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MsFEM à la Crouzeix-Raviart for Highly Oscillatory Elliptic Problems*

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(*In honor of the scientific heritage of Jacques-Louis Lions*)

Abstract We introduce and analyze a multiscale finite element type method (MsFEM) in the vein of the classical Crouzeix-Raviart finite element method that is specifically adapted for highly oscillatory elliptic problems. We illustrate numerically the efficiency of the approach and compare it with several variants of MsFEM.

Keywords Homogenization, Finite elements, Galerkin methods, Highly oscillatory PDE
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1 Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $f \in L^2(\Omega)$ (more regularity on the right-hand side will be needed later on). We consider the problem

$$-\operatorname{div}[A_\varepsilon(x)\nabla u^\varepsilon] = f \text{ in } \Omega, \quad u^\varepsilon = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where A_ε is a highly oscillatory, uniformly elliptic and bounded matrix. To fix the ideas (and this will in fact be a necessary assumption for the analysis which we provide below to hold true), one might think of $A_\varepsilon(x) = A_{\rm per}\big(\frac{x}{\varepsilon}\big)$, where $A_{\rm per}$ is $\mathbb{Z}^d$ periodic. The approach which we introduce here to address problem (1.1) is a multiscale finite element type method (henceforth abbreviated as MsFEM). As any such method, our approach is not restricted to the periodic setting. Only our analysis is. Likewise, we will assume for simplicity of our analysis that the matrices A_ε which we manipulate are symmetric matrices.

Our purpose is to propose and study a specific multiscale finite element method for the problem (1.1), where the Galerkin approximation space is constructed from ideas similar to those by Crouzeix and Raviart in their construction of a classical FEM space [14].

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Recall that the general idea of MsFEM approaches is to construct an approximation space by using precomputed, local functions, that are solutions to the equation under consideration with simple (typically vanishing) right-hand sides. This is in contrast to standard finite element approaches, where the approximation space is based on generic functions, namely piecewise polynomials. To construct our specific multiscale finite element method for the problem (1.1), we recall the classical work of Crouzeix and Raviart [14]. We preserve the main feature of their nonconforming FEM space, i.e., that the continuity accross the edges of the mesh is enforced only in a weak sense by requiring that the average of the jump vanishes on each edge. As shown in Subsection 2.1 below, this "weak" continuity condition leads to some natural boundary conditions for the multiscale basis functions.

Our motivation for the introduction of such finite element functions stems from our wish to address several specific multiscale problems, most of them in a nonperiodic setting, for which implementing flexible boundary conditions on each mesh element is of particular interest. A prototypical situation is that of a perforated medium, where inclusions are not periodically located and where the accuracy of the numerical solution is extremely sensitive to an appropriate choice of values of the finite element basis functions on the boundaries of elements when the latter intersect inclusions. The Crouzeix-Raviart type elements which we construct then provide an advantageous flexibility. Additionally, when the problem under consideration is not (as (1.1) above) a simple scalar elliptic Poisson problem but a Stokes type problem, it is well-known that the Crouzeix-Raviart approach also allows — in the classical setting — for directly encoding the incompressibility constraint in the finite element space. This property will be preserved for the approach which we introduce here in the multiscale context. We will not proceed further in this direction and refer the interested reader to our forthcoming publication (see [25]) for more details on this topic and related issues.

Of course, our approach is not the only possible one to address the category of problems we consider. Sensitivity of the numerical solution upon the choice of boundary condition set for the multiscale finite element basis functions is a classical issue. Formally, it may be easily understood. In a one-dimensional situation (see for instance [26] for a formalization of this argument), the error committed by using a multiscale finite element type approach entirely comes from the error committed in the bulk of each element, because it is easy to make the numerical solution agree with the exact solution on nodes. In dimensions greater than one, however, it is impossible to match the finite dimensional approximation on the boundary of elements with the exact, infinite dimensional trace of the exact solution on this boundary. A second source of numerical error thus follows from this. And the derivation of variants of MsFEM type approaches can be seen as the quest to solve the issue of inappropriate boundary conditions on the boundaries of mesh elements.

Many tracks, each of which leads to a specific variant of the general approach, have been followed to address the issue. The simplest choice (see [21–22]) is to use linear boundary conditions, as in the standard P1 finite element method. This yields a multiscale finite element space consisting of continuous functions. The use of nonconforming finite elements is an attractive alternative, leading to more accurate and more flexible variants of the method. The work [12] uses Raviart-Thomas finite elements for a mixed formulation of a highly oscillatory elliptic problem similar to that considered in the present article. Many contributions such as [1–2, 5, 7] present variants and follow-up of this work. For non-mixed formulations, we mention the well-known oversampling method (giving birth to nonconforming finite elements, see [16, 20–21]). We also mention the work [11], where a variant of the classical MsFEM approach (i.e., without oversampling) is presented. Basis functions also satisfy Dirichlet linear boundary conditions on the boundaries of the finite elements, but continuity accross the edges is only enforced at the midpoint of the edges, as in the approach suggested by Crouzeix and Raviart [14]. Note that this approach, although also inspired by the work [14], differs from ours in the sense that we do not impose any Dirichlet boundary conditions when constructing the basis functions (see Subsection 2.1 below for more details).

In the context of an HMM-type method, we mention the works [3–4] for the computation of an approximation of the coarse scale solution. An excellent review of many of the existing approaches is presented in [6], and for the general development of MsFEM (as of 2009) we refer to [15].

Our purpose here is to propose yet another possibility, which may be useful in specific contexts. Results for problems of type (1.1), although good, will not be spectacularly good. However, the ingredients which we employ here to analyze the approach and the structure of our proof will be very useful when studying the same Crouzeix-Raviart type approach for a specific setting of particular interest: the case for perforated domains. In that case, we will show in [25] how extremely efficient our approach is.

Our article is articulated as follows. We outline our approach in Section 2 and state the corresponding error estimate, for the periodic setting, in Section 3 (Theorem 3.1). The subsequent two sections are devoted to the proof of the main error estimate. We recall some elementary facts and tools of numerical analysis in Section 4, and turn to the actual proof of Theorem 3.1 in Section 5. Section 6 presents some numerical comparisons between the approach which we introduce here and some existing MsFEM type approaches.

2 Presentation of Our MsFEM Approach

Throughout this article, we assume that the ambient dimension is $d = 2$ or $d = 3$ and that Ω is a polygonal (resp. polyhedral) domain. We define a mesh $\mathcal{T}_H$ on Ω, i.e., a decomposition of Ω into polygons (resp. polyhedra) each of diameter at most H, and denote $\mathcal{E}_H$ the set of all the internal edges (or faces) of $\mathcal{T}_H$. We assume that the mesh does not have any hanging nodes. Otherwise stated, each internal edge (resp. face) is shared by exactly two elements of the mesh. In addition, $\mathcal{T}_H$ is assumed to be a regular mesh in the following sense: for any mesh element $T \in \mathcal{T}_H$, there exists a smooth one-to-one and onto mapping $K : \overline{T} \to T$, where $\overline{T} \subset \mathbb{R}^d$ is the reference element (a polygon, resp. a polyhedron, of fixed unit diameter) and $\|\nabla K\|_{L^\infty} \le CH$, $\|\nabla K^{-1}\|_{L^\infty} \le CH^{-1}$, C being some universal constant independent of T, to which we will refer as the regularity parameter of the mesh. To avoid some technical complications, we also assume that the mapping K corresponding to each $T \in \mathcal{T}_H$ is affine on every edge (resp. face) of $\partial\overline{T}$. In the following and to fix the ideas, we will have in mind the two-dimensional situation and a mesh consisting of triangles, which satisfies the minimum angle condition to ensure that the mesh is regular in the sense defined above (see [10, Section 4.4]). We will repeatedly use the notations and terminologies (triangle, edge, $\cdots$) of this setting, although the analysis carries over to quadrangles if $d = 2$, or to tetrahedra and parallelepipeda if $d = 3$.

The bottom line of our multiscale finite element method à la Crouzeix-Raviart is, as for the classical version of the method, to require the continuity of the (here highly oscillatory) finite element basis functions only in the sense of averages on the edges, rather than to require the continuity at the nodes (which is for instance the case in the oversampling variant of the MsFEM). In doing so, we expect more flexibility, and therefore better approximation properties in delicate cases.

2.1 Construction of the MsFEM basis functions

Functional spaces We introduce the functional space

$$W_H = \Big\{ u \in L^2(\Omega) \text{ such that } u|_T \in H^1(T) \text{ for any } T \in \mathcal{T}_H, \\ \int_e [[u]] = 0 \text{ for all } e \in \mathcal{E}_H \text{ and } u = 0 \text{ on } \partial\Omega \Big\},$$

where $[[u]]$ denotes the jump of u over an edge. We next introduce its subspace

$$W_H^0 = \Big\{ u \in W_H \text{ such that } \int_e u = 0 \text{ for all } e \in \mathcal{E}_H \Big\}$$

and define the MsFEM space à la Crouzeix-Raviart

$$V_H = \{ u \in W_H \text{ such that } a_H(u, v) = 0 \text{ for all } v \in W_H^0 \}$$

as the orthogonal complement of W_H^0 in W_H, where by orthogonality we mean orthogonality for the scalar product defined by

$$a_H(u, v) = \sum_{T \in \mathcal{T}_H} \int_T (\nabla v)^{\mathrm{T}} A_\varepsilon(x) \nabla u \, \mathrm{d}x. \tag{2.1}$$

We recall that for simplicity we assume all matrices are symmetric.

Notation For any $u \in W_H$, we henceforth denote by

$$\|u\|_E := \sqrt{a_H(u, u)}$$

the energy norm associated with the form a_H.

"Strong" form To get a more intuitive grasp on the space V_H, we note that any function $u \in V_H$ satisfies, on any element $T \in \mathcal{T}_H$,

$$\int_T (\nabla v)^{\mathrm{T}} A_\varepsilon \nabla u = 0 \quad \text{for all } v \in H^1(T) \quad \text{s.t.} \int_{\Gamma_i} v = 0 \quad \text{for all } i = 1, \cdots, N_\Gamma,$$

where Γ_i (with $i = 1, \cdots, N_\Gamma$) are the N_Γ edges composing the boundary of T (note that, if $\Gamma_i \subset \partial\Omega$, the condition $\int_{\Gamma_i} v = 0$ is replaced by $v = 0$ on Γ_i; this is a convention which we will use throughout our article without explicitly mentioning it). This can be rewritten as

$$\int_T (\nabla v)^{\mathrm{T}} A_\varepsilon \nabla u = \sum_{i=1}^{N_\Gamma} \lambda_i \int_{\Gamma_i} v \quad \text{for all } v \in H^1(T)$$

for some scalar constants $\lambda_1, \cdots, \lambda_{N_\Gamma}$. Hence, the restriction of any $u \in V_H$ to T is a solution to the boundary value problem

$$-\operatorname{div}[A_\varepsilon(x)\nabla u] = 0 \text{ in } T, \quad n \cdot A_\varepsilon \nabla u = \lambda_i \text{ on each } \Gamma_i.$$

The flux along each edge interior to Ω is therefore a constant. This of course defines u only up to an additive constant, which is fixed by the "continuity" condition

$$\int_e [[u]] = 0 \quad \text{for all } e \in \mathcal{E}_H \quad \text{and} \quad u = 0 \text{ on } \partial\Omega. \tag{2.2}$$

Remark 1.1 Observe that, in the case $A_\varepsilon = \mathrm{Id}$, we recover the classical nonconforming finite element spaces:

(1) Crouzeix-Raviart element (see [14]) on any triangular mesh: on each T, $u|_T \in \operatorname{Span}\{1, x, y\}$.

(2) Rannacher-Turek element (see [29]) on any rectangular Cartesian mesh: on each T, $u|_T \in \operatorname{Span}\{1, x, y, x^2 - y^2\}$.

Basis functions We can associate the basis functions of V_H with the internal edges of the mesh as follows. Let e be such an edge and let T_1 and T_2 be the two mesh elements that share that edge e. The basis function ϕ_e associated to e, the support of which is $T_1 \cup T_2$, is constructed as follows. Let us denote the edges composing the boundary of T_k ($k = 1$ or 2) by Γ_i^k (with $i = 1, \cdots, N_\Gamma$), and without loss of generality suppose that $\Gamma_1^1 = \Gamma_1^2 = e$. On each T_k, the function ϕ_e is the unique solution in $H^1(T_k)$ to

$$\begin{aligned} -\operatorname{div}[A_\varepsilon(x)\nabla \phi_e] &= 0 && \text{in } T_k, \\ \int_{\Gamma_i^k} \phi_e &= \delta_{i1} && \text{for } i = 1, \cdots, N_\Gamma, \\ n \cdot A_\varepsilon \nabla \phi_e &= \lambda_i^k && \text{on } \Gamma_i^k,\ i = 1, \cdots, N_\Gamma, \end{aligned}$$

where δ_{i1} is the Kronecker symbol. Note that, for the edge $\Gamma_1^1 = \Gamma_1^2 = e$ shared by the two elements, the value of the flux may be different from one side of the edge to the other one: λ_1^1 may be different from λ_1^2. The existence and the uniqueness of ϕ_e follow from standard analysis arguments.

Decomposition property A specific decomposition property based on the above finite element spaces will be useful in the sequel. Consider some function $u \in W_H$, and introduce $v_H \in V_H$ such that, for any element $T \in \mathcal{T}_H$, we have $v_H \in H^1(T)$, and

$$\begin{aligned} -\operatorname{div}[A_\varepsilon(x)\nabla v_H] &= 0 && \text{in } T, \\ \int_{\Gamma_i} v_H &= \int_{\Gamma_i} u && \text{for } i = 1, \cdots, N_\Gamma, \\ n \cdot A_\varepsilon \nabla v_H &= \lambda_i && \text{on } \Gamma_i,\ i = 1, \cdots, N_\Gamma. \end{aligned}$$

Consider now $v^0 = u - v_H \in W_H$. We see that, for any edge e,

$$\int_e v^0 = \int_e u - \int_e v_H = 0,$$

and thus $v^0 \in W_H^0$. We can hence decompose (in a unique way) any function $u \in W_H$ as the sum $u = v_H + v^0$, with $v_H \in V_H$ and $v^0 \in W_H^0$.

2.2 Definition of the numerical approximation

Using the finite element spaces introduced above, we now define the MsFEM approximation of the solution u^ε to (1.1) as the solution $u_H \in V_H$ to

$$a_H(u_H, v) = \int_\Omega f v \quad \text{for any } v \in V_H, \tag{2.3}$$

where a_H is defined by (2.1). Note that (2.3) is a nonconforming approximation of (1.1), since $V_H \not\subset H^1_0(\Omega)$.

The problem (2.3) is well-posed. Indeed, it is finite dimensional so that it suffices to prove that $f = 0$ implies $u_H = 0$. But $f = 0$ implies, taking $v = u_H$ in (2.3) and using the coercivity of A_ε, that $\nabla u_H = 0$ on every $T \in \mathcal{T}_H$. The continuity condition (2.2) then shows that $u_H = 0$ on Ω.

3 Main Result

The main purpose of our article is to present the numerical analysis of the method outlined in the previous section. To this end, we need to restrict the setting of the approach (stated above for, and indeed applicable to, general matrices A_ε) to the periodic setting. The essential reason for this restriction is that, in the process of the proof of our main error estimate (Theorem 3.1 below), we need to use an accurate description of the asymptotic behavior (as $\varepsilon \to 0$) of the oscillatory solution u^ε. Schematically speaking, our error estimate is established using a triangle inequality of the form

$$\|u^\varepsilon - u_H\| \le \|u^\varepsilon - u^{\varepsilon,1}\| + \|u^{\varepsilon,1} - u_H\|,$$

where $u^{\varepsilon,1}$ is an accurate description of the exact solution u^ε to (1.1), for ε small. Such an accurate description is not available in the completely general setting where the method is applicable. In the periodic setting, however, we do have such a description at our disposal. It is provided by the two-scale expansion of the homogenized solution to the problem. This is the reason why we restrict ourselves to this setting. Some other specific settings could perhaps allow for the same type of analysis, but we will not proceed in this direction. On the other hand, in the present state of our understanding of the problem and to the best of our knowledge of the existing literature, we are not aware of any strategy of proof that could accommodate the fully general oscillatory setting.

Periodic homogenization We henceforth assume that, in (1.1),

$$A_\varepsilon(x) = A_{\rm per}\left(\frac{x}{\varepsilon}\right), \tag{3.1}$$

where $A_{\rm per}$ is $\mathbb{Z}^d$ periodic (and of course bounded and uniformly elliptic). It is then well-known (see the classical textbooks [8, 13, 24], and also [17] for a general, numerically oriented presentation) that the solution u^ε to (1.1) converges, weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$, to the solution $u^\star$ to

$$-\mathrm{div}(A^\star_{\rm per} \nabla u^\star) = f \quad \text{in } \Omega, \quad u^\star = 0 \quad \text{on } \partial\Omega, \tag{3.2}$$

with the homogenized matrix given by, for any $1 \le i, j \le d$,

$$(A^\star_{\rm per})_{ij} = \int_{(0,1)^d} (e_i + \nabla w_{e_i}(y))^{\rm T} A_{\rm per}(y)(e_j + \nabla w_{e_j}(y)) {\rm d}y,$$

where, for any $p \in \mathbb{R}^d$, w_p is the unique (up to the addition of a constant) solution to the corrector problem associated to the periodic matrix $A_{\rm per}$:

$$-{\rm div}[A_{\rm per}(p + \nabla w_p)] = 0, \quad w_p \text{ is } \mathbb{Z}^d\text{-periodic.} \tag{3.3}$$

The corrector functions allow to compute the homogenized matrix, and to obtain a convergence result in the H^1 strong norm. Indeed, introduce

$$u^{\varepsilon,1}(x) = u^\star(x) + \varepsilon \sum_{i=1}^d w_{e_i}\left(\frac{x}{\varepsilon}\right) \frac{\partial u^\star}{\partial x_i}(x). \tag{3.4}$$

Then, we have the following proposition.

Proposition 3.1 *Suppose that the dimension is $d > 1$, that the solution $u^\star$ to* (3.2) *belongs to $W^{2,\infty}(\Omega)$ and that, for any $p \in \mathbb{R}^d$, the corrector w_p solution to* (3.3) *belongs to $W^{1,\infty}(\mathbb{R}^d)$. Then*

$$\|u^\varepsilon - u^{\varepsilon,1}\|_{H^1(\Omega)} \le C\sqrt{\varepsilon}\|\nabla u^\star\|_{W^{1,\infty}(\Omega)} \tag{3.5}$$

for a constant C independent of ε and $u^\star$.

We refer to [24, p. 28] for a proof of this result. Note that, in dimension $d = 1$, the rate of convergence of $u^\varepsilon - u^{\varepsilon,1}$ to 0 is even better.

Error estimate We are now in a position to state our main result.

Theorem 3.1 *Let u^ε be the solution to* (1.1) *for a matrix A_ε given by* (3.1). *We furthermore assume that*

$$A_{\rm per} \textit{ is Hölder continuous} \tag{3.6}$$

and that the solution $u^\star$ to (3.2) *belongs to $C^2(\overline{\Omega})$. Let u_H be the solution to* (2.3). *We have*

$$\|u^\varepsilon - u_H\|_E \le CH\|f\|_{L^2(\Omega)} + C\left(\sqrt{\varepsilon} + H + \sqrt{\frac{\varepsilon}{H}}\right)\|\nabla u^\star\|_{C^1(\overline{\Omega})}, \tag{3.7}$$

where the constant C is independent of H, ε, f and $u^\star$.

Two remarks are in order, first on the necessity of our assumption (3.6), and next on the comparison with other, well established variants of MsFEM.

Remark 3.1 (On the regularity of $A_{\rm per}$) We recall that, under assumption (3.6), the solution w_p to (3.3) (with, say, zero mean) satisfies, for any $p \in \mathbb{R}^d$,

$$w_p \in C^{1,\delta}(\mathbb{R}^d) \quad \text{for some } \delta > 0. \tag{3.8}$$

We refer to [19, Theorem 8.22 and Corollary 8.36]. Thus, assumption (3.6) implies that $w_p \in W^{1,\infty}(\mathbb{R}^d)$, which in turn is a useful assumption in Proposition 3.1. The regularity (3.8) is also a useful ingredient in the proof of Theorem 3.1 (see (5.11) and (5.14)).

Remark 3.2 (Comparison with other approaches) It is useful to compare our error estimate (3.7) with similar estimates for some existing MsFEM-type approaches in the literature. The classical MsFEM from [22] (by "classical", we mean the method using basis functions satisfying linear boundary conditions on each element) yields an exactly similar majoration in terms of $\sqrt{\varepsilon} + H + \sqrt{\frac{\varepsilon}{H}}$. It is claimed in [22] that the same majoration also holds for the MsFEM-O variant. This variant (in the form presented in [22]) is restricted to the two-dimensional setting. It uses boundary conditions provided by the solution to the oscillatory ordinary differential equation obtained by taking the trace of the original equation (1.1) on the edge considered.

The famous variant of MsFEM using oversampling (see [16, 21]) gives a slightly better estimation, in terms of $\sqrt{\varepsilon} + H + \frac{\varepsilon}{H}$. The best estimation which we are aware of is obtained by using a Petrov-Galerkin variant of MsFEM with oversampling (see [23]). It bounds the error from above by $\sqrt{\varepsilon} + H + \varepsilon$, but this only holds in the regime $\frac{\varepsilon}{H} \leq C^{te}$ and for a sufficiently (possibly prohibitively) large oversampling ratio. All these comparisons show that the method which we present here is guaranteed to be accurate, although not spectacularly accurate, for the equation (1.1) considered. An actually much better behavior will be observed in practice, in particular for the case of a perforated domain that we study in [25].

A comparison with other, related but slightly different in spirit approaches, can also be of interest. The approaches [27–28] yield an error estimate better than that obtained with the oversampling variant of MsFEM. The computational cost is however larger, owing to the large size of the oversampling domain employed.

4 Some Classical Ingredients for Our Analysis

Before we get to the proof of our main result, Theorem 3.1, we first need to collect here some standard results. These include trace theorems, Poincaré-type inequalities, error estimates for nonconforming finite elements and eventually convergences of oscillating functions. With a view to next using these results for our proof, we actually need not only to recall them but also, for some of them, to make explicit the dependency of the constants appearing in the various estimates upon the size of the domain (which will be taken, in practice, as an element of the mesh, of diameter H). Of course, these results are standard, and their proof is recalled here only for the sake of completeness.

First we recall the definition, borrowed from [18, Definition B.30], of the $H^{1/2}$ space.

Definition 4.1 *For any open domain $\omega \subset \mathbb{R}^n$ and any $u \in L^2(\omega)$, we define the norm*

$$\|u\|^2_{H^{1/2}(\omega)} := \|u\|^2_{L^2(\omega)} + |u|^2_{H^{1/2}(\omega)},$$

where

$$|u|^2_{H^{1/2}(\omega)} := \int_\omega \int_\omega \frac{|u(x) - u(y)|^2}{|x-y|^{n+1}} \mathrm{d}x\mathrm{d}y,$$

and define the space

$$H^{1/2}(\omega) := \{u \in L^2(\omega),\ \|u\|_{H^{1/2}(\omega)} < \infty\}.$$

4.1 Reference element

We first work on the reference element $\overline{T}$, with edges $\overline{e} \subset \partial\overline{T}$ (we recall that our terminology and notation suggest that, to fix the ideas, we have in mind triangles in two dimensions). By the standard trace theorem, we know that there exists C, such that

$$\forall v \in H^1(\overline{T}), \quad \forall \overline{e} \subset \partial\overline{T}, \quad \|v\|_{H^{1/2}(\overline{e})} \leq C\|v\|_{H^1(\overline{T})}. \tag{4.1}$$

In addition, we have the following result.

Lemma 4.1 *There exists C (depending only on the reference mesh element), such that*

$$\forall v \in H^1(\overline{T}) \text{ with } \int_{\overline{e}} v = 0 \text{ for some } \overline{e} \subset \partial\overline{T}, \quad \|v\|_{H^1(\overline{T})} \leq C\|\nabla v\|_{L^2(\overline{T})}. \tag{4.2}$$

The proof follows from the following result (see [18, Lemma A.38]).

Lemma 4.2 (Petree-Tartar) *Let X, Y and Z be three Banach spaces. Let $A \in \mathcal{L}(X, Y)$ be an injective operator and let $T \in \mathcal{L}(X, Z)$ be a compact operator. If there exists $c > 0$, such that $c\|x\|_X \leq \|Ax\|_Y + \|Tx\|_Z$, then* $\mathrm{Im}(A)$ *is closed. Equivalently, there exists $\alpha > 0$, such that*

$$\forall x \in X, \quad \alpha\|x\|_X \leq \|Ax\|_Y.$$

Proof of Lemma 4.1 Consider an edge $\overline{e} \subset \partial\overline{T}$. We apply Lemma 4.2 with $Z = L^2(\overline{T})$, $Y = (L^2(\overline{T}))^d$,

$$X = \left\{v \in H^1(\overline{T}) \text{ with } \int_{\overline{e}} v = 0\right\}$$

equipped with the norm $H^1(\overline{T})$, $Av = \nabla v$ (which is indeed injective on X), and $Tv = v$ (which is indeed compact from X to Z). Lemma 4.2 readily yields the bound (4.2) after taking the maximum over all edges $\overline{e}$.

4.2 Finite element of size *H*

We will repeatedly use the following Poincaré inequality.

Lemma 4.3 *There exists C (depending only on the regularity of the mesh) independent of H such that, for any $T \in \mathcal{T}_H$,*

$$\forall v \in H^1(T) \text{ with } \int_e v = 0 \quad \text{for some } e \subset \partial T, \quad \|v\|_{L^2(T)} \leq CH\|\nabla v\|_{L^2(T)}. \tag{4.3}$$

Proof To convey the idea of the proof in a simple case, we first assume that the actual mesh element T considered is homothetic to the reference mesh element $\overline{T}$ with a ratio H. We introduce $v_H(x) = v(Hx)$ defined on the reference element. We hence have $v(x) = v_H\left(\frac{x}{H}\right)$. Thus,

$$\|v\|^2_{L^2(T)} = \int_T v^2(x)\mathrm{d}x = \int_T v_H^2\left(\frac{x}{H}\right)\mathrm{d}x = H^d \int_{\overline{T}} v_H^2(y)\mathrm{d}y$$

and

$$\|\nabla v\|^2_{L^2(T)} = \int_T |\nabla v(x)|^2\mathrm{d}x = H^{-2}\int_T \left|\nabla v_H\left(\frac{x}{H}\right)\right|^2\mathrm{d}x = H^{d-2}\int_{\overline{T}} |\nabla v_H(y)|^2\mathrm{d}y.$$

We now use Lemma 4.1, and conclude that

$$\|v\|^2_{L^2(T)} = H^d\|v_H\|^2_{L^2(\overline{T})} \leq CH^d\|\nabla v_H\|^2_{L^2(\overline{T})} = CH^2\|\nabla v\|^2_{L^2(T)},$$

which is (4.3) in this simple case. To obtain (4.3) in full generality, we have to slightly adapt the above argument. We shall use, here and throughout the proof of the subsequent lemmas, the notation $A \sim B$ when the two quantities A and B satisfy $c_1 A \leq B \leq c_2 A$ with the constants c_1 and c_2 depending only on the regularity parameter of the mesh. Let us recall that for all $T \in \mathcal{T}_H$, there exists a smooth one-to-one and onto mapping $K : \overline{T} \to T$ satisfying $\|\nabla K\|_{L^\infty} \leq CH$ and $\|\nabla K^{-1}\|_{L^\infty} \leq CH^{-1}$. We now introduce $v_H(x) = v(K(x))$ defined on the reference element. We hence have

$$\|v\|^2_{L^2(T)} = \int_T v^2(x)\mathrm{d}x = \int_T v_H^2(K^{-1}(x))\mathrm{d}x \sim H^d\int_{\overline{T}} v_H^2(y)\mathrm{d}y$$

and

$$\begin{aligned}\|\nabla v\|^2_{L^2(T)} = \int_T |\nabla v(x)|^2\mathrm{d}x &\sim H^{-2}\int_T |\nabla v_H(K^{-1}(x))|^2\mathrm{d}x \\ &\sim H^{d-2}\int_{\overline{T}} |\nabla v_H(y)|^2\mathrm{d}y.\end{aligned}$$

Using Lemma 4.1 (note that $\int_{\overline{e}} v_H(y)\mathrm{d}y = 0$ since the mapping K is affine on the edges, hence, is of constant Jacobian on $\overline{e}$), we obtain

$$\|v\|^2_{L^2(T)} \sim H^d\|v_H\|^2_{L^2(\overline{T})} \leq CH^d\|\nabla v_H\|^2_{L^2(\overline{T})} \leq CH^2\|\nabla v\|^2_{L^2(T)},$$

which is the bound (4.3).

We also have the following trace results.

Lemma 4.4 *There exists C (depending only on the regularity of the mesh) such that, for any $T \in \mathcal{T}_H$ and any edge $e \subset \partial T$, we have*

$$\forall v \in H^1(T), \quad \|v\|^2_{L^2(e)} \leq C(H^{-1}\|v\|^2_{L^2(T)} + H\|\nabla v\|^2_{L^2(T)}). \tag{4.4}$$

Under the additional assumption that $\int_e v = 0$, we have

$$\|v\|^2_{L^2(e)} \leq CH\|\nabla v\|^2_{L^2(T)}. \tag{4.5}$$

If $\int_e v = 0$ and $H \leq 1$, then

$$\|v\|^2_{H^{1/2}(e)} \leq C\|\nabla v\|^2_{L^2(T)}. \tag{4.6}$$

These bounds are classical results (see [10, p. 282]). We provide here a proof for the sake of completeness.

Proof of Lemma 4.4 We proceed as in the proof of Lemma 4.3 and use the same notation. We use $v_H(x) = v(K(x))$ defined on the reference element. We have

$$\|v\|^2_{L^2(e)} = \int_e v^2(x)\mathrm{d}x = \int_e v_H^2(K^{-1}(x))\mathrm{d}x \sim H^{d-1}\int_{\overline{e}} v_H^2(y)\mathrm{d}y = H^{d-1}\|v_H\|^2_{L^2(\overline{e})}.$$

By a standard trace inequality, we obtain

$$\begin{aligned}\|v\|^2_{L^2(e)} &\leq CH^{d-1}(\|v_H\|^2_{L^2(\overline{T})} + \|\nabla v_H\|^2_{L^2(\overline{T})})\\ &\leq CH^{d-1}\Big(\frac{1}{H^d}\|v\|^2_{L^2(T)} + \frac{1}{H^{d-2}}\|\nabla v\|^2_{L^2(T)}\Big),\end{aligned}$$

where we have used some ingredients of the proof of Lemma 4.3. This shows that (4.4) holds.

We now turn to (4.5):

$$\begin{aligned}\|v\|^2_{L^2(e)} \sim H^{d-1}\|v_H\|^2_{L^2(\overline{e})} &\leq CH^{d-1}\|v_H\|^2_{H^1(\overline{T})} \leq CH^{d-1}\|\nabla v_H\|^2_{L^2(\overline{T})}\\ &\leq CH\|\nabla v\|^2_{L^2(T)},\end{aligned}$$

where we have used (4.1)–(4.2). This proves (4.5).

We eventually establish (4.6). We first observe, using Definition 4.1 with the domain $\omega \equiv e \subset \mathbb{R}^{d-1}$, that

$$\begin{aligned}|v|^2_{H^{1/2}(e)} &= \int_e\int_e \frac{|v(x)-v(y)|^2}{|x-y|^d}\mathrm{d}x\mathrm{d}y\\ &\sim \frac{1}{H^d}\int_e\int_e \frac{|v_H(K^{-1}(x)) - v_H(K^{-1}(y))|^2}{|K^{-1}(x)-K^{-1}(y)|^d}\mathrm{d}x\mathrm{d}y\\ &\sim H^{d-2}\int_{\overline{e}}\int_{\overline{e}} \frac{|v_H(x)-v_H(y)|^2}{|x-y|^d}\mathrm{d}x\mathrm{d}y\\ &\sim H^{d-2}\,|v_H|^2_{H^{1/2}(\overline{e})}.\end{aligned}$$

Hence, using (4.1)–(4.2) and since $H \leq 1$,

$$\begin{aligned}\|v\|^2_{H^{1/2}(e)} &= \|v\|^2_{L^2(e)} + |v|^2_{H^{1/2}(e)} \sim H^{d-1}\|v_H\|^2_{L^2(\overline{e})} + H^{d-2}|v_H|^2_{H^{1/2}(\overline{e})}\\ &\leq CH^{d-2}\|v_H\|^2_{H^{1/2}(\overline{e})} \leq CH^{d-2}\|v_H\|^2_{H^1(\overline{T})}\\ &\leq CH^{d-2}\|\nabla v_H\|^2_{L^2(\overline{T})} \sim C\|\nabla v\|^2_{L^2(T)}.\end{aligned}$$

This proves (4.6) and concludes the proof of Lemma 4.4.

The following result is a direct consequence of (4.5) and (4.6).

Corollary 4.1 *Consider an edge $e \in \mathcal{E}_H$, and let $T_e \subset \mathcal{T}_H$ denote all the triangles sharing this edge. There exists C (depending only on the regularity of the mesh), such that*

$$\forall v \in W_H, \quad \|\,[[v]]\,\|^2_{L^2(e)} \leq CH\sum_{T\in T_e}\|\nabla v\|^2_{L^2(T)}. \tag{4.7}$$

If $H \leq 1$, then

$$\forall v \in W_H, \quad \| [[v]] \|^2_{H^{1/2}(e)} \leq C \sum_{T \in T_e} \|\nabla v\|^2_{L^2(T)}. \tag{4.8}$$

Proof We introduce $c_e = |e|^{-1} \int_e v$, which is well-defined since $\int_e [[v]] = 0$. On each side of the edge, the function $v - c_e$ has zero average on that edge. Hence, using (4.5), we have

$$\begin{aligned}
\| [[v]] \|^2_{L^2(e)} &= \| [[v - c_e]] \|^2_{L^2(e)} = \|(v_1 - c_e) - (v_2 - c_e)\|^2_{L^2(e)} \\
&\leq 2\|v_1 - c_e\|^2_{L^2(e)} + 2\|v_2 - c_e\|^2_{L^2(e)} \\
&\leq CH(\|\nabla v_1\|^2_{L^2(T_1)} + \|\nabla v_2\|^2_{L^2(T_2)}) \\
&= CH \sum_{T \in T_e} \|\nabla v\|^2_{L^2(T)},
\end{aligned}$$

where we have used the notation $v_1 = v|_{T_1}$. The proof of (4.8) follows a similar pattern, using (4.6).

4.3 Error estimate for nonconforming FEM

The error estimate which we establish in Section 5 is essentially based on a Céa-type (or Strang-type) lemma extended to nonconforming finite element methods. We state this standard estimate in the actual context we work in (but again emphasize that it is of course completely general in nature).

Lemma 4.5 (see [10, Lemma 10.1.7]) *Let u^ε be the solution to* (1.1) *and u_H be the solution to* (2.3)*. Then*

$$\|u^\varepsilon - u_H\|_E \leq \inf_{v \in V_H} \|u^\varepsilon - v\|_E + \sup_{v \in V_H \setminus \{0\}} \frac{|a_H(u^\varepsilon - u_H, v)|}{\|v\|_E}. \tag{4.9}$$

The first term in (4.9) is the usual best approximation error already present in the classical Céa Lemma. This term measures how accurately the space V_H (or, in general, any approximation space) approximates the exact solution u^ε. The second term of (4.9) measures how the nonconforming setting affects the result. This term would vanish if V_H were a subset of $H^1_0(\Omega)$.

4.4 Integrals of oscillatory functions

We shall also need the following result.

Lemma 4.6 *Let $e \in \mathcal{E}_H$, T_1 and T_2 be the two elements adjacent to e and $\tau \in \mathbb{R}^d$, $|\tau| \leq 1$, be a vector tangent (i.e., parallel) to e. Then, for any function $u \in H^1(T_1 \cup T_2)$, any $v \in W_H$ and any $J \in C^1(\mathbb{R}^d)$, we have*

$$\begin{aligned}
&\Big| \int_e u(x) \, [[v(x)]] \, \tau \cdot \nabla J\Big(\frac{x}{\varepsilon}\Big) \Big| \\
&\leq C \sqrt{\frac{\varepsilon}{H}} \, \|J\|_{C^1(\mathbb{R}^d)} \sum_{T = T_1, T_2} |v|_{H^1(T)} (\|u\|_{L^2(T)} + H|u|_{H^1(T)})
\end{aligned} \tag{4.10}$$

with a constant C which depends only on the regularity of the mesh.

As will be clear from the proof below, the fact that we consider in the above left-hand side the jump of v, rather than v itself, is not essential. A similar estimate holds for the quantity $\int_e u(x)\, v(x)\, \tau \cdot \nabla J\big(\frac{x}{\varepsilon}\big)$, where u and v are any functions of regularity $H^1(T)$ for some $T \in \mathcal{T}_H$ and e is an edge of ∂T.

Proof of Lemma 4.6 Let c_e be the average of v over e and denote $v_j = v|_{T_j}$. Since $[[v]] = (v_1 - c_e) - (v_2 - c_e)$, we obviously have

$$\Big| \int_e u(x)\, [[v(x)]]\, \tau \cdot \nabla J\Big(\frac{x}{\varepsilon}\Big) \Big| \le \sum_{j=1}^{2} \Big| \int_e u(x)\, (v_j(x) - c_e)\, \tau \cdot \nabla J\Big(\frac{x}{\varepsilon}\Big) \Big|. \tag{4.11}$$

Fix j. We first recall that there exists a one-to-one and onto mapping $K : \overline{T} \to T_j$ from the reference element $\overline{T}$ onto T_j satisfying $\|\nabla K\|_{L^\infty} \le CH$ and $\|\nabla K^{-1}\|_{L^\infty} \le CH^{-1}$. In particular, there exists an edge $\overline{e}$ of $\overline{T}$ such that $K(\overline{e}) = e$. We introduce the functions $u_H(y) = u(K(y))$, $v_H(y) = v_j(K(y)) - c_e$ defined on the reference element, and observe that

$$\begin{aligned} &\int_e u(x)\, (v_j(x) - c_e)\, \tau \cdot \nabla J\Big(\frac{x}{\varepsilon}\Big) \mathrm{d}x \\ &\sim H^{d-1} \int_{\overline{e}} u_H(y) v_H(y)\, \tau \cdot \nabla J\Big(\frac{K(y)}{\varepsilon}\Big) \mathrm{d}y. \end{aligned} \tag{4.12}$$

We now claim that

$$\begin{aligned} &\Big| \int_{\overline{e}} u_H(y) v_H(y)\, \tau \cdot \nabla J\Big(\frac{K(y)}{\varepsilon}\Big) \mathrm{d}y \Big| \\ &\le C \sqrt{\frac{\varepsilon}{H}} \|J\|_{C^1(\mathbb{R}^d)} \|u_H\|_{H^{1/2}(\overline{e})} \|v_H\|_{H^{1/2}(\overline{e})}. \end{aligned} \tag{4.13}$$

This inequality is obtained by interpolation. Suppose indeed, in the first step, that u_H and v_H belong to $H^1(\overline{e})$. Using that the mapping K is affine on the edges and thus is of constant gradient, we first see that

$$\begin{aligned} &\int_{\overline{e}} u_H(y) v_H(y)\, \tau \cdot \nabla J\Big(\frac{K(y)}{\varepsilon}\Big) \mathrm{d}y \\ &= C \frac{\varepsilon}{H} \int_{\overline{e}} u_H(y) v_H(y)\, \tau \cdot \nabla \Big[J\Big(\frac{K(y)}{\varepsilon}\Big) \Big] \mathrm{d}y. \end{aligned} \tag{4.14}$$

By integration by parts, we next observe that

$$\begin{aligned} &\frac{\varepsilon}{H} \int_{\overline{e}} u_H(y) v_H(y)\, \tau \cdot \nabla \Big[J\Big(\frac{K(y)}{\varepsilon}\Big) \Big] \mathrm{d}y \\ &= \frac{\varepsilon}{H} \int_{\partial\overline{e}} u_H(y) v_H(y)\, \tau \cdot \nu\, J\Big(\frac{K(y)}{\varepsilon}\Big) \mathrm{d}y \\ &\quad - \frac{\varepsilon}{H} \int_{\overline{e}} J\Big(\frac{K(y)}{\varepsilon}\Big) \tau \cdot \nabla (u_H(y) v_H(y)) \mathrm{d}y, \end{aligned} \tag{4.15}$$

where ν is the outward normal unit vector to $\partial\overline{e}$ tangent to $\overline{e}$. Collecting (4.14)–(4.15), and using the Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned}&\Big|\int_{\overline{e}} u_H(y)v_H(y)\,\tau\cdot\nabla J\Big(\frac{K(y)}{\varepsilon}\Big)\mathrm{d}y\Big|\\ &\le C\frac{\varepsilon}{H}\|J\|_{C^0(\mathbb{R}^d)}[\|u_H\|_{L^2(\partial\overline{e})}\|v_H\|_{L^2(\partial\overline{e})}+2\|u_H\|_{H^1(\overline{e})}\|v_H\|_{H^1(\overline{e})}]\\ &\le C\frac{\varepsilon}{H}\|J\|_{C^0(\mathbb{R}^d)}\|u_H\|_{H^1(\overline{e})}\|v_H\|_{H^1(\overline{e})},\end{aligned} \tag{4.16}$$

where the last inequality above follows from the trace inequality which is valid with a constant C depending only on $\overline{e}$. On the other hand, for u_H and v_H that only belong to $L^2(\overline{e})$, we obviously have

$$\begin{aligned}&\Big|\int_{\overline{e}} u_H(y)v_H(y)\,\tau\cdot\nabla J\Big(\frac{K(y)}{\varepsilon}\Big)\mathrm{d}y\Big|\\ &\le \|\nabla J\|_{C^0(\mathbb{R}^d)}\|u_H\|_{L^2(\overline{e})}\|v_H\|_{L^2(\overline{e})}.\end{aligned} \tag{4.17}$$

By interpolation between (4.16)–(4.17) (see [9, Theorem 4.4.1]), we obtain (4.13).

The sequel of the proof is easy. Collecting (4.12)–(4.13), we deduce that

$$\begin{aligned}&\Big|\int_e u(x)\,(v_j(x)-c_e)\,\tau\cdot\nabla J\Big(\frac{x}{\varepsilon}\Big)\mathrm{d}x\Big|\\ &\le CH^{d-\frac{3}{2}}\sqrt{\varepsilon}\,\|J\|_{C^1(\mathbb{R}^d)}\|u_H\|_{H^{1/2}(\overline{e})}\|v_H\|_{H^{1/2}(\overline{e})}\\ &\le CH^{d-\frac{3}{2}}\sqrt{\varepsilon}\,\|J\|_{C^1(\mathbb{R}^d)}\|u_H\|_{H^1(\overline{T})}\|\nabla v_H\|_{L^2(\overline{T})},\end{aligned} \tag{4.18}$$

where we have used in the last line the trace inequality (4.1) and Lemma 4.1 for v_H (recall that $\int_{\overline{e}} v_H=0$, since, on the one hand, $\int_e v_j-c_e=0$ and, on the other hand, the mapping K is affine on $\overline{e}$, and hence is of constant gradient).

To return to norms on the actual element T_j rather than on the reference element $\overline{T}$, we use the following relations, established in the proof of Lemma 4.3:

$$\begin{aligned}&\|u\|_{L^2(T_j)}\sim H^{\frac{d}{2}}\|u_H\|_{L^2(\overline{T})},\\ &|u|_{H^1(T_j)}\sim H^{\frac{d}{2}-1}|u_H|_{H^1(\overline{T})},\\ &|v_j|_{H^1(T_j)}\sim H^{\frac{d}{2}-1}|v_H|_{H^1(\overline{T})}.\end{aligned}$$

We then infer from (4.18) that

$$\begin{aligned}&\Big|\int_e u(x)(v_j(x)-c_e)\,\tau\cdot\nabla J\Big(\frac{x}{\varepsilon}\Big)\Big|\\ &\le CH^{d-\frac{3}{2}}\sqrt{\varepsilon}\,\|J\|_{C^1(\mathbb{R}^d)}[H^{-\frac{d}{2}}\|u\|_{L^2(T_j)}\\ &\quad+H^{-\frac{d}{2}+1}|u|_{H^1(T_j)}]H^{-\frac{d}{2}+1}|v_j|_{H^1(T_j)}\\ &\le C\sqrt{\frac{\varepsilon}{H}}\,\|J\|_{C^1(\mathbb{R}^d)}[\|u\|_{L^2(T_j)}+H|u|_{H^1(T_j)}]|v_j|_{H^1(T_j)}.\end{aligned}$$

Inserting this bound in (4.11) for $j=1$ and 2 yields the desired bound (4.10).

5 Proof of the Main Error Estimate

Now that we have reviewed a number of classical ingredients, we are in the position, in this section, to prove our main result, Theorem 3.1.

As announced above, our proof is based on the estimate (4.9) provided by Lemma 4.5. To bound both terms in the right-hand side of (4.9), we will use the following result, the proof of which is postponed until Subsection 5.2.

Lemma 4.7 *Under the same assumptions as those of Theorem* 3.1*, we have that, for any* $v \in W_H$,

$$\Big| \sum_{T \in \mathcal{T}_H} \int_{\partial T} v \Big(A_{\rm per}\Big(\frac{x}{\varepsilon}\Big) \nabla u^\varepsilon \Big) \cdot n \Big| \leq C \Big(\sqrt{\varepsilon} + H + \sqrt{\frac{\varepsilon}{H}} \Big) \|v\|_E \, \|\nabla u^\star\|_{C^1(\overline{\Omega})}, \tag{5.1}$$

where the constant C *is independent of* H, ε, f, $u^\star$ *and* v.

Remark 4.1 A more precise estimate is given in the course of the proof (see (5.23)).

5.1 Proof of Theorem 3.1

Momentarily assuming Lemma 4.7, we now prove our main result.

We argue on estimate (4.9) provided by Lemma 4.5. In the right-hand side of (4.9), we first bound the nonconforming error (the second term). Let $v \in V_H$. We use the definition (2.1) of a_H and (2.3) to compute:

$$\begin{aligned} a_H(u^\varepsilon - u_H, v) &= \sum_{T \in \mathcal{T}_H} \int_T (\nabla v)^{\rm T} A_{\rm per}\Big(\frac{x}{\varepsilon}\Big) \nabla u^\varepsilon - \int_\Omega f v \\ &= \sum_{T \in \mathcal{T}_H} \int_{\partial T} v \Big(A_{\rm per}\Big(\frac{x}{\varepsilon}\Big) \nabla u^\varepsilon \Big) \cdot n \\ &\quad - \sum_{T \in \mathcal{T}_H} \int_T v \,{\rm div}\, \Big(A_{\rm per}\Big(\frac{x}{\varepsilon}\Big) \nabla u^\varepsilon \Big) - \int_\Omega f v \\ &= \sum_{T \in \mathcal{T}_H} \int_{\partial T} v \Big(A_{\rm per}\Big(\frac{x}{\varepsilon}\Big) \nabla u^\varepsilon \Big) \cdot n, \end{aligned}$$

using (1.1) and the regularity of u^ε. Observing that, by definition, $v \in V_H \subset W_H$, we can use Lemma 4.7 to majorize the above right-hand side. We obtain

$$\sup_{v \in V_H \setminus \{0\}} \frac{|a_H(u^\varepsilon - u_H, v)|}{\|v\|_E} \leq C \Big(\sqrt{\varepsilon} + H + \sqrt{\frac{\varepsilon}{H}} \Big) \|\nabla u^\star\|_{C^1(\overline{\Omega})}. \tag{5.2}$$

We now turn to the best approximation error (the first term of the right-hand side of (4.9)). As shown at the end of Subsection 2.1, we can decompose $u^\varepsilon \in H^1_0(\Omega) \subset W_H$ as

$$u^\varepsilon = v_H + v^0, \quad v_H \in V_H, \quad v^0 \in W^0_H.$$

We may compute, again using (1.1) and the regularity of u^ε, that

$$\begin{aligned}
\|u^\varepsilon - v_H\|_E^2 &= a_H(u^\varepsilon - v_H, u^\varepsilon - v_H) \\
&= a_H(u^\varepsilon - v_H, v^0) \quad \text{(by definition of } v^0\text{)} \\
&= a_H(u^\varepsilon, v^0) \quad \text{(by orthogonality of } V_H \text{ with } W_H^0\text{)} \\
&= \sum_{T\in\mathcal{T}_H} \int_T (\nabla v^0)^{\mathrm{T}} A_{\mathrm{per}}\Big(\frac{x}{\varepsilon}\Big)\nabla u^\varepsilon \\
&= \sum_{T\in\mathcal{T}_H} \int_{\partial T} v^0 \Big(A_{\mathrm{per}}\Big(\frac{x}{\varepsilon}\Big)\nabla u^\varepsilon\Big)\cdot n + \sum_{T\in\mathcal{T}_H} \int_T v^0 f. \qquad (5.3)
\end{aligned}$$

Since $v^0 \in W_H^0$, we may use (4.3) and bound the second term of the right-hand side of (5.3) as follows:

$$\begin{aligned}
\Big|\sum_{T\in\mathcal{T}_H} \int_T v^0 f\Big| &\le \sum_{T\in\mathcal{T}_H} \|v^0\|_{L^2(T)}\|f\|_{L^2(T)} \quad \text{(Cauchy Schwarz inequality)} \\
&\le CH \sum_{T\in\mathcal{T}_H} \|\nabla v^0\|_{L^2(T)}\|f\|_{L^2(T)} \\
&\le CH\|v^0\|_E\,\|f\|_{L^2(\Omega)}, \qquad (5.4)
\end{aligned}$$

where we have used in the last line the Cauchy Schwarz inequality and an equivalence of norms. The first term of the right-hand side of (5.3) is bounded by using Lemma 4.7 (since $v^0 \in W_H^0 \subset W_H$), which yields

$$\Big|\sum_{T\in\mathcal{T}_H} \int_{\partial T} v^0 \Big(A_{\mathrm{per}}\Big(\frac{x}{\varepsilon}\Big)\nabla u^\varepsilon\Big)\cdot n\Big| \le C\Big(\sqrt{\varepsilon} + H + \sqrt{\frac{\varepsilon}{H}}\Big)\|v^0\|_E\,\|\nabla u^\star\|_{C^1(\overline{\Omega})}. \qquad (5.5)$$

Inserting (5.4)–(5.5) in the right-hand side of (5.3), we deduce that

$$\|u^\varepsilon - v_H\|_E^2 \le CH\|v^0\|_E\,\|f\|_{L^2(\Omega)} + C\Big(\sqrt{\varepsilon} + H + \sqrt{\frac{\varepsilon}{H}}\Big)\|v^0\|_E\,\|\nabla u^\star\|_{C^1(\overline{\Omega})}.$$

Since $v^0 = u^\varepsilon - v_H$, we may factor out $\|v^0\|_E$ and obtain

$$\|u^\varepsilon - v_H\|_E \le CH\|f\|_{L^2(\Omega)} + C\Big(\sqrt{\varepsilon} + H + \sqrt{\frac{\varepsilon}{H}}\Big)\|\nabla u^\star\|_{C^1(\overline{\Omega})}.$$

By the definition of the infimum, we of course have $\inf_{v\in V_H} \|u^\varepsilon - v\|_E \le \|u^\varepsilon - v_H\|_E$, and thus

$$\inf_{v\in V_H} \|u^\varepsilon - v\|_E \le CH\|f\|_{L^2(\Omega)} + C\Big(\sqrt{\varepsilon} + H + \sqrt{\frac{\varepsilon}{H}}\Big)\|\nabla u^\star\|_{C^1(\overline{\Omega})}. \qquad (5.6)$$

Inserting (5.2) and (5.6) in the right-hand side of (4.9), we obtain the desired bound (3.7). This concludes the proof of Theorem 3.1.

5.2 Proof of Lemma 4.7

We now establish Lemma 4.7, actually the key step of the proof of Theorem 3.1.

Let $v \in W_H$. Using (1.1) and (3.2), and inserting in the term we are estimating the approximation $u^{\varepsilon,1}$ defined by (3.4) of the exact solution u^ε, we write

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_H}\int_{\partial T} v\Big(A_{\rm per}\Big(\frac{x}{\varepsilon}\Big)\nabla u^\varepsilon\Big)\cdot n \\
&= -\sum_{T\in\mathcal{T}_H}\int_T vf + \sum_{T\in\mathcal{T}_H}\int_T (\nabla v)^{\rm T} A_{\rm per}\Big(\frac{x}{\varepsilon}\Big)\nabla u^\varepsilon \\
&= \sum_{T\in\mathcal{T}_H}\int_T v\,{\rm div}(A^\star_{\rm per}\nabla u^\star) + \sum_{T\in\mathcal{T}_H}\int_T (\nabla v)^{\rm T} A_{\rm per}\Big(\frac{x}{\varepsilon}\Big)\nabla(u^\varepsilon - u^{\varepsilon,1}) \\
&\quad + \sum_{T\in\mathcal{T}_H}\int_T (\nabla v)^{\rm T} A_{\rm per}\Big(\frac{x}{\varepsilon}\Big)\nabla u^{\varepsilon,1} \\
&= \sum_{T\in\mathcal{T}_H}\int_{\partial T} v(A^\star_{\rm per}\nabla u^\star)\cdot n + \sum_{T\in\mathcal{T}_H}\int_T (\nabla v)^{\rm T} A_{\rm per}\Big(\frac{x}{\varepsilon}\Big)\nabla(u^\varepsilon - u^{\varepsilon,1}) \\
&\quad + \sum_{T\in\mathcal{T}_H}\int_T (\nabla v)^{\rm T}\Big(A_{\rm per}\Big(\frac{x}{\varepsilon}\Big)\nabla u^{\varepsilon,1} - A^\star_{\rm per}\nabla u^\star\Big) \\
&= {\rm A} + {\rm B} + {\rm C}.
\end{aligned} \tag{5.7}$$

We now successively bound the three terms A, B and C in the right-hand side of (5.7). Loosely speaking:

(1) The first term A is macroscopic in nature and would be present for the analysis of a classical Crouzeix-Raviart type method. It will eventually contribute for $O(H)$ to the overall estimate (5.1) (and thus to (3.7)).

(2) The second term B is independent from the discretization: it is an "infinite dimensional" term, the size of which, namely $O(\sqrt{\varepsilon})$, is entirely controlled by the quality of approximation of u^ε by $u^{\varepsilon,1}$. It is the term for which we specifically need to put ourselves in the periodic setting.

(3) The third term C would likewise go to zero if the size of the mesh were much larger than the small coefficient ε; it will contribute for the $O\big(\sqrt{\frac{\varepsilon}{H}}\big)$ term in the estimate (5.1).

Step 1 Bound on the first term of (5.7) We first note that

$$\sum_{T\in\mathcal{T}_H}\int_{\partial T} v(A^\star_{\rm per}\nabla u^\star)\cdot n = \sum_{e\in\mathcal{E}_H}\int_e [[v]](A^\star_{\rm per}\nabla u^\star)\cdot n.$$

We now use arguments that are standard in the context of Crouzeix-Raviart finite elements (see [10, p. 281]). Introducing, for each edge e, the constant $c_e = |e|^{-1}\int_e (A^\star_{\rm per}\nabla u^\star)\cdot n$, and using $\int_e [[v]] = 0$ with $v \in W_H$, we write

$$\begin{aligned}
&\Big|\sum_{T\in\mathcal{T}_H}\int_{\partial T} v(A^\star_{\rm per}\nabla u^\star)\cdot n\Big| \\
&= \Big|\sum_{e\in\mathcal{E}_H}\int_e [[v]](A^\star_{\rm per}\nabla u^\star)\cdot n\Big| \\
&\le \sum_{e\in\mathcal{E}_H}\Big|\int_e [[v]]((A^\star_{\rm per}\nabla u^\star)\cdot n - c_e)\Big|
\end{aligned}$$

$$\leq \sum_{e \in \mathcal{E}_H} \| [[v]] \|_{L^2(e)} \ \|(A^\star_{\rm per} \nabla u^\star) \cdot n - c_e\|_{L^2(e)}$$
$$\leq \Big[\sum_{e \in \mathcal{E}_H} \| [[v]] \|^2_{L^2(e)} \Big]^{\frac{1}{2}} \ \Big[\sum_{e \in \mathcal{E}_H} \|(A^\star_{\rm per} \nabla u^\star) \cdot n - c_e\|^2_{L^2(e)} \Big]^{\frac{1}{2}},$$

successively using the continuous and discrete Cauchy-Schwarz inequalities in the last two lines. We now use (4.5) and (4.7) to respectively estimate the two factors in the above right-hand side. Doing so, we obtain

$$\Big| \sum_{T \in \mathcal{T}_H} \int_{\partial T} v (A^\star_{\rm per} \nabla u^\star) \cdot n \Big|$$
$$\leq C \Big[\sum_{e \in \mathcal{E}_H} H \sum_{T \in \mathcal{T}_e} \|\nabla v\|^2_{L^2(T)} \Big]^{\frac{1}{2}} \ \Big[\sum_{\substack{e \in \mathcal{E}_H \\ \text{choose one } T \, \in \, \mathcal{T}_e}} H \|\nabla^2 u^\star\|^2_{L^2(T)} \Big]^{\frac{1}{2}}.$$

We hence have that

$$\Big| \sum_{T \in \mathcal{T}_H} \int_{\partial T} v (A^\star_{\rm per} \nabla u^\star) \cdot n \Big|$$
$$\leq C \Big[H \sum_{T \in \mathcal{T}_H} \|\nabla v\|^2_{L^2(T)} \Big]^{\frac{1}{2}} \ \Big[\sum_{T \in \mathcal{T}_H} H \|\nabla^2 u^\star\|^2_{L^2(T)} \Big]^{\frac{1}{2}}$$
$$\leq C H \ \|v\|_E \ \|\nabla^2 u^\star\|_{L^2(\Omega)}. \tag{5.8}$$

Step 2 Bound on the second term of (5.7) We note that

$$\Big| \sum_{T \in \mathcal{T}_H} \int_T (\nabla v)^{\rm T} A_{\rm per}\Big(\frac{x}{\varepsilon}\Big) \nabla (u^\varepsilon - u^{\varepsilon,1}) \Big|$$
$$\leq \|A_{\rm per}\|_{L^\infty} \sum_{T \in \mathcal{T}_H} \|\nabla v\|_{L^2(T)} \, \|\nabla (u^\varepsilon - u^{\varepsilon,1})\|_{L^2(T)}$$
$$\leq C \|v\|_E \, \|\nabla (u^\varepsilon - u^{\varepsilon,1})\|_{L^2(\Omega)}$$
$$\leq C \sqrt{\varepsilon} \, \|v\|_E \, \|\nabla u^\star\|_{W^{1,\infty}(\Omega)}, \tag{5.9}$$

eventually using (3.5).

Step 3 Bound on the third term of (5.7) To start with, we differentiate $u^{\varepsilon,1}$ defined by (3.4):

$$\nabla u^{\varepsilon,1}(x) = \sum_{i=1}^d \partial_i u^\star(x) \Big(e_i + \nabla w_{e_i}\Big(\frac{x}{\varepsilon}\Big)\Big) + \varepsilon \sum_{i=1}^d w_{e_i}\Big(\frac{x}{\varepsilon}\Big) \partial_i \nabla u^\star(x).$$

The third term of (5.7) thus writes

$$\sum_{T \in \mathcal{T}_H} \int_T (\nabla v)^{\rm T} \Big(A_{\rm per}\Big(\frac{x}{\varepsilon}\Big) \nabla u^{\varepsilon,1} - A^\star_{\rm per} \nabla u^\star \Big)$$
$$= \varepsilon \sum_{i=1}^d \sum_{T \in \mathcal{T}_H} \int_T (\nabla v)^{\rm T} A_{\rm per}\Big(\frac{x}{\varepsilon}\Big) \partial_i \nabla u^\star(x) \ w_{e_i}\Big(\frac{x}{\varepsilon}\Big)$$
$$+ \sum_{T \in \mathcal{T}_H} \sum_{i=1}^d \int_T (\nabla v)^{\rm T} \partial_i u^\star(x) \, G_i\Big(\frac{x}{\varepsilon}\Big), \tag{5.10}$$

where we have introduced the vector fields

$$G_i(x) = A_{\rm per}(x)(e_i + \nabla w_{e_i}(x)) - A^\star_{\rm per} e_i, \quad 1 \le i \le d,$$

which are all $\mathbb{Z}^d$ periodic, divergence free and of zero mean. In addition, in view of the assumption (3.6), which implies (3.8), we see that

$$G_i \text{ is Hölder continuous.} \tag{5.11}$$

We now successively bound the two terms of the right-hand side of (5.10). The first term is quite straightforward to bound. Using Cauchy-Schwarz inequalities and that $w_p \in L^\infty$ (see (3.8)), we simply observe that

$$\begin{aligned} &\left| \varepsilon \sum_{i=1}^d \sum_{T \in \mathcal{T}_H} \int_T (\nabla v)^{\rm T} A_{\rm per}\left(\frac{x}{\varepsilon}\right) \partial_i \nabla u^\star(x) \ w_{e_i}\left(\frac{x}{\varepsilon}\right) \right| \\ &\le d\, \varepsilon \, \|A_{\rm per}\|_{L^\infty} \max_i \|w_{e_i}\|_{L^\infty} \sum_{T \in \mathcal{T}_H} \|\nabla v\|_{L^2(T)} \|\nabla^2 u^\star\|_{L^2(T)} \\ &\le C \varepsilon \|v\|_E \|\nabla^2 u^\star\|_{L^2(\Omega)}. \end{aligned} \tag{5.12}$$

The rest of the proof is actually devoted to bounding the second term of the right-hand side of (5.10), a task that requires several estimations. We first use a classical argument already exposed in [24, p. 27]. The vector field G_i being $\mathbb{Z}^d$ periodic, divergence free and of zero mean, there exists (see [24, p. 6]) a skew symmetric matrix $J^i \in \mathbb{R}^{d \times d}$, such that

$$\forall 1 \le \alpha \le d, \quad [G_i]_\alpha = \sum_{\beta=1}^d \frac{\partial [J^i]_{\beta\alpha}}{\partial x_\beta} \tag{5.13}$$

and

$$J^i \in (H^1_{\rm loc}(\mathbb{R}^d))^{d \times d}, \quad J^i \text{ is } \mathbb{Z}^d\text{-periodic}, \quad \int_{(0,1)^d} J^i = 0.$$

In the two-dimensional setting, an explicit expression can be written. We indeed have

$$J^i(x_1, x_2) = \begin{pmatrix} 0 & -\tau^i(x_1,x_2) \\ \tau^i(x_1,x_2) & 0 \end{pmatrix}, \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

with

$$\tau^i(x_1,x_2) = \tau^i(0) + \int_0^1 (x_2 [G_i]_1(tx) - x_1 [G_i]_2(tx)) {\rm d}t,$$

where $\tau^i(0)$ satisfies $\int_{(0,1)^2} \tau^i = 0$. In view of (5.11), we in particular have that

$$J^i \in (C^1(\mathbb{R}^d))^{d \times d}. \tag{5.14}$$

A better regularity (namely $J^i \in (C^{1,\delta}(\mathbb{R}^d))^{d \times d}$ for some $\delta > 0$) actually holds, but we will not need it henceforth.

The same regularity (5.14) can be also proven in any dimension $d \ge 3$, although in a less straightforward manner. Indeed, the components of J^i constructed in [24, p. 7] using the Fourier series can be seen to satisfy the equation

$$-\Delta [J^i]_{\beta\alpha} = \frac{\partial [G_i]_\beta}{\partial x_\alpha} - \frac{\partial [G_i]_\alpha}{\partial x_\beta},$$

complemented with periodic boundary conditions. Hence the function $[J^i]_{\beta\alpha}$, as well as its gradient, is continuous due to the regularity (5.11) and general results on elliptic equations (see [19, Section 4.5]).

In view of (5.13), we see that the α-th coordinate of the vector $\partial_i u^\star(\cdot)\, G_i\left(\frac{\cdot}{\varepsilon}\right)$ reads

$$\begin{aligned}\left[\partial_i u^\star(x)\, G_i\left(\frac{x}{\varepsilon}\right)\right]_\alpha &= \sum_{\beta=1}^d \frac{\partial [J^i]_{\beta\alpha}}{\partial x_\beta}\left(\frac{x}{\varepsilon}\right)\partial_i u^\star(x) \\ &= \varepsilon \sum_{\beta=1}^d \frac{\partial}{\partial x_\beta}\left([J^i]_{\beta\alpha}\left(\frac{x}{\varepsilon}\right)\partial_i u^\star(x)\right) - \varepsilon \sum_{\beta=1}^d [J^i]_{\beta\alpha}\left(\frac{x}{\varepsilon}\right)\partial_{i\beta} u^\star(x) \\ &= \varepsilon[\widetilde{B}_i^\varepsilon(x)]_\alpha - \varepsilon[B_i^\varepsilon(x)]_\alpha, \end{aligned} \tag{5.15}$$

where the vector fields $\widetilde{B}_i^\varepsilon(x) \in \mathbb{R}^d$ and $B_i^\varepsilon(x) \in \mathbb{R}^d$ are defined, for any $1 \le \alpha \le d$, by

$$[B_i^\varepsilon(x)]_\alpha = \sum_{\beta=1}^d [J^i]_{\beta\alpha}\left(\frac{x}{\varepsilon}\right)\partial_{i\beta} u^\star(x) \quad \text{and} \quad [\widetilde{B}_i^\varepsilon(x)]_\alpha = \sum_{\beta=1}^d \frac{\partial}{\partial x_\beta}\left([J^i]_{\beta\alpha}\left(\frac{x}{\varepsilon}\right)\partial_i u^\star(x)\right).$$

The vector field $\widetilde{B}_i^\varepsilon$ is divergence free as J^i is a skew symmetric matrix.

The second term of the right-hand side of (5.10) thus reads

$$\begin{aligned}&\sum_{T\in\mathcal{T}_H}\sum_{i=1}^d \int_T (\nabla v)^{\mathrm{T}} \partial_i u^\star(x)\, G_i\left(\frac{x}{\varepsilon}\right) \\ &= \varepsilon \sum_{T\in\mathcal{T}_H}\sum_{i=1}^d \int_T (\nabla v(x))^{\mathrm{T}} (\widetilde{B}_i^\varepsilon(x) - B_i^\varepsilon(x)) \\ &= \varepsilon \sum_{T\in\mathcal{T}_H}\sum_{i=1}^d \int_{\partial T} v(x)\widetilde{B}_i^\varepsilon(x)\cdot n - \varepsilon \sum_{T\in\mathcal{T}_H}\sum_{i=1}^d \int_T (\nabla v(x))^{\mathrm{T}} B_i^\varepsilon(x), \end{aligned} \tag{5.16}$$

successively using (5.15) and an integration by parts of the former term and the divergence-free property of $\widetilde{B}_i^\varepsilon$. An upper bound for the second term can easily be obtained, given that $J^i \in (L^\infty(\mathbb{R}^d))^{d\times d}$ (see (5.14)):

$$\begin{aligned}\left|\varepsilon \sum_{T\in\mathcal{T}_H}\sum_{i=1}^d \int_T (\nabla v(x))^{\mathrm{T}} B_i^\varepsilon(x)\right| &= \left|\varepsilon \sum_{T\in\mathcal{T}_H}\sum_{i,\alpha,\beta=1}^d \int_T \partial_\alpha v(x)[J^i]_{\beta\alpha}\left(\frac{x}{\varepsilon}\right)\partial_{i\beta} u^\star(x)\right| \\ &\le d^3\, \varepsilon \max_i \|J^i\|_{L^\infty} \sum_{T\in\mathcal{T}_H} \|\nabla v\|_{L^2(T)} \|\nabla^2 u^\star\|_{L^2(T)} \\ &\le C\varepsilon \|v\|_E\, \|\nabla^2 u^\star\|_{L^2(\Omega)}. \end{aligned} \tag{5.17}$$

We are now left with bounding the first term of the right-hand side of (5.16), which reads

$$\begin{aligned}&\varepsilon \sum_{T\in\mathcal{T}_H}\sum_{i=1}^d \int_{\partial T} v(x)\widetilde{B}_i^\varepsilon(x)\cdot n \\ &= \varepsilon \sum_{e\in\mathcal{E}_H}\sum_{i=1}^d \int_e [[v(x)]]\, \widetilde{B}_i^\varepsilon(x)\cdot n\end{aligned}$$

$$
\begin{aligned}
&= \varepsilon \sum_{e\in\mathcal{E}_H} \sum_{i,\alpha,\beta=1}^{d} \int_e [[v(x)]]\, n_\alpha \frac{\partial}{\partial x_\beta}\Big([J^i]_{\beta\alpha}\Big(\frac{x}{\varepsilon}\Big) \partial_i u^\star(x)\Big) \\
&= \sum_{e\in\mathcal{E}_H} \sum_{i,\alpha,\beta=1}^{d} \int_e [[v(x)]]\, n_\alpha \frac{\partial [J^i]_{\beta\alpha}}{\partial x_\beta}\Big(\frac{x}{\varepsilon}\Big) \partial_i u^\star(x) \\
&\quad + \varepsilon \sum_{e\in\mathcal{E}_H} \sum_{i,\alpha,\beta=1}^{d} \int_e [[v(x)]]\, n_\alpha [J^i]_{\beta\alpha}\Big(\frac{x}{\varepsilon}\Big) \partial_{i\beta} u^\star(x).
\end{aligned} \tag{5.18}
$$

Our final task is to successively bound the two terms of the right-hand side of (5.18).

We begin with the first term. Considering an edge e, we recast the contribution of that edge to the first term of the right-hand side of (5.18) as follows, using the skew symmetry of J:

$$
\begin{aligned}
&\sum_{i,\alpha,\beta=1}^{d} \int_e [[v(x)]]\, n_\alpha \frac{\partial [J^i]_{\beta\alpha}}{\partial x_\beta}\Big(\frac{x}{\varepsilon}\Big) \partial_i u^\star(x) \\
&= \sum_{\substack{i,\alpha,\beta=1\\ \beta>\alpha}}^{d} \int_e [[v(x)]]\Big(n_\alpha \frac{\partial [J^i]_{\beta\alpha}}{\partial x_\beta} - n_\beta \frac{\partial [J^i]_{\beta\alpha}}{\partial x_\alpha}\Big)\Big(\frac{x}{\varepsilon}\Big) \partial_i u^\star(x) \\
&= \sum_{\substack{i,\alpha,\beta=1\\ \beta>\alpha}}^{d} \int_e [[v(x)]] (\tau_{\alpha\beta}\cdot \nabla [J^i]_{\beta\alpha})\Big(\frac{x}{\varepsilon}\Big) \partial_i u^\star(x),
\end{aligned} \tag{5.19}
$$

where $\tau_{\alpha\beta} \in \mathbb{R}^d$ is the vector with α-th component set to $-n_\beta$, β-th component set to n_α, and all other components set to 0. Obviously, $\tau_{\alpha\beta}$ is parallel to e. We can thus use Lemma 4.6, and infer from (5.19) that

$$
\begin{aligned}
&\Big| \sum_{i,\alpha,\beta=1}^{d} \int_e [[v(x)]]\, n_\alpha \frac{\partial [J^i]_{\beta\alpha}}{\partial x_\beta}\Big(\frac{x}{\varepsilon}\Big) \partial_i u^\star(x)\Big| \\
&\leq C \sqrt{\frac{\varepsilon}{H}} \sum_{i,\alpha,\beta=1}^{d} \| [J^i]_{\beta\alpha} \|_{C^1(\mathbb{R}^d)} \sum_{T\in T_e} |v|_{H^1(T)} (\|\partial_i u^\star\|_{L^2(T)} + H|\partial_i u^\star|_{H^1(T)}).
\end{aligned}
$$

Using the regularity (5.14) of J^i, we deduce that the first term of the right-hand side of (5.18) satisfies

$$
\begin{aligned}
&\Big| \sum_{e\in\mathcal{E}_H} \sum_{i,\alpha,\beta=1}^{d} \int_e [[v(x)]]\, n_\alpha \frac{\partial [J^i]_{\beta\alpha}}{\partial x_\beta}\Big(\frac{x}{\varepsilon}\Big) \partial_i u^\star(x)\Big| \\
&\leq C \sqrt{\frac{\varepsilon}{H}} \Big[\sum_{e\in\mathcal{E}_H} \sum_{T\in T_e} \|\nabla v\|^2_{L^2(T)} \Big]^{\frac{1}{2}} \\
&\quad \times \Big[\sum_{e\in\mathcal{E}_H} \sum_{T\in T_e} \|\nabla u^\star\|^2_{L^2(T)} + H^2 \|\nabla^2 u^\star\|^2_{L^2(T)} \Big]^{\frac{1}{2}} \\
&\leq C \sqrt{\frac{\varepsilon}{H}}\, \|v\|_E\, \|\nabla u^\star\|_{L^2(\Omega)} + C\sqrt{\varepsilon H}\, \|v\|_E\, \|\nabla^2 u^\star\|_{L^2(\Omega)}.
\end{aligned} \tag{5.20}
$$

We next turn to the second term of the right-hand side of (5.18), which satisfies

$$\begin{aligned}
&\Big|\varepsilon \sum_{e\in\mathcal{E}_H} \sum_{i,\alpha,\beta=1}^{d} \int_e [[v(x)]]\, n_\alpha [J^i]_{\beta\alpha}\Big(\frac{x}{\varepsilon}\Big) \partial_{i\beta} u^\star(x)\Big| \\
&\leq d^3\, \varepsilon \Big(\max_i \|J^i\|_{C^0(\mathbb{R}^d)}\Big) \|\nabla^2 u^\star\|_{C^0(\overline{\Omega})} \sum_{e\in\mathcal{E}_H} \|\,[[v]]\,\|_{L^1(e)} \\
&\leq C\varepsilon\, \|\nabla^2 u^\star\|_{C^0(\overline{\Omega})} \Big[\sum_{e\in\mathcal{E}_H} \|\,[[v]]\,\|^2_{L^2(e)}\Big]^{\frac{1}{2}} \Big[\sum_{e\in\mathcal{E}_H} \|1\|^2_{L^2(e)}\Big]^{\frac{1}{2}} \\
&\leq C\varepsilon\, \|\nabla^2 u^\star\|_{C^0(\overline{\Omega})} \Big[\sum_{e\in\mathcal{E}_H} H \sum_{T\in T_e} \|\nabla v\|^2_{L^2(T)}\Big]^{\frac{1}{2}} \\
&\quad \times \Big[\sum_{\substack{e\in\mathcal{E}_H \\ \text{choose one } T\,\in\, T_e}} H^{-1} \|1\|^2_{L^2(T)}\Big]^{\frac{1}{2}} \\
&\leq C\varepsilon\, \|\nabla^2 u^\star\|_{C^0(\overline{\Omega})}\, \|v\|_E\, |\Omega|^{\frac{1}{2}},
\end{aligned} \tag{5.21}$$

where we have used (4.7) of Corollary 4.1 and (4.4) of Lemma 4.4.

Collecting the estimates (5.10), (5.12), (5.16)–(5.18) and (5.20)–(5.21), we bound the third term of (5.7):

$$\begin{aligned}
&\Big|\sum_{T\in\mathcal{T}_H} \int_T (\nabla v)^{\mathrm{T}} \Big(A_{\mathrm{per}}\Big(\frac{x}{\varepsilon}\Big)\nabla u^{\varepsilon,1} - A^\star_{\mathrm{per}} \nabla u^\star\Big)\Big| \\
&\leq C\sqrt{\varepsilon\, H} \|v\|_E \|\nabla^2 u^\star\|_{L^2(\Omega)} + C\sqrt{\frac{\varepsilon}{H}}\, \|v\|_E\, \|\nabla u^\star\|_{L^2(\Omega)} \\
&\quad + C\varepsilon \|\nabla^2 u^\star\|_{C^0(\overline{\Omega})}\, \|v\|_E,
\end{aligned} \tag{5.22}$$

where C is independent of ε, H, v and $u^\star$ (but depends on Ω).

Step 4 Conclusion Inserting (5.8)–(5.9) and (5.22) in (5.7), we obtain

$$\begin{aligned}
&\Big|\sum_{T\in\mathcal{T}_H} \int_{\partial T} v\Big(A_{\mathrm{per}}\Big(\frac{x}{\varepsilon}\Big)\nabla u^\varepsilon\Big)\cdot n\Big| \\
&\leq C\sqrt{\varepsilon} \|v\|_E (\|\nabla u^\star\|_{W^{1,\infty}(\Omega)} + \sqrt{\varepsilon} \|\nabla^2 u^\star\|_{C^0(\overline{\Omega})}) \\
&\quad + C(H + \sqrt{\varepsilon\, H})\, \|v\|_E \|\nabla^2 u^\star\|_{L^2(\Omega)} + C\sqrt{\frac{\varepsilon}{H}}\, \|v\|_E\, \|\nabla u^\star\|_{L^2(\Omega)},
\end{aligned} \tag{5.23}$$

which yields the desired bound (5.1). This concludes the proof of Lemma 4.7.

6 Numerical Illustrations

For our numerical tests, we consider (1.1) on the domain $\Omega = (0,1)^2$, with the right-hand side $f(x,y) = \sin(x)\sin(y)$.

First test-case We first choose the highly oscillatory matrix

$$A_\varepsilon(x,y) = a_\varepsilon(x,y)\,\mathrm{Id}_2, \quad a_\varepsilon(x,y) = 1 + 100\cos^2(150x)\sin^2(150y) \tag{6.1}$$

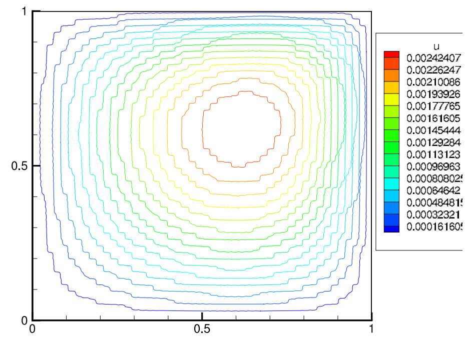

Figure 1 Reference solution for (1.1) with the choice (6.1).

in (1.1). This matrix coefficient is periodic, with period $\varepsilon = \frac{\pi}{150} \approx 0.02$. The reference solution u^ε (computed on a fine mesh 1024×1024 of Ω) is shown in Figure 1.

We show in Figure 2 the relative errors between the fine scale solution u^ε and its approximation provided by various MsFEM type approaches, as a function of the coarse mesh size H.

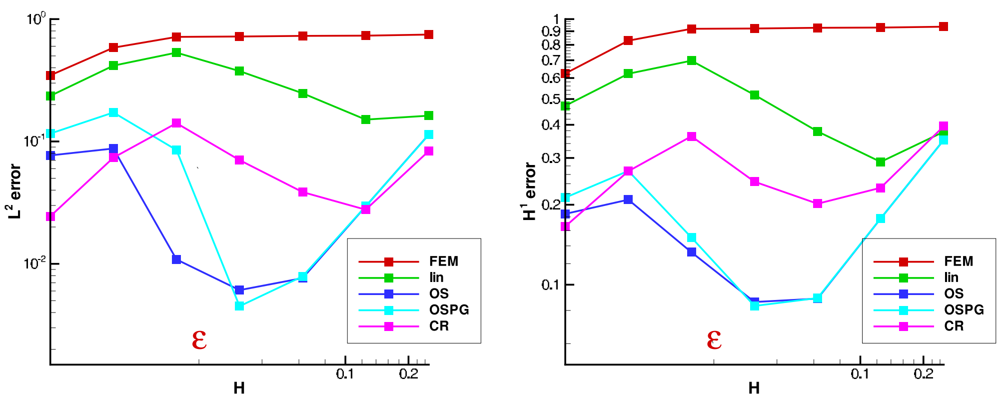

Figure 2 Test-case (6.1): relative errors (in L^2 (left) and H^1-broken (right) norms) with various approaches: FEM — the standard Q1 finite elements, lin — MsFEM with linear boundary conditions, OS — MsFEM with oversampling, OSPG — Petrov-Galerkin MsFEM with oversampling, CR — the MsFEM Crouzeix-Raviart approach we propose.

Our approach is systematically more accurate than the standard (meaning, without the oversampling technique) MsFEM approach. In addition, we see that, for large H, our approach yields an error smaller than or comparable to the best other methods. Likewise, when H is small (but not sufficiently small for the standard FEM approach to be accurate), our approach is also more accurate than the other approaches. For intermediate values of H, our approach is however less accurate than approaches using oversampling (for which we used an oversampling ratio equal to 2). Note that this will no longer be the case for the problem on a perforated domain considered in [25]. Note also that our approach is slightly less expensive than the approaches using oversampling (in terms of computations of the highly oscillatory basis functions) and, much more importantly, has no adjustable parameter.

A comparison with the MsFEM-O variant (described in Remark 3.2) has also been performed but is not included in the figures below. On the particular case considered in this article, we have observed that this approach seems to perform very well. However, it is not clear, in general, whether this approach yields systematically more accurate results than the other MsFEM variants. A more comprehensive assessment of this variant will be performed for the case of perforated domains in [25].

Higher contrast We now consider the cases

$$A_\varepsilon(x,y) = a_\varepsilon(x,y)\,\mathrm{Id}_2, \quad a_\varepsilon(x,y) = 1 + 10^3 \cos^2(150x)\sin^2(150y) \tag{6.2}$$

and

$$A_\varepsilon(x,y) = a_\varepsilon(x,y)\,\mathrm{Id}_2, \quad a_\varepsilon(x,y) = 1 + 10^4 \cos^2(150x)\sin^2(150y) \tag{6.3}$$

in (1.1). In comparison with (6.1), we have increased the contrast by a factor 10 or 100, respectively. Results are shown in Figure 3, top and bottom rows respectively.

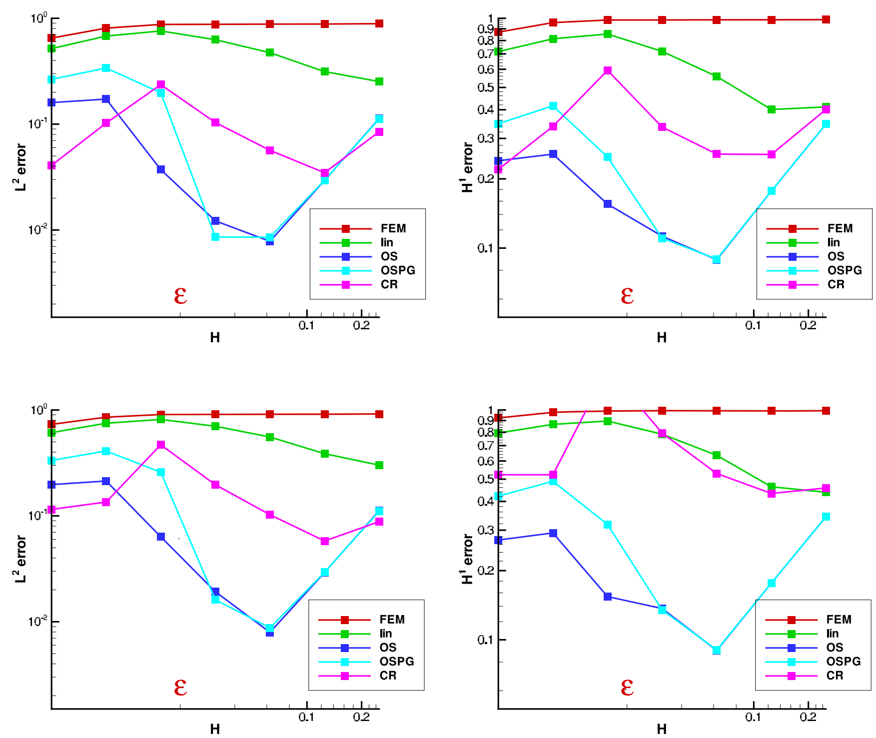

Figure 3 Test-cases (6.2) (top row) and (6.3) (bottom row) for higher contrasts: relative errors (in L^2 (left) and H^1-broken (right) norms) with various approaches: FEM — the standard Q1 finite elements, lin — MsFEM with linear boundary conditions, OS — MsFEM with oversampling, OSPG — Petrov-Galerkin MsFEM with oversampling, CR — the MsFEM Crouzeix-Raviart approach we propose.

We see that the relative quality of the different approaches is not sensitive to the contrast (at least when the latter does not exceed 10^3). Of course, each method provides an approximation of u^ε that is less accurate than in the case (6.1). However, all methods seem to equally suffer from a higher contrast.

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