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Periodic Homogenization for Inner Boundary Conditions with Equi-valued Surfaces: the Unfolding Approach

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(*In honor of the scientific heritage of Jacques-Louis Lions*)

Abstract Making use of the periodic unfolding method, the authors give an elementary proof for the periodic homogenization of the elastic torsion problem of an infinite 3-dimensional rod with a multiply-connected cross section as well as for the general electro-conductivity problem in the presence of many perfect conductors (arising in resistivity well-logging). Both problems fall into the general setting of equi-valued surfaces with corresponding assigned total fluxes. The unfolding method also gives a general corrector result for these problems.

Keywords Periodic homogenization, Elastic torsion, Equi-valued surfaces, Resistivity well-logging, Periodic unfolding method
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1 Introduction

The periodic unfolding method was introduced in [4] (see also [5]). It gave an elementary proof for the classical periodic homogenization problem, including the case with several micro-scales (a detailed account and proofs can be found in [5]).

In this paper, we show how it can be applied to the periodic homogenization of the general problem of equi-valued surfaces with corresponding assigned total fluxes. Two examples of this type of problems are the elastic torsion problem of an infinite 3-dimensional rod with a multiply-connected cross section (where the equations are set in a 2-dimensional domain) and, in any dimension, the electro-conductivity problem in the presence of many isolated perfect conductors.

In the linear elastic torsion problem (see [13, 15] for the setting of the problem), the material is an infinite cylindrical bar with a 2-dimensional cross-section Ω^* obtained from a bounded open set Ω perforated by a finite number of regular closed subsets (which have a nonempty

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interior) $S^1, S^2, \cdots$ (see Figure 1). The stress function of the elastic material is shown to be the solution to the following problem:

$$
\begin{aligned}
&\varphi \in H_0^1(\Omega) \quad \text{with } f_{|S^j} \text{ (an unknown constant) for each } j,\\
&-\Delta\varphi = 1 \quad \text{in } \Omega^*,\\
&\int_{\partial S^j} \frac{\partial \varphi}{\partial n} \mathrm{d}\sigma(x) = |S^j| \quad \text{(the measure of } S^j\text{) for each } j.
\end{aligned}
\tag{1.1}
$$

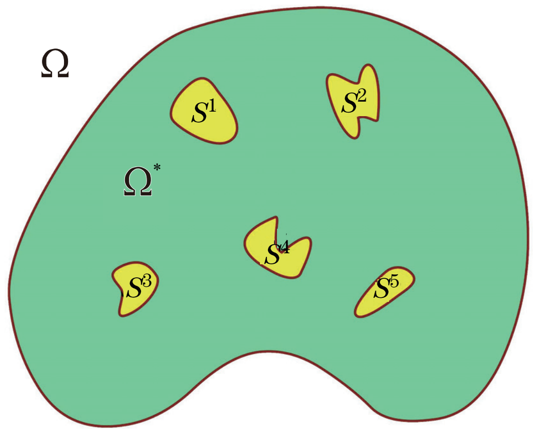


Figure 1 A 2-dimensional cross-section Ω^* obtained from a bounded open set Ω perforated by a finite number of regular closed subsets (which have a nonempty interior) $S^1, S^2, \cdots$.

The electric conductivity problem arising in resistivity well-logging is set in any dimension with the same type of geometry (Ω, Ω^* and S^j's). The conductivity tensor A can vary with the position in Ω^*, the right-hand side is an L^2 function f defined on Ω^*, and the total fluxes on the ∂S^j are given numbers g^j.

$$
\begin{aligned}
&\varphi \in H_0^1(\Omega) \quad \text{with } f_{|S^j} \text{ (an unknown constant) for each } j,\\
&-\operatorname{div}(A(x)\nabla\varphi(x)) = f \quad \text{in } \Omega^*,\\
&\int_{\partial S^j} \frac{\partial \varphi}{\partial \nu_A} \mathrm{d}\sigma(x) = g^j \quad \text{for each } j.
\end{aligned}
\tag{1.2}
$$

Here we refer to [14–15] written by Li et al. on the subject. They also include an exposé of the torsion problem.

Here, we consider the periodic homogenization for these problems. We refer to [7] for the first proof of the elastic torsion problem (via extension operators and oscillating test functions), where regularity assumptions are made for the boundary of the inclusions. In [3], this question is addressed, but still with some geometric conditions and by the same use of oscillating test functions.

One advantage of the unfolding method is that it requires no regularity for the boundary of the inclusions whatsoever. Actually, there is no need to introduce surface integrals, except if one wants to see the "usual" strong formulation, valid for Lipschitz boundaries. Another advantage of the method is that an immediate consequence is a corrector result which is completely general (without the need for extra regularity).

The plan is as follows. In Section 1, we introduce the notations, and set the approximate problem as one which encompasses both the aforementioned problems. Section 2 gives a brief summary of the results of the periodic unfolding method. In Section 3, we establish the convergence to the unfolded problem. In Section 4, we obtain the homogenized limit. Section 5 is devoted to the convergence of the energy and the construction of correctors. In the last section, we consider variants of the problem, and state the corresponding results.

General notations (1) In this work, ε indicates the generic element of a bounded subset of $\mathbb{R}^*_+$ in the closure of which 0 lies. Convergence of ε to 0 is understood in this set. Also, c and C denote generic constants, which do not depend upon ε.

(2) As usual, 1_D denotes the characteristic function of the set D.

(3) For a measurable set D in $\mathbb{R}^n$, $|D|$ denotes its Lebesgue measure.

(4) For simplicity, the notation $L^p(\mathcal{O})$ will be used for both scalar and vector-valued functions defined on the set $\mathcal{O}$, since no ambiguity will arise.

2 Setting of the Problem

We use the general framework of [5] and the notations therein.

Let $\mathbf{b} = (b_1, \cdots, b_n)$ be a basis of $\mathbb{R}^n$. We denote by

$$\mathcal{G} = \Big\{\xi \in \mathbb{R}^n \,\Big|\, \xi = \sum_{i=1}^n k_i b_i,\ (k_1, \cdots, k_n) \in \mathbb{Z}^n\Big\} \tag{2.1}$$

the group of macroscopic periods for the problem.

Let Y be the open parallelotope generated by the basis $\mathbf{b}$, i.e.,

$$\Big\{y \in \mathbb{R}^n \,\Big|\, y = \sum_{i=1}^n y_i b_i,\ (y_1, \cdots, y_n) \in (0,1)^n\Big\}. \tag{2.2}$$

More generally, Y can be any bounded connected subset of $\mathbb{R}^n$ with Lipschitz boundary, having the paving property with respect to the group $\mathcal{G}$.

For $z \in \mathbb{R}^n$, $[z]_Y$ denotes the unique (up to a set of measure zero) integer combination $\sum\limits_{j=1}^n k_j b_j$ of the periods, such that $z - [z]_Y$ belongs to Y. Now set (see Figure 2)

$$\{z\}_Y = z - [z]_Y \in Y \quad \text{a.e. for } z \in \mathbb{R}^n.$$

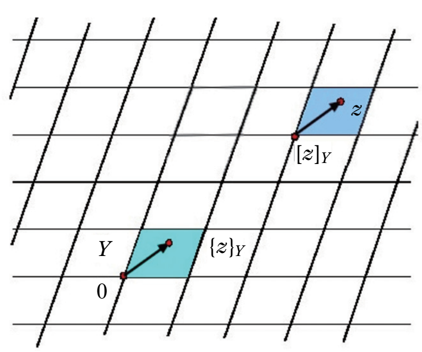


Figure 2 Definition of $[z]_Y$ and $\{z\}_Y$.

Let S be a given compact subset in Y, which is the finite disjoint union of S^j for $j = 1, \cdots, J$ (with the same property). The only condition on the S^j's is that they are pair-wise separated with the strictly positive measure. For the two examples, we consider that the sets S^j are naturally assumed to be connected (although this is not necessary for the treatment given here). The set $Y \setminus S$ is denoted by Y^* (see Figure 3).

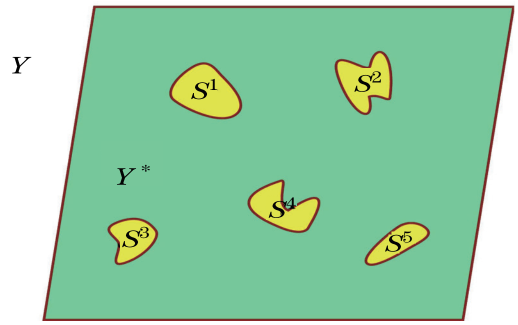


Figure 3 The sets Y and Y^*.

Let now Ω be an open bounded subset of $\mathbb{R}^n$. We define the "holes" in Ω as follows.

Definition 2.1

$$
\begin{aligned}
&S_\varepsilon \doteq \Big\{x \in \Omega, \Big\{\frac{x}{\varepsilon}\Big\}_Y \in S\Big\},\\
&S_\varepsilon^j \doteq \Big\{x \in \Omega, \Big\{\frac{x}{\varepsilon}\Big\}_Y \in S^j\Big\},\\
&\Omega_\varepsilon \doteq \Omega \setminus S_\varepsilon.
\end{aligned}
\tag{2.3}
$$

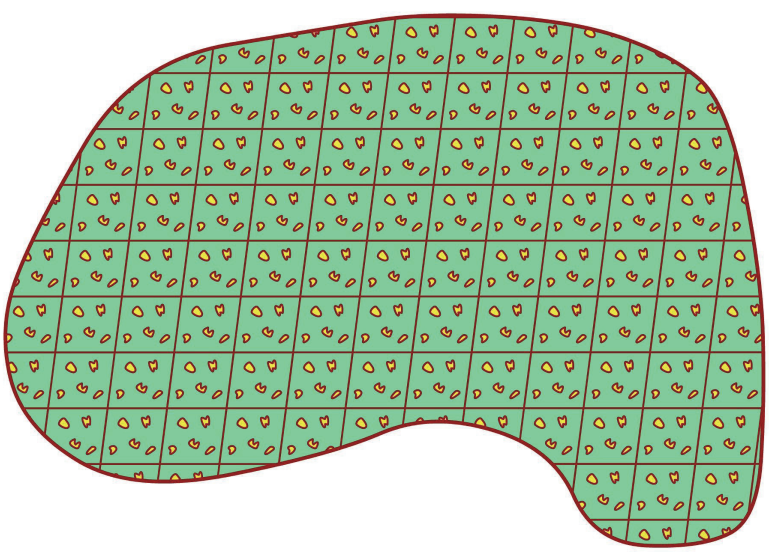


Figure 4 The sets Ω_ε (in green) and S_ε (in yellow).

Remark 2.1 It is well-known that the characteristic function of the sets S_ε^j converges weakly-$*$ to $\frac{|S^j|}{|Y|}$ in $L^\infty(\Omega)$. This is a simple consequence of the properties of the unfolding operator given in the next section.

In this paper, we consider a boundary value problem, which generalizes both cases and does not require consideration of surface integrals. It applies as long as the "inclusions" S_ε^j have the strictly positive measure (so that requiring the restriction of an H_0^1 function to each of them to an arbitrary constant almost everywhere makes sense).

We make no regularity assumption regarding the sets S^j. We only make the natural assumption that they are well separated from each other and from ∂Y in Y (if some are not well separated, then they should be merged into a single one).

Whenever needed, the functions in $H_0^1(\Omega)$ will be extended by 0 in the whole of $\mathbb{R}^n$ (where they belong to $H^1(\mathbb{R}^n)$). Similarly, the functions of $L^2(\Omega)$ will be extended by 0 to the whole of $\mathbb{R}^n$.

We introduce the following two families of subspaces, where $\mathcal{G}_\varepsilon$ denotes the elements $\xi \in \mathcal{G}$, such that the corresponding cell $\varepsilon\xi + \varepsilon Y$ intersects Ω.

Definition 2.2

$$\begin{aligned} W_0^\varepsilon \doteq \{ & v \in H_0^1(\Omega);\ \forall \xi \in \mathcal{G}_\varepsilon \textit{ and } j \in \{1, \cdots, J\}\ v_{|\varepsilon\xi+\varepsilon S^j} \textit{ is a constant function,} \\ & \textit{the value of which depends on } (\xi, j)\}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} L_\varepsilon \doteq \{ & w \in L^2(\Omega);\ \forall \xi \in \mathcal{G}_\varepsilon \textit{ and } j \in \{1, \cdots, J\}\ w_{|\varepsilon\xi+\varepsilon S^j} \textit{ is a constant function,} \\ & \textit{the value of which depends on } (\xi, j)\}. \end{aligned} \tag{2.5}$$

Note that a condition, such as $v_{|\varepsilon\xi+\varepsilon S^j}$ being a constant function, is taken in the sense of almost everywhere, which makes sense because each $\varepsilon\xi + \varepsilon S^j$ is of positive measure.

It also follows that W_0^ε is a closed subspace of $H_0^1(\Omega)$. On the other hand, clearly, L_ε is a finite dimensional subspace of $L^2(\Omega)$. Note that L_ε is the image of $L^2(\Omega)$ under the local average map $\mathcal{M}_\varepsilon$ (defined below).

Concerning the conductivity matrix field A^ε, it is assumed to belong to $M(\alpha, \beta, \Omega_\varepsilon)$ which is traditionally defined as follows.

Definition 2.3 *Let $\alpha, \beta \in \mathbb{R}$, such that $0 < \alpha < \beta$. $M(\alpha, \beta, \mathcal{O})$ denotes the set of the $n \times n$ matrices $B = B(x)$, $B = (b_{ij})_{1 \le i,j \le n} \in (L^\infty(\mathcal{O}))^{n\times n}$, such that*

$$(B(x)\lambda, \lambda) \ge \alpha|\lambda|^2, \quad |B(x)\lambda| \le \beta|\lambda| \quad \textit{for any } \lambda \in \mathbb{R}^n \textit{ and a.e. on } \mathcal{O}.$$

Let f_ε be given in $L^2(\Omega_\varepsilon)$, and g_ε^j in L^ε for $j = 1, \cdots, J$.

The problem we consider is given in the variational form as

$$(\mathrm{P}_\varepsilon) \begin{cases} \text{Find } u_\varepsilon \text{ in } W_0^\varepsilon, \text{ such that } \forall w \in W_0^\varepsilon, \\ \displaystyle\int_{\Omega_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla w \mathrm{d}x = \int_{\Omega_\varepsilon} f_\varepsilon\, w \mathrm{d}x + \varepsilon^n \sum_{\substack{\xi \in \mathcal{G}_\varepsilon \\ j=1,\cdots,J}} g_\varepsilon^j{}_{|_{\varepsilon\xi+\varepsilon Y}} w_{|_{\varepsilon\xi+\varepsilon S^j}}. \end{cases} \tag{2.6}$$

Using the obvious formula

$$\varepsilon^n \sum_{\xi\in\mathcal{G}} g^j_\varepsilon|_{\varepsilon\xi+\varepsilon Y} w|_{\varepsilon\xi+\varepsilon S^j} = \frac{1}{|S^j|}\int_{S^j_\varepsilon} g^j_\varepsilon(x)\, w(x)\mathrm{d}x, \tag{2.7}$$

this problem can be written as

$$(\mathrm{P}_\varepsilon)\begin{cases} \text{Find } u_\varepsilon \text{ in } W^\varepsilon_0, \text{ such that } \forall w\in W^\varepsilon_0, \\ \displaystyle\int_{\Omega_\varepsilon} A^\varepsilon \nabla u_\varepsilon\, \nabla w \mathrm{d}x = \int_\Omega F_\varepsilon\, w\mathrm{d}x, \end{cases} \tag{2.8}$$

where

$$F_\varepsilon \doteq f_\varepsilon\, 1_{\Omega_\varepsilon} + \sum_{\substack{\xi\in\mathcal{G}_\varepsilon \\ j=1,\cdots,J}} \frac{1}{|S^j|} g^j_\varepsilon(x) 1_{S^j_\varepsilon}.$$

This problem has a unique solution by the Lax-Milgram theorem applied in the space W^ε_0 because of Poincaré's inequality in $H^1_0(\Omega)$.

The "strong" formulation of problem (P_ε), assuming at least Lipschitz regularity for the boundaries involved, is

$$\begin{cases} \text{Find } u_\varepsilon \in W^\varepsilon_0, \text{ such that } -\operatorname{div}(A^\varepsilon\nabla u_\varepsilon) = f_\varepsilon \text{ in } \Omega_\varepsilon, \\ \forall \xi\in\mathcal{G}_\varepsilon,\ \forall j=1,\cdots,J,\ \langle A^\varepsilon \nabla u_\varepsilon\cdot n, 1\rangle_{H^{-\frac12},\, H^{\frac12}(\varepsilon\xi+\varepsilon\partial S^j)} = \varepsilon^n g^j_\varepsilon|_{\varepsilon\xi+\varepsilon Y}, \end{cases} \tag{2.9}$$

where $n(x)$ is the outward unit normal to $\varepsilon\xi+\varepsilon\partial S^j$. Under the regularity assumption above, it is classical that the duality pairing makes sense, because, since f_ε belongs to $L^2(\Omega_\varepsilon)$, $A^\varepsilon\nabla u_\varepsilon\cdot n$ is an element of $H^{-\frac12}(\varepsilon\xi+\varepsilon\partial S^j)$. The pairing is often (somewhat incorrectly) written as

$$\int_{\varepsilon\xi+\varepsilon\partial S^j} A^\varepsilon\nabla u_\varepsilon\cdot n(x)\mathrm{d}\sigma(x).$$

Making use of the Lax-Milgram theorem, one gets the following estimate.

Proposition 2.1 *There is a constant C depending only upon α and the Poincaré constant for $H^1_0(\Omega)$ (but not upon ε), such that, for every ε,*

$$|u_\varepsilon|_{H^1_0(\Omega)} \le C|F_\varepsilon|_{L^2(\Omega_\varepsilon)} = C\Big(|f_\varepsilon|^2_{L^2(\Omega_\varepsilon)} + \sum_{\substack{\xi\in\mathcal{G}_\varepsilon \\ j=1,\cdots,J}} \varepsilon^2 |g^j_\varepsilon(x)_{|S^j_\varepsilon}|^2\Big)^{\frac12}. \tag{2.10}$$

Consequently, assume that the right-hand side of (2.10) is bounded, so is the sequence $\{u_\varepsilon\}$ in $H^1_0(\Omega)$.

The homogenization problem is to investigate the weak convergence of this sequence and the possible problem satisfied by its limit under suitable assumptions on the data.

Remark 2.2 The sequence $\{u_\varepsilon\}$ in $H^1_0(\Omega)$ does not usually converge strongly in $H^1_0(\Omega)$, because, when it does, its limit is the zero function. More generally, if v_ε is in W^ε_0 and converges strongly to v_0 in $H^1_0(\Omega)$, then $v_0=0$. The proof is elementary. Going to the limit for the product $0\equiv 1_{S_\varepsilon}\nabla v_\varepsilon$, which, as a consequence of the assumptions, converges weakly to $\frac{|S|}{|Y|}\nabla v_0$, implies that $\nabla v_0\equiv 0$, and hence the result is obtained.

3 A Brief Summary of the Unfolding Method in Fixed Domains

We recall the following notations used in [5]:

$$\widehat{\Omega}_\varepsilon = \text{interior}\Big\{ \bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \overline{Y}) \Big\}, \quad \Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon, \tag{3.1}$$

where

$$\Xi_\varepsilon = \{\xi \in \mathcal{G}, \varepsilon(\xi + Y) \subset \Omega\} \tag{3.2}$$

(see Figure 5). The set $\widehat{\Omega}_\varepsilon$ is the interior of the largest union of $\varepsilon(\xi + \overline{Y})$ cells included in Ω. Here, the set Ξ_ε is slightly smaller than $\mathcal{G}_\varepsilon$ as defined previously.

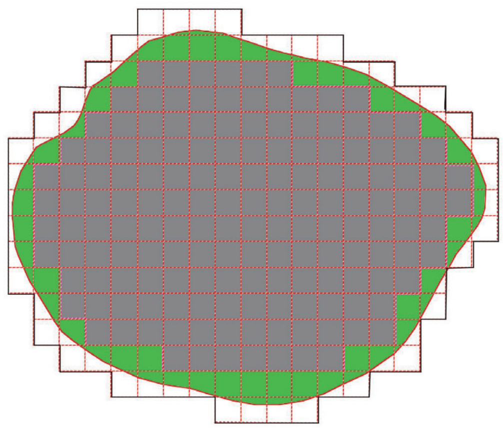


Figure 5 The sets $\widehat{\Omega}_\varepsilon$ (in grey) and Λ_ε (in green).

Definition 3.1 *For ϕ Lebesgue-measurable on $\widehat{\Omega}_\varepsilon$, the unfolding operator $\mathcal{T}_\varepsilon$ is defined as follows:*

$$\mathcal{T}_\varepsilon(\phi)(x,y) = \begin{cases} \phi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big]_Y + \varepsilon y\Big), & \text{a.e. for } (x,y) \in \widehat{\Omega}_\varepsilon \times Y, \\ 0, & \text{a.e. for } (x,y) \in \Lambda_\varepsilon \times Y. \end{cases} \tag{3.3}$$

The properties of the unfolding operator are summarized here.

Theorem 3.1 *Let p belong to $[1, +\infty)$.*
(i) *$\mathcal{T}_\varepsilon$ is linear continuous from $L^p(\Omega)$ to $L^p(\Omega \times Y)$. Its norm is bounded by $|Y|^{\frac{1}{p}}$.*
(ii) *For every w in $L^1(\Omega)$,*

$$\int_\Omega w(x)\mathrm{d}x = \frac{1}{|Y|}\int_{\Omega\times Y} \mathcal{T}_\varepsilon(w)(x,y)\mathrm{d}x\mathrm{d}y + \int_{\Lambda_\varepsilon} w(x)\mathrm{d}x.$$

(iii) *Let $\{w_\varepsilon\}$ be a sequence in $L^p(\Omega)$, such that $w_\varepsilon \to w$ strongly in $L^p(\Omega)$. Then*

$$\mathcal{T}_\varepsilon(w_\varepsilon) \to w \quad \text{strongly in } L^p(\Omega \times Y).$$

(iv) *Let* $\{w_\varepsilon\}$ *be bounded in* $L^p(\Omega)$, *and suppose that the corresponding* $\mathcal{T}_\varepsilon(w_\varepsilon)$ (*which is bounded in* $L^p(\Omega\times Y)$) *converges weakly to* $\widehat{w}$ *in* $L^p(\Omega\times Y)$. *Then*

$$w_\varepsilon \rightharpoonup \mathcal{M}_Y(\widehat{w}) = \frac{1}{|Y|}\int_Y \widehat{w}(\,\cdot\,,y)\mathrm{d}y \quad \text{weakly in } L^p(\Omega).$$

Here, the operator $\mathcal{M}_Y$ is the average over Y.

Definition 3.2 *The operator* $\mathcal{M}_\varepsilon \doteq \mathcal{M}_Y \circ \mathcal{T}_\varepsilon$ *is the local average operator. It assigns to a function, which is integrable on* Ω *its local average* (*associated with the* ε*-cells* $\varepsilon\xi+\varepsilon Y$).

Theorem 3.2 *Let* $\{w_\varepsilon\}$ *be in* $W^{1,p}(\Omega)$ *with* $p\in(1,+\infty)$, *and assume that* $\{w_\varepsilon\}$ *is a bounded sequence in* $W^{1,p}(\Omega)$. *Then, there exist a subsequence* (*still denoted by* $\{\varepsilon\}$) *and functions* w *in* $W^{1,p}(\Omega)$ *and* $\widehat{w}$ *in* $L^p(\Omega;W^{1,p}_{\mathrm{per}}(Y))$ *with* $\mathcal{M}_Y(\widehat{w})\equiv 0$, *such that*

$$\begin{aligned} \mathcal{T}_\varepsilon(w_\varepsilon) &\rightharpoonup w \quad\ \ \text{weakly in } L^p(\Omega;W^{1,p}(Y)),\\ \mathcal{T}_\varepsilon(\nabla w_\varepsilon) &\rightharpoonup \nabla w+\nabla_y\widehat{w} \quad \text{weakly in } L^p(\Omega\times Y). \end{aligned} \tag{3.4}$$

Furthermore, the sequence $\frac{1}{\varepsilon}(\mathcal{T}_\varepsilon(w_\varepsilon)-\mathcal{M}_\varepsilon(w_\varepsilon))$ *converges weakly in* $L^p(\Omega;W^{1,p}(Y))$ *to* $y_M\cdot\nabla w$ $+\widehat{w}$, *where* $y_M \doteq y-\mathcal{M}_Y(y)$.

Here, $W^{1,p}_{\mathrm{per}}(Y)$ denotes the space of the functions in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, which are $\mathcal{G}$-periodic. It is a closed subspace of $W^{1,p}(Y)$, and is endowed with the corresponding norm.

We end this section by recalling the notion of the averaging operator $\mathcal{U}_\varepsilon$. This operator is the adjoint of $\mathcal{T}_\varepsilon$ and maps $L^p(\Omega\times Y)$ into the space $L^p(\Omega)$.

Definition 3.3 *For* p *in* $[1,+\infty]$, *the averaging operator* $\mathcal{U}_\varepsilon : L^p(\Omega\times Y)\mapsto L^p(\Omega)$ *is defined as follows:*

$$\mathcal{U}_\varepsilon(\Phi)(x) = \begin{cases} \dfrac{1}{|Y|}\displaystyle\int_Y \Phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y+\varepsilon z,\Big\{\frac{x}{\varepsilon}\Big\}_Y\Big)\mathrm{d}z, & \text{a.e. for } x\in\widehat{\Omega}_\varepsilon,\\ 0, & \text{a.e. for } x\in\Lambda_\varepsilon. \end{cases}$$

The main properties of $\mathcal{U}_\varepsilon$ are listed in the next proposition.

Proposition 3.1 (Properties of $\mathcal{U}_\varepsilon$) *Suppose that* p *is in* $[1,+\infty)$.
(i) *The averaging operator is linear and continuous from* $L^p(\Omega\times Y)$ *to* $L^p(\Omega)$, *and*

$$\|\mathcal{U}_\varepsilon(\Phi)\|_{L^p(\Omega)} \le |\ Y\ |^{-\frac{1}{p}}\ \|\Phi\|_{L^p(\Omega\times Y)}.$$

(ii) *If* φ *is independent of* y, *and belongs to* $L^p(\Omega)$, *then*

$$\mathcal{U}_\varepsilon(\varphi)\to\varphi \quad \text{strongly in } L^p(\Omega).$$

(iii) *Let* $\{\Phi_\varepsilon\}$ *be a bounded sequence in* $L^p(\Omega\times Y)$, *such that* $\Phi_\varepsilon\rightharpoonup\Phi$ *weakly in* $L^p(\Omega\times Y)$. *Then*

$$\mathcal{U}_\varepsilon(\Phi_\varepsilon)\rightharpoonup\mathcal{M}_Y(\Phi) = \frac{1}{|Y|}\int_Y \Phi(\,\cdot\,,y)\mathrm{d}y \quad \text{weakly in } L^p(\Omega).$$

In particular, for every $\Phi \in L^p(\Omega \times Y)$,

$$\mathcal{U}_\varepsilon(\Phi) \rightharpoonup \mathcal{M}_Y(\Phi) \quad \textit{weakly in } L^p(\Omega).$$

(iv) *Suppose that* $\{w_\varepsilon\}$ *is a sequence in* $L^p(\Omega)$. *Then, the following assertions are equivalent:*

(a) $\mathcal{T}_\varepsilon(w_\varepsilon) \to \widehat{w}$ *strongly in* $L^p(\Omega \times Y)$ *and* $\int_{\Lambda_\varepsilon} |w_\varepsilon|^p \mathrm{d}x \to 0$,

(b) $w_\varepsilon - \mathcal{U}_\varepsilon(\widehat{w}) \to 0$ *strongly in* $L^p(\Omega)$.

We complete this section with a somewhat unusual convergence property involving the averaging operator $\mathcal{U}_\varepsilon$ which is applied in Theorem 6.3.

Proposition 3.2 *For* $p \in [1, +\infty)$, *suppose that* α *is in* $L^p(\Omega)$ *and* β *in* $L^\infty(\Omega; L^p(Y))$. *Then, the product* $\mathcal{U}_\varepsilon(\alpha)\mathcal{U}_\varepsilon(\beta)$ *belongs to* $L^p(\Omega)$ *and*

$$\mathcal{U}_\varepsilon(\alpha\beta) - \mathcal{U}_\varepsilon(\alpha)\mathcal{U}_\varepsilon(\beta) \to 0 \quad \textit{strongly in } L^p(\Omega). \tag{3.5}$$

4 The Unfolded Limit Problem

In order to use the unfolding operator $\mathcal{T}_\varepsilon$, in S_ε, we extend f_ε by zero and A^ε by αI without changing the notation. This implies that $\mathcal{T}_\varepsilon(f_\varepsilon)|_{\Omega\times S} \equiv 0$, and similarly, $\mathcal{T}_\varepsilon(A^\varepsilon)|_{\widehat{\Omega}_\varepsilon \Omega\times S} \equiv \alpha I$.

Let G_ε be defined as $G_\varepsilon \doteq \sum\limits_{j=1,\cdots,J} \frac{1}{|S^j|} g_\varepsilon^j \, 1_{S_\varepsilon^j}$.

We make the following assumptions concerning the data, for ε converging to 0:

$$\text{(H)} \quad \begin{cases} \mathcal{T}_\varepsilon(A^\varepsilon) \text{ converges in measure (or a.e.) in } \Omega \times Y \text{ to } A^0, \\ \mathcal{T}_\varepsilon(f_\varepsilon) \text{ converges weakly to } f_0 \text{ in } L^2(\Omega \times Y), \\ g_\varepsilon^j \text{ converges weakly to } g_0^j \text{ in } L^2(\Omega) \text{ for } j = 1, \cdots, J. \end{cases} \tag{4.1}$$

Note that from the definition of A^ε and f_ε, f_0 vanish on $\Omega \times S$ while $A^0|_{\Omega\times S} \equiv \alpha I$. Since A^ε belongs to $M(\alpha, \beta, \Omega_\varepsilon)$, it follows that A^0 belongs to $M(\alpha, \beta, \Omega \times Y)$.

It follows from the last hypothesis that $\mathcal{T}_\varepsilon(G_\varepsilon)$ converges weakly in $L^2(\Omega \times Y)$ to the function $G_0 \doteq \sum\limits_{j=1,\cdots,J} \frac{1}{|S^j|} g_0^j(x) \, 1_{S^j}(y)$ (it is actually an equivalence). It also implies that $\mathcal{T}_\varepsilon(F_\varepsilon)$ converges weakly in the same space to $F_0 \doteq f_0 + G_0$.

Proposition 4.1 *Under hypothesis* (H), *up to a subsequence* (*which we still denote by* $\{\varepsilon\}$), *there exist* $u_0 \in H_0^1(\Omega)$ *and* $\widehat{u} \in L^2(\Omega; H^1_{\mathrm{per}}(Y))$, *such that*

$$\begin{aligned} &\mathcal{T}_\varepsilon(u_\varepsilon) \rightharpoonup u_0 \ \textit{weakly in } L^2(\Omega; H^1(Y)), \\ &\mathcal{T}_\varepsilon(\nabla u_\varepsilon) \rightharpoonup \nabla u_0 + \nabla_y \widehat{u} \ \textit{weakly in } L^2(\Omega \times Y), \\ &\frac{1}{\varepsilon}(\mathcal{T}_\varepsilon(u_\varepsilon) - \mathcal{M}_\varepsilon(u_\varepsilon)) \ \textit{converges weakly in } L^2(\Omega; H^1(Y)) \textit{ to } y_M \cdot \nabla u_0 + \widehat{u}, \\ &\eta_\varepsilon \doteq \mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_\varepsilon) \rightharpoonup \eta_0 \doteq A^0(\nabla u_0 + \nabla_y \widehat{u}) \ \textit{weakly in } L^2(\Omega \times Y), \\ &\eta_0 \ \textit{vanishes almost everywhere in } \Omega \times S, \\ &y_M \cdot \nabla u_0 + \widehat{u} \ \textit{is independent of } y \textit{ on each } \Omega \times S^j, \ j = 1, \cdots, J. \end{aligned} \tag{4.2}$$

Proof The existence of a subsequence of u_0 and $\widehat{u}$ satisfying the first three conditions follows from Theorem 3.2. The next convergence follows from the fact that convergence in measure (or a.e.) is a multiplier for strong as well as for weak convergence in $L^2(\Omega \times Y)$. The last property follows from the fact that $u_{\varepsilon|\varepsilon\xi+\varepsilon S^j}$ is independent of x, which implies that $\varepsilon^{-1}(\mathcal{T}_\varepsilon(u_\varepsilon) - \mathcal{M}_\varepsilon(u_\varepsilon))|_{\Omega\times S^j}$ is a function only of x. This property is preserved by weak limit, and holds for $y_M \cdot \nabla u_0 + \widehat{u}$. A similar proof implies that η_0 vanishes a.e. on $\Omega \times S$.

From now on, we use the notation η_ε for $A^\varepsilon(\nabla u_\varepsilon)$ (and η_0 for $A^0(\nabla u_0 + \nabla_y \widehat{u})$), so that $\mathcal{T}_\varepsilon(\eta_\varepsilon)$ converges weakly to η_0.

Definition 4.1 *Let $\mathbf{H}^S_{\rm per}$ denote the subspace of $H^1_{\rm per}(Y)$, consisting of functions which are constant on each S^j (with independent constants for each j).*

Proposition 4.2 *For almost every $x \in \Omega$, and for every $\Phi \in L^2(\Omega; \mathbf{H}^S_{\rm per})$, the vector field η_0 satisfies*

$$\int_{\Omega\times Y} \eta_0(x,y) \cdot \nabla_y \Phi(y) \mathrm{d}y = 0. \tag{4.3}$$

The corresponding strong formulation, under regularity assumptions, is

$$\begin{cases} -\mathrm{div}_y \eta_0 = 0 \quad \text{in } \mathcal{D}'(Y^*), \\ \langle \eta_0 \cdot n, 1 \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}(\partial S^j)} = 0 \quad \text{for } j = 1, \cdots, J, \\ \text{ and periodicity conditions of the normal flux of } \eta_0 \text{ on opposite faces of } \partial Y. \end{cases} \tag{4.4}$$

Proof Let w be fixed in $\mathcal{D}(\Omega)$, ψ in $\mathcal{D}(Y) \cap \mathbf{H}^S_{\rm per}$ and ϕ in $\mathcal{C}^\infty_{\rm per}(\overline{Y})$ vanishing on S.

The function v_ε defined as

$$v_\varepsilon(x) \doteq \varepsilon\Big(\mathcal{M}_\varepsilon(w)(x)\psi\Big(\Big\{\frac{x}{\varepsilon}\Big\}_Y\Big) + w(x)\phi\Big(\Big\{\frac{x}{\varepsilon}\Big\}_Y\Big)\Big)$$

belongs to the space W^ε_0, since ψ vanishes near ∂Y. Furthermore, it converges to zero uniformly. Using v_ε as a test function in Problem (P_ε) gives, for ε going to zero according to the established subsequence,

$$\int_{\Omega_\varepsilon} A^\varepsilon \nabla u_\varepsilon \ \nabla v_\varepsilon \mathrm{d}x \to 0. \tag{4.5}$$

The gradient of v_ε is given by

$$\nabla v_\varepsilon(x) \equiv \mathcal{M}_\varepsilon(w)(x)\nabla_y\psi\Big(\Big\{\frac{x}{\varepsilon}\Big\}_Y\Big) + w(x)\nabla_y\phi\Big(\Big\{\frac{x}{\varepsilon}\Big\}_Y\Big) + \varepsilon\nabla w(x)\phi\Big(\Big\{\frac{x}{\varepsilon}\Big\}_Y\Big).$$

Consequently,

$$\mathcal{T}_\varepsilon(\nabla v_\varepsilon) = \mathcal{M}_\varepsilon(w)(x)\nabla_y\psi(y) + \mathcal{T}_\varepsilon(w)\nabla_y\phi(y) + \varepsilon\mathcal{T}_\varepsilon(\nabla w)\phi(y).$$

From the fact that $\mathcal{T}_\varepsilon(w)$ and $\mathcal{M}_\varepsilon(w)$ both converge uniformly to w (in $\Omega \times Y$ and Ω, respectively), it follows that

$$\mathcal{T}_\varepsilon(\nabla v_\varepsilon) \to w(x)\nabla_y(\psi(y) + \phi(y)).$$

Applying Theorem 3.1(ii) to the left-hand side of (4.5), this gives

$$\int_{\Omega_\varepsilon} A^\varepsilon \nabla u_\varepsilon \ \nabla v_\varepsilon \mathrm{d}x = \frac{1}{|Y|} \int_{\Omega\times Y} \mathcal{T}_\varepsilon(\eta_\varepsilon)(x,y)\mathcal{T}_\varepsilon(\nabla v_\varepsilon)\mathrm{d}x\mathrm{d}y \to 0.$$

By Proposition 4.1, this implies

$$\int_{\Omega\times Y} \eta_0(x,y)\, w(x) \nabla_y (\psi(y)+\phi(y))\mathrm{d}x\mathrm{d}y = 0.$$

Now, by a partition of the unity argument, every $\Psi \in \mathcal{C}^\infty_{\text{per}}(\overline{Y}) \cap \mathbf{H}^S_{\text{per}}$ can be written as a sum of a function ψ of the first case and a function ϕ of the second case. Therefore, for every $w \in \mathcal{D}(\Omega)$ and every $\Psi \in \mathcal{C}^\infty_{\text{per}}(\overline{Y}) \cap \mathbf{H}^S_{\text{per}}$, (4.3) is satisfied. Finally, by a totality argument, (4.3) holds for all $\Phi \in L^2(\Omega; \mathbf{H}^S_{\text{per}})$, since η_0 belongs to $L^2(\Omega\times Y)$.

We now turn to the task of obtaining a relevant formula for $\int_{\Omega\times Y} \eta_0 \nabla w \,\mathrm{d}x\mathrm{d}y$ for w in $H^1_0(\Omega)$. To this end, we use the following lemma (in [3, Proposition 2.1], a similar agument is used but only to obtain the first statement).

Lemma 4.1 *Let Ψ be in $C^\infty_{\text{per}}(Y)$ with $\Psi \equiv 1$ on S. For every w in $\mathcal{D}(\Omega)$ and every ε, there exists a v_ε in W^ε_0, such that, as $\varepsilon \to 0$,*

$$\begin{cases} v_\varepsilon \textit{ converges uniformly to } w \textit{ in } \Omega \textit{ as well as weakly in } H^1_0(\Omega), \\ \mathcal{T}_\varepsilon(\nabla) \textit{ converges strongly in } L^2(\Omega\times Y) \textit{ to } \nabla w - \nabla_y((y_M\cdot\nabla w)\Psi(y)). \end{cases} \tag{4.6}$$

Proof It is clear that the function $x \mapsto w(x)\big(1-\Psi\big(\big\{\frac{x}{\varepsilon}\big\}_Y\big)\big) + \mathcal{M}_\varepsilon(w)\Psi\big(\big\{\frac{x}{\varepsilon}\big\}_Y\big)\big)$ belongs to the space W^ε_0. Furthermore, note that

$$\mathcal{T}_\varepsilon(v_{\varepsilon)} = \mathcal{T}_\varepsilon(w)(1-\Psi(y)) + \mathcal{M}_\varepsilon(w)\Psi(y) \to w(1-\Psi+\Psi) = w \quad \text{ uniformly in } \Omega\times Y.$$

Similarly,

$$\begin{aligned} \nabla v_\varepsilon &= \nabla w\Big(1-\Psi\Big(\Big\{\frac{\cdot}{\varepsilon}\Big\}_Y\Big) - \frac{1}{\varepsilon}(w-\mathcal{M}_\varepsilon(w))\Big(\nabla_y\Psi\Big(\Big\{\frac{\cdot}{\varepsilon}\Big\}_Y\Big), \\ \mathcal{T}_\varepsilon(\nabla v_\varepsilon) &= \mathcal{T}_\varepsilon(\nabla w)(1-\Psi(y)) - \frac{1}{\varepsilon}(\mathcal{T}_\varepsilon(w)-\mathcal{M}_\varepsilon(w))\nabla_y\Psi(y). \end{aligned} \tag{4.7}$$

We can now show that the latter one converges strongly in $L^2(\Omega\times Y)$.

Indeed, it is enough to show the strong convergence of $\frac{1}{\varepsilon}(\mathcal{T}_\varepsilon(w)-\mathcal{M}_\varepsilon(w))$ in the same space. We claim that it converges to $y_M\cdot\nabla w$. Indeed, set

$$z_\varepsilon \doteq \frac{1}{\varepsilon}(\mathcal{T}_\varepsilon(w)-\mathcal{M}_\varepsilon(w)) - y_M\cdot\nabla w.$$

Clearly, $\nabla_y z_\varepsilon = \mathcal{T}_\varepsilon(\nabla w) - \nabla w$, which converges strongly to zero in $L^2(\Omega\times Y)$. By the Poincaré-Wirtinger inequality in Y, and since $\mathcal{M}_Y(z_\varepsilon) \equiv 0$, z_ε itself converges to 0 in $L^2(\Omega; H^1(Y))$.

We conclude with the identity $\nabla w(\Psi) + (y_M\cdot\nabla w)\nabla_y\Psi = \nabla_y((y_M\cdot\nabla w)\Psi(y))$.

Choosing such a v_ε as a test function in Problem (P_ε) gives

$$\int_\Omega \eta_\varepsilon \nabla v_\varepsilon \mathrm{d}x = \int_\Omega \big(f_\varepsilon 1_{\Omega_\varepsilon} + G_\varepsilon\big) v_\varepsilon \mathrm{d}x. \tag{4.8}$$

Unfolding the right-hand side of this relation gives the convergence

$$\frac{1}{|Y|}\int_{\Omega\times Y}(\mathcal{T}_\varepsilon(f_\varepsilon)+\mathcal{T}_\varepsilon(G_\varepsilon))\mathcal{T}_\varepsilon(v_\varepsilon)\mathrm{d}x\mathrm{d}y \to \frac{1}{|Y|}\int_{\Omega\times Y} F_0(x,y)\,w(x)\,\mathrm{d}x\mathrm{d}y. \tag{4.9}$$

The left-hand side of (4.8) is also unfolded to obtain the convergence

$$\frac{1}{|Y|}\int_{\Omega\times Y}\mathcal{T}_\varepsilon(\eta_\varepsilon)\mathcal{T}_\varepsilon(\nabla v_\varepsilon)\mathrm{d}x\mathrm{d}y \to \frac{1}{|Y|}\int_{\Omega\times Y}\eta_0(\nabla w-\nabla_y((y_M\cdot\nabla w)\Psi(y)))\mathrm{d}x\mathrm{d}y. \tag{4.10}$$

Regrouping (4.9) and (4.10), we get

$$\begin{aligned}&\frac{1}{|Y|}\int_{\Omega\times Y}\eta_0(\nabla w-\nabla_y((y_M\cdot\nabla w)\Psi(y)))\mathrm{d}x\mathrm{d}y\\ =\;&\frac{1}{|Y|}\int_{\Omega\times Y}F_0(x,y)\,w(x)\,\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{4.11}$$

By a density argument, this still holds for every w in $H_0^1(\Omega)$.

We have proved the following proposition.

Proposition 4.3 *For every* $w\in H_0^1(\Omega)$,

$$\begin{aligned}&\frac{1}{|Y|}\int_{\Omega\times Y}\eta_0(\nabla w-\nabla_y((y_M\cdot\nabla w)\Psi(y)))\mathrm{d}x\mathrm{d}y\\ =\;&\frac{1}{|Y|}\int_{\Omega\times Y}F_0(x,y)\,w(x)\,\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{4.12}$$

Remark 4.1 (1) Because η_0 satisfies (4.3), formula (4.12) does not depend upon the choice of Ψ. Indeed, for such another $\widehat{\Psi}$ for a.e. $x\in\Omega$, by (4.3), one has

$$\int_Y \eta_0\nabla_y((y_M\cdot\nabla w)(\Psi-\widehat{\Psi}))\mathrm{d}y=0. \tag{4.13}$$

(2) If ∂S is assumed regular, in view of (4.3), the term

$$\frac{1}{|Y|}\int_{\Omega\times Y}\eta_0\nabla_y((y_M\cdot\nabla w)\Psi)\mathrm{d}x\mathrm{d}y$$

can be interpreted as

$$\frac{1}{|Y|}\int_\Omega \langle\eta_0\cdot n, y_M\cdot\nabla w\rangle_{H^{-\frac12},\,H^{\frac12}(\partial S)}\mathrm{d}x,$$

because Ψ is identically 1 on S.

Definition 4.2 *Let* W *be the following space*:

$$\begin{aligned}W=\{&(w,\widehat{w}); w\in H_0^1(\Omega), \widehat{w}\in H^1_{\mathrm{per}}(Y), \mathcal{M}_Y(\widehat{w})=0,\\ &\widehat{w}+(y_M\cdot\nabla w)|_{S^j} \textit{ is a constant (depending on } j\textit{) for } j=1,\cdots,J\}.\end{aligned} \tag{4.14}$$

It is a closed subspace of $H_0^1(\Omega)\times H^1_{\mathrm{per}}(Y)$, *and hence it is a Hilbert space.*

We can now state the limit unfolded problem.

Theorem 4.1 *Under Hypothesis* (H), *the whole sequence* $\{u_e\}$ *converges weakly in* $H_0^1(\Omega)$ *to a function* u_0. *There also exists a* $\widehat{u}$ *in* $L^2(\Omega; H^1_{\rm per}(Y))$, *such that* $(u_0, \widehat{u})$ *is the unique solution to the following problem:*

$$\begin{cases} \textit{Find } (u_0,\widehat{u}) \in W, \ \textit{such that } \forall (w,\widehat{w}) \in W, \\ \dfrac{1}{|Y|}\displaystyle\int_{\Omega\times Y} A^0(\nabla u_0 + \nabla_y \widehat{u})(\nabla w + \nabla_y \widehat{w})\mathrm{d}x\mathrm{d}y \\ \quad = \dfrac{1}{|Y|}\displaystyle\int_{\Omega\times Y} F_0(x,y)\, w(x)\,\mathrm{d}x\mathrm{d}y. \end{cases} \tag{4.15}$$

Proof It has already been established that $(u_0, \widehat{u})$ belongs to the space W. Combining Propositions 4.2 and 4.3 gives that for all $w \in H_0^1(\Omega), \Phi \in L^2(\Omega; \mathbf{H}^S_{\rm per}(Y))$, the following holds:

$$\begin{aligned} &\frac{1}{|Y|}\int_{\Omega\times Y} A^0(\nabla u_0 + \nabla_y \widehat{u})(\nabla w + \nabla_y(\Phi - (y_M\cdot\nabla w)\Psi))\mathrm{d}x\mathrm{d}y \\ = &\frac{1}{|Y|}\int_{\Omega\times Y} F_0(x,y)\,w(x)\,\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{4.16}$$

Every element $(w, \widehat{w})$ of W can be written in the form

$$\begin{aligned} &(w, \Phi - (y_M\cdot\nabla w)\Psi - \mathcal{M}_Y(\Phi - (y_M\cdot\nabla w)\Psi)) \\ &\text{with } \nabla_y\widehat{w} = \nabla_y(\Phi - (y_M\cdot\nabla w)\Psi), \end{aligned} \tag{4.17}$$

with (w, Φ) in $H_0^1(\Omega)\times L^2(\Omega;\mathbf{H}^S_{\rm per}(Y))$, by setting $\Phi \doteq \widehat{w} + (y_M.\nabla w)\Psi$. This shows that (4.15) is equivalent to (4.16).

To prove the existence and the uniqueness of the solution, we show that the Lax-Milgram theorem applies to (4.15). It is enough to show that the bilinear form on the left-hand side of (4.15) is coercive.

Since A^0 belongs to $M(\alpha, \beta, \Omega\times Y)$, it follows that

$$\frac{1}{|Y|}\int_{\Omega\times Y} A^0(\nabla u_0 + \nabla_y\widehat{u})(\nabla u_0 + \nabla_y\widehat{u})\mathrm{d}x\mathrm{d}y \geq \frac{\alpha}{|Y|}\|\nabla u_0 + \nabla_y\widehat{u}\|^2_{L^2(\Omega\times Y)}.$$

But since u_0 is independent of y and $\widehat{u}$ is Y-periodic, the latter one is just

$$\alpha\Big(\|\nabla u_0\|^2_{L^2(\Omega)} + \frac{1}{|Y|}\|\nabla_y\widehat{u}\|^2_{L^2(\Omega\times Y)}\Big).$$

One concludes by using the Poincaré inequality in $H_0^1(\Omega)$ and the Poincaré-Wirtinger inequality in $H^1_{\rm per}(Y)$ (since $\mathcal{M}_Y(\widehat{u}) = 0$).

5 The Homogenized Limit Problem

For a given vector $\lambda\in\mathbb{R}^n$, consider the cell-problem for a.e. $x\in\Omega$,

$$\begin{cases} \text{Find } \chi_\lambda \in H^1_{\rm per}(Y), \text{ such that } \mathcal{M}_Y(\chi_\lambda) = 0, \\ (\chi_\lambda + y_M.\lambda)_{|S^j} \text{ is independent of } y \text{ for } j = 1,\cdots,J, \\ \displaystyle\int_Y A^0(x,y)(\nabla_y\chi_\lambda(y) + \lambda)\nabla_y\varphi(y)\mathrm{d}y = 0, \quad \forall\varphi\in\mathbf{H}^S_{\rm per}. \end{cases} \tag{5.1}$$

This problem itself is not variational. Introducing a fixed function Ψ in $\mathcal{D}(Y)$ with $\Psi_{|S} \equiv 1$ and $\mathcal{M}_Y(y_M\Psi) = 0$, the function $U_\lambda \doteq \chi_\lambda + (y\cdot\lambda)\Psi$ belongs to $\mathbf{H}^S_{\rm per}$ with $\mathcal{M}_Y(U_\lambda) = 0$, and is the unique solution to the following variational problem in the same space:

$$\begin{cases} \text{Find } U_\lambda \in \mathbf{H}^S_{\rm per}, \text{ such that } \mathcal{M}_Y(U_\lambda) = 0, \\ \displaystyle\int_Y A^0(x,y)(\nabla_y U_\lambda(y))\nabla_y\varphi(y)\mathrm{d}y \\ \quad = \int_Y A^0((\Psi-1)\lambda + (y_M\cdot\lambda)\nabla_y\Psi)\nabla_y\varphi(y)\mathrm{d}y, \quad \forall\varphi\in\mathbf{H}^S_{\rm per}. \end{cases} \tag{5.2}$$

Note that the Lax-Milgram theorem applies to (5.2).

Once U_λ is obtained, set $\chi_\lambda \doteq U_\lambda - (y\cdot\lambda)\Psi$. We now show that it is independent of the choice of Ψ. Indeed, this corresponds to having uniqueness in (5.1). The difference V of two solutions to (5.1) belongs to $\mathbf{H}^S_{\rm per}$ and satisfies in particular $\int_Y A^0\nabla_y V\,\nabla_y V\mathrm{d}y = 0$, which by ellipticity, implies $\nabla_y V \equiv 0$ so that $V = 0$ $\big($since $\mathcal{M}_Y(V) = 0\big)$.

Using U_λ as a test function in (5.2) for a.e. $x\in\Omega$, it is straightforward to see that

$$|\nabla_y\chi_\lambda|_{L^2(Y)} \le C(\alpha,\beta,Y)|\lambda|, \tag{5.3}$$

with a constant $C(\alpha,\beta,Y)$, which depends only upon α,β and Y.

For simplicity, we write $\chi(\lambda)$ for χ_λ.

Going back to (4.15) with the solution $(u_0,\widehat{u})$, it follows that

$$\widehat{u}(x,y) = \xi(\nabla u_0)\Big(=\sum_{i=1}^n \frac{\partial u_0}{\partial x_i}(x)\,\chi_{e_i}(x,y)\Big). \tag{5.4}$$

Then, (4.15) becomes

$$\begin{cases} \text{Find } u_0\in H^1_0(\Omega), \text{ such that } \forall w\in H^1_0(\Omega), \\ \displaystyle\frac{1}{|Y|}\int_{\Omega\times Y} A^0(\nabla u_0 + \nabla_y\chi(\nabla u_0))\,\nabla w\mathrm{d}x\mathrm{d}y \\ \displaystyle\qquad - \frac{1}{|Y|}\int_{\Omega\times Y} A^0(\nabla u_0 + \nabla_y\chi(\nabla u_0))\nabla_y((y_M\cdot\nabla w)\Psi(y))\mathrm{d}x\mathrm{d}y \\ \displaystyle\quad = \frac{1}{|Y|}\int_{\Omega\times Y} F_0(x,y)\,w(x)\,\mathrm{d}x\mathrm{d}y. \end{cases} \tag{5.5}$$

Definition 5.1 *Set*

$$A^{\rm hom}(\lambda,\mu) \doteq \frac{1}{|Y|}\int_Y A^0(\lambda + \nabla_y\chi_\lambda)(\mu + \nabla_y\chi_\mu)\mathrm{d}y. \tag{5.6}$$

Proposition 5.1 *The homogenized limit problem is*

$$\begin{cases} \textit{Find } u_0\in H^1_0(\Omega), \ \textit{such that } \forall w\in H^1_0(\Omega), \\ \displaystyle\int_\Omega A^{\rm hom}(\nabla u_0)\,\nabla w\mathrm{d}x = \int_\Omega \mathcal{M}_Y(F_0)\,w\mathrm{d}x. \end{cases} \tag{5.7}$$

The matrix field $A^{\rm hom}$ belongs to $M(\alpha,\beta(1+C(\alpha,\beta,Y)),\Omega)$, so that (5.7) is well-posed.

Proof From (5.5), the homogenized problem (5.7) holds with

$$A^{\text{hom}}(\lambda,\mu) \doteq \frac{1}{|Y|}\int_Y A^0(\lambda+\nabla_y\chi_\lambda)(\mu-\nabla_y((y_M\cdot\mu)\Psi))\mathrm{d}y \tag{5.8}$$

for $\lambda,\mu\in\mathbb{R}^n$ and for a.e. $x\in\Omega$. However, by (5.1), since $\chi_\mu+(y_M\cdot\mu)\Psi$ is in $\mathbf{H}^S_{\text{per}}$,

$$\int_Y A^0(\lambda+\nabla_y\chi_\lambda)\nabla_y\chi_\mu\mathrm{d}y = -\int_Y A^0(\lambda+\nabla_y\chi_\lambda)\nabla_y((y_M\cdot\mu)\Psi)\mathrm{d}y,$$

so that

$$A^{\text{hom}}(\lambda,\mu) \doteq \frac{1}{|Y|}\int_Y A^0(\lambda+\nabla_y\chi_\lambda)(\mu+\nabla_y\chi_\mu)\mathrm{d}y, \tag{5.9}$$

which is formula (5.6).

From the coerciveness of A^0, we get for a.e. $x\in\Omega$

$$A^{\text{hom}}(x)(\lambda)\mu \geq \frac{\alpha}{|Y|}|\lambda+\nabla_y\chi_\lambda|^2_{L^2(Y)}.$$

As before, since χ_λ is Y-periodic, $|\lambda+\nabla_y\chi_\lambda|^2_{L^2(Y)} = |Y||\lambda|^2+|\nabla_y\chi_\lambda|^2_{L^2(Y)}$, which shows the α-coerciveness of A^{hom}.

Finally, by (5.3), it follows that $|A^{\text{hom}}(\lambda)\,\mu| \leq (\beta(1+C(\alpha,\beta,Y)))^2|\lambda||\mu|$, which completes the proof.

6 Convergence of the Energy and Correctors

Proposition 6.1 *Under the hypotheses of the preceding sections, the following holds*:

$$\lim_{\varepsilon\to 0}\int_Y A^\varepsilon(\nabla u_\varepsilon)\nabla u_\varepsilon\mathrm{d}x = \int_Y A^{\text{hom}}(\nabla u_0)\nabla u_0\mathrm{d}x. \tag{6.1}$$

Proof By the definition of Problem (P_ε),

$$\int_Y A^\varepsilon(\nabla u_\varepsilon)\nabla u_\varepsilon\mathrm{d}x = \int_\Omega F_\varepsilon\, u_\varepsilon\mathrm{d}x.$$

Using the established convergences, it is straighforward to see that the right-hand side, once unfolded, converges to $\int_\Omega \mathcal{M}_Y(F_0)\,u_0\mathrm{d}x$. We conclude by comparing with (7.23).

Corollary 6.1 *The following strong convergence holds*:

$$\begin{cases} \mathcal{T}_\varepsilon(\nabla u_\varepsilon)\to\nabla u_0+\nabla_y\widehat{u} \quad \textit{strongly in } L^2(\Omega\times Y), \\ \displaystyle\int_{\Lambda_\varepsilon}|\nabla u_\varepsilon|^2\mathrm{d}x\to 0. \end{cases} \tag{6.2}$$

Proof By definition of A^{hom},

$$\int_Y A^{\text{hom}}(\nabla u_0)\nabla u_0\mathrm{d}x = \frac{1}{|Y|}\int_{\Omega\times Y} A^0(\nabla u_0+\nabla_y\widehat{u})(\nabla u_0+\nabla_y\widehat{u})\mathrm{d}x\mathrm{d}y.$$

On the other hand, by the coercivity of A^ε,

$$\int_\Omega A^\varepsilon(\nabla u_\varepsilon)\nabla u_\varepsilon \mathrm{d}x \geq \frac{1}{|Y|}\int_{\Omega\times Y^*}\mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\mathrm{d}x\mathrm{d}y + \alpha\int_{\Lambda_\varepsilon}|\nabla u_\varepsilon|^2\mathrm{d}x. \tag{6.3}$$

Therefore, by (6.1),

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\frac{1}{|Y|}\int_{\Omega\times Y^*}\mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\mathrm{d}x\mathrm{d}y\\ &\leq \frac{1}{|Y|}\int_{\Omega\times Y}A^0(\nabla u_0+\nabla_y\widehat{u})(\nabla u_0+\nabla_y\widehat{u})\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{6.4}$$

Since $\mathcal{T}_\varepsilon(A^{\varepsilon)}$ converges a.e. to A^0, and $\mathcal{T}_\varepsilon(\nabla u_\varepsilon)$ converges weakly to $\nabla u_0+\nabla_y\widehat{u}$ in $L^2(\Omega\times Y)$, if follows from [6, Lemma 4.9] that

$$\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\to\nabla u_0+\nabla_y\widehat{u}\quad\text{ strongly in } L^2(\Omega\times Y).$$

In turn, this, together with (6.3) implies

$$\int_{\Lambda_\varepsilon}|\nabla u_\varepsilon|^2\mathrm{d}x\to 0.$$

Classically in the unfolding method, the convergences of Corollary 6.1 imply the existence of a corrector as follows.

Corollary 6.2 *Under the hypotheses of the preceding sections, as $\varepsilon\to 0$,*

$$|\nabla u_\varepsilon-\nabla u_0-U_\varepsilon(\nabla_y\widehat{u})|_{L^2(\Omega)}\to 0. \tag{6.5}$$

Making use of formula (5.4) for $\widehat{u}$ and Proposition 3.2, we get the following result.

Corollary 6.3 *Under the hypotheses of the preceding sections, as $\varepsilon\to 0$, the following strong convergence holds:*

$$\Big\|\nabla u_\varepsilon-\nabla u_0-\sum_{i=1}^n\mathcal{U}_\varepsilon\Big(\frac{\partial u_0}{\partial x_i}\Big)\mathcal{U}_\varepsilon(\nabla_y\chi_i)\Big\|_{L^2(\Omega)}\to 0. \tag{6.6}$$

In the case where the matrix field A does not depend on x, the following corrector result holds:

$$\Big\|u_\varepsilon-u_0-\varepsilon\sum_{i=1}^n\mathcal{Q}_\varepsilon\Big(\frac{\partial u_0}{\partial x_i}\Big)\chi_i\Big(\Big\{\frac{\cdot}{\varepsilon}\Big\}_Y\Big)\Big\|_{H^1(\Omega)}\to 0. \tag{6.7}$$

Proof By construction, for $i=1,\cdots,n$, the function χ_i belongs to $L^\infty(\Omega;H^1(Y))$. By (6.5),

$$\|\nabla u_\varepsilon-\mathcal{U}_\varepsilon(\nabla u_0+\nabla_y\widehat{u}_0)\|_{L^2(\Omega)}\to 0. \tag{6.8}$$

By Proposition 3.1(ii), this implies

$$\Big\|\nabla u_\varepsilon-\nabla u_0-\sum_{i=1}^n\mathcal{U}_\varepsilon\Big(\frac{\partial u_0}{\partial x_i}\nabla_y\chi_i\Big)\Big\|_{L^2(\Omega)}\to 0. \tag{6.9}$$

Hence (6.6) follows directly from Proposition 3.2.

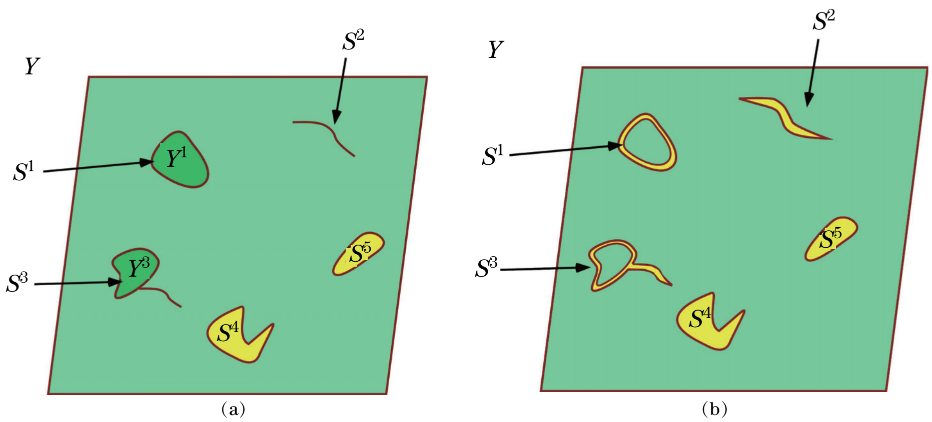
Convergence (6.7) follows from (6.6) as in [5].

7 Other Connected Problems

7.1 Cracks and fissures

One can consider the case when some of the sets S^j are cracks or fissures, i.e., they are Lipschitz submanifolds of codimension one in Y. The case of a submanifold without boundary corresponds to the case of the boundary of a compact subset Y^j in Y. The case of a submanifold with a boundary corresponds to a crack in Y. A combination of the two can also occur (see Figure 6(a) for the various cases).

Each of these cases can be seen as limits of thick inclusions (see Figure 6(b)).



Figures 6 The various cases of cracks and fissures (S^1, S^2 and S^3) as limits of thick inclusions.

The corresponding conditions for regular cracks (which have two sides) in the strong form are as follows:

$$\begin{cases} \text{The solution } u \text{ is an unknown constant on each fissure } \varepsilon\xi + \varepsilon S^j, \\ \displaystyle\int_{\varepsilon\xi+\varepsilon S^j} \left[\frac{\partial u}{\partial \nu_A}\right]_{\varepsilon\xi+\varepsilon S^j} \mathrm{d}\sigma(x) = g^j_{\varepsilon\,|\varepsilon\xi+\varepsilon Y}, \quad \text{a given number}, \end{cases} \tag{7.1}$$

where $\left[\frac{\partial u}{\partial \nu_A}\right]_{\varepsilon\xi+\varepsilon S^j}$ denotes the sum of the two outward conormal derivatives from both sides of the crack (it can be considered as the jump of the conormal derivative across the fissure $\varepsilon\xi + \varepsilon S^j$, and hence it is denoted by the notation).

In the definition of the space W_0^ε, there is the requirement that the functions be constant almost everywhere with respect to the surface measure on the fissures (equivalently the $(n-1)$-dimensional Hausdorff measure). In the variational formulation, the term associated with the fissure $\varepsilon\xi + \varepsilon S^j$ is

$$\varepsilon^{n-1} g^j_{\varepsilon\,|\varepsilon\xi+\varepsilon Y} w_{|\varepsilon\xi+\varepsilon S^j} = \frac{1}{|S^j|}\int_{\varepsilon\xi+\varepsilon S^j} g^j_\varepsilon\, w \mathrm{d}\sigma(x).$$

From then on, the proofs are the same, and the statements of the results are modified in an obvious way.

The homogenized matrix field is given by the same definition (5.6), where the ξ_λ are given as solutions to (5.1). The homogenized problem is (7.23). The only modification is in the definition of the space $\mathbf{H}^S_{\mathrm{per}}$, where the conditions on the fissures are taken in the sense of traces (i.e., almost everywhere for the corresponding surface measures).

7.2 The global conductor 1

In the global conductor case, the situation is the same as in the previous cases, but all the conductors are somehow connected, so that the solution takes the same unknown constant value on all of S_ε. The problem is therefore set in the smaller subspace

$$W^\varepsilon_{0c} \doteq \{w \in H^1_0(\Omega); w|_{S_\varepsilon} \text{ is constant}\}, \tag{7.2}$$

and is defined for given A^ε and f_ε as before and for a given real number g_ε as

$$(\widetilde{\mathcal{P}}_\varepsilon)\quad \begin{cases} \text{Find } u_\varepsilon \in W^\varepsilon_{0c}, \text{ such that for all } w \in W^\varepsilon_{0c}, \\ \displaystyle\int_\Omega A^\varepsilon \nabla u_\varepsilon \nabla w \mathrm{d}x = \int_\Omega f_\varepsilon w \mathrm{d}x + g_\varepsilon w|_{S_\varepsilon}. \end{cases} \tag{7.3}$$

It is easy to see that if f_ε is bounded in $H^{-1}(\Omega)$, and g_ε is bounded in $\mathbb{R}$, so is u_ε in $H^1_0(\Omega)$. By compactness of the Sobolev embedding, it follows that $\{u_\varepsilon\}$ is compact in $L^2(\Omega)$. Since 1_{S_ε} converges weakly-$*$ in $L^\infty(\Omega)$ to $\theta \doteq \frac{|S|}{|Y|} > 0$, in view of the identity $u_\varepsilon 1_{S_\varepsilon} \equiv C_\varepsilon (\in \mathbb{R})$, which converges weakly in $L^2(\Omega)$, it follows that the whole sequence $\{u_\varepsilon\}$ converges to a constant (namely, $\theta^{-1} \lim C_\varepsilon$). But the only constant in $H^1_0(\Omega)$ is 0. Therefore, u_ε also converges weakly to 0 in $H^1_0(\Omega)$, and C_ε converges to 0.

Here we use the obvious variant of Theorem 3.2, where the sequence $\frac{1}{\varepsilon}(\mathcal{T}_\varepsilon(u_\varepsilon) - C_\varepsilon)$ instead of $\frac{1}{\varepsilon}(\mathcal{T}_\varepsilon(u_\varepsilon) - \mathcal{M}_\varepsilon(u_\varepsilon))$ is used to obtain the limit $\widehat{u}$. This is valid because of the existence of the corresponding Poincaré-Wirtinger inequality in the space

$$H^S(Y) \doteq \{\psi \in H^1(Y) \text{ with } \psi_{|S} \equiv 0\}. \tag{7.4}$$

Proposition 7.1 *There exists a positive real number C_{P}, such that for every $\psi \in H^S(Y)$,*

$$|\psi|_{L^2(Y)} \le C_{\mathrm{P}} |\nabla_y \psi|_{L^2(Y)}. \tag{7.5}$$

Proof This is a straightforward consequence of the existence of a Poincaré-Wirtinger constant C_{PW} for $H^1(Y)$, which implies that for every $\psi \in H^1(Y)$,

$$|\psi - M_Y(\psi)|_{L^2(Y)} \le C_{\mathrm{PW}} |\nabla_y \psi|_{L^2(Y)}.$$

Assuming that ψ vanishes on S and taking the average over S (which is a positive Lebesgue measure) imply $|M_Y(\psi)|_{L^2(Y)} = |M_Y(\psi)|\, |Y|^{\frac{1}{2}} \le C_{\mathrm{PW}} |\nabla_y \psi|_{L^2(Y)}$. Combining with the previous inequality, this implies inequality (7.5) with $C_{\mathrm{P}} = 2\, C_{\mathrm{PW}}$.

We denote by $H^S_{\mathrm{per}}(Y)$ the subspace of $H^S(Y)$ consisting of its Y-periodic elements.

It then follows that, up to a subsequence, $\frac{1}{\varepsilon}(\mathcal{T}_\varepsilon(u_\varepsilon) - C_\varepsilon)$ converges weakly to some $\widehat{u}$ in $L^2(\Omega; H^1(Y))$. Furthermore, $\widehat{u}$ belongs to $L^2(\Omega; H^S_{\mathrm{per}}(Y))$. Under the same hypothesis on $\mathcal{T}_\varepsilon(A^{\varepsilon)}$ as before, $\mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_\varepsilon)$ converges weakly to $\eta_0 \doteq A^0 \nabla_y \widehat{u}$ in $L^2(\Omega \times Y)$.

Considering as before a $w \in \mathcal{D}(\Omega)$ and a ϕ in $\mathcal{C}^\infty_{\mathrm{per}}(\overline{Y})$ which vanishes on S, one gets

$$\int_{\Omega \times Y} \eta_0(x,y)\, w(x) \nabla_y \phi(y)\, \mathrm{d}x\mathrm{d}y = 0.$$

By the same totality argument,

$$\int_{\Omega\times Y} \eta_0(x,y)\cdot\nabla_y\Phi(y)\mathrm{d}y = 0$$

holds for every $\Phi \in L^2(\Omega; H^1_{\mathrm{per}})$, which vanishes on $\Omega\times S$. However, $\widehat{u}$ is itself in $L^2(\Omega; H^1_{\mathrm{per}})$, and therefore, one concludes that $\nabla_y\widehat{u} \equiv 0$, and since $\widehat{u}$ vanishes on $\Omega\times S$, this implies that $\widehat{u}$ itself is 0.

The interesting question is to determine the next term in the expansion of u_ε in powers of ε. This requires more estimates, both for C_e and for $|\nabla u_\varepsilon|_{L^2(\Omega)}$.

If S_ε intersects $\partial\Omega$ on a set of non-zero capacity (which may well happen quite often), then clearly, $C_\varepsilon = 0$ (see Remark 7.3 below). When this is not the case, we use the well-known Hardy inequality in $H^1_0(\Omega)$, which requires that $\partial\Omega$ be Lipschitz. Denote by $\delta(x)$ the distance of x to $\partial\Omega$, and by Ω^d the set $\{x\in\Omega, \delta(x) < d\}$. The Hardy inequality states that there exists a constant C_H independent of d, such that

$$\forall w \in H^1_0(\Omega),\ \forall d > 0 \text{ and } d \text{ is small enough},\quad \Big|\frac{w}{\delta}\Big|_{L^2(\Omega^d)} \le C_H|\nabla w|_{L^2(\Omega^d)}. \tag{7.6}$$

Choosing $d = d_\varepsilon$ so that Ω^{d_ε} contains the boundary layer Λ_ε as well as all neighboring ε-cells, and applying the Hardy inequality to u_ε, one gets

$$\frac{|C_\varepsilon|}{d_\varepsilon}|S_\varepsilon\cap\Omega^{d_\varepsilon}|^{\frac12} \le \Big|\frac{u_\varepsilon}{\delta}\Big|_{L^2(\Omega^d)} \le C_H|\nabla u_\varepsilon|_{L^2(\Omega^{d_\varepsilon})}. \tag{7.7}$$

Since $\partial\Omega$ is Lipschitz, $|\Omega^{d_\varepsilon}|$ is of order d_ε, and d_ε is of order ε, it follows that $|S_\varepsilon\cap\Omega^{d_\varepsilon}|$, which is of order $\theta\,|\Omega^{d_\varepsilon}|$, is itself of order ε. This implies that

$$|C_\varepsilon| \le c\,\varepsilon^{\frac12}|\nabla u_\varepsilon|_{L^2(\Omega^{d_\varepsilon})}. \tag{7.8}$$

A similar computation gives

$$|u_\varepsilon|_{L^2(\Omega^{d_\varepsilon})} \le c\,\varepsilon|\nabla u_\varepsilon|_{L^2(\Omega^{d_\varepsilon})}. \tag{7.9}$$

We return to the first estimate of $|\nabla u_\varepsilon|_{L^2(\Omega)}$, letting F_ε denote $f_\varepsilon 1_{\Omega_\varepsilon} + g_\varepsilon 1_{S_\varepsilon}$ and using the unfolding formula as follows:

$$\alpha|\nabla u_\varepsilon|^2_{L^2(\Omega)} \le \int_{\Lambda_\varepsilon} F_\varepsilon u_\varepsilon\mathrm{d}x + \int_{\Omega\setminus\Lambda_\varepsilon} F_\varepsilon(u_\varepsilon - C_\varepsilon)\mathrm{d}x + C_\varepsilon\int_{\Omega\setminus\Lambda_\varepsilon} F_\varepsilon\mathrm{d}x. \tag{7.10}$$

By the Proposition 7.1, it follows that

$$\begin{aligned}|u_\varepsilon - C_\varepsilon|_{L^2(\Omega\setminus\Lambda_\varepsilon)} &= |\mathcal{T}_\varepsilon(u_\varepsilon) - C_\varepsilon|_{L^2(\Omega\times Y)} \le C|\nabla_y\mathcal{T}_\varepsilon(u_\varepsilon)|_{L^2(\Omega\times Y)}\\ &= C\varepsilon|\mathcal{T}_\varepsilon(\nabla u_\varepsilon)|_{L^2(\Omega\times Y)} = C\varepsilon|\nabla u_\varepsilon|_{L^2(\Omega)}.\end{aligned} \tag{7.11}$$

Since Ω^{d_ε} contains Λ_ε, combining (7.8)–(7.9) and (7.11) with (7.10) gives

$$\alpha|\nabla u_\varepsilon|_{L^2(\Omega)} \le c\Big(\varepsilon|f_\varepsilon|_{L^2(\Omega)} + \varepsilon^{\frac12}\Big|\int_{\Omega\setminus\Lambda_\varepsilon} F_\varepsilon\mathrm{d}x\Big|\Big). \tag{7.12}$$

Suppose that $|F_\varepsilon|_{L^2(\Omega)}$ is bounded. Then, by (7.12), $|\nabla u_\varepsilon|_{L^2(\Omega)}$ is an $O(\varepsilon^{\frac{1}{2}})$, so that by (7.8), C_ε is an $O(\varepsilon)$. Together with (7.9) and (7.11), this shows that $|u_\varepsilon|_{L^2(\Omega)}$ is also an $O(\varepsilon)$. Inequality (7.11) also implies that $|u_\varepsilon - C_\varepsilon|_{L^2_{\rm loc}(\Omega)}$ is an $O(\varepsilon^{\frac{3}{2}})$. These estimates do not suffice to obtain the next term in the expansion if C_ε is not zero. We need the extra hypothesis

$$(\mathrm{H}_0) \text{ For the values of } \varepsilon, \text{ such that } C_\varepsilon \neq 0, \ \Big|\int_{\Omega\setminus\Lambda_\varepsilon} F_\varepsilon \mathrm{d}x\Big| \text{ is an } O(\varepsilon^{\frac{1}{2}}). \tag{7.13}$$

Proposition 7.2 *Under* (H_0), $|u_\varepsilon|_{H^1_0(\Omega)}$ *is an* $O(\varepsilon)$. *Furthermore,* $|u_\varepsilon|_{L^2(\Omega)}$ *and* C_ε *are* $O(\varepsilon^{\frac{3}{2}})$ *and* $|u_\varepsilon - C_\varepsilon|_{L^2_{\rm loc}(\Omega)}$ *is an* $O(\varepsilon^2)$.

Proof This is a consequence of (7.12), and then of (7.8)–(7.9) and (7.11).

Remark 7.1 Since $\partial\Omega$ is assumed Lipschitz, it follows that $\big|\int_{\Lambda_\varepsilon} F_\varepsilon \mathrm{d}x\big|$ is bounded above by $|F_\varepsilon|_{L^2(\Omega)}|\Lambda_\varepsilon|^{\frac{1}{2}}$, which is itself an $O(\varepsilon^{\frac{1}{2}})$. Consequently, in the condition (H_0) above, $\big|\int_{\Omega\setminus\Lambda_\varepsilon} F_\varepsilon \mathrm{d}x\big|$ can be replaced by $\big|\int_\Omega F_\varepsilon \mathrm{d}x\big|$.

One can now apply the variant of Theorem 3.2 to the sequence $U_\varepsilon \doteq \frac{u_\varepsilon}{\varepsilon}$. Up to a subsequence, there exist two functions U_0 in $H^1_0(\Omega)$ and $\widehat{U}$ in $L^2(\Omega; H^S_{\rm per}(Y))$, such that

$$\begin{aligned}
&U_\varepsilon \text{ converges weakly to } U_0 \text{ in } H^1_0(\Omega),\\
&\mathcal{T}_\varepsilon(\nabla U_\varepsilon) \text{ converges weakly to } \nabla U_0 + \nabla_y \widehat{U} \text{ in } L^2(\Omega\times Y),\\
&\frac{1}{\varepsilon}(\mathcal{T}_\varepsilon(U_\varepsilon) - \varepsilon^{-1}C_\varepsilon) \text{ converges weakly to } y_M\cdot\nabla U_0 + \widehat{U} \text{ in } L^2(\Omega; H^S(Y)).
\end{aligned} \tag{7.14}$$

Because of Proposition 7.2, a simplification $U_0 \equiv 0$ occurs.

Consequently, $\frac{1}{\varepsilon}(\mathcal{T}_\varepsilon(U_\varepsilon) - \varepsilon^{-1}C_\varepsilon)$ converges weakly to $\widehat{U}$ in $L^2(\Omega; H^S(Y))$.

We complete the assumptions with

$$(\widehat{\mathrm{H}}) \ \begin{cases}
\mathcal{T}_\varepsilon(A^\varepsilon) \text{ converges in measure (or a.e.) in } \Omega\times Y \text{ to } A^0,\\
\mathcal{T}_\varepsilon(f_\varepsilon) \text{ converges weakly to } f_0 \text{ in } L^2(\Omega\times Y^*),\\
g_\varepsilon \text{ converges to } g_0 \text{ in } \mathbb{R}.
\end{cases}$$

Note that under the condition of Remark 7.1, $\int_{\Omega\times Y^*} f_0(x,y)\,\mathrm{d}x\mathrm{d}y + g_0|\Omega|\,|S| = 0$.

Theorem 7.1 *Under assumptions* (H_0) *and* $(\widehat{\mathrm{H}})$, $\mathcal{T}_\varepsilon\big(\frac{1}{\varepsilon^2}(u_\varepsilon - C_\varepsilon)\big)$ *converges weakly in* $L^2(\Omega; H^S(Y))$ *to* $\widehat{U}$, *which is the unique solution of the following variational problem:*

$$\begin{cases}
\widehat{U} \in L^2(\Omega; H^S_{\rm per}(Y)),\ \forall \Psi \in L^2(\Omega; H^S_{\rm per}(Y)),\\
\displaystyle\int_{\Omega\times Y} A^0(x,y)\nabla_y\widehat{U}(x,y)\,\nabla_y\Psi(x,y)\,\mathrm{d}x\mathrm{d}y = \int_{\Omega\times Y} f_0(x,y)\Psi(x,y)\,\mathrm{d}x\mathrm{d}y.
\end{cases} \tag{7.15}$$

Proof Consider a ψ in $\mathcal{C}^\infty_{\rm per}(\overline{Y})$, which vanishes on S. Again let ϕ be in $\mathcal{D}(\Omega)$. Then, $w_\varepsilon(x) \doteq \varepsilon\,\phi(x)\psi\big(\big\{\frac{x}{\varepsilon}\big\}_Y\big)$ is in the space W^ε_{0c}, so it is an acceptable test function. As in the previous computation, this gives at the limit

$$\int_{\Omega\times Y} A^0(x,y)\nabla_y\widehat{U}(x,y)\,\phi(x)\nabla_y\psi(y)\,\mathrm{d}x\mathrm{d}y = \int_{\Omega\times Y} f_0(x,y)\phi(x)\psi(y)\,\mathrm{d}x\mathrm{d}y.$$

By the usual totality argument, this implies (7.15). The existence and uniqueness of $\widehat{U}$ follow from the application of the Lax-Milgram theorem.

The strong formulation of Problem (7.15) is as follows: $\widehat{U} \in L^2(\Omega; H^S_{\mathrm{per}}(Y))$ and

$$-\operatorname{div}_y(A^0(x,y)\nabla_y \widehat{U}(x,y)) = f_0(x,y) \quad \text{for a.e. } (x,y) \in \Omega \times Y^*.$$

The result obtained here can be seen as a sort of expansion of order 2 in ε.

From the weak convergence of $\mathcal{T}_\varepsilon\big(\frac{1}{\varepsilon^2}(u_\varepsilon - C_\varepsilon)\big)$ to $\widehat{U}$ in $L^2(\Omega; H^S(Y))$, it follows that $\frac{1}{\varepsilon^2}(u_\varepsilon - C_\varepsilon)$ converges weakly to $\mathcal{M}_Y(\widehat{U})$ in $L^2(\Omega)$.

These are not completely satisfactory, because we do not know if, in general, C_ε is of order ε^2 itself.

Remark 7.2 If one assumes a stronger condition than (H_0), namely, $\big|\int_\Omega F_\varepsilon \mathrm{d}x\big|$ is an $o(\varepsilon^{\frac{1}{2}})$, then one can show the convergence of the energy, which implies that

$$\mathcal{T}_\varepsilon\Big(\frac{\nabla u_\varepsilon}{\varepsilon}\Big) \text{ converges strongly in } L^2(\Omega \times Y) \text{ to } \nabla_y \widehat{U}. \tag{7.16}$$

In turn, this gives the following corrector result:

$$\nabla u_\varepsilon = \varepsilon U_\varepsilon(\nabla_y \widehat{U}) + o_{L^2(\Omega}(\varepsilon).$$

There remains a boundary layer, if one wants a corrector for u_ε itself, as well as the open question of the behavior of $\frac{C_\varepsilon}{\varepsilon^2}$.

Remark 7.3 In the case where S_ε intersects $\partial\Omega$ in a set of non-zero capacity, it follows that C_ε is 0. Assuming that this is the case for all ε's of the sequence, this problem reduces to that of a homogeneous Dirichlet condition in S_ε. The corresponding problem was originally studied for the Stokes system by Tartar in the appendix of [18] and for the Laplace equation by Lions [16]. The nonlinear case was later studied in [9].

7.3 The global conductor 2

The presentation of the previous section is not necessarily realistic because of the unpredictability of the fact that S_ε intersects the boundary of Ω. It may be more realistic to consider that the conductor is restricted to a compact subset $\overline{\Omega}_0$ of Ω, i.e., $S^0_\varepsilon \doteq S_\varepsilon \cap \overline{\Omega}_0$. We make this hypothesis in this section. We shall also assume that $\partial\Omega_0$ is a null set for the Lebesgue measure, and denote $\Omega \setminus \overline{\Omega}_0$ by Ω_1 (see Figure 7).

In this case, the variational space for the original problem is

$$W^{S_\varepsilon}_{0c} \doteq \{w \in H^1_0(\Omega); w_{|S^0_\varepsilon} \text{ is a constant}\}. \tag{7.17}$$

The variational formulation is for given A^ε and f_ε as before and for a given real number g_ε as

$$(\widetilde{\mathcal{P}}_\varepsilon) \quad \begin{cases} \text{Find } u_\varepsilon \in W^{S_\varepsilon}_{0c}, \text{ such that for all } w \in W^{S_\varepsilon}_{0c}, \\ \displaystyle\int_\Omega A^\varepsilon \nabla u_\varepsilon \nabla w \mathrm{d}x = \int_\Omega f_\varepsilon w \mathrm{d}x + g_\varepsilon w|_{S^0_\varepsilon}. \end{cases} \tag{7.18}$$

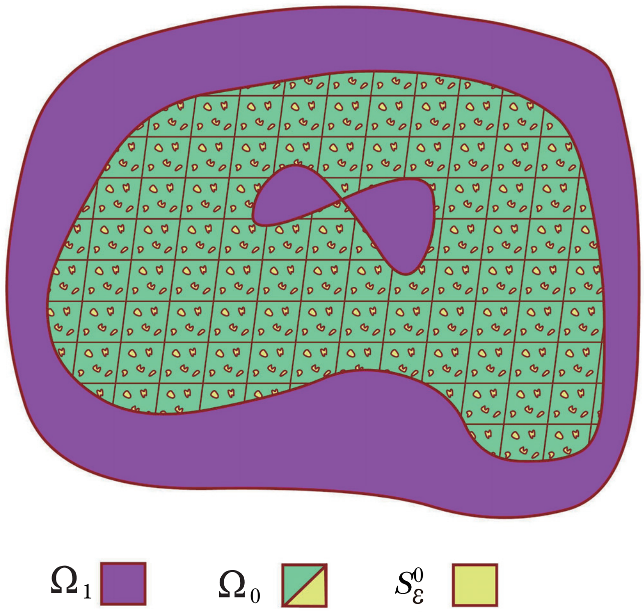


Figure 7 The global conductor.

It is easy to see that if f_ε is bounded in $L^2(\Omega)$ (actually $H^{-1}(\Omega)$ is enough!), and g_ε is bounded in $\mathbb{R}$, so is u_ε in $H^1_0(\Omega)$. By compactness of the Sobolev embedding, it follows that $\{u_\varepsilon\}$ is compact in $L^2(\Omega)$. Since $1_{S^0_\varepsilon}$ converges weakly-$*$ in $L^\infty(\Omega)$ to $\theta \doteq \frac{|S|}{|Y|}1_{\overline{\Omega}_0} > 0$, and in view of the identity $u_\varepsilon 1_{S^0_\varepsilon} \equiv C_\varepsilon \in \mathbb{R}$, which converges weakly in $L^2(\Omega)$, it follows that the whole sequence $\{u_\varepsilon 1_{\overline{\Omega}_0}\}$ converges to a constant $\big($namely, $\frac{|Y|}{|S|}\lim C_\varepsilon\big)$. Consequently, every weak limit point of $\{u_\varepsilon\}$ is constant on $\overline{\Omega}_0$.

By Theorem 3.2, it follows that, up to a subsequence, u_ε converges weakly to some u_0 in $H^1_0(\Omega)$ and $\frac{1}{\varepsilon}(\mathcal{T}_\varepsilon(u_\varepsilon) - \mathcal{M}_Y(u_\varepsilon))$ converges weakly to some $\widehat{u}$ in $L^2(\Omega; H^1(Y))$. Furthermore, $\widehat{u}$ belongs to $L^2(\Omega; H^1_{\mathrm{per}}(Y))$ and $\mathcal{M}_Y(\widehat{u}) = 0$. Also note that, as before, $(y_M \cdot \nabla u_0 + \widehat{u})_{|\Omega_0 \times S}$ is a constant. Since u_0 is a constant on Ω_0, this reduces to that $\widehat{u}_{|\Omega_0\times S}$ is a constant.

Let $\mathcal{H}$ denote the subspace of $H^1_0(\Omega)$, consisting of the functions, which are constant a.e. in $\overline{\Omega}_0$.

Now, the pair $(u_0, \widehat{u})$ belongs to the space

$$\begin{aligned} \mathcal{W} = \{(w, \widehat{w}); w \in \mathcal{H},\ \widehat{w} \in L^2(\Omega; H^1_{\mathrm{per}}(Y)),\\ \mathcal{M}_Y(\widehat{w}) = 0 \text{ and } \widehat{w}_{|\overline{\Omega}_0\times S} \text{ is a constant}\}. \end{aligned} \tag{7.19}$$

Under the same hypothesis on $\mathcal{T}_\varepsilon(A^{\varepsilon)}$ as before, $\mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_\varepsilon)$ converges weakly to $\eta_0 \doteq A^0\nabla_y\widehat{u}$ in $L^2(\Omega \times Y)$ (of course, η_0 vanishes on $\overline{\Omega}_0 \times S$).

We assume that g_ε converges to g_0 in $\mathbb{R}$, while $\mathcal{T}_\varepsilon(f_\varepsilon)$ converges weakly to f_0 in $L^2(\Omega \times Y)$ (they vanish on $\overline{\Omega}_0 \times S$). Consequently, $F_\varepsilon \doteq \mathcal{T}_\varepsilon(f_\varepsilon) + g_\varepsilon 1_{\overline{\Omega}_0\times S}$ converges weakly to $F_0 \doteq f_0 + g_0 1_{\overline{\Omega}_0\times S}$ in $L^2(\Omega \times Y)$.

Now let $(w, \widehat{w})$ be an element of $\mathcal{W} \cap (\mathcal{D}(\Omega) \times \mathcal{D}(\Omega; C^1(\overline{Y})))$, and set as the test function $\varphi_\varepsilon(x) \doteq w(x) + \varepsilon\widehat{w}\big(x, \{\frac{x}{\varepsilon}\}_Y\big)$.

Making use of the unfolding formula and using the standard density argument, one can get the unfolded limit problem as follows:

$$\begin{cases} \text{Find } (u_0, \widehat{u}) \in \mathcal{W}, \text{ such that } \forall (w, \widehat{w}) \in W, \\ \dfrac{1}{|Y|} \displaystyle\int_{\Omega \times Y} A^0 (\nabla u_0 + \nabla_y \widehat{u})(\nabla w + \nabla_y \widehat{w}) \mathrm{d}x\mathrm{d}y \\ \quad = \dfrac{1}{|Y|} \displaystyle\int_{\Omega \times Y} F_0(x, y)\, w(x)\, \mathrm{d}x\mathrm{d}y. \end{cases} \tag{7.20}$$

Since ∇u_0 vanishes in Ω_0, this implies that $\widehat{u}$ is also zero on $\Omega_0 \times Y$.

Therefore, the left-hand side in (7.20) is computed only on $\Omega_1 \times Y$.

For a given vector $\lambda \in \mathbb{R}^n$, the cell-problem is defined for a.e. $x \in \Omega_1$ as

$$\begin{cases} \text{Find } \chi_\lambda \in H^1_{\text{per}}(Y), \text{ such that } \mathcal{M}_Y(\chi_\lambda) = 0, \\ \displaystyle\int_Y A^0(x, y)(\lambda + \nabla_y \chi_\lambda(y)) \nabla_y \varphi(y) \mathrm{d}y = 0, \quad \forall \varphi \in \mathbf{H}^S_{\text{per}}. \end{cases} \tag{7.21}$$

This is actually the "standard" corrector for periodic homogenization in Ω_1. The corresponding homogenized matrix is the standard one, namely,

$$A^{\text{hom}}(\lambda, \mu) \doteq \frac{1}{|Y|} \int_Y A^0 (\lambda + \nabla_y \chi_\lambda)(\mu + \nabla_y \chi_\mu) \mathrm{d}y. \tag{7.22}$$

Proposition 7.3 *The homogenized limit problem takes the following form*:

$$\begin{cases} \textit{Find } u_0 \in \mathcal{H}, \textit{ such that } \forall w \in \mathcal{H}, \\ \displaystyle\int_{\Omega_1} A^{\text{hom}} (\nabla u_0)\, \nabla w \mathrm{d}x = \int_\Omega \mathcal{M}_Y(F_0)\, w \mathrm{d}x. \end{cases} \tag{7.23}$$

The strong formulation can be given here, and if $\partial\Omega_0$ is Lipschitz as a problem on Ω_1,

$$\begin{cases} -\mathrm{div}(A^{\text{hom}} \nabla u_0) = \mathcal{M}_Y(f_0) \quad \text{ in } \Omega_1, \\ u_{0|\partial\Omega_0} \text{ is an unknown constant}, \\ \displaystyle\int_{\partial\Omega_0} \frac{\partial u_0}{\partial \nu_{A^{\text{hom}}}} \mathrm{d}\sigma(x) = \int_{\Omega_0} \mathcal{M}_Y(F_0) \mathrm{d}x. \end{cases} \tag{7.24}$$

Remark 7.4 (1) The convergence of the energy holds in this situation, which implies the existence of a corrector.

(2) This result holds in the more general situation, where the sequence of matrix fields A^ε H-converges to A^{hom} in Ω_1 (for the definition and properties of H-convergence, see [17]).

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References

[1] Bellieud, M., Torsion effects in elastic composites with high contrast, *SIAM J. Math. Anal.*, **41**(6), 2009/2010, 2514–2553.

[2] Braides, A. and Garroni, A., Homogenization of nonlinear periodic media with stiff and soft inclusions, *Math. Models Methods Appl. Sci.*, **5**(4), 1995, 543–564.

[3] Briane, M., Homogenization of the torsion problem and the Neumann problem in nonregular periodically perforated domains, *Math. Models Methods Appl. Sci.*, **7**(6), 1997, 847–870.

[4] Cioranescu, D., Damlamian, A. and Griso, G., Periodic unfolding and homogenization, *C. R. Acad. Sci. Paris, Série* 1, **335**, 2002, 99–104.

[5] Cioranescu, D., Damlamian, A. and Griso, G., The periodic unfolding method in homogenization, *SIAM J. Math. Anal.*, **40**(4), 2008, 1585–1620.

[6] Cioranescu, D., Damlamian, A., Donato, P., et al., The periodic unfolding method in domains with holes, *SIAM J. Math. Anal.*, **44**(2), 2012, 718–760.

[7] Cioranescu, D. and Paulin, J. S. J., Homogenization in open sets with holes, *J. Math. Anal. Appl.*, **71**, 1979, 590–607.

[8] Cioranescu, D. and Paulin, J. S. J., Homogenization of Reticulated Structures, Applied Mathematical Sciences, **136**, Springer-Verlag, New York, 1999.

[9] Donato, P. and Picard, C., Convergence of Dirichlet problems for monotone operators in a class of porous media, *Ricerche Mat.*, **49**, 2000, 245–268.

[10] Gianni, G. D. M., and Murat, F., Asymptotic behaviour and correctors for Dirichlet problems in perforated domains with homogeneous monotone operators, *Annali Della Scuola Normale Superiore di Pisa-Classe di Scienze, Sér.* 4, **24**(2), 1997, 239–290.

[11] De Arcangelis, R., Gaudiello, A. and Paderni, G., Some cases of homogenization of linearly coercive gradients constrained variational problems, *Math. Models Meth. Appl. Sci.*, **6**, 1996, 901–940.

[12] Griso, G., Error estimate and unfolding for periodic homogenization, *Asymptot. Anal.*, **40**, 2004, 269–286.

[13] Lanchon, H., Torsion éastoplastique d'un arbre cylindrique de section simplement ou multiplement connexe, *J. Méanique*, **13**, 1974, 267–320.

[14] Li, T. T. and Tan, Y. J., Mathematical problems and methods in resistivity well-loggings, *Surveys Math. Indust.*, **5**(3), 1995, 133–167.

[15] Li, T. T., Zheng, S. M., Tan, Y. Y. and Shen, W. X., Boundary value problems with equivalued surface and resistivity well-logging, Pitman Research Notes in Mathematics Series, **382**, Longman, Harlow, 1998.

[16] Lions, J. L., Some methods in the mathematical analysis of systems and their control, Science Press, Beijing; Gordon and Breach, Science Publishers, New York, 1981.

[17] Murat, F. and Tartar, L., H-Convergence, Topics in the mathematical modelling of composite materials, Progr. Nonlinear Differential Equations Appl., **31**, R. V. Kohn (ed), Birkhäser, Boston, 1997, 21–43.

[18] Sanchez-Palencia, E., Nonhomogeneous media and vibration theory, Lecture Notes in Physics, **127**, Springer-Verlag, New York, 1980.