Chin. Ann. Math.
34B(1), 2013, 99–112
DOI: 10.1007/s11401-012-0756-6

Chinese Annals of Mathematics, Series B

Sharp Interpolation Inequalities on the Sphere: New Methods and Consequences*

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(*In honor of the scientific heritage of Jacques-Louis Lions*)

Abstract This paper is devoted to various considerations on a family of sharp interpolation inequalities on the sphere, which in dimension greater than 1 interpolate between Poincaré, logarithmic Sobolev and critical Sobolev (Onofri in dimension two) inequalities. The connection between optimal constants and spectral properties of the Laplace-Beltrami operator on the sphere is emphasized. The authors address a series of related observations and give proofs based on symmetrization and the ultraspherical setting.

Keywords Sobolev inequality, Interpolation, Gagliardo-Nirenberg inequality, Logarithmic Sobolev inequality, Heat equation
2000 MR Subject Classification 26D10, 46E35, 58E35

1 Introduction

The following interpolation inequality holds on the sphere:

$$\frac{p-2}{d}\int_{\mathbb{S}^d}|\nabla u|^2\mathrm{d}\mu+\int_{\mathbb{S}^d}|u|^2\mathrm{d}\mu\ge\Big(\int_{\mathbb{S}^d}|u|^p\mathrm{d}\mu\Big)^{\frac{2}{p}},\quad \forall\, u\in \mathrm{H}^1(\mathbb{S}^d,\mathrm{d}\mu) \tag{1.1}$$

for any $p\in(2,2^*]$ with $2^*=\frac{2d}{d-2}$ if $d\ge 3$, and for any $p\in(2,\infty)$ if $d=2$. In (1.1), $\mathrm{d}\mu$ is the uniform probability measure on the d-dimensional sphere, that is, the measure induced by Lebesgue's measure on $\mathbb{S}^d\subset\mathbb{R}^{d+1}$, up to a normalization factor such that $\mu(\mathbb{S}^d)=1$.

Such an inequality was established by Bidaut-Véron and Véron [21] in the more general context of compact manifolds with uniformly positive Ricci curvature. Their method is based on the Bochner-Lichnerowicz-Weitzenböck formula and the study of the set of solutions to an elliptic equation, which is seen as a bifurcation problem and contains the Euler-Lagrange equation associated to the optimality case in (1.1). Later, in [12], Beckner gave an alternative proof based on Legendre's duality, the Funk-Hecke formula, proved in [27, 31], and the expression of

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*Project supported by ANR grants CBDif and NoNAP, the ECOS project (No. C11E07), the Chilean research grants Fondecyt (No. 1090103), Fondo Basal CMM-Chile, Project Anillo ACT-125 CAPDE and the National Science Foundation (No. DMS-0901304).

some optimal constants found by Lieb [33]. Bakry, Bentaleb and Fahlaoui in a series of papers based on the carré du champ method and mostly devoted to the ultraspherical operator showed a result which turns out to give yet another proof, which is anyway very close to the method of [21]. Their computations allow to slightly extend the range of the parameter p (see [7–8, 14–20] and [34, 37] for earlier related works).

In all computations based on the Bochner-Lichnerowicz-Weitzenböck formula, the choice of exponents in the computations appears somewhat mysterious. The seed for such computations can be found in [28]. Our purpose is on one hand to give alternative proofs, at least for some ranges of the parameter p, which do not rely on such a technical choice. On the other hand, we also aim at simplifying the existing proofs (see Section 3.2).

Inequality (1.1) is remarkable for several reasons as follows:

(1) It is optimal in the sense that 1 is the optimal constant. By Hölder's inequality, we know that $\|u\|_{\mathrm{L}^2(\mathbb{S}^d)} \le \|u\|_{\mathrm{L}^p(\mathbb{S}^d)}$, so that the equality case can only be achieved by functions, which are constants a.e. Of course, the main issue is to prove that the constant $\frac{p-2}{d}$ is optimal, which is one of the classical issues of the so-called A-B problem, for which we primarily refer to [30].

(2) If $d \ge 3$, the case $p = 2^*$ corresponds to the Sobolev's inequality. Using the stereographic projection as in [33], we easily recover Sobolev's inequality in the Euclidean space $\mathbb{R}^d$ with the optimal constant and obtain a simple characterization of the extremal functions found by Aubin and Talenti [5, 36–37].

(3) In the limit $p \to 2$, one obtains the logarithmic Sobolev inequality on the sphere, while by taking $p \to \infty$ if $d = 2$, one recovers Onofri's inequality (see [25] and Corollary 2.1 below).

Exponents are not restricted to $p > 2$. Consider indeed the functional

$$\mathcal{Q}_p[u] := \frac{p-2}{d}\,\frac{\int_{\mathbb{S}^d} |\nabla u|^2 \mathrm{d}\mu}{\left(\int_{\mathbb{S}^d} |u|^p \mathrm{d}\mu\right)^{\frac{2}{p}} - \int_{\mathbb{S}^d} |u|^2 \mathrm{d}\mu}$$

for $p \in [1,2) \cup (2, 2^*]$ if $d \ge 3$, or $p \in [1,2) \cup (2,\infty)$ if $d = 2$, and

$$\mathcal{Q}_2[u] := \frac{2}{d}\,\frac{\int_{\mathbb{S}^d} |\nabla u|^2 \mathrm{d}\mu}{\int_{\mathbb{S}^d} |u|^2 \log\left(\frac{|u|^2}{\int_{\mathbb{S}^d} |u|^2 \mathrm{d}\mu}\right) \mathrm{d}\mu}$$

for any $d \ge 1$. Because $\mathrm{d}\mu$ is a probability measure, $\left(\int_{\mathbb{S}^d} |u|^p \mathrm{d}\mu\right)^{\frac{2}{p}} - \int_{\mathbb{S}^d} |u|^2 \mathrm{d}\mu$ is nonnegative if $p > 2$, nonpositive if $p \in [1, 2)$, and equal to zero if and only if u is constant a.e. Denote by $\mathcal{A}$ the set of $\mathrm{H}^1(\mathbb{S}^d, \mathrm{d}\mu)$ functions, which are not a.e. constants. Consider the infimum

$$\mathcal{I}_p := \inf_{u \in \mathcal{A}} \mathcal{Q}_p[u] \,. \tag{1.2}$$

With these notations, we can state a slight result more general than the one of (1.1), which goes as follows and also covers the range $p \in [1, 2]$.

Theorem 1.1 *With the above notations, $\mathcal{I}_p = 1$ for any $p \in [1, 2^*]$ if $d \ge 3$, or any $p \in [1, \infty)$ if $d = 1, 2$.*

As already explained above, in the case $(2, 2^*]$, the above theorem was proved first in [21, Corollary 6.2], and then in [12], by using the previous results of Lieb [33] and the Funk-Hecke formula (see [27, 31]). The case $p = 2$ was covered in [12]. The whole range $p \in [1, 2^*]$ was covered in the case of the ultraspherical operator in [19–20]. Here we give alternative proofs for various ranges of p, which are less technical and interesting in themselves, as well as some extensions.

Notice that the case $p=1$ can be written as

$$\int_{\mathbb{S}^d} |\nabla u|^2 \mathrm{d}\mu \geq d\Big[\int_{\mathbb{S}^d} |u|^2 \mathrm{d}\mu - \Big(\int_{\mathbb{S}^d} |u| \mathrm{d}\mu\Big)^2\Big], \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, \mathrm{d}\mu),$$

which is equivalent to the usual Poincaré inequality

$$\int_{\mathbb{S}^d} |\nabla u|^2 \mathrm{d}\mu \geq d \int_{\mathbb{S}^d} |u-\overline{u}|^2 \mathrm{d}\mu, \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, \mathrm{d}\mu) \quad \text{with } \overline{u} = \int_{\mathbb{S}^d} u \mathrm{d}\mu\,.$$

See Remark 2.1 for more details. The case $p=2$ provides the logarithmic Sobolev inequality on the sphere. It holds as a consequence of the inequality for $p \neq 2$ (see Corollary 1.1).

For $p \neq 2$, the existence of a minimizer of

$$u \mapsto \int_{\mathbb{S}^d} |\nabla u|^2 \mathrm{d}\mu + \frac{\mathrm{d}\mathcal{I}_p}{p-2}\big[\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\big]$$

in $\big\{u \in \mathrm{H}^1(\mathbb{S}^d, \mathrm{d}\mu) : \int_{\mathbb{S}^d} |u|^p \mathrm{d}\mu = 1\big\}$ is easily achieved by variational methods, and will be taken for granted. The compactness for either $p \in [1,2)$ or $2 < p < 2^*$ is indeed classical, while the case $p = 2^*$, $d \geq 3$ can be studied by concentration-compactness methods. If a function $u \in \mathrm{H}^1(\mathbb{S}^d, \mathrm{d}\mu)$ is optimal for (1.1) with $p \neq 2$, then it solves the Euler-Lagrange equation

$$-\Delta_{\mathbb{S}^d} u = \frac{\mathrm{d}\,\mathcal{I}_p}{p-2}\big[\|u\|^{2-p}_{\mathrm{L}^p(\mathbb{S}^d)}\, u^{p-1} - u\big], \tag{1.3}$$

where $\Delta_{\mathbb{S}^d}$ denotes the Laplace-Beltrami operator on the sphere $\mathbb{S}^d$.

In any case, it is possible to normalize the $\mathrm{L}^p(\mathbb{S}^d)$-norm of u to 1 without restriction because of the zero homogeneity of $\mathcal{Q}_p$. It turns out that the optimality case is achieved by the constant function, with value $u \equiv 1$ if we assume $\int_{\mathbb{S}^d} |u|^p \mathrm{d}\mu = 1$, in which case the inequality degenerates because both sides are equal to 0. This explains why the dimension d shows up here: the sequence $(u_n)_{n\in\mathbb{N}}$, satisfying

$$u_n(x) = 1 + \frac{1}{n}\, v(x)$$

with $v \in \mathrm{H}^1(\mathbb{S}^d, \mathrm{d}\mu)$, such that $\int_{\mathbb{S}^d} v \mathrm{d}\mu = 0$, is indeed minimizing if and only if

$$\int_{\mathbb{S}^d} |\nabla v|^2 \mathrm{d}\mu \geq d \int_{\mathbb{S}^d} |v|^2 \mathrm{d}\mu,$$

and the equality case is achieved if v is an optimal function for the above Poincaré inequality, i.e., a function associated to the first non-zero eigenvalue of the Laplace-Beltrami operator $-\Delta_{\mathbb{S}^d}$ on the sphere $\mathbb{S}^d$. Up to a rotation, this means

$$v(\xi) = \xi_d, \quad \forall\, \xi = (\xi_0,\, \xi_1, \cdots, \xi_d) \in \mathbb{S}^d \subset \mathbb{R}^{d+1},$$

since $-\Delta_{\mathbb{S}^d} v = \mathrm{d} v$. Recall that the corresponding eigenspace of $-\Delta_{\mathbb{S}^d}$ is d-dimensional and is generated by the composition of v with an arbitrary rotation.

1.1 The logarithmic Sobolev inequality

As the first classical consequence of (1.2), we have a logarithmic Sobolev inequality. This result is rather classical. Related forms of the result can be found, for instance, in [9] or in [3].

Corollary 1.1 *Let $d \ge 1$. For any $u \in \mathrm{H}^1(\mathbb{S}^d, \mathrm{d}\mu) \setminus \{0\}$, we have*

$$\int_{\mathbb{S}^d} |u|^2 \log\Big(\frac{|u|^2}{\int_{\mathbb{S}^d} |u|^2 \mathrm{d}\mu}\Big)\mathrm{d}\mu \le \frac{2}{d}\int_{\mathbb{S}^d} |\nabla u|^2 \mathrm{d}\mu.$$

Moreover, the constant $\frac{2}{d}$ is sharp.

Proof The inequality is achieved by taking the limit as $p \to 2$ in (1.2). To see that the constant $\frac{2}{d}$ is sharp, we can observe that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{S}^d} |1+\varepsilon\, v|^2 \log\Big(\frac{|1+\varepsilon\, v|^2}{\int_{\mathbb{S}^d} |1+\varepsilon\, v|^2 \mathrm{d}\mu}\Big)\mathrm{d}\mu = 2\int_{\mathbb{S}^d} |v-\overline{v}|^2 \mathrm{d}\mu$$

with $\overline{v} = \int_{\mathbb{S}^d} v\,\mathrm{d}\mu$. The result follows by taking $v(\xi) = \xi_d$.

2 Extensions

2.1 Onofri's inequality

In the case of dimension $d = 2$, (1.1) holds for any $p > 2$, and we recover Onofri's inequality by taking the limit $p \to \infty$. This result is standard in the literature (see for instance [12]). For completeness, let us give a statement and a short proof.

Corollary 2.1 *Let $d = 1$ or $d = 2$. For any $v \in \mathrm{H}^1(\mathbb{S}^d, \mathrm{d}\mu)$, we have*

$$\int_{\mathbb{S}^d} \mathrm{e}^{v-\overline{v}}\mathrm{d}\mu \le \mathrm{e}^{\frac{1}{2d}\int_{\mathbb{S}^d} |\nabla v|^2 \mathrm{d}\mu},$$

where $\overline{v} = \int_{\mathbb{S}^d} v\,\mathrm{d}\mu$ is the average of v. Moreover, the constant $\frac{1}{2d}$ in the right-hand side is sharp.

Proof In dimension $d = 1$ or $d = 2$, (1.1) holds for any $p > 2$. Take $u = 1 + \frac{v}{p}$ and consider the limit as $p \to \infty$. We observe that

$$\int_{\mathbb{S}^d} |\nabla u|^2 \mathrm{d}\mu = \frac{1}{p^2}\int_{\mathbb{S}^d} |\nabla v|^2 \mathrm{d}\mu \quad \text{and} \quad \lim_{p\to\infty}\int_{\mathbb{S}^d} |u|^p \mathrm{d}\mu = \int_{\mathbb{S}^d} \mathrm{e}^v \mathrm{d}\mu,$$

so that

$$\Big(\int_{\mathbb{S}^d} |u|^p \mathrm{d}\mu\Big)^{\frac{2}{p}} - 1 \sim \frac{2}{p}\log\Big(\int_{\mathbb{S}^d} \mathrm{e}^v \mathrm{d}\mu\Big) \quad \text{and} \quad \int_{\mathbb{S}^d} |u|^2 \mathrm{d}\mu - 1 \sim \frac{2}{p}\int_{\mathbb{S}^d} v\,\mathrm{d}\mu\ .$$

The conclusion holds by passing to the limit $p \to \infty$ in (1.1). Optimality is once more achieved by considering $v = \varepsilon\, v_1$, $v_1(\xi) = \xi_d$, $d = 1$ and Taylor expanding both sides of the inequality in terms of $\varepsilon > 0$ small enough. Notice indeed that $-\Delta_{\mathbb{S}^d} v_1 = \lambda_1\, v_1$ with $\lambda_1 = d$, so that

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \varepsilon^2\,\|\nabla v_1\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \varepsilon^2 d\,\|v_1\|^2_{\mathrm{L}^2(\mathbb{S}^d)},$$

$\int_{\mathbb{S}^d} v_1 \mathrm{d}\mu = \overline{v}_1 = 0$, and

$$\int_{\mathbb{S}^d} \mathrm{e}^{v-\overline{v}}\mathrm{d}\mu - 1 \sim \frac{\varepsilon^2}{2}\int_{\mathbb{S}^d} |v-\overline{v}|^2 \mathrm{d}\mu = \frac{1}{2}\,\varepsilon^2\,\|v_1\|^2_{\mathrm{L}^2(\mathbb{S}^d)}.$$

2.2 Interpolation and a spectral approach for $p \in (1,2)$

In [10], Beckner gave a method to prove interpolation inequalities between the logarithmic Sobolev and the Poincaré inequalities in the case of a Gaussian measure. Here we shall prove that the method extends to the case of the sphere and therefore provides another family of interpolating inequalities, in a new range: $p \in [1,2)$, again with optimal constants. For further considerations on inequalities that interpolate between the Poincaré and the logarithmic Sobolev inequalities, we refer to [1–2, 9–10, 23–24, 27, 33] and the references therein.

Our purpose is to extend (1.1) written as

$$\frac{1}{d}\int_{\mathbb{S}^d}|\nabla u|^2 \mathrm{d}\mu \ge \frac{\left(\int_{\mathbb{S}^d}|u|^p \mathrm{d}\mu\right)^{\frac{2}{p}} - \int_{\mathbb{S}^d}|u|^2 \mathrm{d}\mu}{p-2}, \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, \mathrm{d}\mu) \tag{2.1}$$

to the case $p \in [1,2)$. Let us start with a remark.

Remark 2.1 At least for any nonnegative function v, using the fact that μ is a probability measure on $\mathbb{S}^d$, we may notice that

$$\int_{\mathbb{S}^d}|v-\overline{v}|^2 \mathrm{d}\mu = \int_{\mathbb{S}^d}|v|^2 \mathrm{d}\mu - \left(\int_{\mathbb{S}^d} v \,\mathrm{d}\mu\right)^2$$

can be rewritten as

$$\int_{\mathbb{S}^d}|v-\overline{v}|^2 \mathrm{d}\mu = \frac{\int_{\mathbb{S}^d}|v|^2 \mathrm{d}\mu - \left(\int_{\mathbb{S}^d}|v|^p \mathrm{d}\mu\right)^{\frac{2}{p}}}{2-p}$$

for $p=1$. Hence this extends (1.1) to the case $q=1$. However, as already noticed for instance in [1], the inequality

$$\int_{\mathbb{S}^d}|v|^2 \mathrm{d}\mu - \left(\int_{\mathbb{S}^d}|v| \mathrm{d}\mu\right)^2 \le \frac{1}{d}\int_{\mathbb{S}^d}|\nabla v|^2 \mathrm{d}\mu$$

also means that, for any $c \in \mathbb{R}$,

$$\int_{\mathbb{S}^d}|v+c|^2 \mathrm{d}\mu - \left(\int_{\mathbb{S}^d}|v+c| \mathrm{d}\mu\right)^2 \le \frac{1}{d}\int_{\mathbb{S}^d}|\nabla v|^2 \mathrm{d}\mu.$$

If v is bounded from below a.e. with respect to μ and $c > -\operatorname{ess\,inf}_{\mu} v$, so that $v + c > 0$ μ a.e., and the left-hand side is

$$\int_{\mathbb{S}^d}|v+c|^2 \mathrm{d}\mu - \left(\int_{\mathbb{S}^d}|v+c| \mathrm{d}\mu\right)^2 = c^2 + 2\,c\int_{\mathbb{S}^d} v\,\mathrm{d}\mu + \int_{\mathbb{S}^d}|v|^2 \mathrm{d}\mu - \left(c + \int_{\mathbb{S}^d} v\,\mathrm{d}\mu\right)^2 = \int_{\mathbb{S}^d}|v-\overline{v}|^2 \mathrm{d}\mu,$$

so that the inequality is the usual Poincaré inequality. By density, we recover that (2.1) written for $p=1$ exactly amounts to Poincaré inequality written not only for $|v|$, but also for any $v \in \mathrm{H}^1(\mathbb{S}^d, \mathrm{d}\mu)$.

Next, using the method introduced by Beckner [10] in the case of a Gaussian measure, we are in the position to prove (2.1) for any $p \in (1,2)$, knowing that the inequality holds for $p=1$ and $p=2$.

Proposition 2.1 *Inequality* (2.1) *holds for any* $p \in (1,2)$ *and any* $d \ge 1$. *Moreover,* d *is the optimal constant.*

Proof Optimality can be checked by Taylor expanding $u = 1 + \varepsilon\, v$ at order two in terms of $\varepsilon > 0$ as in the case $p=2$ (the logarithmic Sobolev inequality). To establish the inequality itself, we may proceed in two steps.

Step 1 (Nelson's Hypercontractivity Result) Although the result can be established by direct methods, we follow here the strategy of Gross [29], which proves the equivalence of the optimal hypercontractivity result and the optimal logarithmic Sobolev inequality.

Consider the heat equation of $\mathbb{S}^d$, namely,

$$\frac{\partial f}{\partial t} = \Delta_{\mathbb{S}^d} f$$

with the initial data $f(t=0,\,\cdot\,)=u \in L^{\frac{2}{p}}(\mathbb{S}^d)$ for some $p \in (1,2]$, and let $F(t) := \|f(t,\,\cdot\,)\|_{\mathrm{L}^{p(t)}(\mathbb{S}^d)}$. The key computation goes as follows:

$$\begin{aligned}\frac{F'}{F} &= \frac{\mathrm{d}}{\mathrm{d}t}\log F(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{1}{p(t)}\,\log\Big(\int_{\mathbb{S}^d}|f(t,\,\cdot\,)|^{p(t)}\mathrm{d}\mu\Big)\Big]\\ &= \frac{p'}{p^2\,F^p}\Big[\int_{\mathbb{S}^d} v^2\log\Big(\frac{v^2}{\int_{\mathbb{S}^d}v^2\mathrm{d}\mu}\Big)\mathrm{d}\mu + 4\,\frac{p-1}{p'}\int_{\mathbb{S}^d}|\nabla v|^2\mathrm{d}\mu\Big]\end{aligned}$$

with $v := |f|^{\frac{p(t)}{2}}$. Assuming that $4\,\frac{p-1}{p'} = \frac{2}{d}$, that is,

$$\frac{p'}{p-1} = 2d,$$

we find that

$$\log\Big(\frac{p(t)-1}{p-1}\Big) = 2dt,$$

if we require that $p(0) = p < 2$. Let $t_* > 0$ satisfy $p(t_*) = 2$. As a consequence of the above computation, we have

$$\|f(t_*,\,\cdot\,)\|_{\mathrm{L}^2(\mathbb{S}^d)} \le \|u\|_{\mathrm{L}^{\frac{2}{p}}(\mathbb{S}^d)}, \quad \text{if } \frac{1}{p-1} = \mathrm{e}^{2dt_*}. \tag{2.2}$$

Step 2 (Spectral Decomposition) Let $u = \sum_{k\in\mathbb{N}} u_k$ be a decomposition of the initial datum on the eigenspaces of $-\Delta_{\mathbb{S}^d}$, and denote by $\lambda_k = k\,(d+k-1)$ the ordered sequence of the eigenvalues: $-\Delta_{\mathbb{S}^d} u_k = \lambda_k\,u_k$ (see for instance [20]). Let $a_k = \|u_k\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$. As a straightforward consequence of this decomposition, we know that $\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \sum_{k\in\mathbb{N}} a_k$, $\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \sum_{k\in\mathbb{N}} \lambda_k\,a_k$ and

$$\|f(t_*,\,\cdot\,)\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \sum_{k\in\mathbb{N}} a_k\,\mathrm{e}^{-2\,\lambda_k\,t_*}.$$

Using (2.2), it follows that

$$\frac{\big(\int_{\mathbb{S}^d}|u|^p\mathrm{d}\mu\big)^{\frac{2}{p}} - \int_{\mathbb{S}^d}|u|^2\mathrm{d}\mu}{p-2} \le \frac{\big(\int_{\mathbb{S}^d}|u|^2\mathrm{d}\mu\big) - \int_{\mathbb{S}^d}|f(t_*,\,\cdot\,)|^2\mathrm{d}\mu}{2-p} = \frac{1}{2-p}\sum_{k\in\mathbb{N}^*}\lambda_k\,a_k\,\frac{1-\mathrm{e}^{-2\,\lambda_k\,t_*}}{\lambda_k}.$$

Notice that $\lambda_0 = 0$ so that the term corresponding to $k=0$ can be omitted in the series. Since $\lambda \mapsto \frac{1-\mathrm{e}^{-2\,\lambda\,t_*}}{\lambda}$ is decreasing, we can bound $\frac{1-\mathrm{e}^{-2\,\lambda_k\,t_*}}{\lambda_k}$ from above by $\frac{1-\mathrm{e}^{-2\,\lambda_1\,t_*}}{\lambda_1}$ for any $k \ge 1$. This proves that

$$\frac{\big(\int_{\mathbb{S}^d}|u|^p\mathrm{d}\mu\big)^{\frac{2}{p}} - \int_{\mathbb{S}^d}|u|^2\mathrm{d}\mu}{p-2} \le \frac{1-\mathrm{e}^{-2\,\lambda_1\,t_*}}{(2-p)\,\lambda_1}\sum_{k\in\mathbb{N}^*}\lambda_k\,a_k = \frac{1-\mathrm{e}^{-2\,\lambda_1\,t_*}}{(2-p)\,\lambda_1}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}.$$

The conclusion follows easily if we notice that $\lambda_1 = d$ and $e^{-2\lambda_1 t_*} = p - 1$, so that

$$\frac{1 - e^{-2\lambda_1 t_*}}{(2-p)\,\lambda_1} = \frac{1}{d}.$$

The optimality of this constant can be checked as in the case $p > 2$ by a Taylor expansion of $u = 1 + \varepsilon\, v$ at order two in terms of $\varepsilon > 0$ small enough.

3 Symmetrization and the Ultraspherical Framework

3.1 A reduction to the ultraspherical framework

We denote by $(\xi_0, \xi_1, \cdots, \xi_d)$ the coordinates of an arbitrary point $\xi \in \mathbb{S}^d$ with $\sum_{i=0}^{d} |\xi_i|^2 = 1$. The following symmetry result is a kind of folklore in the literature, and we can see [5, 33, 11] for various related results.

Lemma 3.1 *Up to a rotation, any minimizer of* (1.2) *depends only on* ξ_d.

Proof Let u be a minimizer for $\mathcal{Q}_p$. By writing u in (1.1) in spherical coordinates $\theta \in [0, \pi]$, $\varphi_1, \varphi_2, \cdots, \varphi_{d-1} \in [0, 2\pi)$ and using decreasing rearrangements (see, for instance, [24]), it is not difficult to prove that among optimal functions, there is one which depends only on θ. Moreover, the equality in the rearrangement inequality means that u has to depend on only one coordinate, i.e., $\xi_d = \sin\theta$.

Let us observe that the problem on the sphere can be reduced to a problem involving the ultraspherical operator as follows:

(1) Using Lemma 3.1, we know that (1.1) is equivalent to

$$\frac{p-2}{d}\int_0^\pi |v'(\theta)|^2 \mathrm{d}\sigma + \int_0^\pi |v(\theta)|^2 \mathrm{d}\sigma \ge \Big(\int_0^\pi |v(\theta)|^p \mathrm{d}\sigma\Big)^{\frac{2}{p}}$$

for any function $v \in \mathrm{H}^1([0,\pi], \mathrm{d}\sigma)$, where

$$\mathrm{d}\sigma(\theta) := \frac{(\sin\theta)^{d-1}}{Z_d}\mathrm{d}\theta \quad \text{with } Z_d := \sqrt{\pi}\,\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})}.$$

(2) The change of variables $x = \cos\theta$ and $v(\theta) = f(x)$ allows to rewrite the inequality as

$$\frac{p-2}{d}\int_{-1}^1 |f'|^2\, \nu \mathrm{d}\nu_d + \int_{-1}^1 |f|^2 \mathrm{d}\nu_d \ge \Big(\int_{-1}^1 |f|^p \mathrm{d}\nu_d\Big)^{\frac{2}{p}},$$

where $\mathrm{d}\nu_d$ is the probability measure defined by

$$\nu_d(x)\mathrm{d}x = \mathrm{d}\nu_d(x) := Z_d^{-1}\, \nu^{\frac{d}{2}-1}\mathrm{d}x \quad \text{with } \nu(x) := 1 - x^2,\ Z_d = \sqrt{\pi}\,\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})}\ .$$

We also want to prove the result in the case $p < 2$, to obtain the counterpart of Theorem 1.1 in the ultraspherical setting. On $[-1, 1]$, consider the probability measure $\mathrm{d}\nu_d$, and define

$$\nu(x) := 1 - x^2\,,$$

so that $\mathrm{d}\nu_d = Z_d^{-1}\, \nu^{\frac{d}{2}-1}\mathrm{d}x$. We consider the space $\mathrm{L}^2((-1,1), \mathrm{d}\nu_d)$ with the scalar product

$$\langle f_1, f_2\rangle = \int_{-1}^1 f_1\, f_2 \mathrm{d}\nu_d,$$

and use the notation

$$\|f\|_p = \Big(\int_{-1}^1 f^p \mathrm{d}\nu_d\Big)^{\frac{1}{p}}.$$

On $\mathrm{L}^2((-1,1),\mathrm{d}\nu_d)$, we define the self-adjoint ultraspherical operator by

$$\mathcal{L}\,f := (1-x^2)\,f'' - d\,x\,f' = \nu\,f'' + \frac{d}{2}\,\nu'\,f',$$

which satisfies the identity

$$\langle f_1, \mathcal{L}\,f_2\rangle = -\int_{-1}^1 f_1'\,f_2'\,\nu \mathrm{d}\nu_d.$$

Then the result goes as follows.

Proposition 3.1 *Let $p \in [1,2^*]$, $d \ge 1$. Then we have*

$$-\,\langle f, \mathcal{L}\,f\rangle = \int_{-1}^1 |f'|^2\,\nu \mathrm{d}\nu_d \ge d\,\frac{\|f\|_p^2 - \|f\|_2^2}{p-2}, \quad \forall\, f \in \mathrm{H}^1([-1,1],\mathrm{d}\nu_d), \tag{3.1}$$

if $p \neq 2$; and

$$-\,\langle f, \mathcal{L}\,f\rangle = \frac{d}{2}\int_{-1}^1 |f|^2\,\log\Big(\frac{|f|^2}{\|f\|_2^2}\Big)\mathrm{d}\nu_d,$$

if $p = 2$.

We may notice that the proof in [21] requires $d \ge 2$, while the case $d = 1$ is also covered in [12]. In [20], the restriction $d \ge 2$ was removed by Bentaleb et al. Our proof is inspired by [21] and also [14, 17], but it is a simplification (in the particular case of the ultraspherical operator) in the sense that only integration by parts and elementary estimates are used.

3.2 A proof of Proposition 3.1

Let us start with some preliminary observations. The operator $\mathcal{L}$ does not commute with the derivation, but we have the relation

$$\Big[\frac{\partial}{\partial x}, \mathcal{L}\Big]\,u = (\mathcal{L}\,u)' - \mathcal{L}\,u' = -2\,x\,u'' - d\,u'.$$

As a consequence, we obtain

$$\langle \mathcal{L}\,u, \mathcal{L}\,u\rangle = -\int_{-1}^1 u'\,(\mathcal{L}\,u)'\,\nu \mathrm{d}\nu_d = -\int_{-1}^1 u'\,\mathcal{L}\,u'\,\nu \mathrm{d}\nu_d + \int_{-1}^1 u'\,(2\,x\,u'' + d\,u')\,\nu \mathrm{d}\nu_d,$$
$$\langle \mathcal{L}\,u, \mathcal{L}\,u\rangle = \int_{-1}^1 |u''|^2\,\nu^2 \mathrm{d}\nu_d - d\,\langle u, \mathcal{L}\,u\rangle$$

and

$$\int_{-1}^1 (\mathcal{L}\,u)^2 \mathrm{d}\nu_d = \langle \mathcal{L}\,u, \mathcal{L}\,u\rangle = \int_{-1}^1 |u''|^2\,\nu^2 \mathrm{d}\nu_d + d\int_{-1}^1 |u'|^2\,\nu \mathrm{d}\nu_d. \tag{3.2}$$

On the other hand, a few integrations by parts show that

$$\Big\langle \frac{|u'|^2}{u}\,\nu \mathcal{L}\,u\Big\rangle = \frac{d}{d+2}\int_{-1}^1 \frac{|u'|^4}{u^2}\,\nu^2 \mathrm{d}\nu_d - 2\,\frac{d-1}{d+2}\int_{-1}^1 \frac{|u'|^2\,u''}{u}\,\nu^2 \mathrm{d}\nu_d, \tag{3.3}$$

where we have used the fact that $\nu\,\nu'\,\nu_d = \frac{2}{d+2}\,(\nu^2\,\nu_d)'$.

Let $p \in (1,2) \cup (2,2^*)$. In $H^1([-1,1],\mathrm{d}\nu_d)$, now consider a minimizer f for the functional

$$f \mapsto \int_{-1}^1 |f'|^2\;\nu \mathrm{d}\nu_d - d\,\frac{\|f\|_p^2 - \|f\|_2^2}{p-2} =: \mathcal{G}[f],$$

made of the difference of the two sides in (3.1). The existence of such a minimizer can be proved by classical minimization and compactness arguments. Up to a multiplication by a constant, f satisfies the Euler-Lagrange equation

$$-\frac{p-2}{\mathrm{d}}\,\mathcal{L}\,f + f = f^{p-1}.$$

Let β be a real number to be fixed later and define u by $f = u^\beta$, such that

$$\mathcal{L}\,f = \beta\,u^{\beta-1}\left(\mathcal{L}\,u + (\beta-1)\,\frac{|u'|^2}{u}\,\nu\right).$$

Then u is a solution to

$$-\mathcal{L}\,u - (\beta-1)\,\frac{|u'|^2}{u}\,\nu + \lambda\,u = \lambda\,u^{1+\beta\,(p-2)} \quad \text{with } \lambda := \frac{d}{(p-2)\,\beta}.$$

If we multiply the equation for u by $\frac{|u'|^2}{u}\,\nu$ and integrate, we get

$$-\int_{-1}^1 \mathcal{L}\,u\,\frac{|u'|^2}{u}\,\nu \mathrm{d}\nu_d - (\beta-1)\int_{-1}^1 \frac{|u'|^4}{u^2}\,\nu^2 \mathrm{d}\nu_d + \lambda\int_{-1}^1 |u'|^2\;\nu \mathrm{d}\nu_d = \lambda\int_{-1}^1 u^{\beta\,(p-2)}\,|u'|^2\;\nu \mathrm{d}\nu_d.$$

If we multiply the equation for u by $-\mathcal{L}\,u$ and integrate, we get

$$\int_{-1}^1 (\mathcal{L}\,u)^2 \mathrm{d}\nu_d + (\beta-1)\int_{-1}^1 \mathcal{L}\,u\,\frac{|u'|^2}{u}\;\nu \mathrm{d}\nu_d + \lambda\int_{-1}^1 |u'|^2\;\nu \mathrm{d}\nu_d = (\lambda+d)\int_{-1}^1 u^{\beta\,(p-2)}\,|u'|^2\;\nu \mathrm{d}\nu_d.$$

Collecting terms, we find that

$$\int_{-1}^1 (\mathcal{L}\,u)^2 \mathrm{d}\nu_d + \left(\beta+\frac{d}{\lambda}\right)\int_{-1}^1 \mathcal{L}\,u\,\frac{|u'|^2}{u}\;\nu \mathrm{d}\nu_d + (\beta-1)\left(1+\frac{d}{\lambda}\right)\int_{-1}^1 \frac{|u'|^4}{u^2}\,\nu^2 \mathrm{d}\nu_d - d\int_{-1}^1 |u'|^2\;\nu \mathrm{d}\nu_d = 0.$$

Using (3.2)–(3.3), we get

$$\begin{aligned}
&\int_{-1}^1 |u''|^2\;\nu^2 \mathrm{d}\nu_d + \left(\beta+\frac{d}{\lambda}\right)\left[\frac{d}{d+2}\int_{-1}^1 \frac{|u'|^4}{u^2}\;\nu^2 \mathrm{d}\nu_d - 2\,\frac{d-1}{d+2}\int_{-1}^1 \frac{|u'|^2\,u''}{u}\;\nu^2 \mathrm{d}\nu_d\right]\\
&+ (\beta-1)\left(1+\frac{d}{\lambda}\right)\int_{-1}^1 \frac{|u'|^4}{u^2}\,\nu^2 \mathrm{d}\nu_d = 0,
\end{aligned}$$

that is,

$$\mathsf{a}\int_{-1}^1 |u''|^2\;\nu^2 \mathrm{d}\nu_d + 2\,\mathsf{b}\int_{-1}^1 \frac{|u'|^2\,u''}{u}\;\nu^2 \mathrm{d}\nu_d + \mathsf{c}\int_{-1}^1 \frac{|u'|^4}{u^2}\;\nu^2 \mathrm{d}\nu_d = 0, \tag{3.4}$$

where

$$\begin{aligned}
&\mathsf{a} = 1,\\
&\mathsf{b} = -\left(\beta+\frac{d}{\lambda}\right)\frac{d-1}{d+2},\\
&\mathsf{c} = \left(\beta+\frac{d}{\lambda}\right)\frac{d}{d+2} + (\beta-1)\left(1+\frac{d}{\lambda}\right).
\end{aligned}$$

Using $\frac{d}{\lambda} = (p-2)\,\beta$, we observe that the reduced discriminant

$$\delta = \mathsf{b}^2 - \mathsf{a}\,\mathsf{c} < 0$$

can be written as

$$\delta = A\,\beta^2 + B\,\beta + 1 \quad \text{with } A = (p-1)^2\,\frac{(d-1)^2}{(d+2)^2} - p + 2 \text{ and } B = p - 3 - \frac{d\,(p-1)}{d+2}.$$

If $p < 2^*$, $B^2 - 4\,A$ is positive, and therefore it is possible to find β, such that $\delta < 0$.

Hence, if $p < 2^*$, we have shown that $\mathcal{G}[f]$ is positive unless the three integrals in (3.4) are equal to 0, that is, u is constant. It follows that $\mathcal{G}[f] = 0$, which proves (3.1) if $p \in (1,2)\cup(2,2^*)$. The cases $p = 1$, $p = 2$ (see Corollary 1.1) and $p = 2^*$ can be proved as limit cases. This completes the proof of Proposition 3.1.

4 A Proof Based on a Flow in the Ultraspherical Setting

Inequality (3.1) can be rewritten for $g = f^p$, i.e., $f = g^\alpha$ with $\alpha = \frac{1}{p}$, as

$$-\,\langle f, \mathcal{L}\,f\rangle = -\,\langle g^\alpha, \mathcal{L}\,g^\alpha\rangle =: \mathcal{I}[g] \ge d\,\frac{\|g\|_1^{2\,\alpha} - \|g^{2\,\alpha}\|_1}{p-2} =: \mathcal{F}[g].$$

4.1 Flow

Consider the flow associated to $\mathcal{L}$, that is,

$$\frac{\partial g}{\partial t} = \mathcal{L}\,g, \tag{4.1}$$

and observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|g\|_1 = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\,\|g^{2\,\alpha}\|_1 = -\,2\,(p-2)\,\langle f, \mathcal{L}\,f\rangle = 2\,(p-2)\int_{-1}^{1} |f'|^2\;\nu\mathrm{d}\nu_d,$$

which finally gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[g(t,\cdot)] = -\frac{d}{p-2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|g^{2\,\alpha}\|_1 = -\,2d\,\mathcal{I}[g(t,\cdot)].$$

4.2 Method

If (3.1) holds, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[g(t,\cdot)] \le -\,2d\,\mathcal{F}[g(t,\cdot)], \tag{4.2}$$

and thus we prove

$$\mathcal{F}[g(t,\cdot)] \le \mathcal{F}[g(0,\cdot)]\,\mathrm{e}^{-\,2dt}, \quad \forall\,t \ge 0.$$

This estimate is actually equivalent to (3.1) as shown by estimating $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[g(t,\cdot)]$ at $t = 0$.

The method based on the Bakry-Emery approach amounts to establishing first that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}[g(t,\cdot)] \le -\,2d\,\mathcal{I}[g(t,\cdot)] \tag{4.3}$$

and proving (4.2) by integrating the estimates on $t \in [0, \infty)$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{F}[g(t,\cdot)] - \mathcal{I}[g(t,\cdot)]) \geq 0$$

and $\lim_{t\to\infty}(\mathcal{F}[g(t,\cdot)] - \mathcal{I}[g(t,\cdot)]) = 0$, this means that

$$\mathcal{F}[g(t,\cdot)] - \mathcal{I}[g(t,\cdot)] \leq 0, \quad \forall\, t \geq 0,$$

which is precisely (3.1) written for $f(t,\cdot)$ for any $t \geq 0$ and in particular for any initial value $f(0,\cdot)$.

The equation for $g = f^p$ can be rewritten in terms of f as

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f + (p-1)\, \frac{|f'|^2}{f}\, \nu.$$

Hence, we have

$$-\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} |f'|^2\, \nu \mathrm{d}\nu_d = \frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\langle f, \mathcal{L}\, f\rangle = \langle \mathcal{L}\, f, \mathcal{L}\, f\rangle + (p-1)\left\langle \frac{|f'|^2}{f}\nu, \mathcal{L}\, f\right\rangle.$$

4.3 An inequality for the Fisher information

Instead of proving (3.1), we will established the following stronger inequality, for any $p \in (2, 2^\sharp]$, where $2^\sharp := \frac{2\mathrm{d}^2+1}{(d-1)^2}$:

$$\langle \mathcal{L}\, f, \mathcal{L}\, f\rangle + (p-1)\left\langle \frac{|f'|^2}{f}\,\nu, \mathcal{L}\, f\right\rangle + d\,\langle f, \mathcal{L}\, f\rangle \geq 0. \tag{4.4}$$

Notice that (3.1) holds under the restriction $p \in (2, 2^\sharp]$, which is stronger than $p \in (2, 2^*]$. We do not know whether the exponent $2^\sharp$ in (4.4) is sharp or not.

4.4 Proof of (4.4)

Using (3.2)–(3.3) with $u = f$, we find that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} |f'|^2\, \nu \mathrm{d}\nu_d + 2d\int_{-1}^{1} |f'|^2\, \nu \mathrm{d}\nu_d \\ &= -2\int_{-1}^{1}\Big(|f''|^2 + (p-1)\,\frac{d}{d+2}\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\frac{|f'|^2\, f''}{f}\Big)\,\nu^2 \mathrm{d}\nu_d\,.\end{aligned}$$

The right-hand side is nonpositive, if

$$|f''|^2 + (p-1)\,\frac{d}{d+2}\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\frac{|f'|^2\, f''}{f}$$

is pointwise nonnegative, which is granted if

$$\left[(p-1)\,\frac{d-1}{d+2}\right]^2 \leq (p-1)\,\frac{d}{d+2},$$

a condition which is exactly equivalent to $p \leq 2^\sharp$.

4.5 An improved inequality

For any $p \in (2, 2^\sharp)$, we can write that

$$
\begin{aligned}
&|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}\\
&= \alpha\,|f''|^2 + \frac{p-1}{d+2}\left|\frac{d-1}{\sqrt{d}}\,f'' - \sqrt{d}\,\frac{|f'|^2}{f}\right|^2 \geq \alpha\,|f''|^2,
\end{aligned}
$$

where

$$
\alpha := 1 - (p-1)\,\frac{(d-1)^2}{d\,(d+2)}
$$

is positive. Now, using the Poincaré inequality

$$
\int_{-1}^{1} |f''|^2 \mathrm{d}\nu_{d+4} \geq (d+2) \int_{-1}^{1} |f' - \overline{f'}|^2 \mathrm{d}\nu_{d+2},
$$

where

$$
\overline{f'} := \int_{-1}^{1} f' \mathrm{d}\nu_{d+2} = -d \int_{-1}^{1} x\,f \mathrm{d}\nu_d,
$$

we obtain an improved form of (4.4), namely,

$$
\langle \mathcal{L}\,f, \mathcal{L}\,f \rangle + (p-1)\Big\langle \frac{|f'|^2}{f}\nu, \mathcal{L}\,f \Big\rangle + [d + \alpha\,(d+2)]\;\langle f, \mathcal{L}\,f \rangle \geq 0,
$$

if we can guarantee that $\overline{f'} \equiv 0$ along the evolution determined by (4.1). This is the case if we assume that $f(x) = f(-x)$ for any $x \in [-1, 1]$. Under this condition, we find that

$$
\int_{-1}^{1} |f'|^2\;\nu \mathrm{d}\nu_d \geq [d + \alpha\,(d+2)]\,\frac{\|f\|_p^2 - \|f\|_2^2}{p-2}.
$$

As a consequence, we also have

$$
\int_{\mathbb{S}^d} |\nabla u|^2 \mathrm{d}\mu + \int_{\mathbb{S}^d} |u|^2 \mathrm{d}\mu \geq \frac{d + \alpha\,(d+2)}{p-2}\Big(\int_{\mathbb{S}^d} |u|^p \mathrm{d}\mu\Big)^{\frac{2}{p}}
$$

for any $u \in \mathrm{H}^1(\mathbb{S}^d, \mathrm{d}\mu)$, such that, using spherical coordinates,

$$
u(\theta, \varphi_1, \varphi_2, \cdots, \varphi_{d-1}) = u(\pi - \theta, \varphi_1, \varphi_2, \cdots, \varphi_{d-1}), \quad \forall (\theta, \varphi_1, \varphi_2, \cdots, \varphi_{d-1}) \in [0, \pi] \times [0, 2\pi)^{d-1}\,.
$$

4.6 One more remark

The computation is exactly the same if $p \in (1, 2)$, and henceforth we also prove the result in such a case. The case $p = 1$ is the limit case corresponding to the Poincaré inequality

$$
\int_{-1}^{1} |f'|^2 \mathrm{d}\nu_{d+2} \geq d\Big(\int_{-1}^{1} |f|^2 \mathrm{d}\nu_d - \Big|\int_{-1}^{1} f \mathrm{d}\nu_d\Big|^2\Big)
$$

and arises as a straightforward consequence of the spectral properties of $\mathcal{L}$. The case $p = 2$ is achieved as a limiting case. It gives rise to the logarithmic Sobolev inequality (see, for instance, [34]).

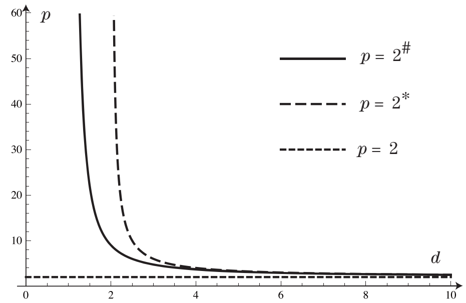

Figure 1 Plot of $d \mapsto 2^\sharp = \frac{2d^2+1}{(d-1)^2}$ and $d \mapsto 2^* = \frac{2d}{d-2}$.

4.7 Limitation of the method

The limitation $p \le 2^\sharp$ comes from the pointwise condition

$$h := |f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f} \ge 0.$$

Can we find special test functions f, such that this quantity can be made negative? Which are admissible, such that $h\,\nu^2$ is integrable? Notice that at $p = 2^\sharp$, we have that $f(x) = |x|^{1-d}$, such that $h \equiv 0$, but such a function or functions obtained by slightly changing the exponent, are not admissible for larger values of p.

By proving that there is contraction of $\mathcal{I}$ along the flow, we look for a condition which is stronger than one of asking that there is contraction of $\mathcal{F}$ along the flow. It is therefore possible that the limitation $p \le 2^\sharp$ is intrinsic to the method.

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