d}y,$$

$$\langle \mathbf{A}^y \mathbf{u}, \mathbf{u} \rangle = \iint_{\mathcal{M}} \Big[\widetilde{v}_0 v_y v + g\phi_y v + \widetilde{v}_0 u_y u + \frac{g}{\widetilde{\phi}_0} \widetilde{v}_0 \phi_y \phi + g v_y \phi \Big] \mathrm{d}x\mathrm{d}y.$$

Then

$$\begin{aligned} \langle \mathbf{A}^x \mathbf{u}, \mathbf{u} \rangle &= \frac{\widetilde{u}_0}{2} \iint_{\mathcal{M}} \Big[(u^2)_x + (v^2)_x + \frac{g}{\widetilde{\phi}_0} (\phi^2)_x \Big] \mathrm{d}x\mathrm{d}y + \iint_{\mathcal{M}} g(\phi u)_x \mathrm{d}x\mathrm{d}y \\ &= \frac{\widetilde{u}_0}{2} \int_0^{L_2} \Big[u^2 + v^2 + \frac{g}{\widetilde{\phi}_0} \phi^2 \Big]_{x=0}^{x=L_1} \mathrm{d}y + \int_0^{L_2} [g(\phi u)]_{x=0}^{x=L_1} \mathrm{d}y. \end{aligned} \tag{3.8}$$

Recall that $\mathbf{u} = \mathbf{0}$ at $x = 0$. Also the assumption (3.1) yields that

$$\frac{\widetilde{u}_0}{2} u^2 + \frac{\widetilde{u}_0}{2} g \frac{\phi^2}{\widetilde{\phi}_0} + g\phi u$$

is pointwise positive. Therefore, we infer from (3.8) that $\langle \mathbf{A}^x \mathbf{u}, \mathbf{u} \rangle \geq 0$. A similar computation provides $\langle \mathbf{A}^y \mathbf{u}, \mathbf{u} \rangle \geq 0$ (since $\widetilde{v}_0^2 > g\widetilde{\phi}_0$). In view of (3.7), the proof of Lemma 3.1 is complete.

Remark 3.1 The fact that the boundary and initial value problem (3.2)–(3.4) is well-posed is a recent result proved in [12]. The proof relies on the semigroup theory and necessitates in particular proving (by approximation) that $\langle \mathbf{A}\mathbf{u}, \mathbf{u} \rangle \geq 0$ for all $\mathbf{u} \in L^2(\mathcal{M})^3$, such that $\mathbf{A}\mathbf{u} \in L^2(\mathcal{M})^3$, and $\mathbf{u}$ satisfies (3.3). The fact that (3.3) makes sense for such $\mathbf{u}$'s results from a trace theorem also proved in [12].

3.2 Multilevel finite-volume spatial discretization

3.2.1 Finite-volume discretization

We decompose $\mathcal{M} = (0, L_1) \times (0, L_2)$ into $3N_1 \times 3N_2$ rectangles denoted by $(k_{i,j})_{1 \leq i \leq 3N_1, 1 \leq j \leq 3N_2}$ of size $\Delta x \times \Delta y$ with $3N_1 \Delta x = L_1$ and $3N_2 \Delta y = L_2$.

For $0 \leq i \leq 3N_1$ and for $0 \leq j \leq 3N_2$, let

$$x_{i+\frac{1}{2}} = i\Delta x \quad \text{and} \quad y_{j+\frac{1}{2}} = j\Delta y.$$

Then the rectangles $(k_{i,j})$ are, for $1 \leq i \leq 3N_1, 1 \leq j \leq 3N_2$,

$$k_{i,j} = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}).$$

We also define the center (x_i, y_j) of each cell k_{ij},

$$\begin{cases} x_i = \dfrac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}) = (i-1)\Delta x + \dfrac{\Delta x}{2}, & 1 \leq i \leq 3N_1, \\ y_j = \dfrac{1}{2}(y_{j-\frac{1}{2}} + y_{j+\frac{1}{2}}) = (j-1)\Delta y + \dfrac{\Delta y}{2}, & 1 \leq j \leq 3N_2. \end{cases}$$

For the boundary conditions, we add fictitious cells on the west and south sides,

$$k_{0,j} = (-\Delta x, 0) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}), \qquad \text{centered at } \Big(x_0 = -\frac{\Delta x}{2}, y_j\Big),\ 1 \le j \le 3N_2$$

and

$$k_{i,0} = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (-\Delta y, 0), \qquad \text{centered at } \Big(x_i, y_0 = -\frac{\Delta y}{2}\Big),\ 1 \le i \le 3N_1.$$

The finite-volume scheme is found by integrating the equations (3.2) over each control volume $(k_{i,j})_{1\le i\le 3N_1, 1\le j\le 3N_2}$. The first equation yields for $1 \le i \le 3N_1, 1 \le j \le 3N_2$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\Delta x\Delta y}\iint_{k_{i,j}} u(x,y,t)\mathrm{d}x\mathrm{d}y + \frac{\widetilde{u}_0}{\Delta x\Delta y}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}[u(x_{i+\frac{1}{2}},y,t) - u(x_{i-\frac{1}{2}},y,t)]\mathrm{d}y\\
&+ \frac{\widetilde{v}_0}{\Delta x\Delta y}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}[u(x,y_{j+\frac{1}{2}},t) - u(x,y_{j-\frac{1}{2}},t)]\mathrm{d}x\\
&+ \frac{g}{\Delta x\Delta y}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}[\phi(x_{i+\frac{1}{2}},y,t) - \phi(x_{i-\frac{1}{2}},y,t)]\mathrm{d}y = \int_{k_{i,j}} f_u(x,y,t)\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Let us denote

$$\begin{aligned}
V_h = \{ &\text{ the space of step functions constant on } \ k_{i,j},\ 0 \le i \le 3N_1,\ 0 \le j \le 3N_2\\
&\text{with } \ w_{|k_{i,j}} = w_{i,j} \text{ and } w_{0,j} = w_{i,0} = 0\}.
\end{aligned}$$

We approximate the unknown $\mathbf{u} = (u, v, \phi)$ with $\mathbf{u}_h \simeq \mathbf{u}_h(t) \in (V_h)^3 = \mathbf{V}_h$, and use an upwind scheme for the fluxes, since $\widetilde{u}_0 > 0$ and $\widetilde{v}_0 > 0$,

$$\begin{aligned}
\mathbf{u}(x_{i+\frac{1}{2}}, y, t) \simeq \mathbf{u}_{i,j}(t), &\quad y \in [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}],\\
\mathbf{u}(x, y_{j+\frac{1}{2}}, t) \simeq \mathbf{u}_{i,j}(t), &\quad x \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}].
\end{aligned}$$

This gives the following semi-discrete equations for $1 \le i \le 3N_1$ and $1 \le j \le 3N_2$:

$$\begin{cases}
\dfrac{\mathrm{d}}{\mathrm{d}t}u_{i,j} + \widetilde{u}_0\dfrac{u_{i,j} - u_{i-1,j}}{\Delta x} + \widetilde{v}_0\dfrac{u_{i,j} - u_{i,j-1}}{\Delta y} + g\dfrac{\phi_{i,j} - \phi_{i-1,j}}{\Delta x} = f_{u,i,j},\\
\dfrac{\mathrm{d}}{\mathrm{d}t}v_{i,j} + \widetilde{u}_0\dfrac{v_{i,j} - v_{i-1,j}}{\Delta x} + \widetilde{v}_0\dfrac{v_{i,j} - v_{i,j-1}}{\Delta y} + g\dfrac{\phi_{i,j} - \phi_{i,j-1}}{\Delta y} = f_{v,i,j},\\
\dfrac{\mathrm{d}}{\mathrm{d}t}\phi_{i,j} + \widetilde{u}_0\dfrac{\phi_{i,j} - \phi_{i-1,j}}{\Delta x} + \widetilde{v}_0\dfrac{\phi_{i,j} - \phi_{i,j-1}}{\Delta y}\\
\quad + \widetilde{\phi}_0\Big(\dfrac{u_{i,j} - u_{i-1,j}}{\Delta x} + \dfrac{v_{i,j} - v_{i,j-1}}{\Delta y}\Big) = f_{\phi,i,j},\\
\mathbf{u}_{0,j} = \mathbf{u}_{i,0} = 0,\\
\mathbf{u}_{i,j}(0) = \mathbf{u}^0_{i,j},
\end{cases} \tag{3.9}$$

where $\mathbf{f} = (f_u, f_v, f_\phi)$, $\mathbf{u^0} = (u^0, v^0, \phi^0)$ and

$$\mathbf{f}_{i,j}(t) = \frac{1}{\Delta x\Delta y}\int_{k_{i,j}} \mathbf{f}(x,y,t)\mathrm{d}x\mathrm{d}y, \quad \mathbf{u}^0_{i,j} = \frac{1}{\Delta x\Delta y}\int_{k_{i,j}} \mathbf{u}^0(x,y)\mathrm{d}x\mathrm{d}y. \tag{3.10}$$

Let us introduce the finite difference operators

$$\delta_{1h}g_h = \frac{1}{\Delta x}(g_{i,j} - g_{i-1,j}) \quad \text{on } k_{i,j},$$

$$\delta_{2h} g_h = \frac{1}{\Delta y}(g_{i,j} - g_{i,j-1}) \quad \text{on } k_{i,j}.$$

We can now define in an obvious way, based on (3.9) the finite difference operator $\mathbf{A}_h = (A_{1h}, A_{2h}, A_{3h})$, operating on $\mathbf{V}_h$

$$\begin{cases} A_{1h}\mathbf{u}_h = \widetilde{u}_0 \delta_{1h} u_h + \widetilde{v}_0 \delta_{2h} u_h + g\delta_{1h}\phi_h, \\ A_{2h}\mathbf{u}_h = \widetilde{u}_0 \delta_{1h} v_h + \widetilde{v}_0 \delta_{2h} v_h + g\delta_{2h}\phi_h, \\ A_{3h}\mathbf{u}_h = \widetilde{u}_0 \delta_{1h} \phi_h + \widetilde{v}_0 \delta_{2h} \phi_h + \widetilde{\phi}_0 \delta_{1h} u_h + \widetilde{\phi}_0 \delta_{2h} v_h \end{cases} \tag{3.11}$$

and its decomposition $\mathbf{A}_h = \mathbf{A}_h^x + \mathbf{A}_h^y$, to be used later on,

$$\begin{cases} A_h^x \mathbf{u}_h = (\widetilde{u}_0 \delta_{1h} u_h + g\delta_{1h}\phi_h, \widetilde{u}_0 \delta_{1h} v_h, \widetilde{u}_0 \delta_{1h}\phi_h + \widetilde{\phi}_0 \delta_{1h} u_h), \\ A_h^x \mathbf{u}_h = (\widetilde{v}_0 \delta_{2h} u_h, \widetilde{v}_0 \delta_{2h} v_h + g\delta_{2h}\phi_h, \widetilde{v}_0 \delta_{2h}\phi_h + \widetilde{\phi}_0 \delta_{2h} v_h). \end{cases} \tag{3.12}$$

Those are the discrete versions of $\mathbf{A}, A_1, A_2, A_3, \mathbf{A}^x, \mathbf{A}^y$.

We can now check that $\mathbf{A}_h$, the discrete version of $\mathbf{A}$, is positive like $\mathbf{A}$.

Lemma 3.2 *For all* $\mathbf{u}_h = (u_h, v_h, \phi_h) \in \mathbf{V}_h$, *we have*

$$\langle \mathbf{A}_h \mathbf{u}_h, \mathbf{u}_h \rangle \geq 0, \tag{3.13}$$

where $\langle \cdot, \cdot \rangle$ *is the scalar product on* $L^2(\mathcal{M})^3$, *given by* (3.6).

Proof We write

$$\langle \mathbf{A}_h \mathbf{u}_h, \mathbf{u}_h \rangle = \langle A_h^x \mathbf{u}_h, \mathbf{u}_h \rangle + \langle A_h^y \mathbf{u}_h, \mathbf{u}_h \rangle, \tag{3.14}$$

where

$$\begin{aligned} \langle A_h^x \mathbf{u}_h, \mathbf{u}_h \rangle &= (\widetilde{u}_0 \delta_{1h} u_h, u_h) + (g\delta_{1h}\phi_h, u_h) + (\widetilde{u}_0 \delta_{1h} v_h, v_h) \\ &\quad + \frac{g}{\widetilde{\phi}_0}(\widetilde{u}_0 \delta_{1h}\phi_h, \phi_h) + g(\delta_{1h} u_h, \phi_h), \\ \langle A_h^y \mathbf{u}_h, \mathbf{u}_h \rangle &= (\widetilde{v}_0 \delta_{2h} u_h, u_h) + (g\delta_{2h}\phi_h, v_h) + (\widetilde{v}_0 \delta_{2h} v_h, v_h) \\ &\quad + \frac{g}{\widetilde{\phi}_0}(\widetilde{v}_0 \delta_{2h}\phi_h, \phi_h) + g(\delta_{2h} v_h, \phi_h). \end{aligned}$$

We first remark that

$$\begin{aligned} (\widetilde{u}_0 \delta_{1h} u_h, u_h) &= \frac{\widetilde{u}_0}{2} \Delta y \sum_{i=1}^{3N_1} \sum_{j=1}^{3N_2} (|u_{i,j}|^2 - |u_{i-1,j}|^2 + |u_{i,j} - u_{i-1,j}|^2) \\ &= \frac{\widetilde{u}_0}{2} \Delta y \sum_{j=1}^{3N_2} \Big(|u_{3N_{1},j}|^2 + \sum_{i=1}^{3N_1} |u_{i,j} - u_{i-1,j}|^2 \Big). \end{aligned} \tag{3.15}$$

Then we write

$$(\phi_{i,j} - \phi_{i-1,j}) u_{i,j} + (u_{i,j} - u_{i-1,j}) \phi_{i,j} = u_{i,j}\phi_{i,j} - u_{i-1,j}\phi_{i-1,j} + (u_{i,j} - u_{i-1,j})(\phi_{i,j} - \phi_{i-1,j}).$$

Using these two formulas, we obtain

$$\begin{aligned}
&(\widetilde{u}_0\delta_{1h}u_h,u_h)+\frac{g}{\widetilde{\phi}_0}(\widetilde{u}_0\delta_{1h}\phi_h,\phi_h)+(g\delta_{1h}\phi_h,u_h)+g(\delta_{1h}u_h,\phi_h)\\
=&\frac{\widetilde{u}_0}{2}\Delta y\sum_j\Big(|u_{3N_1,j}|^2+\frac{g}{\widetilde{\phi}_0}|\phi_{3N_1,j}|^2\Big)\\
&+\Delta y\frac{\widetilde{u}_0}{2}\sum_{i,j}|u_{i,j}-u_{i-1,j}|^2+\Delta y\frac{g\widetilde{u}_0}{2\widetilde{\phi}_0}\sum_{i,j}|\phi_{i,j}-\phi_{i-1,j}|^2\\
&+g\Delta y\sum_{i,j}(u_{i,j}-u_{i-1,j})(\phi_{i,j}-\phi_{i-1,j})+g\Delta y\sum_j u_{3N_1,j}\phi_{3N_1,j}.
\end{aligned}$$

Since $\widetilde{u}_0>0$ and $\widetilde{u}_0^2>g\widetilde{\phi}_0$, the expressions

$$\frac{\widetilde{u}_0}{2}|u_{i,j}-u_{i-1,j}|^2+\frac{g\widetilde{u}_0}{2\widetilde{\phi}_0}|\phi_{i,j}-\phi_{i-1,j}|^2+g(u_{i,j}-u_{i-1,j})(\phi_{i,j}-u\phi_{i-1,j})$$

and

$$\frac{\widetilde{u}_0}{2}|u_{3N_1,j}|^2+\frac{g\widetilde{u}_0}{2\widetilde{\phi}_0}|\phi_{3N_1,j}|^2+gu_{3N_1,j}\phi_{3N_1,j}$$

are positive and the corresponding sums are positive as well.

Finally, using also the analogue of (3.15) for v_h, we conclude that $\langle A_h^x\mathbf{u}_h,\mathbf{u}_h\rangle\geq 0$. Similarly, it can be checked that $\langle A_h^y\mathbf{u}_h,\mathbf{u}_h\rangle\geq 0$. Recalling (3.14), this completes the proof of Lemma 3.2.

In fact, a perusal of the calculations above shows that we have proved the following useful lemma.

Lemma 3.3 *For every $\mathbf{u}_h\in\mathbf{V}_h$,*

$$\begin{cases}
\langle\mathbf{A}_h^x\mathbf{u}_h,\mathbf{u}_h\rangle\geq\kappa_1\Delta y\sum_{j=1}^{3N_2}\Big[|\mathbf{u}_{3N1,j}|^2+\sum_{i=1}^{3N_1}|\mathbf{u}_{i,j}-\mathbf{u}_{i-1,j}|^2\Big],\\
\langle\mathbf{A}_h^y,\mathbf{u}_h,\mathbf{u}_h\rangle\geq\kappa_1\Delta x\sum_{i=1}^{3N_1}\Big[|\mathbf{u}_{i,3N_2}|^2+\sum_{j=1}^{3N_1}|\mathbf{u}_{i,j}-\mathbf{u}_{i,j-1}|^2\Big],
\end{cases}\tag{3.16}$$

where the constant κ_1 depends on $\widetilde{u}_0,\widetilde{v}_0,\widetilde{\phi}_0,g$ and in particular on the positive numbers $\widetilde{u}_0^2-g\widetilde{\phi}_0,\widetilde{v}_0^2-g\widetilde{\phi}_0$.

3.2.2 Multilevel finite-volume discretization

We introduce the coarse mesh consisting of the rectangles K_{lm} $(1\leq l\leq N_1, 1\leq m\leq N_2)$,

$$K_{lm}=\bigcup_{\alpha,\beta=0}^{2}k^*_{3l-\alpha,3m-\beta}=(x_{3l-2-\frac{1}{2}},x_{3l-\frac{1}{2}})\times(y_{3m-2-\frac{1}{2}},y_{3m+\frac{1}{2}}).$$

We also define the fictitious rectangles $K_{0,m},K_{l,0}$ $(l=1,\cdots N_1,\ m=1,\cdots,N_2)$, needed for the implementation of the boundary conditions, and they are defined as above with m or $l=0$.

* Including, strictly speaking, the separation edges.

We introduce the space V_{3h} defined like V_h. If $u_h \in V_h$ and $u_h|_{k_{ij}} = u_{i,j}$, we define for $l = 1, \cdots, N_1,\ m = 1, \cdots, N_2$ the averages as

$$U_{l,m} = \frac{1}{9} \sum_{\alpha,\beta=0}^{2} u_{3l-\alpha,3m-\beta}, \tag{3.17}$$

and the incremental unknowns as

$$Z_{3l-\alpha,3m-\beta} = u_{3l-\alpha,3m-\beta} - U_{l,m}, \tag{3.18}$$

which satisfy of course

$$\sum_{\alpha,\beta=0}^{2} Z_{3l-\alpha,3m-\beta} = 0. \tag{3.19}$$

We note the following algebraic relations (using (3.19)):

$$\sum_{\alpha,\beta=0}^{2} |u_{3l-\alpha,3m-\beta}|^2 = 9|U_{l,m}|^2 + \sum_{\alpha,\beta=0}^{2} |Z_{3l-\alpha,3m-\beta}|^2. \tag{3.20}$$

Multiplying by $\Delta x \Delta y$ and adding for $l = 1, \cdots, N_1,\ m = 1, \cdots, N_2$, we find

$$|u_h|^2 = |U_h|^2 + |Z_h|^2, \tag{3.21}$$

where $|\cdot|$ is still the norm in $L^2(\mathcal{M}), U_h$ is the step function equal to U_{lm} on $K_{l,m}$ and Z_h is the step function equal to $Z_{i,j}$ on $k_{i,j}$.

3.3 Euler implicit time discretization and estimates

We proceed to some extent as in the one-dimensional space. We define a time step Δt with $N_T \Delta t = T$, and set $t_n = n\Delta t$. We denote by

$$\mathbf{u}_h^n = \{\mathbf{u}_{i,j}^n,\ 1 \le i \le 3N_1,\ 1 \le j \le 3N_2\}$$

the discrete unknowns, where $\mathbf{u}_{i,j}^n$ is an expected approximation

$$\mathbf{u}_{i,j}^n \simeq \frac{1}{\Delta x \Delta y} \int_{k_{i,j}} \mathbf{u}(x, y, t_n) \mathrm{d}x \mathrm{d}y.$$

The spatial discretization has been presented in Section 3.2. We will now discretize the shallow water equations in time by using the implicit Euler scheme, and advance equation (3.9) for p steps in time on the fine mesh with a time step of $\frac{\Delta t}{p}$, where p (and q below) are two fixed integers larger than 1.

These steps will bring us, e.g., from t_n to t_{n+1}. We then perform q steps with a time step Δt bringing us from t_{n+1} to t_{n+q+1}. For simplicity, we suppose that N_T is a multiple of $q+1$, and we set $N_q = \frac{N_T}{q+1}$. The steps performed with the time step Δt will use the coarse mesh. We first consider in Section 3.3.1 the p steps performed with mesh $\frac{\Delta t}{p}$ on the fine grid. Then the q steps on the coarse grid are described in Section 3.3.2.

3.3.1 Scheme and estimates on the fine grid

We start from equations (3.9) and write thus for $s = 1, \cdots, p$,

$$\begin{cases} \frac{p}{\Delta t}(u_{i,j}^{n+\frac{s+1}{p}} - u_{i,j}^{n+\frac{s}{p}}) + \widetilde{u}_0\delta_{1h}u_{i,j}^{n+\frac{s+1}{p}} \\ \quad +\widetilde{v}_0\delta_{2h}u_{i,j}^{n+\frac{s+1}{p}} + g\delta_{1h}\phi_{i,j}^{n+\frac{s+1}{p}} = f_{u,i,j}^{n+\frac{s+1}{p}}, \\ \frac{p}{\Delta t}(v_{i,j}^{n+\frac{s+1}{p}} - v_{i,j}^{n+\frac{s}{p}}) + \widetilde{u}_0\delta_{1h}v_{i,j}^{n+\frac{s+1}{p}} \\ \quad +\widetilde{v}_0\delta_{2h}v_{i,j}^{n+(s+\frac{1}{p})} + g\delta_{2h}\phi_{i,j}^{n+\frac{s+1}{p}} = f_{v,i,j}^{n+\frac{s+1}{p}}, \\ \frac{p}{\Delta t}(\phi_{i,j}^{n+\frac{s+1}{p}} - \phi_{i,j}^{n+\frac{s}{p}}) + \widetilde{u}_0\delta_{1h}\phi_{i,j}^{n+\frac{s+1}{p}} \\ \quad +\widetilde{v}_0\delta_{2h}\phi_{i,j}^{n+\frac{s+1}{p}} + \widetilde{\phi}_0(\delta_{1h}u_{i,j}^{n+\frac{s+1}{p}} + \delta_{2h}v_{i,j}^{n+\frac{s+1}{p}}) = f_{\phi,i,j}^{n+\frac{s+1}{p}}. \end{cases} \tag{3.22}$$

With the definition of $\mathbf{A}_h$ introduced in (3.11), equations (3.22) amount to

$$\frac{p}{\Delta t}(\mathbf{u}_h^\tau - \mathbf{u}_h^{\tau-\frac{1}{p}}) + A_h\mathbf{u}_h^\tau = \mathbf{f}_h^\tau. \tag{3.23}$$

Here we have set for simplicity $n + \frac{s+1}{p} = \tau$, $n + \frac{s}{p} = \tau - \frac{1}{p}$, $\mathbf{u}_h^\tau = (u_h^\tau, v_h^\tau, \phi_h^\tau)$, $\mathbf{f}_h^\tau = (f_{u,h}^\tau, f_{v,h}^\tau, f_{\phi,h}^\tau)$.

Taking the scalar product in $\mathbf{V}_h$ of each side of (3.23) with $2\frac{\Delta t}{p}\mathbf{u}^\tau$, we see that

$$\begin{aligned} &|\mathbf{u}_h^\tau|^2 - |\mathbf{u}_h^{\tau-\frac{1}{p}}|^2 + |\mathbf{u}_h^\tau - \mathbf{u}_h^{\tau-\frac{1}{p}}|^2 + 2\frac{\Delta t}{p}\langle \mathbf{A}_h\mathbf{u}_h^\tau, \mathbf{u}_h^\tau\rangle \\ &= \frac{2\Delta t}{p}\langle \mathbf{f}_h{}^\tau, \mathbf{u}_h^\tau\rangle \le \frac{\Delta t}{p}|\mathbf{f}_h{}^\tau|^2 + \frac{\Delta t}{p}|\mathbf{u}_h{}^\tau|^2. \end{aligned} \tag{3.24}$$

Hence thanks to Lemma 3.2 (comparing with (2.19)–(2.25)),

$$|\mathbf{u}_h^{n+\frac{s+1}{p}}|^2 \le \frac{1}{1-\frac{\Delta t}{p}}|\mathbf{u}_h^{n+\frac{s}{p}}|^2 + \frac{1}{1-\frac{\Delta t}{p}}\frac{\Delta t}{p}|\mathbf{f}_h^{n+\frac{s}{p}}|^2, \tag{3.25}$$

and for $\frac{\Delta t}{p} \le \frac{1}{2}$ and $s = 1, \cdots, p$ (comparing with (2.25)),

$$\begin{aligned} &|\mathbf{u}_h^{n+\frac{s}{p}}|^2 \le 4^{\frac{s\Delta t}{p}}\kappa^n(\mathbf{u}^0, \mathbf{f}), \\ &\kappa^n(\mathbf{u}^0, \mathbf{f}) = |\mathbf{u}_h^0|^2 + \int_{n\Delta t}^{(n+1)\Delta t}\int_0^{L_2}\int_0^{L_1}|\mathbf{f}(x,y,t)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}t. \end{aligned}$$

In particular, for $s = p$,

$$|\mathbf{u}_h^{n+1}|^2 \le 4^{\Delta t}\kappa^n(\mathbf{u}^0, \mathbf{f}). \tag{3.26}$$

3.3.2 Scheme and estimates on the coarse grid

We now consider the q a time-steps performed on the coarse grid with a time step Δt.

We discretize the equations (3.9) in time, starting from time $t_{n+1} = (n+1)\Delta t$ using the same scheme as for equations (3.22) but with a coarse mesh (comparing with (2.26)). We obtain

$$\frac{1}{\Delta t}(\mathbf{U}_h^\tau - \mathbf{U}_h^{\tau-1}) + \mathbf{A}_{3h}\mathbf{U}_h^\tau = \mathbf{F}_h^\tau, \tag{3.27}$$

where $\tau = n+s+1,\ s = 1,\cdots,q,\ \mathbf{U}_h^\tau = (U_{u,h}^\tau, U_{v,h}^\tau, U_{\phi,h}^\tau)$ and $\mathbf{U}_h \in \mathbf{V}_{3h}$ has components $\mathbf{U}_{i,j}$ on $K_{i,j}$ $(i = 0,\cdots,N_1,\ j = 0,\cdots,N_2)$. Finally, $\mathbf{F}_h^\tau$ has components $\mathbf{F}_{i,j}^\tau$ on $K_{i,j}$ with

$$\mathbf{F}_{i,j}^\tau = \frac{1}{\Delta t}\frac{1}{9\Delta x\Delta y}\int_{(\tau-1)\Delta t}^{\tau\Delta t}\int_{K_{i,j}} \mathbf{f}(x,y,t)\mathrm{d}x\mathrm{d}y\mathrm{d}t. \tag{3.28}$$

A priori estimates are obtained by taking the scalar product in $\mathbf{V}_{3h}$ of each side of (3.27) with $6\Delta t\mathbf{U}_h^\tau$. We find (comparing with (2.31))

$$|\mathbf{U}_h^\tau|^2 - |\mathbf{U}_h^{\tau-1}|^2 + |\mathbf{U}_h^\tau - \mathbf{U}_h^{\tau-1}|^2 + 2\Delta t(\mathbf{A}_{3h}\mathbf{U}_h^\tau, \mathbf{U}_h^\tau) = 2\Delta t(\mathbf{F}_h^\tau, \mathbf{U}_h^\tau),$$

and in view of Lemma 3.2 (for $\mathbf{A}_{3h}$),

$$\begin{aligned}
|\mathbf{U}_h^\tau|^2 &\le |\mathbf{U}_h^{\tau-1}|^2 + 2\Delta t|\mathbf{F}_h^\tau||\mathbf{U}_h^\tau| \\
&\le \Delta t|\mathbf{U}_h^\tau|^2 + |\mathbf{U}_h^{\tau-1}|^2 + \Delta t|\mathbf{F}_h^\tau|^2, \\
|\mathbf{U}_h^\tau|^2 &\le \frac{1}{1-\Delta t}[|\mathbf{U}_h^{\tau-1}|^2 + |\mathbf{F}_h^\tau|^2] \\
&\le \frac{1}{1-\Delta t}\Big[|\mathbf{U}_h^{\tau-1}|^2 + \int_{(\tau-1)\Delta t}^{\tau\Delta t} |\mathbf{f}(\,\cdot\,,t)|_{L^2}^2\mathrm{d}t\Big].
\end{aligned}$$

Thus, for $\Delta t \le \frac{1}{2}$,

$$|\mathbf{U}_h^\tau|^2 \le 4^{\Delta t}\Big[|\mathbf{U}_h^{\tau-1}|^2 + \int_{(\tau-1)\Delta t}^{\tau\Delta t} |\mathbf{f}(\,\cdot\,,t)|_{L^2}^2\mathrm{d}t\Big]. \tag{3.29}$$

We write the equations (3.29) for $\tau = n+s+1,\ s = 1,\cdots,q$. We multiply the equation for $\tau = n+s+1$ by $4^{(q-s)\Delta t}$ and add these equations for $s = 1,\cdots,q$. We find

$$|\mathbf{U}_h^{n+q+1}|^2 \le 4^{q\Delta t}\Big[|\mathbf{U}_h^{n+1}|^2 + \int_{(n+1)\Delta t}^{(n+q+1)\Delta t} |\mathbf{f}(\cdot,t)|_{L^2}^2\mathrm{d}t\Big]. \tag{3.30}$$

During the steps from $(n+1)\Delta t$ to $(n+q+1)\Delta t$, the $\mathbf{Z}_h$ are frozen. Thus

$$\mathbf{Z}_h^{n+s+1} = \mathbf{Z}_h^{n+1}, \quad s = 1,\cdots,q, \tag{3.31}$$

and we recover the $\mathbf{u}_h^{n+s+1}$ in the form

$$\mathbf{u}_h^{n+s+1} = \mathbf{U}_h^{n+s+1} + \mathbf{Z}_h^{n+1}. \tag{3.32}$$

Then, because of (3.30) and (2.15),

$$|\mathbf{u}_h^{n+q+1}|^2 \le 4^{q\Delta t}\Big[|\mathbf{u}_h^{n+1}|^2 + \int_{(n+1)\Delta t}^{(n+q+1)\Delta t} |\mathbf{f}(\cdot,t)|_{L^2}^2\mathrm{d}t\Big]. \tag{3.33}$$

Combining (3.33) with (3.26), we find

$$|\mathbf{u}_h^{n+q+1}|^2 \le 4^{(q+1)\Delta t}\Big[|\mathbf{u}_h^{n}|^2 + \int_{n\Delta t}^{(n+q+1)\Delta t} |\mathbf{f}(\cdot,t)|_{L^2}^2\mathrm{d}t\Big]. \tag{3.34}$$

We can repeat the procedure for any interval of time $(n\Delta t, (n+q+1)\Delta t),\ n = 1, \cdots, N_q$, and arrive at the stability result

$$\begin{aligned}|\mathbf{u}_h^m|^2 &\le 4^{m\Delta t}\Big[|\mathbf{u}_h^0|^2 + \int_0^{m\Delta t} |\mathbf{f}(\cdot,t)|_{L^2}^2 \mathrm{d}t\Big] \\ &\le 4^T\Big[|\mathbf{u}^0|^2 + \int_0^T |\mathbf{f}(\cdot,t)|_{L^2}^2 \mathrm{d}t\Big] \qquad (3.35)\end{aligned}$$

valid for $m = 1, \cdots, N_q$.

Theorem 3.1 *The multilevel scheme defined by the equations* (3.22) *and* (3.27) *is stable in* $L^\infty(0,T;L^2(\mathcal{M})^3)$ *in the sense of* (3.35).

4 Other Schemes and Other Methods

The coarse grid schemes that we have used in Sections 2 and 3 amount to using the same schemes on the coarse grid as on the fine grid. Another possibility for the coarse grid is to average on each coarse grid the fine grid equations associated with the corresponding fine grids. These schemes are made explicit below. However, the study of the stability of these new schemes appears difficult, and we will only present the study of stability in the one-dimensional case for the simple transport equation (see Section 4.1), and for a one-dimensional shallow water equation (see Section 4.2). Furthermore, the boundary condition will be space periodicity, and the stability analysis is made by the von Neumann method (see [22]).

4.1 The one-dimensional case

We start with the one-dimensional space, and consider the same problem as (2.1), with $f = 0$,

$$\frac{\delta u}{\delta t}(x,t) + \frac{\delta u}{\delta x}(x,t) = 0 \qquad (4.1)$$

for $(x,t) \in (0,L) \times (0,T)$, and with the space periodicity boundary condition, and the initial condition

$$u(x,0) = u^0(x). \qquad (4.2)$$

On the fine grid, we will perform an approximation by the implicit Euler scheme in time and upwind finite-volume in space, so that the scheme will be very much like the one in (2.16) except that the second formula of (2.16) is replaced by the periodicity condition

$$u_0^{n+\frac{s+1}{p}} = u_{3N}^{n+\frac{s+1}{p}}. \qquad (4.3)$$

We perform p steps with a time step $\frac{\Delta t}{p}$ and a space mesh $\Delta x = \frac{L}{3N}$. Then as explained below, we make q steps with a time step Δt and a mesh step $3\Delta x$. Thus we start again with the p steps.

4.1.1 The fine grid scheme with a small time step

The scheme reads

$$\frac{p}{\Delta t}(u_j^\tau - u_j^{\tau-\frac{1}{p}}) + \frac{1}{\Delta x}(u_j^\tau - u_{j-1}^\tau) = 0, \tag{4.4}$$

where $\tau = n + \frac{s}{p}$, $s = 1, \cdots, p$, $j = 1, \cdots, 3N$, u_j^τ is meant to be an approximation of $\frac{1}{\Delta x}\int_{k_j} u(x, \tau\Delta t)\mathrm{d}x$ with $k_j = ((j-1)h, jh)$ and $h = \Delta x$; $u_0^\tau = u_{3N}^\tau$ by periodicity.

We associate with a sequence v_j, and its Fourier transform (see [22, p. 38]) is as follows:

$$\widehat{v}(\xi) = \frac{1}{2\pi}\sum_{j=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}jh\xi} v_j h. \tag{4.5}$$

Below we will consider periodic sequences v_j, $j \in \mathbb{Z}$, $v_{j+3N} = v_j$, $h^* = \frac{2\pi}{3N}$ and define the discrete Fourier coefficients (see [5, 9, 22])

$$\widehat{v}_m = \frac{1}{3N}\sum_{j=1}^{3N} \mathrm{e}^{-\mathrm{i}mjh^*} v_j, \quad m = 1, \cdots, 3N. \tag{4.6}$$

We then have the discrete Parseval formula

$$\sum_{m=1}^{3N} |\widehat{v}_m|^2 = \frac{1}{3N}\sum_{j=1}^{3N} |v_j|^2 \tag{4.7}$$

(see the details in [3, 22]). Note that the sequence $\{\widehat{v}_m\}$ is itself periodic with period $3N$, and if $(\sigma v)_j = v_{j-1}$, then

$$\widehat{\sigma v}_m = \mathrm{e}^{-\mathrm{i}mh^*}\widehat{v}_m. \tag{4.8}$$

Then (4.4) is rewritten as

$$\Big(1 + \frac{\Delta t}{p\Delta x}\Big)u_j^\tau - \frac{\Delta t}{p\Delta x}u_{j-1}^\tau = u_j^{\tau-\frac{1}{p}}, \tag{4.9}$$

that is, for the Fourier transforms defined as in (4.6), where $h^* = \frac{2\pi}{3N}$,

$$\Big(1 + \frac{\Delta t}{p\Delta x}(1 - \mathrm{e}^{-\mathrm{i}mh^*})\Big)\widehat{u}_m^\tau = \widehat{u}_m^{\tau-\frac{1}{p}}, \quad m = 1, \cdots, 3N. \tag{4.10}$$

Hence the amplification factor for the fine mesh is

$$g_{F,m} = \Big[1 + \frac{\Delta t}{p\Delta x}(1 - \mathrm{e}^{-\mathrm{i}mh^*})\Big]^{-1}, \quad m = 1, \cdots, 3N. \tag{4.11}$$

We observe that

$$\begin{aligned}
g_{F,m}^{-1} &= \Big[1 + \frac{\Delta t}{p\Delta x}(1 - \cos(h^* m)) + \mathrm{i}\frac{\Delta t}{p\Delta x}\sin(h^* m)\Big], \\
|g_{F,m}^{-1}|^2 &= \Big[1 + \frac{\Delta t}{p\Delta x}(1 - \cos(h^* m))\Big]^2 + \Big(\frac{\Delta t}{p\Delta x}\Big)^2 \sin^2(h^* m), \\
&= 1 + 2(1 - \cos(h^* m))\Big(\Big(\frac{\Delta t}{p\Delta x}\Big)^2 + \frac{\Delta t}{p\Delta x}\Big).
\end{aligned}$$

We conclude that

$$|g_{F,m}| \leq 1, \quad m = 1, \cdots, 3N. \tag{4.12}$$

Recall that $\tau = n + \frac{s}{p}$, $s = 1, \cdots, p$. Denoting by u_h^τ the piecewise constant function given by $u_h^\tau = u_j^\tau$ on k_j, (4.7) and (4.12) yield

$$\begin{aligned} |u_h^{n+\frac{s}{p}}|^2 &= \sum_{j=1}^{3N} \Delta x |u_j^{n+\frac{s}{p}}|^2 = 3N\Delta x \sum_{m=1}^{3N} |\widehat{u}_m^{n+\frac{s}{p}}|^2 \\ &\leq 3N\Delta x \sum_{m=1}^{3N} |\widehat{u}_m^n|^2 = |u_h^n|^2 \quad \text{for } s = 1, \cdots, q. \end{aligned}$$

In particular, for $s = p$,

$$|u_h^{n+1}|^2 \leq |u_h^n|^2, \tag{4.13}$$

and therefore these steps of the scheme (4.4) on the fine grid are stable for the L^2-norm.

4.1.2 The coarse grid scheme with a “large” time step

Considering first the analogue of (4.4) with a time step Δt and a space mesh Δx, we would write ($\tau = n + s + 1$ now, $s = 1, \cdots, q$)

$$\frac{1}{\Delta t}(u_j^\tau - u_j^{\tau-1}) + \frac{1}{\Delta x}(u_j^\tau - u_{j-1}^\tau) = 0. \tag{4.14}$$

To obtain the scheme with a time step Δt and a space mesh $3\Delta x$, we add (average) the equations (4.14) corresponding to $j = 3l, 3l-1, 3l-2$.

Setting

$$U_l^\tau = \frac{1}{3}(u_{3l}^\tau + u_{3l-1}^\tau + u_{3l-2}^\tau), \tag{4.15}$$

we obtain

$$\frac{1}{\Delta t}(U_l^\tau - U_l^{\tau-1}) + \frac{1}{3\Delta x}(u_{3l}^\tau - u_{3l-3}^\tau) = 0 \tag{4.16}$$

for $l = 1, \cdots, N$.

We elaborate on the $u = U + Z$ decomposition (independent of the time step).

The $u = U + Z$ decomposition

Given the sequence u_j, $j = 1, \cdots, 3N$ ($u_0 = u_{3N}$), we define the sequence

$$U_\ell = \frac{1}{3}\sum_{\alpha=0}^{2} u_{3l-\alpha}, \quad l = 1, \cdots, N, \tag{4.17}$$

and the sequences

$$Z_{3l-\alpha} = u_{3l-\alpha} - U_l, \tag{4.18}$$

$\alpha = 0, 1, 2,\ \ell = 1, \cdots, N$. We observe that

$$\sum_{\alpha=0}^{2} Z_{3l-\alpha} = 0.$$

Now the multistep algorithm that we consider consists in freezing the Z during the step $n + 2, \cdots, n + q + 1$, that is,

$$Z_j^{n+s+1} = Z_j^{n+1}, \quad s = 1, \cdots, q, \quad j = 1, \cdots, 3N, \tag{4.19}$$

so that

$$Z_{3l-\alpha}^{\tau} = Z_{3l-\alpha}^{n+1} = u_{3l-\alpha}^{\tau} - U_l^{\tau}$$

for $\alpha = 0, 1, 2$, $\tau = n + s + 1$, $s = 1, \cdots, q$. Hence $U_l^{\tau} - U_l^{\tau-1} = u_{3\ell-\alpha}^{\tau} - u_{3\ell-\alpha}^{\tau-1}$ for $\alpha = 0, 1, 2$, and for those values of τ. With $\alpha = 0$, (4.16) becomes

$$\frac{1}{\Delta t}(u_{3l}^{\tau} - u_{3l}^{\tau-1}) + \frac{1}{3\Delta x}(u_{3l}^{\tau} - u_{3l-3}^{\tau}) = 0. \tag{4.20}$$

That is, as in (4.9),

$$\Big(1 + \frac{\Delta t}{3\Delta x}\Big)u_{3l}^{\tau} - \frac{\Delta t}{3\Delta x}u_{3l-3}^{\tau} = u_{3l}^{\tau-1}. \tag{4.21}$$

Before we introduce the Fourier transform of (4.21) and the amplification function similar to the g_F, we have to elaborate a bit more on the $u = U + Z$ decomposition at the level of the Fourier transforms.

We write (independent of the time step τ), with $h^* = \frac{2\pi}{3N}$ for $m = 1, \cdots, 3N$,

$$\begin{aligned}\widehat{u}_m &= \frac{1}{3N}\sum_{j=1}^{3N} u_j \mathrm{e}^{-\mathrm{i}h^* jm} \\ &= \frac{1}{3N}\sum_{\ell=1}^{N}(u_{3l}\mathrm{e}^{-3\mathrm{i}h^* lm} + u_{3l-1}\mathrm{e}^{-\mathrm{i}h^*(3l-1)m} + u_{3l-2}\mathrm{e}^{-\mathrm{i}h^*(3l-2)m}).\end{aligned}$$

We now introduce the partial Fourier sum of the type of (4.6),

$$\widehat{u}_{(3l-\alpha),m} = \frac{1}{3N}\sum_{\ell=1}^{N} u_{3l-\alpha}\mathrm{e}^{-\mathrm{i}h^* 3lm}. \tag{4.22}$$

We observe that this partial Fourier sum is periodic in m with period $3N$, and that Parseval relation similar to (4.7) holds,

$$\sum_{m=1}^{3N} |\widehat{u}_{(3l-\alpha),m}|^2 = \frac{1}{3N}\sum_{\ell=1}^{N} |u_{3l-\alpha}|^2, \quad \alpha = 0, 1, 2. \tag{4.23}$$

We can hence write

$$\widehat{u}_m = \widehat{u}_{(3l),m} + \mathrm{e}^{\mathrm{i}h^* m}\widehat{u}_{(3l-1),m} + \mathrm{e}^{2\mathrm{i}h^* m}\widehat{u}_{(3l-2),m}. \tag{4.24}$$

Then

$$\widehat{u}_{(3l-3),m} = \widehat{u}_{(3l),m}\mathrm{e}^{-3\mathrm{i}h^* m}, \tag{4.25}$$

and now (4.21) yields by a partial Fourier transform

$$\Big(1+\frac{\Delta t}{3\Delta x}\Big)\widehat{u}^{\tau}_{(3l),m} - \frac{\Delta t}{3\Delta x}\mathrm{e}^{-3\mathrm{i}h^* m}\widehat{u}^{\tau}_{(3l),m} = \widehat{u}^{\tau-1}_{(3l),m}. \tag{4.26}$$

That is,

$$\widehat{u}^{\tau}_{(3l),m} = g_{C,m}\widehat{u}^{\tau-1}_{(3l),m}, \quad m = 1,\cdots,3N, \tag{4.27}$$

corresponding to the amplification factor $g_{C,m}$ with

$$g^{-1}_{C,m} = 1 + \frac{\Delta t}{3\Delta x}(1-\mathrm{e}^{-3\mathrm{i}h^* m}). \tag{4.28}$$

We can conclude as before that $|g^{-1}_{C,m}| \geq 1$,

$$|g_{C,m}| \leq 1, \quad m = 1,\cdots,3N, \tag{4.29}$$

and thus the scheme (4.21), (4.26) is "stable". Also

$$\widehat{u}^{n+s+1}_{(3l),m} = g^{s}_{C,m}\widehat{u}^{n+1}_{(3l),m}, \quad m = 1,\cdots,3N,\ s = 1,\cdots,q. \tag{4.30}$$

The important point now is that we know nothing about the stability of the u^{τ}_{3l-1}, u^{τ}_{3l-2}, and we have to elaborate more to prove this stability.

In the similar way to (4.24), we write for $m = 1,\cdots,3N$,

$$\widehat{Z}_m = \widehat{Z}_{(3l),m} + \mathrm{e}^{\mathrm{i}h^* m}\widehat{Z}_{(3l-1),m} + \mathrm{e}^{2\mathrm{i}h^* m}\widehat{Z}_{(3l-2),m}. \tag{4.31}$$

The relations

$$u_{3l-\alpha} = U_l + Z_{3l-\alpha}, \quad \alpha = 0,1,2$$

given by the partial discrete Fourier transform for $m = 1,\cdots,3N$,

$$\widehat{u}_{(3l-\alpha),m} = \widehat{U}_{(l),m} + \widehat{Z}_{(3l-\alpha),m}. \tag{4.32}$$

Hence with (4.19) and (4.32),

$$\widehat{u}^{n+s+1}_{(3l-\alpha),m} = \widehat{U}^{n+s+1}_{(l),m} + \widehat{Z}^{n+1}_{(3l-\alpha),m}. \tag{4.33}$$

Using (4.30), we obtain the expression of $\widehat{U}^{n+s+1}_{(l),m}$ for $\alpha = 0$,

$$\widehat{U}^{n+s+1}_{(l),m} = g^{s}_{C,m}\widehat{u}^{n+1}_{(3l),m} - \widehat{Z}^{n+1}_{(3l),m}, \quad m = 1,\cdots,3N. \tag{4.34}$$

There remains to express $\widehat{Z}^{n+1}_{(3l)}$ in terms of the $\widehat{u}^{n+1}_{(3l-\alpha),m}$, $\alpha = 0,1,2$.

We proceed in the physical space, independent of the time step τ, to have

$$U_l = \frac{1}{3}(u_{3l} + u_{3l-1} + u_{3l-2})$$

and

$$\begin{cases} Z_{3l} = u_{3l} - U_l = \dfrac{1}{3}(2u_{3l} - u_{3l-1} - u_{3l-2}), \\ Z_{3l-1} = u_{3l-1} - U_l = \dfrac{1}{3}(2u_{3l-1} - u_{3l} - u_{3l-2}), \\ Z_{3l-2} = u_{3l-2} - U_l = \dfrac{1}{3}(2u_{3l-2} - u_{3l} - u_{3l-1}). \end{cases} \tag{4.35}$$

Thus for the Fourier transforms, for $m = 1, \cdots, 3N$,

$$\begin{cases} \widehat{Z}_{(3l),m} = \dfrac{1}{3}(2\widehat{u}_{(3l),m} - \widehat{u}_{(3l-1),m} - \widehat{u}_{(3l-2),m}), \\ \widehat{Z}_{(3l-1),m} = \dfrac{1}{3}(2\widehat{u}_{(3l-1),m} - \widehat{u}_{(3l),m} - \widehat{u}_{(3l-2),m}), \\ \widehat{Z}_{(3l-2),m} = \dfrac{1}{3}(2\widehat{u}_{(3l-2),m} - \widehat{u}_{(3l),m} - \widehat{u}_{(3l-1),m}). \end{cases} \tag{4.36}$$

This holds in particular at the time step $\tau = n + 1$.

Now we look for the expression of the $\widehat{u}^{n+s+1}_{(3l-\alpha),m}$ $(\alpha = 0, 1, 2)$, in terms of the $\widehat{u}^{n+1}_{(3l-\beta),m}$, that of $\widehat{u}^{n+s+1}_{(3l),m}$ has been already found (see (4.30)).

By (4.32)–(4.34), (4.36) and (4.19),

$$\widehat{u}^{n+s+1}_{(3l-1),m} = (g^s_{C,m} - 1)\widehat{u}^{n+1}_{(3l),m} + \widehat{u}^{n+1}_{(3l-1),m}, \tag{4.37}$$

$$\widehat{u}^{n+s+1}_{(3l-2),m} = (g^s_{C,m} - 1)\widehat{u}^{n+1}_{(3l),m} + \widehat{u}^{n+1}_{(3l-2),m}. \tag{4.38}$$

We rewrite (4.30), (4.37)–(4.38) in matrical form,

$$\begin{pmatrix} \widehat{u}^{n+s+1}_{(3l),m} \\ \widehat{u}^{n+s+1}_{(3l-1),m} \\ \widehat{u}^{n+s+1}_{(3l-2),m} \end{pmatrix} = G^{(s)}_{C,m} \begin{pmatrix} \widehat{u}^{n+1}_{(3l),m} \\ \widehat{u}^{n+1}_{(3l-1),m} \\ \widehat{u}^{n+1}_{(3l-2),m} \end{pmatrix}, \quad m = 1, \cdots, 3N, \tag{4.39}$$

$$G^{(s)}_{C,m} = \begin{pmatrix} g^s_{C,m} & 0 & 0 \\ g^s_{C,m} - 1 & 1 & 0 \\ g^s_{C,m} - 1 & 0 & 1 \end{pmatrix}.$$

The passing from u^{n+1} to u^{n+s+1} is given in the matrical form by (4.39). The stability of the scheme for passing from u^{n+1} to u^{n+s+1} is equivalent to showing that the spectral radius of $G^{(s)}_{C,m}$ is not larger than 1 for $m = 1, \cdots, 3N$. The eigenvalues of $G^{(s)}_{C,m}$ are not larger than 1. These eigenvalues are 1, 1, $g^s_{C,m}$, and we have seen that $|g_{C,m}| \leq 1$.

More precisely, using that the spectral radius of $G^{(s)}_{C,m}$ is less than 1 and (4.23), we have

$$\begin{aligned} |u^{n+s+1}_h|^2 &= \sum_{\alpha=0}^{2} \sum_{\ell=1}^{N} \Delta x |u^{n+s+1}_{3\ell-\alpha}|^2 = 3N\Delta x \sum_{\alpha=0}^{2} \sum_{m=1}^{3N} |\widehat{u}^{n+s+1}_{(3\ell-\alpha),m}|^2 \\ &\leq 3N\Delta x \sum_{\alpha=0}^{2} \sum_{m=1}^{3N} |\widehat{u}^{n+1}_{(3\ell-\alpha),m}|^2 = |u^{n+1}_h|^2 \end{aligned} \tag{4.40}$$

and for $s = q$,

$$|u_h^{n+q+1}| \le |u_h^{n+1}|. \tag{4.41}$$

Combining (4.13) and (4.41), we obtain the stability of the scheme.

Theorem 4.1 *The multilevel scheme defined by the equations* (4.4) *and* (4.16) *is stable in* $L^\infty(0,\infty; L^2(\mathcal{M}))$. *More precisely, for all* n,

$$|u_h^n| \le |u^0|. \tag{4.42}$$

4.2 The linearized 1D shallow water equation

By restriction to 1 dimension, equations (3.2) with $f = 0$ become

$$\begin{cases} \dfrac{\delta u}{\delta t} + \widetilde{u}_0 \dfrac{\delta u}{\delta x} + g\dfrac{\delta \phi}{\delta x} = 0, \\ \dfrac{\delta \phi}{\delta t} + \widetilde{u}_0 \dfrac{\delta \phi}{\delta x} + \widetilde{\phi}_0 \dfrac{\delta u}{\delta x} = 0. \end{cases} \tag{4.43}$$

We assume the background flow $(\widetilde{u}_0, \widetilde{\phi}_0)$ to be supersonic (supercritical), that is,

$$\widetilde{u}_0 > \sqrt{g\widetilde{\phi}_0}. \tag{4.44}$$

The boundary conditions are space periodicity, and the initial conditions are given such that they are similar as (3.4). The time and space meshes are the same as in Sections 2.1 and 2.2.

4.2.1 The Fine grid scheme with a "small" time step

The fine grid mesh scheme reads

$$\begin{cases} \dfrac{p}{\Delta t}(u_j^\tau - u_j^{\tau-\frac{1}{p}}) + \dfrac{\widetilde{u}_0}{\Delta x}(u_j^\tau - u_{j-1}^\tau) + \dfrac{g}{\Delta x}(\phi_j^\tau - \phi_{j-1}^\tau) = 0, \\ \dfrac{p}{\Delta t}(\phi_j^\tau - \phi_j^{\tau-\frac{1}{p}}) + \dfrac{\widetilde{u}_0}{\Delta x}(\phi_j^\tau - \phi_{j-1}^\tau) + \dfrac{\widetilde{\phi}_0}{\Delta x}(u_j^\tau - u_{j-1}^\tau) = 0, \end{cases} \tag{4.45}$$

where $\tau = n + \frac{s}{p}$, $s = 1, \cdots, p$, $j = 1, \cdots, 3N$, $u_0^\tau = u_{3N}^\tau$, $\phi_0^\tau = \phi_{3N}^\tau$ by space periodicity.

We rewrite (4.45) in the form

$$\begin{cases} \Big(1 + \dfrac{\widetilde{u}_0}{p}\dfrac{\Delta t}{\Delta x}\Big)u_j^\tau - \dfrac{\widetilde{u}_0}{p}\dfrac{\Delta t}{\Delta x}u_{j-1}^\tau + \dfrac{g}{p}\dfrac{\Delta t}{\Delta x}(\phi_j^\tau - \phi_{j-1}^\tau) = u_j^{\tau-\frac{1}{p}}, \\ \Big(1 + \dfrac{\widetilde{u}_0}{p}\dfrac{\Delta t}{\Delta x}\Big)\phi_j^\tau - \dfrac{\widetilde{u}_0}{p}\dfrac{\Delta t}{\Delta x}\phi_{j-1}^\tau + \dfrac{\widetilde{\phi}_0}{p}\dfrac{\Delta t}{\Delta x}(u_j^\tau - u_{j-1}^\tau) = \phi_j^{\tau-\frac{1}{p}}. \end{cases} \tag{4.46}$$

From this, we deduce for the Fourier transforms, for $m = 1, \cdots, 3N$,

$$\begin{cases} \Big(1 + \dfrac{\widetilde{u}_0}{p}\dfrac{\Delta t}{\Delta x}(1 - \mathrm{e}^{-imh^*})\Big)\widehat{u}_m^\tau + \dfrac{g}{p}\dfrac{\Delta t}{\Delta x}\widehat{\phi}_m^\tau(1 - \mathrm{e}^{-imh^*}) = \widehat{u}_m^{\tau-\frac{1}{p}}, \\ \Big(1 + \dfrac{\widetilde{u}_0}{p}\dfrac{\Delta t}{\Delta x}(1 - \mathrm{e}^{-imh^*})\Big)\widehat{\phi}_m^\tau + \dfrac{\widetilde{\phi}_0}{p}\dfrac{\Delta t}{\Delta x}\widehat{u}_m^\tau(1 - \mathrm{e}^{-imh^*}) = \widehat{\phi}_m^{\tau-\frac{1}{p}}, \end{cases} \tag{4.47}$$

that is,

$$\begin{pmatrix}\widehat{u}_m^{\tau}\\ \widehat{\phi}_m^{\tau}\end{pmatrix} = G_{F,m}\begin{pmatrix}\widehat{u}_m^{\tau-\frac{1}{p}}\\ \widehat{\phi}_m^{\tau-\frac{1}{p}}\end{pmatrix} \tag{4.48}$$

with

$$G_{F,m}^{-1} = \begin{pmatrix} 1+\dfrac{\widetilde{u}_0}{p}\dfrac{\Delta t}{\Delta x}(1-\mathrm{e}^{-imh^*}) & \dfrac{g}{p}\dfrac{\Delta t}{\Delta x}(1-\mathrm{e}^{-imh^*}) \\ \dfrac{\widetilde{\phi}_0}{p}\dfrac{\Delta t}{\Delta x}(1-\mathrm{e}^{-imh^*}) & 1+\dfrac{\widetilde{u}_0}{p}\dfrac{\Delta t}{\Delta x}(1-\mathrm{e}^{-imh^*}) \end{pmatrix}.$$

The eigenvalues of $G_{F,m}^{-1}$ are easily computed

$$\rho_{\pm,m} = 1 + \Lambda_{\pm}(1-\mathrm{e}^{-imh^*})$$

with

$$\Lambda_{\pm} = \frac{1}{p}\Big(\widetilde{u}_0 \pm \sqrt{g\widetilde{\phi}_0}\Big)\frac{\Delta t}{\Delta x}.$$

We have

$$|\rho_{\pm,m}|^2 = 1 + 2(1-\cos(h^*m))(\Lambda_{\pm}^2 + \Lambda_{\pm}).$$

The condition $\widetilde{u}_0 > \sqrt{g\widetilde{\phi}_0}$ implies $\Lambda_{\pm} > 0$, and thus

$$|\rho_{\pm,m}| \geq 1, \quad m = 1, \cdots, 3N.$$

Hence, setting $\mathbf{u} = (u,\phi)$ (comparing with (4.13)), we have

$$|\mathbf{u}_h^{n+1}|^2 \leq |\mathbf{u}_h^n|^2, \tag{4.49}$$

so that these steps of the small step scheme (4.45) are stable.

4.2.2 The Coarse grid scheme with a "large" time step

We define the cell averages

$$U_l = \frac{1}{3}(u_{3l} + u_{3l-1} + u_{3l-2}),$$

$$\Phi_l = \frac{1}{3}(\phi_{3l} + \phi_{3l-1} + \phi_{3l-2})$$

and the incremental unknowns

$$Z_{3l-\alpha}^u = u_{3l-\alpha} - U_l,$$

$$Z_{3l-\alpha}^{\phi} = \phi_{3l-\alpha} - \Phi_l.$$

The analogue of scheme (4.16) reads

$$\begin{cases} \dfrac{1}{\Delta t}(U_l^{\tau} - U_l^{\tau-1}) + \dfrac{\widetilde{u}_0}{3\Delta x}(u_{3l}^{\tau} - u_{3l-3}^{\tau}) + \dfrac{g}{3\Delta x}(\phi_{3l}^{\tau} - \phi_{3l-3}^{\tau}) = 0, \\ \dfrac{1}{\Delta t}(\Phi_l^{\tau} - \Phi_l^{\tau-1}) + \dfrac{\widetilde{u}_0}{3\Delta x}(\phi_{3l}^{\tau} - \phi_{3l-3}^{\tau}) + \dfrac{\widetilde{\phi}_0}{3\Delta x}(u_{3l}^{\tau} - u_{3l-3}^{\tau}) = 0 \end{cases} \tag{4.50}$$

for $\tau = n+s+1$, $s = 1, \cdots, q$ and $l = 1, \cdots, N$.

Observing as in (4.19) that

$$Z_j^{u,n+s+1} = Z_j^{u,n+1}, \quad Z_j^{\phi,n+s+1} = Z_j^{\phi,n+1} \tag{4.51}$$

for $s = 1, \cdots, q$, $j = 1, \cdots, 3N$ and thus that

$$U_l^\tau - U_l^{\tau-1} = u_{3l}^\tau - u_{3l}^{\tau-1},$$

$$\Phi_l^\tau - \Phi_l^{\tau-1} = \phi_{3l}^\tau - \phi_{3l}^{\tau-1},$$

(4.50) yields

$$\begin{cases} \dfrac{1}{\Delta t}(u_{3l}^\tau - u_{3l}^{\tau-1}) + \dfrac{\widetilde{u}_0}{3\Delta x}(u_{3l}^\tau - u_{3l-3}^\tau) + \dfrac{g}{3\Delta x}(\phi_{3l}^\tau - \phi_{3l-3}^\tau) = 0, \\ \dfrac{1}{\Delta t}(\phi_{3l}^\tau - \phi_{3l}^{\tau-1}) + \dfrac{\widetilde{u}_0}{3\Delta x}(\phi_{3l}^\tau - \phi_{3l-3}^\tau) + \dfrac{\widetilde{\phi}_0}{3\Delta x}(u_{3l}^\tau - u_{3l-3}^\tau) = 0. \end{cases} \tag{4.52}$$

Hence, for the partial Fourier transforms for $m = 1, \cdots, 3N$ (comparing with (4.27)),

$$\begin{pmatrix} \widehat{u}_{(3l),m}^\tau \\ \widehat{\phi}_{(3l),m}^\tau \end{pmatrix} = G_{C,m} \begin{pmatrix} \widehat{u}_{(3l),m}^{\tau-1} \\ \widehat{\phi}_{(3l),m}^{\tau-1} \end{pmatrix} \tag{4.53}$$

with

$$G_{C,m}^{-1} = \begin{pmatrix} 1 + \dfrac{\widetilde{u}_0}{3}\dfrac{\Delta t}{\Delta x}(1 - \mathrm{e}^{-3\mathrm{i}h^* m}) & \dfrac{g}{3}\dfrac{\Delta t}{\Delta x}(1 - \mathrm{e}^{-3\mathrm{i}h^* m}) \\ \dfrac{\widetilde{\phi}_0}{3}\dfrac{\Delta t}{\Delta x}(1 - \mathrm{e}^{-3\mathrm{i}h^* m}) & 1 + \dfrac{\widetilde{u}_0}{3}\dfrac{\Delta t}{\Delta x}(1 - \mathrm{e}^{-3\mathrm{i}h^* m}) \end{pmatrix},$$

where $G_{C,m}^{-1}$ is very similar to $G_{F,m}^{-1}$, and we prove in the same way that its eigenvalues are larger than or equal to 1 in magnitude.

For the moment, we infer from (4.53) that

$$\begin{pmatrix} \widehat{u}_{(3l),m}^{n+s+1} \\ \widehat{\phi}_{(3l),m}^{n+s+1} \end{pmatrix} = G_{C,m}^s \begin{pmatrix} \widehat{u}_{(3l),m}^{n+1} \\ \widehat{\phi}_{(3l),m}^{n+1} \end{pmatrix}. \tag{4.54}$$

Then by (4.54),

$$\begin{aligned} \begin{pmatrix} \widehat{U}_{(l),m}^{n+s+1} \\ \widehat{\Phi}_{(l),m}^{n+s+1} \end{pmatrix} &= \begin{pmatrix} \widehat{u}_{(3l),m}^{n+s+1} \\ \widehat{\phi}_{(3l),m}^{n+s+1} \end{pmatrix} - \begin{pmatrix} \widehat{Z}_{(3l),m}^{u,n+s+1} \\ \widehat{Z}_{(3l),m}^{\phi,n+s+1} \end{pmatrix} \\ &= G_{C,m}^s \begin{pmatrix} \widehat{u}_{(3l),m}^{n+1} \\ \widehat{\phi}_{(3l),m}^{n+1} \end{pmatrix} - \begin{pmatrix} \widehat{Z}_{(3l),m}^{u,n+1} \\ \widehat{Z}_{(3l),m}^{\phi,n+1} \end{pmatrix}. \end{aligned} \tag{4.55}$$

We then need to express $\widehat{u}_{(3l-\alpha),m}^{n+s+1}$, $\widehat{\phi}_{(3l-\alpha),m}^{n+s+1}$ in terms of $\widehat{u}_{(3l-\beta),m}^{n+1}$, $\widehat{\phi}_{(3l-\beta),m}^{n+1}$, $\alpha = 1, 2$, $\beta = 0, 1, 2$. We write as in equations (4.37)–(4.38),

$$\begin{pmatrix} \widehat{u}_{(3l-1),m}^{n+s+1} \\ \widehat{\phi}_{(3l-1),m}^{n+s+1} \end{pmatrix} = (G_{C,m}^s - I) \begin{pmatrix} \widehat{u}_{(3l),m}^{n+1} \\ \widehat{\phi}_{(3l)}^{n+1} \end{pmatrix} + \begin{pmatrix} \widehat{u}_{(3l-1),m}^{n+1} \\ \widehat{\phi}_{(3l-1),m}^{n+1} \end{pmatrix}, \tag{4.56}$$

$$\begin{pmatrix} \widehat{u}^{n+s+1}_{(3l-2),m} \\ \widehat{\phi}^{n+s+1}_{(3l-2),m} \end{pmatrix} = (G^s_{C,m} - I) \begin{pmatrix} \widehat{u}^{n+1}_{(3l),m} \\ \widehat{\phi}^{n+1}_{(3l),m} \end{pmatrix} + \begin{pmatrix} \widehat{u}^{n+1}_{(3l-2),m} \\ \widehat{\phi}^{n+1}_{(3l-2),m} \end{pmatrix}. \tag{4.57}$$

In the end,

$$\begin{pmatrix} \widehat{u}^{n+s+1}_{(3l),m} \\ \widehat{\phi}^{n+s+1}_{(3l),m} \\ \widehat{u}^{n+s+1}_{(3l-1),m} \\ \widehat{\phi}^{n+s+1}_{(3l-1),m} \\ \widehat{u}^{n+s+1}_{(3l-2),m} \\ \widehat{\phi}^{n+s+1}_{(3l-2),m} \end{pmatrix} = \mathcal{G}^{(s)}_{C,m} \begin{pmatrix} \widehat{u}^{n+1}_{(3l),m} \\ \widehat{\phi}^{n+1}_{(3l),m} \\ \widehat{u}^{n+1}_{(3l-1),m} \\ \widehat{\phi}^{n+1}_{(3l-1),m} \\ \widehat{u}^{n+1}_{(3l-2),m} \\ \widehat{\phi}^{n+1}_{(3l-2),m} \end{pmatrix}, \quad m = 1, \cdots, 3N \tag{4.58}$$

with

$$\mathcal{G}^{(s)}_{C,m} = \begin{pmatrix} G^s_{C,m} & 0 & 0 \\ G^s_{C,m} - I & I & 0 \\ G^s_{C,m} - I & 0 & I \end{pmatrix}.$$

All the eigenvalues of $\mathcal{G}^{(s)}_{C,m}$ are less than or equal to 1, which ensures the stability of the scheme (4.50) going from $t = (n+1)\Delta t$ to $t = (n+s+1)\Delta t$.

Then we have

$$|\mathbf{u}^{n+s+1}_h| \le |\mathbf{u}^{n+1}_h|, \quad \text{for } s = 1, \cdots, q. \tag{4.59}$$

Theorem 4.2 *The multilevel scheme defined by equations* (4.45) *and* (4.50) *is stable in* $L^\infty(0, \infty; L^2(\mathcal{M})^2)$*. More precisely, for all* n*,*

$$|\mathbf{u}^n_h| \le |\mathbf{u}^0|. \tag{4.60}$$

References

[1] Adamy, K., Bousquet, A., Faure, S., et al., A multilevel method for finite-volume discretization of the two-dimensional nonlinear shallow-water equations, *Ocean Modelling*, **33**, 2010, 235–256. DOI: 10.1016/j.ocemod.2010.02.006

[2] Adamy, K. and Pham, D., A finite-volume implicit Euler scheme for the linearized shallow water equations: stability and convergence, *Numerical Functional Analysis and Optimization*, **27** (7–8), 2006, 757–783.

[3] Bellanger, M., Traitement du Signal, Dunod, Paris, 2006.

[4] Bousquet, A., Marion, M. and Temam, R., Finite volume multilevel approximation of the shallow water equations II, in preparation.

[5] Canuto, C., Hussaini, M. Y., Quarteroni, A. and Zhang, T. A., Spectral Methods, Evolution to Complex Geometries and Applications to Fluid Dynamics, Scientific Computation, Springer-Verlag, Berlin, 2007.

[6] Chen, Q., Shiue, M. C. and Temam, R., The barotropic mode for the primitive equations, Special issue in memory of David Gottlieb, *Journal of Scientific Computing*, **45**, 2010, 167–199. DOI: 10.1007/s10915-009-9343-8

[7] Chen, Q., Shiue, M. C., Temam, R. and Tribbia, J., Numerical approximation of the inviscid 3D Primitive equations in a limited domain, *Math. Mod. and Num. Anal.* (*M2AN*), **45**, 2012, 619–646. DOI: 10.105/m2an/2011058

[8] Dubois, T., Jauberteau, F. and Temam, R., Dynamic, Multilevel Methods and the Numerical Simulation of Turbulence, Cambridge University Press, Cambridge, 1999.

[9] Dautray, R. and Lions, J. L., Mathematical analysis and numerical methods for science and technology, Springer-Verlag, Berlin, 1990–1992.

[10] Eymard, R., Gallouet, T. and Herbin, R., Finite volume methods, Handbook of Numerical Analysis, P. G. Ciarlet, J. L. Lions (eds.), Vol. VII, North-Holland, Amsterdam, 2002, 713–1020.

[11] Gie, G. M. and Temam, R., Cell centered finite-volume methods using Taylor series expansion scheme without fictitious domains, *International Journal of Numerical Analysis and Modeling*, **7**(1), 2010, 1–29.

[12] Huang, A. and Temam, R., The linearized 2D inviscid shallow water equations in a rectangle: boundary conditions and well-posedness, to appear.

[13] Leveque, R. J., Finite Volume Methods for Hyperbolic Problems, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2002.

[14] Lions, J. L., Temam, R. and Wang, S., Models of the coupled atmosphere and ocean (CAO I), *Computational Mechanics Advances*, **1**, 1993, 5–54.

[15] Lions, J. L., Temam, R. and Wang, S., Numerical analysis of the coupled models of atmosphere and ocean (CAO II), *Computational Mechanics Advances*, **1**, 1993, 55–119.

[16] Lions, J. L., Temam, R. and Wang, S., Splitting up methods and numerical analysis of some multiscale problems, *Computational Fluid Dynamics Journal,* special issue dedicated to A. Jameson, **5**(2), 1996, 157–202.

[17] Marchuk, G. I., Methods of numerical mathematics, 2nd edition, Translated from the Russian by Arthur A. Brown, Applications of Mathematics, **2**, Springer-Verlag, New York, Berlin, 1982.

[18] Marion, M. and Temam, R., Nonlinear Galerkin Methods, *SIAM J. Num. Anal.,* **26**, 1989, 1139–1157.

[19] Marion, M. and Temam, R., Navier-Stokes equations, Theory and Approximation, Handbook of Numerical Analysis, P. G. Ciarlet and J. L. Lions (eds.), North-Holland, Amsterdam, VI, 1998, 503–689.

[20] Rousseau, A., Temam, R. and Tribbia, J., The 3D Primitive Equations in the Absence of Viscosity: Boundary Conditions and Well-Posedness in the Linearized Case, *J. Math. Pures Appl.*, **89**(3), 2008, 297–319. DOI: 10.1016/j.matpur.2007.12.001

[21] Rousseau, A., Temam, R. and Tribbia, J., Boundary value problems for the inviscid primitive equations in limited domains, Computational Methods for the Atmosphere and the Oceans, Handbook of Numerical Analysis, Special Volume, Vol. XIV, R. M. Temam, J. J. Tribbia (Guest Editors), P. G. Ciarlet (Editor), Elsevier, Amsterdam, 2008.

[22] Strikwerda, J. C., Finite Difference Schemes and Partial Differential Equations, 2nd edition, SIAM, philadelphia, PA, 2004.

[23] Temam, R., Inertial manifolds and multigrid methods, *SIAM J. Math. Anal.*, **21**, 1990, 154–178.

[24] Temam, R. and Tribbia, J., Open boundary conditions for the primitive and Boussinesq equations, *J. Atmospheric Sciences*, **60**, 2003, 2647–2660.

[25] Yanenko, N. N., The method of fractional steps, The solution of problems of mathematical physics in several variables, Translated from the Russian by T. Cheron, English translation edited by M. Holt, Springer-Verlag, New York, Heidelberg, 1971.