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Asymptotic Analysis in a Gas-Solid Combustion Model with Pattern Formation*

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(*In honor of the scientific heritage of Jacques-Louis Lions*)

Abstract The authors consider a free interface problem which stems from a gas-solid model in combustion with pattern formation. A third-order, fully nonlinear, self-consistent equation for the flame front is derived. Asymptotic methods reveal that the interface approaches a solution to the Kuramoto-Sivashinsky equation. Numerical results which illustrate the dynamics are presented.

Keywords Asymptotics, Free interface, Kuramoto-Sivashinsky equation, Pseudo-differential operator, Spectral method
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1 Introduction

Combustion phenomena are particularly important for science and industry, as Lions pointed out in his foreword to the special issue of the CNRS "Images des Mathématiques" in 1996 (see [1]). Flames constitute a complex physical system involving fluid dynamics and multistep chemical kinetics (see, e.g., [9]). In the middle of the 20th century, the Russian School, which included Frank-Kamenetskii and Zel'dovich, used formal asymptotics based on large activation energy to write simpler descriptions of such a reactive system. Later, the development of systematic asymptotic techniques during the 1960s opened the way towards revealing an underlying simplicity in many combustion processes. Eventually, the full power of asymptotical analysis has been realized by modern singular perturbation theory. Lions was the first one to put these formalities on a rigorous basis in his seminal monograph "Perturbations singulières dans les problèmes aux limites et en contrôle optimal" (see [14]).

In short, the small perturbation parameter in activation-energy asymptotics is the inverse of the normalized activation energy, the Zel'dovich number β. In the limit $\beta \to +\infty$, the flame front reduces to a free interface. The laminar flames of low-Lewis-number premixtures are known to display diffusive-thermal instability responsible for the formation of a non-steady cellular structure (see [24]), when the Lewis number Le (the ratio of thermal and molecular

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diffusivities) is such that Le $\lesssim 1$. From an asymptotical viewpoint, one combines the limit of large activation energy with the requirement that $\alpha = \frac{1}{2}\beta(1 - \text{Le})$ remains bounded: in the near equidiffusive flame model (or NEF for short), β^{-1} and $1 - \text{Le}$ are asymptotically of the same order of magnitude (see [22]).

A very challenging problem is the derivation of a single equation for the free interface, which may capture most of the dynamics and, as a consequence, yields a reduction of the effective dimensionality of the system. Asymptotical methods are also the main tool: in a set of conveniently rescaled dependent and independent variables, the flame front is asymptotically represented (see [23]) by a solution to the Kuramoto-Sivashinsky (or K-S for short) equation

$$\Phi_\tau + 4\Phi_{\eta\eta\eta\eta} + \Phi_{\eta\eta} + \frac{1}{2}(\Phi_\eta)^2 = 0. \tag{K-S}$$

This equation has received considerable attention from the mathematical community (see [25]), especially for its ability to generate a cellular structure, pattern formation, and chaotic behavior in an appropriate range of parameters (see [12]). We refer to [2–7] for a rigorous mathematical approach to the derivation of (K-S).

In this paper, we consider a model in gas-solid combustion, proposed in [13]. This model was motivated by the experimental studies of Zik and Moses (see [26]) who observed a striking fingering pattern in flames spreading over thin solid fuels. The phenomenon was interpreted in terms of the diffusive instability similar to that occurring in laminar flames of low-Lewis-number premixtures. As we show below, the gas-solid and premixed gas systems share some common asymptotic features, especially the K-S equation.

The free interface system for the scaled temperature θ, the excess enthalpy S, the prescribed flow intensity U (with $0 < U < 1$), and the moving front $x = \xi(t, y)$, is as follows:

$$U\frac{\partial\theta}{\partial x} = \Delta\theta, \quad x < \xi(t, y), \tag{1.1}$$

$$\theta = 1, \quad x \ge \xi(t, y), \tag{1.2}$$

$$\frac{\partial\theta}{\partial t} + U\frac{\partial S}{\partial x} = \Delta S - \alpha\Delta\theta, \quad x \ne \xi(t, y). \tag{1.3}$$

System (1.1)–(1.3) is coupled with the following jump conditions for the normal derivatives of θ and S:

$$\left[\frac{\partial\theta}{\partial n}\right] = -\exp(S), \quad \left[\frac{\partial S}{\partial n}\right] = \alpha\left[\frac{\partial\theta}{\partial n}\right]. \tag{1.4}$$

It is not difficult to show that (1.1)–(1.4) admit a planar traveling wave solution with velocity $-V$, where $V = -U\ln U$. Setting $x' = x + Vt$, the traveling wave solution is given by

$$\overline{\theta}(x') = \begin{cases} \exp(Ux'), & x' \le 0, \\ 1, & x' > 0, \end{cases}$$

$$\overline{S}(x') = \begin{cases} (\alpha - \ln U)Ux'\exp(Ux') + (\ln U)\exp(Ux'), & x' \le 0, \\ \ln U, & x' > 0. \end{cases}$$

As usual, one fixes the moving front. We set

$$\xi(t, y) = -Vt + \varphi(t, y), \quad x' = x - \xi(t, y),$$

where φ is the perturbation of the planar front. In this new framework, the system (1.1)–(1.3) can be written as follows:

$$U\theta_{x'} = \Delta_\varphi \theta, \quad x' < 0, \tag{1.5}$$

$$\theta = 1, \quad x' \geq 0, \tag{1.6}$$

$$\theta_t + (V - \varphi_t)\theta_{x'} + US_{x'} = \Delta_\varphi S - \alpha \Delta_\varphi \theta, \quad x' \neq 0, \tag{1.7}$$

where

$$\Delta_\varphi = (1 + (\varphi_y)^2) D_{x'x'} + D_{yy} - \varphi_{yy} D_{x'} - 2\varphi_y D_{x'y}.$$

The front is now fixed at $x' = 0$. The first jump condition in (1.4) is

$$\sqrt{1 + (\varphi_y)^2}\Big[\frac{\partial \theta}{\partial x'}\Big] = -\exp(S), \tag{1.8}$$

and the second one is

$$\Big[\frac{\partial S}{\partial x'}\Big] = \alpha \Big[\frac{\partial \theta}{\partial x'}\Big]. \tag{1.9}$$

We will consider a quasi-steady version of the model, motivated by the fact that, in similar problems, not far from the instability threshold, the respective time derivatives of the temperature and enthalpy (if any) exhibit a relatively small effect on the solution. The dynamics appears to be essentially driven by the front. We can thus introduce a quasi-steady model replacing (1.5)–(1.7) by

$$U\theta_{x'} = \Delta_\varphi \theta, \quad x' < 0,$$
$$\theta = 1, \quad x' \geq 0,$$
$$(V - \varphi_t)\theta_{x'} + US_{x'} = \Delta_\varphi S - \alpha \Delta_\varphi \theta, \quad x' \neq 0.$$

Next we consider the perturbations of temperature u and enthalpy v,

$$\theta = \overline{\theta} + u, \quad S = \overline{S} + v,$$

and, for simplicity, in the equations satisfied by u, v and φ, we keep only the linear and second-order terms for φ, and the first-order terms for u and v. Writing x instead of x' to avoid a cumbersome notation, some (easy) computations reveal that the triplet (u, v, φ) solves the differential equations

$$Uu_x - \Delta u = (\Delta_\varphi - \Delta)\overline{\theta}, \quad x < 0,$$
$$Vu_x - \Delta(v - \alpha u) + Uv_x - \varphi_t \overline{\theta}_x = (\Delta_\varphi - \Delta)(\overline{S} - \alpha\overline{\theta}), \quad x \neq 0,$$

where $u \equiv 0$ in $[0, +\infty)$, and

$$(\Delta_\varphi - \Delta)\overline{\theta} = (U(\varphi_y)^2 - \varphi_{yy})U\mathrm{e}^{Ux},$$

$$(\Delta_\varphi - \Delta)(\overline{S} - \alpha\overline{\theta}) = \begin{cases} (\varphi_y)^2(\alpha - \ln U)U^2(1 + Ux)\mathrm{e}^{Ux} - \varphi_{yy}(\alpha - \ln U)U^2 x\mathrm{e}^{Ux}, & x < 0, \\ 0, & x > 0. \end{cases}$$

The previous system is endowed with a set of boundary conditions. First, the continuities of θ and S at the front yield the equation

$$u(0^-) = [v] = 0$$

(recall that $u(x) = 0$ for $x \geq 0$). Second, up to the second-order, condition (1.8) gives

$$-U + [u_x] = -(1 + (\varphi_y)^2)^{-\frac{1}{2}} U \mathrm{e}^{v(0)} \sim -\Big(1 - \frac{1}{2}(\varphi_y)^2\Big) U \Big(1 + v(0) + \frac{1}{2}(v(0))^2\Big).$$

By keeping only the first-order for v, we get the condition

$$- u_x(0^-) + Uv(0) = \frac{1}{2}(\varphi_y)^2 U.$$

Finally, the condition $[S_x] = \alpha[\theta_x]$ yields

$$[v_x] = -\alpha u_x(0^-).$$

Summing up, the final system is as follows:

$$\begin{cases} Uu_x - \Delta u = (\Delta_\varphi - \Delta)\overline{\theta}, & x < 0, \\ Vu_x - \Delta(v - \alpha u) + Uv_x - \varphi_t \overline{\theta}_x = (\Delta_\varphi - \Delta)(\overline{S} - \alpha\overline{\theta}), & x \neq 0, \\ u(0^-) = [v] = 0, \\ Uv(0) - u_x(0^-) = \dfrac{1}{2}(\varphi_y)^2 U, \\ [v_x] = -\alpha u_x(0^-). \end{cases} \tag{1.10}$$

Throughout this paper, we will also use the very convenient notation

$$\gamma = \alpha - \ln U.$$

First, our goal is to derive a self-consistent equation for the front φ,

$$\varphi_t = \mathscr{A}(\varphi) + \mathscr{M}((\varphi_y)^2), \tag{1.11}$$

where $\mathscr{A}$ is a third-order, pseudo-differential operator, in contrast to the NEF model in gaseous combustion where the corresponding linear operator is of the second-order (see [6]). Another important feature is that the nonlinear term is also of the third-order, which means that the equation (1.11) is fully nonlinear. Here the spatial domain is a two-dimensional strip $\mathbb{R} \times \big[-\frac{\ell}{2}, \frac{\ell}{2}\big]$ with periodic boundary conditions at $\pm\frac{\ell}{2}$.

Second, we define a small parameter $\varepsilon = \gamma - 1$. The main result of this paper states the precise sense in which the front φ approaches a solution to the Kuramoto-Sivashinsky equation when $\varepsilon \to 0$.

Theorem 1.1 *Let $\Phi_0 \in H^m\big(-\frac{\ell_0}{2}, \frac{\ell_0}{2}\big)$ be a periodic function of period ℓ_0. Further, let Φ be the periodic solution to* (K-S) (*with period ℓ_0*) *on a fixed time interval $[0, T]$, satisfying the initial condition $\Phi(0, \cdot) = \Phi_0$. Then, if m is large enough, there exists an $\varepsilon_0 = \varepsilon_0(T) \in (0, 1)$ such that, for $0 < \varepsilon \leq \varepsilon_0$, (1.11) admits a unique classical solution φ on $[0, \frac{T}{\varepsilon^2 U^2}]$, which is periodic with period $\frac{\ell_0}{\sqrt{\varepsilon}U}$ with respect to y, and satisfies*

$$\varphi(0, y) = \varepsilon U^{-1} \Phi_0(y\sqrt{\varepsilon}U), \quad |y| \leq \frac{\ell_0}{2\sqrt{\varepsilon}U}.$$

Moreover, there exists a positive constant C such that

$$|\varphi(t, y) - \varepsilon U^{-1} \Phi(t\varepsilon^2 U^2, y\sqrt{\varepsilon}U)| \leq C\,\varepsilon^2, \quad 0 \leq t \leq \frac{T}{\varepsilon^2 U^2}, \ |y| \leq \frac{\ell_0}{2\sqrt{\varepsilon}U}$$

for any $\varepsilon \in (0, \varepsilon_0]$.

This paper is organized as follows. In Section 2, we proceed to a formal ansatz in the spirit of [23], defining the rescaled variable $\psi = \varepsilon^{-1} U\varphi$ and expanding $\psi = \psi^0 + \varepsilon\psi^1 + \cdots$. It transpires that ψ^0 verifies (K-S), thanks to an elementary solvability condition.

Section 3 is devoted to the derivation of (1.11), via an explicit computation in the discrete Fourier variable. The asymptotic analysis in the rescaled variables $t = \frac{\tau}{\varepsilon^2 U^2}, y = \frac{\eta}{\sqrt{\varepsilon U}}$ is performed in Section 4. Since the perturbation in (1.11) is singular as $\varepsilon \to 0$, we turn to the equivalent (at fixed $\varepsilon > 0$) fourth-order, fully nonlinear equation (1.12), whose prima facie limit as $\varepsilon \to 0$ is the equation (K-S),

$$\begin{aligned}
&\frac{\partial}{\partial \tau}(\sqrt{I - 4\varepsilon D_{\eta\eta}})\psi \\
&= -4D_{\eta\eta\eta\eta}\psi - D_{\eta\eta}\psi \\
&\quad + \frac{1}{4}\{(I - 4\varepsilon D_{\eta\eta})^{\frac{3}{2}} - 3(I - 4\varepsilon D_{\eta\eta}) - 4(1+\varepsilon)(\sqrt{I - 4\varepsilon D_{\eta\eta}} - I)\}(D_\eta\psi)^2.
\end{aligned} \tag{1.12}$$

We prove a priori estimates, which constitute the key tool to prove the main theorem. Finally, numerical computations which illustrate the dynamics in (1.12) are presented in Section 5.

The local existence in time for (1.1)–(1.4) and the stability issue will be addressed in a forthcoming paper, by using the methods of [4, 8] and [15–20].

Notation 1.1 Given a (smooth enough) function $f : \left(-\frac{\ell}{2}, \frac{\ell}{2}\right) \to \mathbb{C}$, we denote by $\widehat{f}(k)$ its k-th Fourier coefficient, that is, we write

$$f(y) = \sum_{k=0}^{+\infty} \widehat{f}(k) w_k(y), \quad y \in \left(-\frac{\ell}{2}, \frac{\ell}{2}\right),$$

where $\{w_k\}$ is a complete set of (complex valued) eigenfunctions of the operator

$$D_{yy} : H^2\left(-\frac{\ell}{2}, \frac{\ell}{2}\right) \to L^2\left(-\frac{\ell}{2}, \frac{\ell}{2}\right),$$

whose eigenvalues are $0, -\frac{4\pi^2}{\ell^2}, -\frac{4\pi^2}{\ell^2}, -\frac{16\pi^2}{\ell^2}, -\frac{16\pi^2}{\ell^2}, -\frac{36\pi^2}{\ell^2}, \cdots$, and we label as $0 = -\lambda_0(\ell) > -\lambda_1(\ell) = -\lambda_2(\ell) > -\lambda_3(\ell) = -\lambda_4(\ell) > \cdots$. Typically, when no confusion may arise, we simply write λ_k instead of $\lambda_k(\ell)$.

For any $s \geq 0$, we denote by $H^s_\sharp$ the usual Sobolev space of order s consisting of ℓ-periodic (generalized) functions, i.e.,

$$H^s_\sharp = \left\{u = \sum_{k=0}^{+\infty} \widehat{u}(k) w_k : \sum_{k=0}^{+\infty} \lambda_k^s |\widehat{u}(k)|^2 < +\infty\right\}.$$

For $s = 0$, we simply write L^2 instead of $H^0_\sharp$, and we denote by $|\cdot|_2$ the usual L^2-norm.

By the notation $\widehat{f}(x, k)$, we mean the k-th Fourier coefficient of the function $f(x, \cdot)$. A similar notation is used for functions which depend also on the time variable.

2 A Formal Ansatz

The aim of this section is to use a formal asymptotic expansion method, in the spirit of [23]. The small perturbation parameter $\varepsilon > 0$ is defined by

$$\alpha = 1 + \ln U + \varepsilon, \quad \text{i.e.,} \quad \gamma = 1 + \varepsilon. \tag{2.1}$$

Accordingly, we now introduce scaled dependent and independent variables

$$t = \frac{\tau}{\varepsilon^2 U^2}, \quad y = \frac{\eta}{\sqrt{\varepsilon} U}, \quad \varphi = \frac{\varepsilon}{U}\psi, \quad u = \varepsilon^2 u_1, \quad v = \varepsilon^2 v_1, \tag{2.2}$$

and the ansatz

$$u_1 = u_1^0 + \varepsilon u_1^1 + \cdots, \quad v_1 = v_1^0 + \varepsilon v_1^1 + \cdots, \quad \psi = \psi^0 + \varepsilon \psi^1 + \cdots.$$

It is easy to rewrite (1.10) in terms of the rescaled variables. At the zeroth order, it comes that

$$\begin{cases} U(u_1^0)_x - (u_1^0)_{xx} = -U^2 \mathrm{e}^{Ux} \psi^0_{\eta\eta}, & x < 0, \\ V(u_1^0)_x - (v_1^0)_{xx} + (u_1^0)_{xx} + (\ln U)(u_1^0)_{xx} + U(v_1^0)_x = -U^3 x \mathrm{e}^{Ux} \psi^0_{\eta\eta}, & x < 0, \\ u_1^0 = 0, & x \geq 0, \\ (v_1^0)_{xx} - U(v_1^0)_x = 0, & x > 0. \end{cases} \tag{2.3}$$

At $x = 0$, the following conditions should be satisfied:

$$u_1^0(0) = [v_1^0] = 0, \tag{2.4a}$$

$$(u_1^0)_x(0) - U v_1^0(0) = 0, \tag{2.4b}$$

$$[(v_1^0)_x] = -(1 + \ln U)(u_1^0)_x(0). \tag{2.4c}$$

We assume that the functions $x \mapsto \mathrm{e}^{-\frac{Ux}{2}} u_1^0(x)$ and $x \mapsto \mathrm{e}^{-\frac{Ux}{2}} v_1^0(x)$ are bounded in $(-\infty, 0)$ and $\mathbb{R}$, respectively. Note that (2.3) coupled with conditions (2.4a) and (2.4b) is uniquely solvable in the unknowns (u_1^0, v_1^0), by taking ψ^0 as a parameter. It turns out that

$$\begin{aligned} &u_1^0 = U x \mathrm{e}^{Ux} \psi^0_{\eta\eta}, \quad x < 0, \\ &v_1^0 = \mathrm{e}^{Ux} \psi^0_{\eta\eta} + U(\ln U) x \mathrm{e}^{Ux} \psi^0_{\eta\eta} + U^2 x^2 \mathrm{e}^{Ux} \psi^0_{\eta\eta}, \quad x < 0, \\ &u_1^0 = 0, \quad x \geq 0, \\ &v_1^0 = \psi^0_{\eta\eta}, \quad x \geq 0. \end{aligned}$$

One might be tempted to use condition (2.4c) to determine function ψ^0. Unfortunately, whatever ψ^0 is, the triplet (u_1^0, v_1^0, ψ^0) satisfies this condition. As a matter of fact, we are not able to determine uniquely a solution to (2.3)–(2.4c). This situation is not surprising at all in the singular perturbation theory (see [10, 14]). To determine ψ^0, one needs to consider the (linear) problem for the first-order terms in the asymptotic expansion of u_1, v_1 and ψ. As we will show in a while, this problem provides a solvability condition, which is just the missing equation for ψ^0.

The system for (u_1^1, v_1^1, ψ^1) is the following one:

$$\begin{cases} U(u_1^1)_x - (u_1^1)_{xx} - U^2 (u_1^0)_{\eta\eta} = (U(\psi^0_\eta)^2 - U\psi^1_{\eta\eta}) U \mathrm{e}^{Ux}, & x < 0, \\ V(u_1^1)_x - (v_1^1)_{xx} - U^2 (v_1^0)_{\eta\eta} + (u_1^0)_{xx} + (1 + \ln U)((u_1^1)_{xx} + U^2 (u_1^0)_{\eta\eta}) + U(v_1^1)_x & \\ \quad = U^2 \psi^0_\tau \mathrm{e}^{Ux} + (\psi^0_\eta)^2 U^2 \mathrm{e}^{Ux} + (\psi^0_\eta)^2 U^3 x \mathrm{e}^{Ux} - U^3 \psi^1_{\eta\eta} x \mathrm{e}^{Ux} - U^3 \psi^0_{\eta\eta} x \mathrm{e}^{Ux}, & x < 0, \\ (v_1^1)_{xx} + U^2 (v_1^0)_{\eta\eta} - U(v_1^1)_x = 0, & x > 0, \\ u_1^1 = 0, & x \geq 0, \\ u_1^1(0) = [v_1^1] = 0, & \\ U v_1^1(0) - (u_1^1)_x(0) = \dfrac{1}{2} U (\psi^0_\eta)^2, & \\ [(v_1^1)_x] = -(1 + \ln U)(u_1^1)_x(0) - (u_1^0)_x(0). & \end{cases} \tag{2.5}$$

As above, we assume that the functions $x \mapsto \mathrm{e}^{-\frac{Ux}{2}} u_1^1(x)$ and $x \mapsto \mathrm{e}^{-\frac{Ux}{2}} v_1^1(x)$ are bounded in $(-\infty, 0)$ and $\mathbb{R}$, respectively. Using these conditions one can easily show that the more general solutions (u_1^1, v_1^1, ψ^1) to the differential equations and the first boundary condition in (2.5) are given by

$$u_1^1 = Ux\mathrm{e}^{Ux}(\psi^0_{\eta\eta\eta\eta} - (\psi^0_\eta)^2 + \psi^1_{\eta\eta}) - \frac{1}{2}U^2x^2\mathrm{e}^{Ux}\psi^0_{\eta\eta\eta\eta}, \quad x < 0,$$

$$v_1^1 = v_1^1(0)\mathrm{e}^{Ux} + Ax\mathrm{e}^{Ux} + Bx^2\mathrm{e}^{Ux} + Cx^3\mathrm{e}^{Ux}, \quad x < 0,$$

$$u_1^1 = 0, \quad x \geq 0,$$

$$v_1^1 = v_1^1(0) + Ux\psi^0_{\eta\eta\eta\eta}, \quad x \geq 0,$$

where

$$A = U(\ln U)(\psi^1_{\eta\eta} - (\psi^0_\eta)^2 + \psi^0_{\eta\eta\eta\eta}) - U\psi^0_\tau - U(\psi^0_\eta)^2 - 3U\psi^0_{\eta\eta\eta\eta},$$

$$B = U^2\psi^0_{\eta\eta} + U^2\psi^1_{\eta\eta} - U^2(\psi^0_\eta)^2 - \frac{1}{2}U^2(\ln U)\psi^0_{\eta\eta\eta\eta} + \frac{3}{2}U^2\psi^0_{\eta\eta\eta\eta},$$

$$C = -\frac{1}{2}U^3\psi^0_{\eta\eta\eta\eta},$$

and $v_1^1(0)$ is an arbitrary parameter. Hence, (u_1^1, v_1^1) depends on ψ^1. To determine both ψ^1 and $v_1^1(0)$, we use the last two boundary conditions which give

$$-U\psi^0_{\eta\eta\eta\eta} + U(\psi^0_\eta)^2 - U\psi^1_{\eta\eta} + Uv_1^1(0) = \frac{U}{2}(\psi^0_\eta)^2 \tag{2.6}$$

and

$$U\psi^1_{\eta\eta} - Uv_1^1(0) = -U\psi^0_\tau - U\psi^0_{\eta\eta} - 5U\psi^0_{\eta\eta\eta\eta}, \tag{2.7}$$

respectively. Obviously, (2.6)–(2.7) is a linear system for $(v_1^1(0), \psi^1_{\eta\eta})$ with the solvability condition

$$\psi^0_\tau + \psi^0_{\eta\eta} + 4\psi^0_{\eta\eta\eta\eta} + \frac{1}{2}(\psi^0_\eta)^2 = 0.$$

Hence, the K-S equation is the missing equation at the zeroth-order, needed to uniquely determine (u_1^0, v_1^0, ψ^0).

3 A Third-Order Fully Nonlinear Pseudo-Differential Equation for the Front

The aim of this section is the derivation of a self-consistent pseudo-differential equation for the front φ. We rewrite (1.10), namely,

$$\begin{cases} Uu_x - \Delta u = U\mathrm{e}^x(U(\varphi_y)^2 - \varphi_{yy}), & x < 0, \\ Vu_x - \Delta(v - \alpha u) + Uv_x & \\ \quad = U\varphi_t\mathrm{e}^x + (\alpha - \ln U)U^2(1 + Ux)\mathrm{e}^{Ux}(\varphi_y)^2 - U^2(\alpha - \ln U)x\mathrm{e}^{Ux}\varphi_{yy}, & x < 0, \\ Uv_x - \Delta v = 0, & x > 0, \\ u(0) = [v] = 0, & \\ Uv(0) - u_x(0) = \frac{1}{2}U(\varphi_y)^2, & \\ [v_x] = -\alpha u_x(0) & \end{cases} \tag{3.1}$$

in a two-dimensional strip $\mathbb{R}\times\big[-\frac{\ell}{2},\frac{\ell}{2}\big]$, with periodicity in the y variable.

3.1 Computations in the discrete Fourier variable

Throughout this subsection, (u,v,φ) is a sufficiently smooth solution to (3.1) such that the functions

$$(x,y)\mapsto \mathrm{e}^{-\frac{Ux}{2}}u(t,x,y),\quad (x,y)\mapsto \mathrm{e}^{-\frac{Ux}{2}}v(t,x,y)$$

are bounded in $(-\infty,0]\times\big[-\frac{\ell}{2},\frac{\ell}{2}\big]$ and $\mathbb{R}\times\big[-\frac{\ell}{2},\frac{\ell}{2}\big]$, respectively.

We start from the first equation in (3.1), namely,

$$Uu_x-\Delta u=(U(\varphi_y)^2-\varphi_{yy})U\mathrm{e}^{x}, \tag{3.2}$$

and the boundary condition $u(\,\cdot\,,0,\,\cdot\,)=0$. Applying the Fourier transform to both sides of (3.2), we end up with the infinitely many equations

$$U\widehat{u}_x(t,x,k)-\widehat{u}_{xx}(t,x,k)+\lambda_k\widehat{u}(t,x,k)=(U\widehat{(\varphi_y)^2}(t,k)+\lambda_k\widehat{\varphi}(t,k))U\mathrm{e}^{Ux}$$

for $k\geq 0$. For notational convenience, we set $\nu_k=\frac{U}{2}+\frac{1}{2}\sqrt{U^2+4\lambda_k}$ for any $k\geq 0$.

Since u vanishes at $x=0$ and tends to 0 as $x\to-\infty$ not slower than $\mathrm{e}^{\frac{Ux}{2}}$, the modes $\widehat{u}(\,\cdot\,,\cdot,k)$ should enjoy the same properties. Easy computations reveal that

$$\begin{aligned}
&\widehat{u}(t,x,0)=-U\widehat{(\varphi_y)^2}(t,0)x\mathrm{e}^{Ux},\quad x\leq 0,\\
&\widehat{u}(t,x,k)=U(\lambda_k)^{-1}(U\widehat{(\varphi_y)^2}(t,k)+\lambda_k\widehat{\varphi}(t,k))(\mathrm{e}^{Ux}-\mathrm{e}^{\nu_k x}),\quad x\leq 0,\ \ k\geq 1.
\end{aligned}$$

Applying the same arguments to the equation for v jointly with the second and the fourth boundary conditions in (3.1), we obtain that the modes $\widehat{v}(\,\cdot\,,\cdot\,,k)$ are given by

$$\begin{aligned}
\widehat{v}(t,x,0)=&\frac{1}{U}\widehat{\varphi}_t(t,0)+(-\gamma U\widehat{(\varphi_y)^2}(t,0)-U(\ln U)\widehat{(\varphi_y)^2}(t,0)\\
&-\widehat{\varphi_t}(t,0))x\mathrm{e}^{Ux}-\gamma U^2\widehat{(\varphi_y)^2}(t,0)x^2\mathrm{e}^{Ux},\quad x<0,\\
\widehat{v}(t,x,0)=&\frac{1}{U}\widehat{\varphi}_t(t,0),\quad x>0
\end{aligned}$$

and

$$\begin{aligned}
&\widehat{v}(t,x,k)=c_{1,k}\mathrm{e}^{\nu_k x}+A_k\mathrm{e}^{Ux}+B_kx\mathrm{e}^{Ux}+C_kx\mathrm{e}^{\nu_k x},\quad x<0,\\
&\widehat{v}(t,x,k)=c_{2,k}\mathrm{e}^{(U-\nu_k)x},\quad x\geq 0
\end{aligned}$$

for $k\geq 1$, where

$$\begin{aligned}
A_k=&\frac{(\alpha+\gamma)U^2}{\lambda_k}\widehat{(\varphi_y)^2}(t,k)+\alpha U\widehat{\varphi}(t,k)+\frac{U}{\lambda_k}\widehat{\varphi}_t(t,k),\\
B_k=&\frac{\gamma U^3}{\lambda_k}\widehat{(\varphi_y)^2}(t,k)+\gamma U^2\widehat{\varphi}(t,k),\\
C_k=&\frac{\gamma U^3\nu_k}{\lambda_k(U-2\nu_k)}\widehat{(\varphi_y)^2}(t,k)+\frac{\gamma U^2\nu_k}{U-2\nu_k}\widehat{\varphi}(t,k),\\
c_{2,k}=&\Big(\frac{\gamma U^2(U-\nu_k)}{\lambda_k(U-2\nu_k)}+\frac{\gamma U^3}{\lambda_k(U-2\nu_k)}+\frac{\gamma U^3}{(\nu_k-U)(U-2\nu_k)^2}\Big)\widehat{(\varphi_y)^2}(t,k)\\
&+\Big(\frac{\gamma U^2}{U-2\nu_k}+\frac{\gamma U^2\nu_k}{(U-2\nu_k)^2}\Big)\widehat{\varphi}(t,k)+\frac{U(U-\nu_k)}{\lambda_k(U-2\nu_k)}\widehat{\varphi}_t(t,k),\\
c_{1,k}=&c_{2,k}-A_k.
\end{aligned}$$

The equation for the front now comes by the last but one boundary condition in (3.1), which we have not used so far, in the Fourier variable, by taking advantage of the formulas for the modes of $\widehat{u}$ and $\widehat{v}$. It turns out that the equation for the front (in Fourier coordinates) is

$$\begin{aligned}
&\widehat{\varphi}_t(t,0)+\frac{1}{2}U\widehat{(\varphi_y)^2}(t,0)=0,\\
&\frac{U(U-\nu_k)}{\lambda_k(U-2\nu_k)}\widehat{\varphi}_t(t,k)+\Big(\frac{\gamma U^2}{U-2\nu_k}+\frac{\gamma U^2\nu_k}{(U-2\nu_k)^2}+\nu_k-U\Big)\widehat{\varphi}(t,k)\\
&\quad+\Big(\frac{\gamma U^2(U-\nu_k)}{\lambda_k(U-2\nu_k)}+\frac{\gamma U^3}{\lambda_k(U-2\nu_k)}+\frac{\gamma U^3}{(\nu_k-U)(U-2\nu_k)^2}+\frac{U(\nu_k-U)}{\lambda_k}-\frac{1}{2}\Big)\widehat{(\varphi_y)^2}(t,k)=0,
\end{aligned}$$

or, even, in the much more compact form

$$\begin{aligned}
(X_kU)\widehat{\varphi}_t(t,k)&=\frac{1}{4}(U^2-X_k^2)(X_k^2-\gamma U^2)\widehat{\varphi}(t,k)\\
&\quad+\frac{1}{4}(X_k^3-3UX_k^2-4\gamma U^2X_k+4\gamma U^3)\widehat{(\varphi_y)^2}(t,k)\\
&=(-4\lambda_k^2+(\gamma-1)U^2\lambda_k)\widehat{\varphi}(t,k)\\
&\quad+\frac{1}{4}(X_k^3-3UX_k^2-4\gamma U^2X_k+4\gamma U^3)\widehat{(\varphi_y)^2}(t,k)
\end{aligned}\tag{3.3}$$

for any $k\ge 0$, if we set

$$X_k=\sqrt{U^2+4\lambda_k},\quad k\ge 0.$$

Therefore, we have proved the following proposition.

Proposition 3.1 *Let (u,v,φ) be a sufficiently smooth solution to (3.1) such that the functions $(x,y)\mapsto \mathrm{e}^{-\frac{Ux}{2}}u(t,x,y)$ and $(x,y)\mapsto \mathrm{e}^{-\frac{Ux}{2}}v(t,x,y)$ are bounded in $(-\infty,0]\times\big[-\frac{\ell}{2},\frac{\ell}{2}\big]$ and $\mathbb{R}\times\big[-\frac{\ell}{2},\frac{\ell}{2}\big]$, respectively. Then, the interface φ solves the equations (3.3) for any $k\ge 0$.*

3.2 A fourth-order pseudo-differential equation for the front

Let us define the pseudo-differential operators (or Fourier multipliers) $\mathscr{B},\mathscr{L}$ and $\mathscr{F}$ through their symbols, respectively

$$b_k=X_kU,\quad l_k=-4\lambda_k^2+(\gamma-1)\lambda_kU^2,\quad f_k=\frac{1}{4}(X_k^3-3UX_k^2-4\gamma U^2X_k+4\gamma U^3)$$

for any $k\ge 0$. It is easy to see that

$$\begin{aligned}
&\mathscr{B}=U(U^2I-4D_{yy})^{\frac{1}{2}},\\
&\mathscr{F}=\frac{1}{4}(U^2I-4D_{yy})^{\frac{3}{2}}-\frac{3}{4}U(U^2I-4D_{yy})-\gamma U^2\Big(\sqrt{U^2I-4D_{yy}}-U\Big),
\end{aligned}$$

while the realization of $\mathscr{L}$ in L^2 is the operator

$$L=-4D_{yyyy}-(\gamma-1)U^2D_{yy}$$

with $H^4_\sharp$ being a domain.

It follows from Proposition 3.1 that the front φ solves the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{B}(\varphi)=\mathscr{L}(\varphi)+\mathscr{F}((\varphi_y)^2).\tag{3.4}$$

The main feature of (3.4) is that the nonlinear part is rather unusual. Actually, it has a fourth-order leading term, as $\mathscr{L}$ does. Therefore, (3.4) is a fully nonlinear equation. More precisely, we have the following result.

Lemma 3.1 *The operators $\mathscr{B}$ and $\mathscr{F}$ admit bounded realizations $B : H^1_\sharp \to L^2$ and $F : H^3_\sharp \to L^2$, respectively. Moreover, B is invertible.*

Proof A straightforward asymptotic analysis reveals that

$$b_k \sim 2\sqrt{\lambda_k}U, \quad f_k \sim 2\lambda_k^{\frac{3}{2}}$$

as $k \to +\infty$, from which we deduce that $\mathscr{B}$ and $\mathscr{F}$ admit bounded realizations $B : H^1_\sharp \to L^2$ and $F : H^3_\sharp \to L^2$.

Finally, since $b_k \neq 0$ for any $k \geq 0$, it follows that B is invertible.

3.3 The third-order pseudo-differential equation for the front

In view of Lemma 3.1, we may rewrite (3.4) as

$$\varphi_t = \mathscr{B}^{-1}\mathscr{L}(\varphi) + \mathscr{B}^{-1}\mathscr{F}((\varphi_y)^2)$$

or, equivalently, as

$$\varphi_t = \mathscr{A}(\varphi) + \mathscr{M}((\varphi_y)^2). \tag{3.5}$$

We emphasize that (3.5) is a pseudo-differential, fully nonlinear equation of the third-order, since the pseudo-differential operators $\mathscr{A}$ and $\mathscr{M}$ have symbols

$$a_k = \frac{(U^2 - X_k^2)(X_k^2 - \gamma U^2)}{4UX_k} \quad \text{and} \quad m_k = \frac{X_k^3 - 3UX_k^2 - 4\gamma U^2 X_k + 4\gamma U^3}{4UX_k}$$

for $k \geq 0$, respectively. Clearly, any smooth enough solution to (3.4) solves (3.5) as well.

The following result is crucial for the rest of the paper.

Theorem 3.1 *The following properties are satisfied:*

(i) *The realization $A : H^3_\sharp \to L^2$ of $\mathscr{A}$ is a sectorial operator and the sequence (a_k) constitutes its spectrum $\sigma(A)$. In particular, 0 is a simple eigenvalue of A, and the spectral projection Π associated with 0 is given by*

$$\Pi(\psi) = \frac{1}{\ell}\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \psi(y)\mathrm{d}y, \quad \psi \in L^2.$$

Finally, $\sigma(A) \setminus \{0\}$ is contained in the left half-plane $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0\}$ if and only if $\gamma < \gamma_c$.

(ii) *The realization $M : H^2_\sharp \to L^2$ of the operator $\mathscr{M}$ is bounded.*

Proof (i) Let us split

$$a_k = -\frac{2\lambda_k^{\frac{3}{2}}}{U} + \frac{-4\lambda_k^2 + 4\lambda_k^2\sqrt{\frac{U^2}{4\lambda_k} + 1} - \lambda_k U^2 + \lambda_k\gamma U^2}{U\sqrt{U^2 + 4\lambda_k}} =: -\frac{2\lambda_k^{\frac{3}{2}}}{U} + a_{1,k}$$

for any $k \geq 0$. Since

$$a_{1,k} \sim \frac{1}{4}\sqrt{\lambda_k}(2\gamma - 1)U$$

as $k \to +\infty$, if $\gamma \neq \frac{1}{2}$, we can infer that the realization A of operator $\mathscr{A}$ in $H^3_\sharp$ is well-defined. Moreover, since A splits into the sum of two operators A_0 (whose symbol is $(-2\lambda_k^{\frac{3}{2}}U^{-1})$) and A_1, which is a nice perturbation of A_0 (being a bounded operator in $H^1_\sharp$, which is an intermediate space of class $J_{\frac{1}{3}}$ between L^2 and $H^3_\sharp$), in view of [21, Proposition 2.4.1(i)], it is enough to prove that A_0 is a sectorial operator. But this follows immediately from the general abstract results (see, e.g., [11, Chapter 3]), or a direct computation. Indeed, if λ has the positive real part, then the equation $\lambda u - A_0 u = f$ has the unique solution, for any $f \in L^2$,

$$u = R(\lambda, A_0)f = U \sum_{k=0}^{+\infty} \frac{\widehat{f}(k)}{\lambda U + 2\lambda_k^{\frac{3}{2}}}$$

and

$$|R(\lambda, A_0)f|_2^2 = U^2 \sum_{k=0}^{+\infty} \frac{|\widehat{f}(k)|^2}{|\lambda U + 2\lambda_k^{\frac{3}{2}}|^2} \le \frac{1}{|\lambda|^2} \sum_{k=0}^{+\infty} |\widehat{f}(k)|^2 = \frac{1}{|\lambda|^2} |f|_2^2.$$

Proposition 2.1.1 in [21] yields the sectoriality of A_0.

Next we compute the spectrum of the operator A. Since $H^3_\sharp$ is compactly embedded into L^2, $\sigma(A)$ consists of eigenvalues only. We claim that $\sigma(A)$ consists of the elements of the sequence (a_k). Indeed writing the eigenvalue equation in the Fourier variable, we get the infinitely many equations

$$\lambda \widehat{\psi}(k) - a_k \widehat{\psi}(k) = 0, \quad k \ge 0, \tag{3.6}$$

which should be satisfied by the pair λ (the eigenvalue) and ψ (the eigenfunction). It is clear that this system of infinitely many equations admits a non-identically vanishing solution $(\widehat{\psi}(k))$ if and only if λ equals one of the elements of the sequence. The set equality $\sigma(A) = \{a_k : k \ge 0\}$ is thus proved.

Since the sequence (a_k) converges to $-\infty$ as $k \to +\infty$, all the eigenvalues of A are isolated. In particular, 0 is isolated and, again from formula (3.6), we easily see that the eigenspace associated with the eigenvalue $\lambda = 0$ is one-dimensional. To conclude that $\lambda = 0$ is simple, and in view of [21, Propositions A.1.2 and A.2.1], it suffices to prove that it is a simple pole of the resolvent operator. In such a case, the associated spectral projection is the residual at $\lambda = 0$ of $R(\,\cdot\,, A)$.

Clearly, for any $\lambda \notin \sigma(A)$,

$$R(\lambda, A)\zeta = \sum_{k=0}^{+\infty} \frac{1}{\lambda - a_k} \widehat{\zeta}(k) w_k$$

for any $\zeta \in L^2$. Hence,

$$\lambda R(\lambda, A)\zeta = \widehat{\psi}(0) w_0 + \sum_{k=1}^{+\infty} \frac{\lambda}{\lambda - a_k} \widehat{\zeta}(k) w_k =: \Pi\zeta + R_1(\lambda)\zeta.$$

Since $\lambda \ne a_k$ for any $k \ge 1$, and $a_k \to -\infty$ as $k \to +\infty$, there exists a neighborhood of $\lambda = 0$ in which the ratio $\frac{|\lambda|}{|\lambda - a_k|}$ is bounded, uniformly with respect to $k \ge 1$. As a byproduct, in such a neighborhood of $\lambda = 0$, the mapping $\lambda \mapsto R_1(\lambda)$ is bounded with values in $L(L^2)$. This shows that $\lambda = 0$ is a simple pole of operator A.

To conclude the proof of point (i), let us determine the values of γ such that $\sigma(A) \setminus \{0\}$ does not contain nonnegative elements. For this purpose, it suffices to observe that $a_k < 0$ for any $k \ge 1$ if and only if $4\lambda_k + U^2 - \gamma U^2 > 0$ for such k's, which is equivalent to $4\lambda_1 + U^2 - \gamma U^2 > 0$, since (λ_k) is a nondecreasing sequence. Hence, the condition for $\sigma(A) \setminus \{0\}$ be contained in $(-\infty, 0)$, is $\gamma < \gamma_c$, where

$$\gamma_c = 1 + \frac{16\pi^2}{\ell^2 U^2}. \tag{3.7}$$

(ii) As in the proof of Lemma 3.1, it suffices to observe that $m_k \sim \lambda_k U^{-1}$ as $k \to +\infty$.

The linearized stability principle (see, e.g., [21, Section 9.1.1]) and the results in Theorem 3.1 yield the following stability analysis.

Corollary 3.1 *Let* γ_c *be given by* (3.7).
(a) *If* $\gamma < \gamma_c$, *then the null solution to* (3.5) *is* (*orbitally*) *stable, with an asymptotic phase, with respect to sufficiently smooth and small perturbations.*
(b) *If* $\gamma > \gamma_c$, *then the null solution to* (3.5) *is unstable.*

4 Rigorous Asymptotic Derivation of the K-S Equation

The second question that we address is the link between (3.5) and (K-S). As in Section 2, we consider the small perturbation parameter $\varepsilon > 0$ defined by

$$\gamma = 1 + \varepsilon$$

(see (2.1)). Moreover, we perform the same change of dependent and independent variables as in (2.2), namely,

$$t = \frac{\tau}{\varepsilon^2 U^2}, \quad y = \frac{\eta}{\sqrt{\varepsilon} U}, \quad \varphi = \frac{\varepsilon}{U}\psi.$$

The key-idea is to link the small positive parameter ε and the width of the strip which will blow up as $\varepsilon \to 0$. For fixed $\ell_0 > 0$, we take ℓ of the form,

$$\ell_\varepsilon = \frac{\ell_0}{\sqrt{\varepsilon} U}.$$

Hence γ_c (see (3.7)) converges to 1 as $\varepsilon \to 0$.

In view of Corollary 3.1, in order to avoid a trivial dynamics, we assume that $\gamma_c > 1$. This means that we take the bifurcation parameter ℓ_0 larger than 4π and obtain that $\gamma_c \in (1, 1+\varepsilon)$.

For the new variables, $\mathscr{B}$ is replaced by the operator $\mathscr{B}_\varepsilon = U^2\sqrt{I - 4\varepsilon D_{\eta\eta}}$. Lemma 3.1 applies to this operator and guarantees that, for any fixed $\varepsilon > 0$, the realization $B_\varepsilon : H^1_\sharp \to L^2$ of $\mathscr{B}_\varepsilon$ is bounded. However, the perturbation is clearly singular as $\varepsilon \to 0$, since obviously $B_\varepsilon \to U^2 I$. Therefore, it is hopeless to take the limit $\varepsilon \to 0$ in the third-order equation (3.5). Fortunately, the fourth-order equation (3.4) is more friendly, since, after the division by ε^3 and U^3, it comes that

$$\begin{aligned} &\frac{\partial}{\partial \tau}(\sqrt{I - 4\varepsilon D_{\eta\eta}})\psi \\ &= -4D_{\eta\eta\eta\eta}\psi - D_{\eta\eta}\psi \\ &\quad + \frac{1}{4}\{(I - 4\varepsilon D_{\eta\eta})^{\frac{3}{2}} - 3(I - 4\varepsilon D_{\eta\eta}) - 4(1+\varepsilon)(\sqrt{I - 4\varepsilon D_{\eta\eta}} - I)\}(D_\eta \psi)^2, \end{aligned} \tag{4.1}$$

which is the perturbed equation that we are going to study, with periodic boundary conditions at $\eta = \pm\frac{\ell_0}{2}$.

Mimicking (3.4), we rewrite (4.1) in the abstract way,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathscr{B}_\varepsilon \psi = \mathscr{L}\psi + \mathscr{F}_\varepsilon((\psi_\eta)^2), \tag{4.2}$$

where the symbols of the operators $\mathscr{B}_\varepsilon$, $\mathscr{L}$ and $\mathscr{F}_\varepsilon$ are

$$b_{\varepsilon,k} = X_{\varepsilon,k}, \quad s_k = -\lambda_k(4\lambda_k - 1), \quad f_{\varepsilon,k} = \frac{1}{4}(X^3_{\varepsilon,k} - 3X^2_{\varepsilon,k} - 4(1+\varepsilon)X_{\varepsilon,k} + 4 + 4\varepsilon)$$

for any $k \geq 0$, respectively, and

$$X_{\varepsilon,k} = \sqrt{1 + 4\varepsilon\lambda_k}, \quad k \geq 0.$$

Writing (4.2) in the discrete Fourier variable gives infinitely many equations

$$b_{\varepsilon,k}\widehat{\psi}_\tau(\tau, k) = -\lambda_k(4\lambda_k - 1)\widehat{\psi}(\tau, k) + f_{\varepsilon,k}\widehat{(\psi_\eta)^2}(\tau, k)$$

for any $k \geq 0$. Note that the leading terms (namely, at order 0 in ε) of $b_{\varepsilon,k}$ and $f_{\varepsilon,k}$ are 1 and $-\frac{1}{2}$, respectively.

Fix $T > 0$. For $\Phi_0 \in H^m_\sharp$ $(m \geq 4)$, the Cauchy problem

$$\begin{cases} \Phi_\tau(\tau, \eta) = -4\Phi_{\eta\eta\eta\eta}(\tau, \eta) - \Phi_{\eta\eta}(\tau, \eta) - \dfrac{1}{2}(\Phi_\eta(\tau, \eta))^2, & \tau \geq 0,\ |\eta| \leq \frac{\ell_0}{2}, \\ D^k_\eta\Phi\Big(\tau, -\dfrac{\ell_0}{2}\Big) = D^k_\eta\Phi\Big(\tau, \dfrac{\ell_0}{2}\Big), & \tau \geq 0,\ k \leq m-1, \\ \Phi(0, \eta) = \Phi_0(\eta), & |\eta| \leq \frac{\ell_0}{2} \end{cases}$$

admits a unique solution $\Phi \in C([0, T]; H^m_\sharp)$ such that $\Phi_\tau \in C([0, T]; H^{m-4}_\sharp)$ (see, e.g., [6, Appendix B]).

Through Φ, we split $\psi = \Phi + \varepsilon\rho_\varepsilon$. For simplicity, we take zero as the initial condition for ρ_ε, and to avoid cumbersome notation, in the sequel, we write ρ for ρ_ε.

If ψ solves (4.2), then

$$\frac{\partial}{\partial\tau}\mathscr{B}_\varepsilon(\rho) + \mathscr{H}_\varepsilon(\Phi_\tau) = \mathscr{L}(\rho) + \mathscr{M}_\varepsilon((\Phi_\eta)^2) + \varepsilon\mathscr{F}_\varepsilon((\rho_\eta)^2) + 2\mathscr{F}_\varepsilon(\Phi_\eta\rho_\eta), \tag{4.3}$$

where the symbols of the operators $\mathscr{H}_\varepsilon$ and $\mathscr{M}_\varepsilon$ are

$$h_{\varepsilon,k} = \frac{1}{\varepsilon}(X_{\varepsilon,k} - 1), \quad m_{\varepsilon,k} = \frac{1}{4\varepsilon}(X^3_{\varepsilon,k} - 3X^2_{\varepsilon,k} - 4(1+\varepsilon)X_{\varepsilon,k} + 6 + 4\varepsilon)$$

for any $k \geq 0$.

Proposition 4.1 *There exists a positive constant C_* such that the following properties are satisfied for any $\varepsilon \in (0, 1]$:*

(a) *For any $s = 2, 3, \cdots$, the operators $\mathscr{B}_\varepsilon$ and $\mathscr{H}_\varepsilon$ admit bounded realizations B_ε and H_ε, respectively, mapping $H^s_\sharp$ into $H^{s-2}_\sharp$. Moreover,*

$$\|B_\varepsilon\|_{L(H^s_\sharp, H^{s-2}_\sharp)} + \|H_\varepsilon\|_{L(H^s_\sharp, H^{s-2}_\sharp)} \leq C_*.$$

Finally, the operator B_ε is invertible from $H^s_\sharp$ to $H^{s-2}_\sharp$.

(b) *For any $s = 3, 4, \cdots$, the operators $\mathscr{F}_\varepsilon$ and $\mathscr{M}_\varepsilon$ admit bounded realizations F_ε and M_ε, respectively, mapping H^s into H^{s-3}. Moreover,*

$$\|F_\varepsilon\|_{L(H^s_\sharp, H^{s-3}_\sharp)} + \|M_\varepsilon\|_{L(H^s_\sharp, H^{s-3}_\sharp)} \leq C_*.$$

Proof The statement follows from an analysis of the symbols of the operators $\mathscr{B}_\varepsilon$, $\mathscr{F}_\varepsilon$, $\mathscr{H}_\varepsilon$ and $\mathscr{M}_\varepsilon$. Without much effort, one can show that

$$|h_{\varepsilon,k}| \leq 4\lambda_k, \quad |m_{\varepsilon,k}| \leq 2\lambda_k^{\frac{3}{2}} + 25\lambda_k$$

for any $k \geq 0$ and any $\varepsilon \in (0, 1]$. These estimates combined with the formulas $0 \neq b_{\varepsilon,k} = \varepsilon h_{\varepsilon,k} + 1$ and $f_k = \varepsilon m_{\varepsilon,k} - \frac{1}{2}$, for any $k \geq 0$ and any $\varepsilon \in (0, 1]$, yield the assertion.

Instead of studying (3.4), we find it much more convenient to deal with the equation satisfied by $\zeta := \rho_\eta$, i.e.,

$$\frac{\partial}{\partial\tau}\mathscr{B}_\varepsilon(\zeta) + \mathscr{H}_\varepsilon(\Psi_\tau) = \mathscr{L}(\zeta) + \mathscr{M}_\varepsilon((\Psi^2)_\eta) + \varepsilon\mathscr{F}_\varepsilon((\zeta^2)_\eta) + 2\mathscr{F}_\varepsilon((\Psi\zeta)_\eta), \tag{4.4}$$

which we couple with the initial condition $\zeta(0,\,\cdot\,) = 0$. Here, $\Psi = \Phi_\eta$.

4.1 A priori estimates

For any $n = 0, 1, 2, \cdots$ and any $T > 0$, we set

$$X_n(T) = \{\zeta \in C([0,T]; H_\sharp^{4\vee 2n}) \cap C^1([0,T]; L^2) : \zeta_\tau \in C([0,T]; H_\sharp^{2\vee(n+1)})\},$$

where $a \vee b := \max\{a, b\}$.

For any $\varepsilon > 0$, we introduce in $H_\sharp^{\frac{1}{2}}$ the norm

$$\|\zeta\|^2_{\frac{1}{2},\varepsilon} = \sum_{k=0}^{+\infty} \sqrt{1 + 4\varepsilon\lambda_k}\,|\widehat{\zeta}(k)|^2, \quad \zeta \in H_\sharp^{\frac{1}{2}}.$$

Note that, for any fixed $\varepsilon > 0$, $\|\cdot\|_{\frac{1}{2},\varepsilon}$ is a norm, equivalent to the usual norm in $H_\sharp^{\frac{1}{2}}$.

The main result of this subsection is contained in the following theorem, where we set $\Psi_0 = (\Phi_0)_\eta$.

Theorem 4.1 *Fix an integer $n \geq 0$ and $T > 0$. Further, suppose that $\Psi_0 \in H_\sharp^{n+6}$. Then, there exist $\varepsilon_1 = \varepsilon_1(n, T) \in (0, 1)$ and $K_n = K_n(T) > 0$ such that, if $\zeta \in X_n(T_1)$ is a solution on the time interval $[0, T_1]$ to (4.4) for some $T_1 \leq T$, then*

$$\sup_{\tau\in[0,T_1]} \|D_\eta^n \zeta(\tau,\,\cdot\,)\|^2_{\frac{1}{2},\varepsilon} \leq K_n, \tag{4.5}$$

whenever $0 < \varepsilon \leq \varepsilon_1$.

Note that the assumptions on Ψ_0 guarantee that $\Psi \in C([0,T]; H_\sharp^{n+4}) \cap C^1([0,T]; H_\sharp^{n+2})$.

The proof of Theorem 4.1 heavily relies on the following lemma.

Lemma 4.1 *Let A_0, c_0, c_1, c_2, c_3, ε, T_0 be positive constants, and let T_1 be such that $0 < T_1 < T_0$. Further, let f_ε and $A_\varepsilon : [0, T_1] \to \mathbb{R}$ be a positive continuous function and a positive continuously differentiable function respectively such that*

$$\begin{cases} A'_\varepsilon(\tau) + (c_0 - \varepsilon^2(A_\varepsilon(\tau))^2) f_\varepsilon(\tau) \leq c_1 + c_2 A_\varepsilon(\tau) + c_3\varepsilon(A_\varepsilon(\tau))^2, & \tau \in [0, T_1], \\ A_\varepsilon(0) = 0. \end{cases}$$

Then, there exist $\varepsilon_1 = \varepsilon_1(T_0) \in (0, 1)$ and a constant $K = K(T_0)$ such that $A_\varepsilon(\tau) \leq K$ for any $\tau \in [0, T_1]$ and any $\varepsilon \in (0, \varepsilon_1]$.

Proof When f_ε identically vanishes, the proof follows from [2, Lemma 3.1], which shows that we can take

$$\varepsilon_1(T_0) = \frac{3c_2^2}{16c_1c_3(\mathrm{e}^{c_2T_0} - 1)} \quad \text{and} \quad K \leq \frac{4c_1\mathrm{e}^{c_2T_0}}{3c_2}.$$

Let us now consider the general case when f_ε does not identically vanish in $[0, T_1]$. We fix

$$\varepsilon_0 = \varepsilon_0(T_0) \leq \frac{3c_2^2}{16c_1c_3(\mathrm{e}^{c_2T_0} - 1)}$$

such that $9c_0c_2^2 - 12c_1c_2e^{c_2T_0}\varepsilon_0 - 16c_1^2e^{2c_2T_0}\varepsilon_0^2 > 0$, and $\varepsilon \in (0, \varepsilon_0]$. We claim that $c_0 - \varepsilon^2(A_\varepsilon(\tau))^2 > 0$ for any $\tau \in [0, T_1]$.

Let $(0, T_\varepsilon)$ be the largest interval (possibly depending on ε), where $c_0 - \varepsilon A_\varepsilon - \varepsilon^2(A_\varepsilon)^2$ is positive. The existence of this interval is clear, since A_ε vanishes at 0. The positivity of $c_0 - \varepsilon^2(A_\varepsilon)^2$ in $(0, T_\varepsilon)$ shows that $A_\varepsilon' \le c_1 + c_2A_\varepsilon + c_3\varepsilon A_\varepsilon^2$ in such an interval. From the above result we can infer that $A_\varepsilon(\tau) \le \frac{4c_1e^{c_2T_0}}{3c_2}$ for any $\tau \in [0, T_\varepsilon]$, so that $c_0 - \varepsilon^2(A_\varepsilon(T_\varepsilon))^2 > 0$. By the definition of T_ε, this clearly implies that $T_\varepsilon = T_1$.

Proof of Theorem 4.1 Throughout the proof, we assume that $T_1 \le T$ is fixed, and ε and τ are arbitrarily fixed in $(0, 1]$ and in $[0, T_1]$, respectively. Moreover, to avoid cumbersome notations, we denote by c almost all the constants appearing in the estimates. Hence, the exact value of c may change from line to line, but we do not need to follow the constants throughout the estimates. We just need to stress how the estimates depend on ε. As a matter of fact, all the c's are independent not only of ε but also of τ, Ψ and ζ. On the contrary, they may depend on n (and, actually, in most cases they do). Finally, we denote by $K(\Psi)$ a constant, which may depend on n and also on Ψ. As above, $K(\Psi)$ may vary from estimate to estimate.

The first step of the proof consists in multiplying both sides of (4.4) by $(-1)^nD_\eta^{2n}\zeta$, and integrating by parts over $\left(-\frac{\ell_0}{2}, \frac{\ell_0}{2}\right)$. This yields

$$
\begin{aligned}
&\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} B_\varepsilon(\zeta_\tau(\tau,\cdot))(-1)^nD_\eta^{2n}\zeta(\tau,\cdot)\mathrm{d}\eta + 4\int_{\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} |D_\eta^{n+2}\zeta(\tau,\cdot)|^2\mathrm{d}\eta - \int_{\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} |D_\eta^{n+1}\zeta(\tau,\cdot)|^2\mathrm{d}\eta \\
={}& -\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} (H_\varepsilon(\Psi_\tau(\tau,\cdot)) - M_\varepsilon((\Psi^2)_\eta(\tau,\cdot)))(-1)^nD_\eta^{2n}\zeta(\tau,\cdot)\mathrm{d}\eta \\
&+ \varepsilon\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} F_\varepsilon((\zeta^2)_\eta(\tau,\cdot))(-1)^nD_\eta^{2n}\zeta(\tau,\cdot)\mathrm{d}\eta \\
&+ 2\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} F_\varepsilon((\Psi\zeta)_\eta(\tau,\cdot))(-1)^nD_\eta^{2n}\zeta(\tau,\cdot)\mathrm{d}\eta.
\end{aligned}
\tag{4.6}
$$

Using Parseval's formula and the definition of the symbol $b_{\varepsilon,k}$, one can easily show that

$$
\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} B_\varepsilon(\zeta_\tau(\tau,\cdot))(-1)^nD_\eta^{2n}\zeta(\tau,\cdot)\mathrm{d}\eta = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\|D_\eta^n\zeta(\tau,\cdot)\|_{\frac{1}{2},\varepsilon}^2. \tag{4.7}
$$

We now deal with the other terms in (4.6). Integrating n-times by parts and, then, using Poincaré-Wirtinger and Cauchy-Schwarz inequalities, jointly with Proposition 4.1, it is not difficult to show that

$$
\left|\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} (H_\varepsilon(\Psi_\tau(\tau,\eta)) - M_\varepsilon((\Psi^2)_\eta(\tau,\cdot)))(-1)^nD_\eta^{2n}\zeta(\tau,\cdot)\mathrm{d}\eta\right| \le K(\Psi) + |D_\eta^n\zeta(\tau,\cdot)|_2^2 \tag{4.8}
$$

for any $\zeta \in X_n(T_1)$.

Estimating the other two integral terms in the right-hand side of (4.6) demands much more effort. The starting point is the following estimate:

$$
\begin{aligned}
&\left|\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} F_\varepsilon(\chi_\eta(\tau,\cdot))(-1)^nD_\eta^{2n}\zeta(\tau,\cdot)\mathrm{d}\eta\right| \\
&\le c\varepsilon^{\frac{3}{2}}|D_\eta^{n+2}\chi(\tau,\cdot)|_2|D_\eta^{n+2}\zeta(\tau,\cdot)|_2 + c\varepsilon|D_\eta^{n+2}\chi(\tau,\cdot)|_2|D_\eta^{n+1}\zeta(\tau,\cdot)|_2 \\
&\quad + c\sqrt{\varepsilon}|D_\eta^n\chi(\tau,\cdot)|_2|D_\eta^{n+2}\zeta(\tau,\cdot)|_2 + c|D_\eta^n\chi(\tau,\cdot)|_2|D_\eta^{n+1}\zeta(\tau,\cdot)|_2,
\end{aligned}
\tag{4.9}
$$

which holds for any $\chi \in C([0,T_1]; H_\sharp^{4\vee 2n})$. Such a formula follows by observing that

$$\begin{aligned}
&\Big|\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} F_\varepsilon(\chi_\eta(\tau,\cdot))(-1)^n D_\eta^{2n}\zeta(\tau,\cdot)\mathrm{d}\eta\Big| \\
&\le \sum_{k=0}^{+\infty}\lambda_k^n |f_{\varepsilon,k}||\widehat{\chi_\eta}(\tau,k)||\widehat{\zeta}(\tau,k)| \\
&\le c\sum_{k=0}^{+\infty}\lambda_k^n(\varepsilon^{\frac32}\lambda_k^{\frac32}+\varepsilon\lambda_k+\varepsilon^{\frac12}\lambda_k^{\frac12}+1)|\widehat{\chi_\eta}(\tau,k)||\widehat{\zeta}(\tau,k)|.
\end{aligned}$$

Then, by using Young inequality, we estimate the terms in the round brackets.

Now, we plug $\chi=\zeta^2$ into (4.9), and use the estimates

$$|D_\eta^{n+2}(\zeta(\tau,\cdot))^2|_2 \le c(|D_\eta^{n+2}\zeta(\tau,\cdot)|_2|D_\eta^n\zeta(\tau,\cdot)|_2+|D_\eta^{n+1}\zeta(\tau,\cdot)|_2^2), \tag{4.10}$$
$$|D_\eta^{n}(\zeta(\tau,\cdot))^2|_2 \le c|D_\eta^n\zeta(\tau,\cdot)|_2^2, \tag{4.11}$$

which can be obtained by using the Poincaré-Wirtinger inequality and Leibniz formula, the Cauchy-Schwarz inequality. Again, by using the Poincaré-Wirtinger inequality, we obtain

$$\begin{aligned}
&\Big|\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} F_\varepsilon((\zeta^2)_\eta(\tau,\cdot))(-1)^n D_\eta^{2n}\zeta(\tau,\cdot)\mathrm{d}\eta\Big| \\
&\le c\varepsilon^{\frac32}|D_\eta^n\zeta(\tau,\cdot)|_2|D_\eta^{n+2}\zeta(\tau,\cdot)|_2^2+c\varepsilon^{\frac32}|D_\eta^{n+1}\zeta(\tau,\cdot)|_2^2|D_\eta^{n+2}\zeta(\tau,\cdot)|_2 \\
&\quad + c\varepsilon|D_\eta^n\zeta_\eta(\tau,\cdot)|_2|D_\eta^{n+1}\zeta(\tau,\cdot)|_2|D_\eta^{n+2}\zeta(\tau,\cdot)|_2+c\varepsilon|D_\eta^{n+1}\zeta(\tau,\cdot)|_2^3 \\
&\quad + c\varepsilon^{\frac12}(1+\varepsilon)|D_\eta^n\zeta(\tau,\cdot)|_2^2|D_\eta^{n+2}\zeta(\tau,\cdot)|_2+c|D_\eta^n\zeta(\tau,\cdot)|_2^2|D_\eta^{n+1}\zeta(\tau,\cdot)|_2 \\
&\le c\varepsilon^{\frac32}|D_\eta^n\zeta(\tau,\cdot)|_2|D_\eta^{n+2}\zeta(\tau,\cdot)|_2^2+c\varepsilon^{\frac32}|D_\eta^{n+1}\zeta(\tau,\cdot)|_2^2|D_\eta^{n+2}\zeta(\tau,\cdot)|_2 \\
&\quad + c\varepsilon|D_\eta^{n+1}\zeta(\tau,\cdot)|_2^2|D_\eta^{n+2}\zeta(\tau,\cdot)|_2+c\varepsilon|D_\eta^{n+1}\zeta(\tau,\cdot)|_2^2|D_\eta^{n+2}\zeta(\tau,\cdot)|_2 \\
&\quad + c\varepsilon^{\frac12}|D_\eta^n\zeta(\tau,\cdot)|_2^2|D_\eta^{n+2}\zeta(\tau,\cdot)|_2+c|D_\eta^n\zeta(\tau,\cdot)|_2^2|D_\eta^{n+2}\zeta(\tau,\cdot)|_2 \\
&\le c\varepsilon^{\frac32}|D_\eta^n\zeta(\tau,\cdot)|_2|D_\eta^{n+2}\zeta(\tau,\cdot)|_2^2+c\varepsilon^2|D_\eta^{n+1}\zeta(\tau,\cdot)|_2^4+c\varepsilon|D_\eta^{n+2}\zeta(\tau,\cdot)|_2^2 \\
&\quad + c|D_\eta^n\zeta(\tau,\cdot)|_2^4+c|D_\eta^{n+2}\zeta(\tau,\cdot)|_2^2 \\
&\le c\varepsilon^{\frac32}|D_\eta^n\zeta(\tau,\cdot)|_2|D_\eta^{n+2}\zeta(\tau,\cdot)|_2^2+c\varepsilon^2|D_\eta^{n+1}\zeta(\tau,\cdot)|_2^4 \\
&\quad + c|D_\eta^{n+2}\zeta(\tau,\cdot)|_2^2+c|D_\eta^n\zeta(\tau,\cdot)|_2^4.
\end{aligned}\tag{4.12}$$

In the similar way, using the estimate $|D_\eta^m(\Psi\zeta)(\tau,\cdot)|_2\le c|D_\eta^m\zeta(\tau,\cdot)|_2|D_\eta^m\Psi(\tau,\cdot)|_2$ (with $m\in\{n,n+2\}$) in place of (4.10)–(4.11), from (4.9), we get

$$\begin{aligned}
&\Big|\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} F_\varepsilon((\Psi\zeta)_\eta(\tau,\cdot))(-1)^n D_\eta^{2n}\zeta(\tau,\cdot)\mathrm{d}\eta\Big| \\
&\le c\varepsilon^{\frac32}|D_\eta^{n+2}\Psi(\tau,\cdot)|_2|D_\eta^{n+2}\zeta(\tau,\cdot)|_2^2+c\varepsilon|D_\eta^{n+2}\Psi(\tau,\cdot)|_2|D_\eta^{n+1}\zeta(\tau,\cdot)|_2^2 \\
&\quad + c\varepsilon|D_\eta^{n+2}\Psi(\tau,\cdot)|_2|D_\eta^{n+2}\zeta(\tau,\cdot)|_2^2+c|D_\eta^{n}\Psi(\tau,\cdot)|_2|D_\eta^{n}\zeta(\tau,\cdot)|_2^2 \\
&\quad + c\varepsilon|D_\eta^{n}\Psi(\tau,\cdot)|_2|D_\eta^{n+2}\zeta(\tau,\cdot)|_2^2+c\delta^{-1}|D_\eta^{n}\Psi(\tau,\cdot)|_2^2|D_\eta^{n}\zeta(\tau,\cdot)|_2^2+c\delta|D_\eta^{n+2}\zeta(\tau,\cdot)|_2^2
\end{aligned}\tag{4.13}$$

for any $\delta>0$. We just mention the inequality

$$|D^n\Psi(\tau,\cdot)|_2|D^n\zeta(\tau,\cdot)|_2|D^{n+1}\zeta(\tau,\cdot)|_2\le c\delta^{-1}|D^n\Psi(\tau,\cdot)|_2^2|D^n\zeta(\tau,\cdot)|_2^2+\delta|D^{n+1}\zeta(\tau,\cdot)|_2^2,$$

obtained by means of Young and Poincaré-Wirtinger inequalities, which we use to estimate one of the intermediate terms appearing in the proof of (4.13).

Now, taking $c\delta = \frac{5}{2}$, we get the estimate

$$\Big|\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} F_\varepsilon((\Psi\zeta)_\eta(\tau,\cdot))(-1)^n D_\eta^{2n}\zeta(\tau,\cdot)\mathrm{d}\eta\Big|$$
$$\le K(\Psi)(\varepsilon|D_\eta^{n+2}\zeta(\tau,\cdot)|_2^2 + \varepsilon|D_\eta^{n+1}\zeta(\tau,\cdot)|_2^2 + |D_\eta^n\zeta(\tau,\cdot)|_2^2) + \frac{5}{2}|D_\eta^{n+2}\zeta(\tau,\cdot)|_2^2. \qquad (4.14)$$

From (4.6)–(4.8), (4.12), (4.14) and the interpolative inequality

$$|D_\eta^{n+1}\zeta(\tau,\cdot)|_2^2 \le |D_\eta^n\zeta(\tau,\cdot)|_2^2 + \frac{1}{4}|D_\eta^{n+2}\zeta(\tau,\cdot)|_2^2,$$

we can infer that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\|D_\eta^n\zeta(\tau,\cdot)\|_{\frac{1}{2},\varepsilon}^2 + (1-\varepsilon K(\Psi) - c\varepsilon^{\frac{5}{2}}|D_\eta^n\zeta(\tau,\cdot)|_2)|D_\eta^{n+2}\zeta(\tau,\cdot)|_2^2\\
&\le K(\Psi) + K(\Psi)|D_\eta^n\zeta(\tau,\cdot)|_2^2 + \varepsilon K(\Psi)|D_\eta^{n+1}\zeta(\tau,\cdot)|_2^2 + c\varepsilon|D_\eta^n\zeta(\tau,\cdot)|_2^4 + c\varepsilon^3|D_\eta^{n+1}\zeta(\tau,\cdot)|_2^4\\
&\le K(\Psi) + K(\Psi)|D_\eta^n\zeta(\tau,\cdot)|_2^2 + \varepsilon K(\Psi)|D_\eta^{n+2}\zeta(\tau,\cdot)|_2^2\\
&\quad + c\varepsilon|D_\eta^n\zeta(\tau,\cdot)|_2^4 + c\varepsilon^3|D_\eta^n\zeta(\tau,\cdot)|_2^2|D_\eta^{n+2}\zeta(\tau,\cdot)|_2^2,
\end{aligned}$$

which we can rewrite in the form

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\|D_\eta^n\zeta(\tau,\cdot)\|_{\frac{1}{2},\varepsilon}^2 + (2-\varepsilon K(\Psi) - c\varepsilon^2\|D_\eta^n\zeta(\tau,\cdot)\|_{\frac{1}{2},\varepsilon}^2)|D_\eta^{n+2}\zeta(\tau,\cdot)|_2^2\\
&\le K(\Psi) + K(\Psi)\|D_\eta^n\zeta(\tau,\cdot)\|_{\frac{1}{2},\varepsilon}^2 + c\varepsilon\|D_\eta^n\zeta(\tau,\cdot)\|_{\frac{1}{2},\varepsilon}^4,
\end{aligned}$$

by estimating $2\|D_\eta^n\zeta(\tau,\cdot)\|_{\frac{1}{2},\varepsilon} \le \|D_\eta^n\zeta(\tau,\cdot)\|_{\frac{1}{2},\varepsilon}^2 + 1$ and recalling that $\varepsilon \in (0,1]$.

Up to replacing $(0,1]$ by a smaller interval $(0,\varepsilon_0]$, we can assume that $\varepsilon K(\Psi) < 1$ for any $\varepsilon \in (0,\varepsilon_0]$. Hence, applying Lemma 4.1 with

$$\begin{aligned}
&c_0 = 1, \quad c_1 = K(\Psi), \quad c_2 = K(\Psi), \quad c_3 = c,\\
&A_\varepsilon(\tau) = \|D_\eta^n\zeta(\tau,\cdot)\|_{\frac{1}{2},\varepsilon}^2, \quad f_\varepsilon(\tau) = |D_\eta^{n+2}\zeta(\tau,\cdot)|_2^2,
\end{aligned}$$

we complete the proof.

Now, taking advantage of the previous a priori estimates, which can be extended also to variational solutions ζ_N to (4.4) belonging to the space spanned by the functions $w_1,\cdots,w_N$ (with constants independent of $N \in \mathbb{N}$), and using the classical Faedo-Galerkin method, the following result can be proved.

Theorem 4.2 *Fix $T > 0$. Then, there exists an $\varepsilon_0(T) > 0$ such that, for any $0 < \varepsilon \le \varepsilon_0(T)$,* (4.4) *has a unique classical solution ζ on $[0,T]$, vanishing at $\tau = 0$.*

4.2 Proof of Theorem 1.1

Since the unique solution ζ to (4.4) is the candidate to be the η-derivative of the solution ρ to (4.3), ρ should split into the sum $\rho(\tau,\eta) = (\mathscr{P}(\zeta))(\tau,\eta) + \upsilon(\tau)$ for some scalar valued function υ, where

$$(\mathscr{P}(\zeta))(\tau,\eta) = \int_{-\frac{\ell_0}{2}}^{\eta}\zeta(s)\mathrm{d}s - \frac{1}{2}\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}\zeta(s)\Big(1-\frac{2s}{\ell_0}\Big)\mathrm{d}s.$$

Imposing that ρ in the previous form is a solution to (4.3) and projecting along $\Pi(L^2)$, we see that ρ is a solution to (4.3) if and only if υ solves the following Cauchy problem:

$$\begin{cases} \dfrac{\mathrm{d}\upsilon}{\mathrm{d}\tau} = -\Pi(H_\varepsilon(\Phi_\tau)) - \dfrac{1}{2}\varepsilon\Pi(\zeta^2) - \Pi(\Phi_\eta\zeta), \\ \upsilon(0) = 0. \end{cases}$$

Since this problem has in fact a unique solution, and $\mathscr{P}(\zeta) + \upsilon$ vanishes at $\tau = 0$, we conclude that problem (4.3) is uniquely solvable.

To complete the proof, we should show that there exists an $M > 0$ such that

$$\sup_{\substack{\tau\in[0,T] \\ \eta\in\left[-\frac{\ell_0}{2},\frac{\ell_0}{2}\right]}} |\rho(\tau,\eta)| \le M \tag{4.15}$$

uniformly in $0 < \varepsilon \le \varepsilon_0(T)$. Once this estimate is proved, coming back from (4.3) to (1.11), we see that the latter one has a unique classical solution $\varphi : [0, \frac{T}{\varepsilon^2 U^2}] \times \mathbb{R} \to \mathbb{R}$, which is periodic (with respect to the spatial variable) with period $\ell_\varepsilon = \frac{\ell_0}{\sqrt{\varepsilon}U}$, and satisfies

$$\varphi(0, \cdot\,) = \varepsilon U^{-1}\Phi_0(\sqrt{\varepsilon}U\cdot\,),$$

as well as the estimate

$$\|\varphi(t, \cdot\,) - \varepsilon U^{-1}\Phi(t\varepsilon^2U^2, \cdot\sqrt{\varepsilon}U)\|_{C([-\frac{\ell_\varepsilon}{2},\frac{\ell_\varepsilon}{2}])} \le \frac{\varepsilon^2 M}{U}, \quad t \in [0, T_\varepsilon],$$

as is claimed.

So, let us prove (4.15). For this purpose, it is enough to use the a priori estimate (4.5) jointly with the Poincaré-Wirtinger inequality to estimate ζ, and to use (4.5) to estimate υ. This completes the proof of Theorem 1.1.

5 Numerical Experiments

In this section, we intend to solve numerically (4.1) for small positive ε and illustrate the convergence to the solution to (K-S).

In order to reformulate (4.1) on the interval $[0, 2\pi]$ with periodic boundary conditions, we set $x = \frac{\eta}{2\widetilde{\ell_0}}$, where $\widetilde{\ell_0} = \frac{\ell_0}{4\pi}$. It comes that

$$\begin{aligned} &\frac{\partial}{\partial\tau}\left(\sqrt{I - \frac{\varepsilon}{\widetilde{\ell_0}^2}D_{xx}}\right)\psi \\ &= -\frac{1}{4\widetilde{\ell_0}^4}D_{xxxx}\psi - \frac{1}{4\widetilde{\ell_0}^2}D_{xx}\psi \\ &\quad + \frac{1}{16\widetilde{\ell_0}^2}\left\{\left(I - \frac{\varepsilon}{\widetilde{\ell_0}^2}D_{xx}\right)^{\frac{3}{2}} - 3\left(I - \frac{\varepsilon}{\widetilde{\ell_0}^2}D_{xx}\right) - 4(1+\varepsilon)\left(\sqrt{I - \frac{\varepsilon}{\widetilde{\ell_0}^2}D_{xx}} - I\right)\right\}(D_x\psi)^2. \end{aligned}$$

Next, we define the bifurcation parameter $\beta = 4\widetilde{\ell_0}^2$ as in [7, 12]. After multiplication by β^2, it comes that

$$\begin{aligned} &\frac{\partial}{\partial\tau}(\sqrt{\beta^4 - 4\varepsilon\beta^3 D_{xx}})\psi \\ &= -4D_{xxxx}\psi - \beta D_{xx}\psi \\ &\quad + \frac{\beta}{4}\left\{\left(I - \frac{4\varepsilon}{\beta}D_{xx}\right)^{\frac{3}{2}} - 3\left(I - \frac{4\varepsilon}{\beta}D_{xx}\right) - 4(1+\varepsilon)\left(\sqrt{I - \frac{4\varepsilon}{\beta}D_{xx}} - I\right)\right\}(D_x\psi)^2. \end{aligned}$$

Finally, we rescale the time by setting $t = \frac{\tau}{\beta^2}$,

$$\begin{aligned}&\frac{\partial}{\partial t}\Big(\sqrt{I - \frac{4\varepsilon}{\beta}D_{xx}}\Big)\psi \\ &= -4D_{xxxx}\psi - \beta D_{xx}\psi \\ &\quad + \frac{\beta}{4}\Big\{\Big(I - \frac{4\varepsilon}{\beta}D_{xx}\Big)^{\frac{3}{2}} - 3\Big(I - \frac{4\varepsilon}{\beta}D_{xx}\Big) - 4(1+\varepsilon)\Big(\sqrt{I - \frac{4\varepsilon}{\beta}D_{xx}} - I\Big)\Big\}(D_x\psi)^2.\end{aligned}$$

By setting $\varepsilon' = \frac{\varepsilon}{\beta}$, with the prime being omitted hereafter, we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}(\sqrt{I - 4\varepsilon D_{xx}})\psi \\ &= -4D_{xxxx}\psi - \beta D_{xx}\psi \\ &\quad + \frac{\beta}{4}\{(I - 4\varepsilon D_{xx})^{\frac{3}{2}} - 3(I - 4\varepsilon D_{xx}) - 4(1+\varepsilon)(\sqrt{I - 4\varepsilon D_{xx}} - I)\}(D_x\psi)^2. \qquad (5.1)\end{aligned}$$

The initial condition is given by $\psi(0, \cdot) = \psi_0$, where ψ_0 is periodic with period 2π. Note that, in contrast to [7, 12], we do not subtract the drift.

(5.1) in the discrete Fourier variable gives

$$\begin{aligned}&\frac{\partial}{\partial t}(\sqrt{1 + 4\varepsilon k^2})\widehat{\psi}(t,k) \\ &= -4k^4\widehat{\psi}(t,k) + \beta k^2\widehat{\psi}(t,k) \\ &\quad + \frac{\beta}{4}\{(1 + 4\varepsilon k^2)^{\frac{3}{2}} - 3(1 + 4\varepsilon k^2) - 4(1+\varepsilon)(\sqrt{1 + 4\varepsilon k^2} - 1)\}\widehat{(\psi_x)^2}(t,k).\end{aligned}$$

We use a backward-Euler scheme for the first-order time derivative to treat implicitly all the linear terms and to treat explicitly the nonlinear terms. The implicit treatment of the fourth- and second-order terms reduces the stability constraint, while the explicit treatment of the nonlinear terms avoids the expensive process of solving nonlinear equations at each time step. For simplicity, in the rest of this section, we use the notation $\widehat{f}_k$ instead of $\widehat{f}(k)$. It comes that

$$\begin{aligned}&(\sqrt{1 + 4\varepsilon k^2})\frac{\widehat{\psi}_k^{n+1} - \widehat{\psi}_k^n}{\Delta t} \\ &= -4k^4\widehat{\psi}_k^{n+1} + \beta k^2\widehat{\psi}_k^{n+1} \\ &\quad + \frac{\beta}{4}\{(1 + 4\varepsilon k^2)^{\frac{3}{2}} - 3(1 + 4\varepsilon k^2) - 4(1+\varepsilon)(\sqrt{1 + 4\varepsilon k^2} - 1)\}\{[(\psi_x)^n]^2\}_k,\end{aligned}$$

where $\{(\psi_x)^2\}_k$ represents the k-th Fourier coefficient of $(\psi_x)^2$. This method is of the first order with respect to time. From the previous equation, it is easy to compute the k-th Fourier coefficient $\widehat{\psi}_k^{n+1}$. One gets

$$\begin{aligned}\widehat{\psi}_k^{n+1} &= ((1 + 4\varepsilon k^2)^{\frac{1}{2}} + 4k^4\Delta t - \beta k^2\Delta t)^{-1}\Big((1 + 4\varepsilon k^2)^{\frac{1}{2}}\widehat{\psi}_k^n \\ &\quad + \frac{\beta\Delta t}{4}\{(1 + 4\varepsilon k^2)^{\frac{3}{2}} - 3(1 + 4\varepsilon k^2) - 4(1+\varepsilon)[(1 + 4\varepsilon k^2)^{\frac{1}{2}} - 1]\}\{[(\psi_x)^n]^2\}_k\Big). \qquad (5.2)\end{aligned}$$

Practical calculations hold in the spectral space. We use an additional FFT to recover the physical nodal values ψ_j from $\widehat{\psi}_k$, where j stands for the division node in the physical space.

The numerical tests aim at checking the behavior of the solutions to (5.1) for values of ε close to 0, and comparing them to those for the Kuramoto-Sivashinsky equation. In Figures 1–4 and 5–8, we plot consecutive front positions computed by using (5.2), taking $\beta = 10$ and 20 respectively, and giving to ε the following values: 0.1, 0.01, 0.001 and 0 (which correspond to the equation (K-S)).

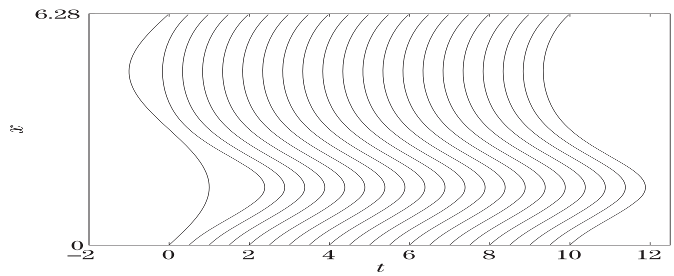

Figure 1 Front propagation with $\beta = 10$, $\varepsilon = 0.1$ and $\psi_0(x) = \sin(x)$.

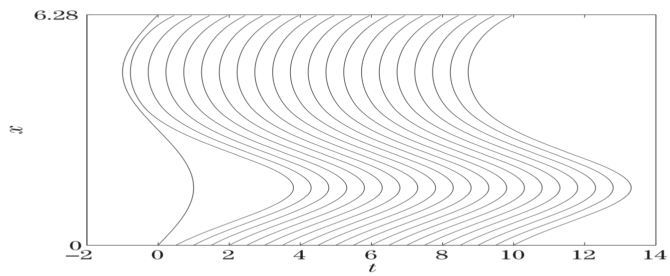

Figure 2 Front propagation with $\beta = 10$, $\varepsilon = 0.01$ and $\psi_0(x) = \sin(x)$.

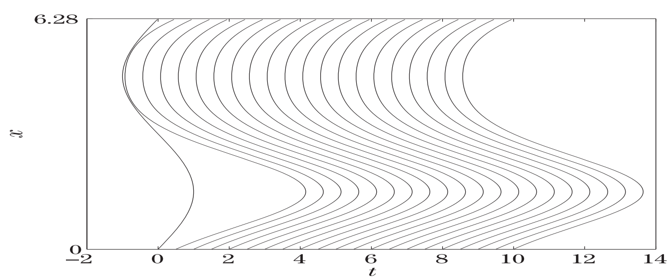

Figure 3 Front propagation with $\beta = 10$, $\varepsilon = 0.001$ and $\psi_0(x) = \sin(x)$.

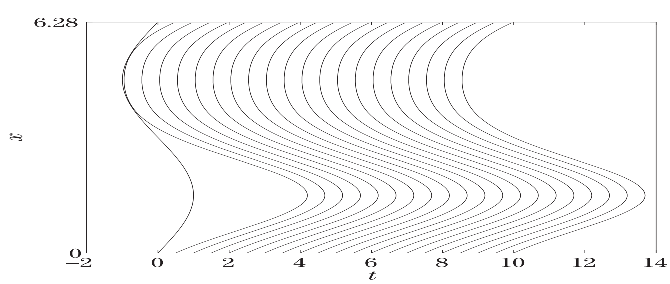

Figure 4 Front propagation with $\beta = 10$, $\varepsilon = 0$ and $\psi_0(x) = \sin(x)$.

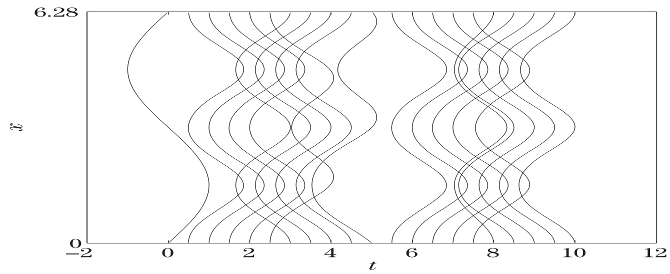

Figure 5 Front propagation with $\beta = 20$, $\varepsilon = 0.1$ and $\psi_0(x) = \sin(x)$.

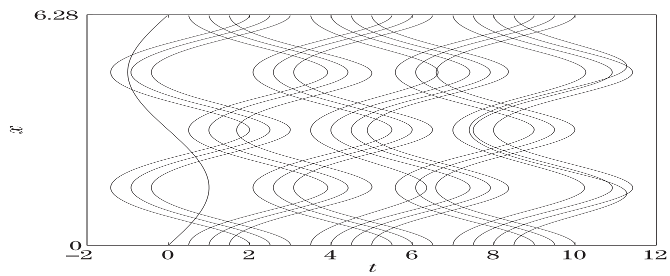

Figure 6 Front propagation with $\beta = 20$, $\varepsilon = 0.01$ and $\psi_0(x) = \sin(x)$.

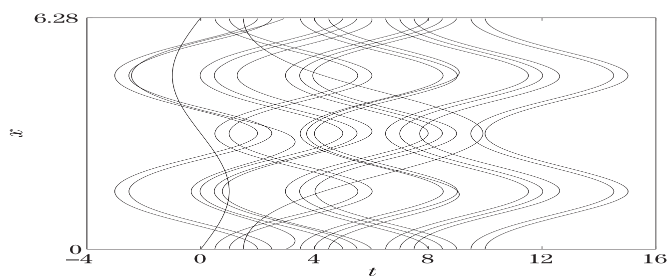

Figure 7 Front propagation with $\beta = 20$, $\varepsilon = 0.001$ and $\psi_0(x) = \sin(x)$.

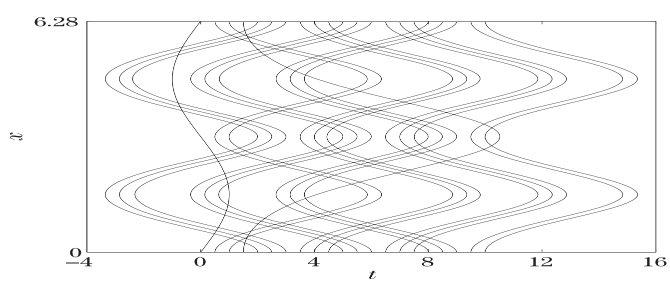

Figure 8 Front propagation with $\beta = 20$, $\varepsilon = 0$ and $\psi_0(x) = \sin(x)$.

We now investigate the dynamics of (5.1) with respect to the parameter β. For this purpose, we fix $\varepsilon = 0.001$.

The numerical simulations confirm that, as for the equation (K-S), 0 turns out to be a global attractor for the solution to (5.1), for any $\beta \in [1, 4]$. A non-trivial attractor is expected for larger β's. In Figures 9–12, we can see the front evolutions generated by (5.2) with $\beta = 30$, 60 for two different initial conditions. In all the figures below, the periodic orbit is clearly observed.

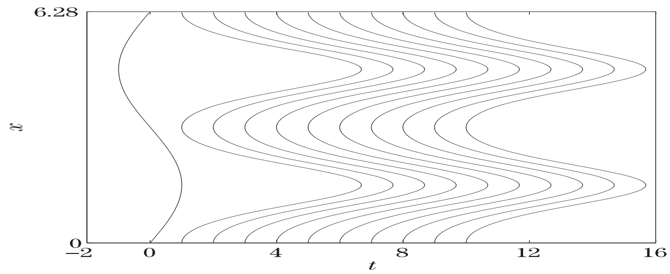

Figure 9 Front propagation with $\beta = 30$, $\varepsilon = 0.001$ and $\psi_0(x) = \sin(x)$.

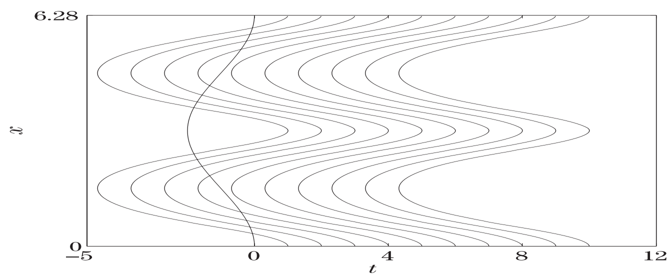

Figure 10 Front propagation with $\beta = 30$, $\varepsilon = 0.001$ and $\psi_0(x) = \cos(x)$.

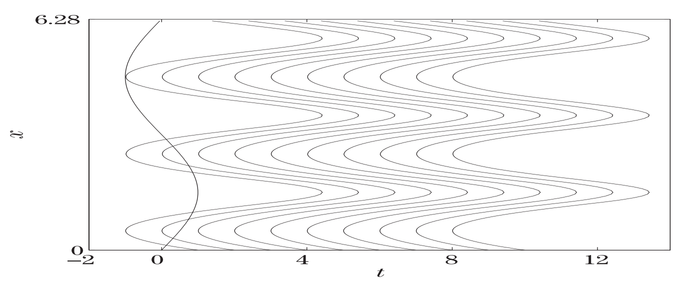

Figure 11 Front propagation with $\beta = 60$, $\varepsilon = 0.001$ and $\psi_0(x) = \sin(x)$.

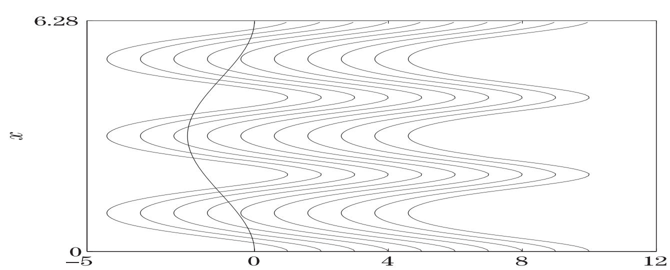

Figure 12 Front propagation with $\beta = 60$, $\varepsilon = 0.001$, and $\psi_0(x) = \cos(x)$.

Summing up, our numerical tests confirm that (5.1) preserves the same structure as the equation (K-S). Larger β generates an even richer dynamics (see Figure 13) where the front propagation is captured from a computation with $\beta = 108$. As predicted in [12], the front evolves toward an essentially quadrimodal global attractor.

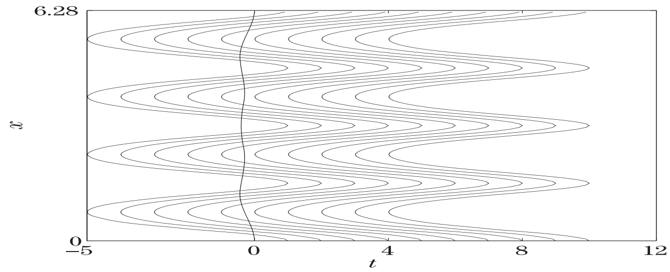

Figure 13 Front propagation with $\beta = 108,\ \varepsilon = 0.0001$ and $\psi_0(x) = 0.1(\cos(x) + \cos(2x) + \cos(3x))$.

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