Chin. Ann. Math.
34B(2), 2013, 277–294
DOI: 10.1007/s11401-013-0762-3

Chinese Annals of Mathematics, Series B
© The Editorial Office of CAM and Springer-Verlag Berlin Heidelberg 2013

Increasing Powers in a Degenerate Parabolic Logistic Equation*

José Francisco RODRIGUES[1]   Hugo TAVARES[1]

(*In honor of the scientific heritage of Jacques-Louis Lions*)

Abstract The purpose of this paper is to study the asymptotic behavior of the positive solutions of the problem

$$\partial_t u - \Delta u = au - b(x)u^p \quad \text{in } \Omega \times \mathbb{R}^+, \quad u(0) = u_0, \quad u(t)|_{\partial\Omega} = 0,$$

as $p \to +\infty$, where Ω is a bounded domain, and $b(x)$ is a nonnegative function. The authors deduce that the limiting configuration solves a parabolic obstacle problem, and afterwards fully describe its long time behavior.

Keywords Parabolic logistic equation, Obstacle problem, Positive solution, Increasing power, Subsolution and supersolution
2000 MR Subject Classification 35B40, 35B09, 35K91

1 Introduction

In this paper, we are interested in the study of the parabolic problem

$$\begin{cases} \partial_t u - \Delta u = au - b(x)u^p & \text{in } Q := \Omega \times (0, +\infty), \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $a > 0$, $p > 1$, $b \in L^\infty(\Omega)$ is a nonnegative function, and Ω is a bounded domain with a smooth boundary. Such system arises in population dynamics, where u denotes the population density of given species, subject to a logistic-type law.

It is well-known that under these assumptions and for very general u_0's, (1.1) admits a unique global positive solution $u_p = u_p(x,t)$. In fact, in order to deduce the existence result, one can make the change of variables $v = \mathrm{e}^{-at}u$, and deduce that v satisfies $\partial_t v - \Delta v + b(x)\mathrm{e}^{pat}v^p = 0$. As $v \mapsto b(x)\mathrm{e}^{pat}|v|^{p-1}v$ is monotone nondecreasing, the theory of monotone operators (see [1–2]) immediately provides the existence of the solution of the problem in v, and hence also for (1.1).

Manuscript received September 14, 2012.
[1]Department of Mathematics and CMAF, Universidade de Lisboa, Avenida Professor Gama Pinto 2, 1649-003 Lisboa, Portugal. E-mail: rodrigue@ptmat.fc.ul.pt htavares@ptmat.fc.ul.pt
*Project supported by Fundação para a Ciência e a Tecnologia (FCT) (No. PEst OE/MAT/UI0209/2011). The second author was also supported by an FCT grant (No. SFRH/BPD/69314/201).

One of our main interests is the study of the solution u_p as $p \to +\infty$. As we will see, in the limit we will obtain a parabolic obstacle problem, and afterwards fully describe its asymptotic limit as $t \to +\infty$.

This study is mainly inspired by the works of Dancer et al [3–5], where the stationary version of (1.1) is addressed. Let us describe their results in detail. Consider the elliptic problem

$$-\Delta u = au - b(x)u^p, \quad u \in H_0^1(\Omega). \tag{1.2}$$

For each domain $\omega \subseteq \mathbb{R}^N$, denote by $\lambda_1(\omega)$ the first eigenvalue of $-\Delta$ in $H_0^1(\omega)$. Assuming $b \in C(\overline{\Omega})$, the study is divided into two cases as follows: The so-called nondegenerate case $\big($where $\min_{\overline{\Omega}} b(x) > 0\big)$ and the degenerate one (where $\Omega_0 := \mathrm{int}\{x \in \Omega : \ b(x) = 0\} \neq \emptyset$ with a smooth boundary).

In the nondegenerate case, it is standard to check that (1.2) has a positive solution if and only if $a > \lambda_1(\Omega)$ (see [6, Lemma 3.1, Theorem 3.5]). For each $a > \lambda_1(\Omega)$ fixed, then in [4] it is shown that $u_p \to w$ in $C^1(\overline{\Omega})$ as $p \to +\infty$, where w is the unique solution of the obstacle-type problem

$$-\Delta w = aw\chi_{\{w<1\}}, \quad w > 0, \quad w|_{\partial\Omega} = 0, \quad \|w\|_\infty = 1. \tag{1.3}$$

It is observed in [3] that u is also the unique positive solution of the variational inequality

$$w \in \mathbb{K} : \int_\Omega \nabla w \cdot \nabla(v - w)\mathrm{d}x \geqslant \int_\Omega aw(v - w)\mathrm{d}x, \quad \forall v \in \mathbb{K}, \tag{1.4}$$

where

$$\mathbb{K} = \{w \in H_0^1(\Omega) : \ w \leqslant 1 \text{ a.e. in } \Omega\}.$$

In the degenerate case, on the other hand, the problem (1.2) has a positive solution if and only if $a \in (\lambda_1(\Omega), \lambda_1(\Omega_0))$. For such a's, assuming that $\Omega_0 \Subset \Omega$, if we combine the results in [4–5], we see that $u_p \to w$ in $L^q(\Omega)$ for every $q \geqslant 1$, where w is the unique nontrivial nonnegative solution of

$$w \in \mathbb{K}_0 : \int_\Omega \nabla w \cdot \nabla(v - w)\mathrm{d}x \geqslant \int_\Omega aw(v - w)\mathrm{d}x, \quad \forall v \in \mathbb{K}_0 \tag{1.5}$$

with

$$\mathbb{K}_0 = \{w \in H_0^1(\Omega) : \ w \leqslant 1 \text{ a.e. in } \Omega \setminus \Omega_0\}.$$

The uniqueness result is the subject of the paper [5]. Therefore, whenever $b(x) \neq 0$, the term $b(x)u^p$ strongly penalizes the points where $u_p > 1$, forcing the limiting solution to be below the obstacle 1 at such points.

Our first aim is to extend these conclusions to the parabolic case (1.1). While doing this, our concern is also to relax some of the assumptions considered in the previous papers, namely, the continuity of b as well as the condition of Ω_0 being in the interior of Ω. In view of that, consider the following conditions for b:

(b1) $b \in L^\infty(\Omega)$.

(b2) There exists Ω_0, an open domain with a smooth boundary, such that

$$\begin{aligned}&b(x) = 0 \quad \text{a.e.} \quad \text{on } \Omega_0,\\ &\forall\, \Omega' \Subset \Omega \setminus \Omega_0 \text{ open}, \quad \exists\, \underline{b} > 0 \text{ such that } b(x) \geqslant \underline{b} \ \text{ a.e. in } \Omega'.\end{aligned}$$

Observe that in (b2), $\Omega_0 = \emptyset$ is allowed, and $\overline{\Omega}_0$ may intersect $\partial\Omega$. Continuous functions with regular nodal sets or characteristic functions of open smooth domains are typical examples of functions satisfying (b1)–(b2). As for the initial data, we consider

(H1) $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$,

(H2) $0 \leqslant u_0 \leqslant 1$ a.e. in $\Omega \setminus \Omega_0$.

Our first main result is the following.

Theorem 1.1 *Assume that b satisfies* (b1)–(b2)*, and u_0 satisfies* (H1)–(H2)*. Then there exists a function u such that, given $T > 0$, $u \in L^\infty(0, T; H_0^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$ and*

$$\begin{aligned}u_p &\to u &&\textit{strongly in } L^2(0, T; H_0^1(\Omega)),\\ \partial_t u_p &\rightharpoonup \partial_t u &&\textit{weakly in } L^2(Q_T).\end{aligned}$$

Moreover, u is the unique solution of the following problem:

For a.e. $t > 0$, $u(t) \in \mathbb{K}_0$,

$$\int_\Omega \partial_t u(t)(v - u(t))\mathrm{d}x + \int_\Omega \nabla u(t) \cdot \nabla(v - u(t))\mathrm{d}x \geqslant \int_\Omega a u(t)(v - u(t))\mathrm{d}x \tag{1.6}$$

for every $v \in \mathbb{K}_0$, with the initial condition $u(0) = u_0$.

Next, we turn to the long time behavior of the solution of (1.6).

Theorem 1.2 *Suppose that b satisfies* (b1)–(b2)*. Take u_0 verifying* (H1)–(H2)*. Fix $a \in (\lambda_1(\Omega), \lambda_1(\Omega_0))$. Let u be the unique positive solution of* (1.6) *and w be the unique nontrivial nonnegative solution of* (1.5)*. Then $\|w\|_\infty = 1$ and*

$$u(t) \to w \quad \textit{strongly in } H_0^1(\Omega), \ \textit{as } t \to +\infty.$$

Moreover, if $a < \lambda_1(\Omega)$, then $\|u(t)\|_{H_0^1(\Omega)} \to 0$; and if $a \geqslant \lambda_1(\Omega_0)$, then both $\|u(t)\|_\infty$ and $\|u(t)\|_{H_0^1(\Omega)}$ go to $+\infty$ as $t \to +\infty$.

We remark that in the case $\Omega_0 = \emptyset$, we let $\lambda_1(\Omega_0) := +\infty$, and $a \geqslant \lambda_1(\Omega_0)$ is a empty condition. The case $a = \lambda_1(\Omega)$ is the subject of Remark 4.1.

Under some stronger regularity assumptions on b, u_0 and Ω_0, it is known (see [6, Theorem 3.7] or [7, Theorem 2.2]) that $u_p(t, x)$ converges to the unique positive solution of (1.2) whenever $a \in (\lambda_1(\Omega), \lambda_1(\Omega_0))$. Hence in this situation, if we combine all this information together with the results obtained in this paper, then we can conclude that the following diagram commutes:

$$\begin{array}{ccc}
\begin{array}{c} u_p(x,t) \text{ positive solution of} \\ \partial_t u - \Delta u = au - b(x)u^p \end{array} & \xrightarrow{p\to+\infty} & u(x,t) \text{ solution of (1.6)} \\
\Big\downarrow {\scriptstyle t\to+\infty} & & \Big\downarrow {\scriptstyle t\to+\infty} \\
\begin{array}{c} u_p \in H^1_0(\Omega) \text{ positive solution of} \\ -\Delta u = au - b(x)u^p \end{array} & \xrightarrow{p\to+\infty} & \begin{array}{c} w \in H^1_0(\Omega) \text{ nontrivial} \\ \text{nonnegative solution of (1.5)} \end{array}
\end{array}$$

The proof of Theorem 1.1 uses a different approach with respect to the works of Dancer et al. While in [4], the authors use fine properties of functions in Sobolev spaces, here we follow some of the ideas presented in the works [8–9], and show that a uniform bound on the quantity

$$\iint_{Q_T} b(x)u_p^{p+1}\mathrm{d}x\mathrm{d}t \quad \text{for each } T>0,$$

implies that $u(t) \in \mathbb{K}_0$ for a.e. $t > 0$ (see the key Lemma 2.4 ahead). As for the proof of Theorem 1.2, the most difficult part is to show that when $a \in (\lambda_1(\Omega), \lambda_1(\Omega_0))$, $u_p(x,t)$ does not go to the trivial solution of (1.5). The key point here is to construct a subsolution of (1.1) independent of p. It turns out that to do this one needs to get a more complete understanding of the nondegenerate case, and to have a stronger convergence of u_p to u as $p \to +\infty$. So we dedicate a part of this paper to the study of this case. To state the results, we start by defining for each $0 < t_1 < t_2$ and $Q_{t_1,t_2} := \Omega \times (t_1,t_2)$, the spaces $C^{1,0}_\alpha(\overline{Q}_{t_1,t_2})$ and $W^{2,1}_q(Q_{t_1,t_2})$. For $q \geqslant 1$, the space $W^{2,1}_q(Q_{t_1,t_2})$ is the set of elements in $L^q(Q_{t_1,t_2})$ with partial derivatives $\partial_t u$, $D_x u$, $D^2_x u$ in $L^q(Q_{t_1,t_2})$. It is a Banach space equipped with the norm

$$\begin{aligned}\|u\|_{2,1;q,Q_{t_1,t_2}} = \|u\|_{L^q(Q_{t_1,t_2})} + \|D_x u\|_{L^q(Q_{t_1,t_2})} \\ + \|D^2_x u\|_{L^q(Q_{t_1,t_2})} + \|\partial_t u\|_{L^q(Q_{t_1,t_2})}.\end{aligned}$$

For each $\alpha \in (0,1)$, $C^{1,0}_\alpha(\overline{Q}_{t_1,t_2})$ is the space of Hölder functions u in $\overline{Q}_{t_1,t_2}$ with exponents α in the x-variable, $\frac{\alpha}{2}$ in the t-variable and with $D_x u$ satisfying the same property. More precisely, defining the Hölder semi-norm

$$[u]_{\alpha,Q_{t_1,t_2}} := \sup\left\{\frac{|u(x,t)-u(x',t')|}{|x-x'|^\alpha + |t-t'|^{\frac{\alpha}{2}}},\ x,x' \in \overline{\Omega},\ t,t' \in [t_1,t_2],\ (x,t) \neq (x',t')\right\},$$

we have that

$$\begin{aligned}&C^{1,0}_\alpha(\overline{Q}_{t_1,t_2}) \\ &:= \{u:\ \|u\|_{C^{1,0}_\alpha(\overline{Q}_{t_1,t_2})} := \|u\|_{L^\infty(Q_{t_1,t_2})} + \|D_x u\|_{L^\infty(Q_{t_1,t_2})} + [u]_{\alpha,Q_{t_1,t_2}} + [D_x u]_{\alpha,Q_{t_1,t_2}} < +\infty\}.\end{aligned}$$

Recall that we have the following embedding for every $0 \leqslant t_1 < t_2$ (see [10, Lemmas II.3.3, II.3.4]):

$$W^{2,1}_q(Q_{t_1,t_2}) \hookrightarrow C^{1,0}_\alpha(\overline{Q}_{t_1,t_2}), \quad \forall 0 \leqslant \alpha < 1 - \frac{N+2}{q}. \tag{1.7}$$

In the nondegenerate case, we have the following result.

Theorem 1.3 *Suppose that b satisfies* (b1) *and the condition as follows:*

(b2′) *there exists $b_0 > 0$, such that $b(x) \geqslant b_0$ for a.e. $x \in \Omega$.*

Let u_0 satisfy (H1) *and $0 \leqslant u_0 \leqslant 1$ for a.e. $x \in \Omega$. Then, in addition to the conclusions of Theorem* 1.1*, we have that*

$$u_p \to u \quad \text{strongly in } C^{1,0}_\alpha(\overline{Q}_{t_1,t_2}),\ \text{weakly in } W^{2,1}_q(Q_{t_1,t_2}),\ \text{as } p \to +\infty$$

for every $\alpha \in (0,1)$, $q \geqslant 1$ and $0 < t_1 < t_2$. Moreover, u is the unique solution of

$$\partial_t u - \Delta u = au\chi_{\{u<1\}} \quad \text{in } Q, \quad u(0) = u_0, \quad \|u\|_\infty \leqslant 1. \tag{1.8}$$

In this case, as $t \to +\infty$, we also obtain a convergence result for the coincidence sets $\{u(x,t) = 1\}$.

Theorem 1.4 *Suppose that b satisfies* (b1)–(b2′)*. Take u_0 satisfying* (H1) *and $0 \leqslant u_0 \leqslant 1$ for a.e. $x \in \Omega$. Fix $a > \lambda_1(\Omega)$. Let u be the unique solution of* (1.8)*, and w be the unique solution of* (1.3)*. Then, as $t \to +\infty$,*

$$u(t) \to w \quad \text{strongly in } H^1_0(\Omega) \cap H^2(\Omega)$$

and

$$\chi_{\{u=1\}}(t) \to \chi_{\{w=1\}} \quad \text{strongly in } L^q(\Omega), \quad \forall q \geqslant 1. \tag{1.9}$$

The structure of this paper is as follows. In Section 2, we prove Theorem 1.1, while in Section 3, Theorem 1.3 is treated. Finally, in Section 4, we use the strong convergence up to the boundary of Ω obtained in the latter theorem to prove Theorem 1.4, and afterwards, we use it combined with a subsolution argument to prove Theorem 1.2.

We end this introduction by pointing out some other works concerning this type of asymptotic limit. The generalization of [4] for the p-Laplacian case was performed in [11]. In [8–9], elliptic problems of the type

$$-\Delta u + f(x,u)|f(x,u)|^p = g(x)$$

were treated, while in the works by Grossi et al. [12–13], and Bonheure and Serra [14], the authors dealt with the asymptotics study of problems of the type

$$-\Delta u + V(|x|)u = u^p,$$

as $p \to +\infty$ in a ball or an annulus both with Neumann and Dirichlet boundary conditions.

2 The General Case: Proof of Theorem 1.1

To make the presentation more structured, we split our proof into several lemmas. We start by showing a very simple comparison principle which is an easy consequence of the monotonicity of the operator $u \mapsto |u|^{p-1}u$.

Lemma 2.1 *Suppose that u is a solution of* (1.1)*. Take v a supersolution, satisfying*

$$\partial_t v - \Delta v \geqslant av - b(x)v^p \quad \text{in } Q_T,$$
$$v(0) = v_0, \quad v(t)|_{\partial\Omega} = 0$$

with $u_0 \leqslant v_0$. Then $u(x,t) \leqslant v(x,t)$ a.e. On the other hand, if v is a subsolution satisfying

$$\partial_t v - \Delta v \leqslant av - b(x)v^p \quad \text{in } Q_T,$$
$$v(0) = v_0, \quad v(t)|_{\partial\Omega} = 0$$

with $v_0 \leqslant u_0$, then $v(x,t) \leqslant u(x,t)$.

Proof The proof is quite standard, but we include it here only for the sake of completeness. In the case v is a supersolution, we have

$$\partial_t(u-v) - \Delta(u-v) + b(x)(u^p - v^p) \leqslant a(u-v).$$

Multiplying this by $(u(t)-v(t))^+$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega [(u(t)-v(t))^+]^2 \mathrm{d}x + \int_\Omega |\nabla(u(t)-v(t))^+|^2 \mathrm{d}x \\ &+ \int_\Omega b(x)(u^p(t) - v^p(t))(u(t)-v(t))^+ \mathrm{d}x \\ \leqslant\ & a\int_\Omega [(u(t)-v(t))^+]^2 \mathrm{d}x.\end{aligned}$$

As $b(x)(u^p - v^p)(u-v)^+ \geqslant 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega [(u(t)-v(t))^+]^2 \mathrm{d}x \leqslant 2a\int_\Omega [(u(t)-v(t))^+]^2 \mathrm{d}x,$$

whence

$$\int_\Omega [(u(t)-v(t))^+]^2 \mathrm{d}x \leqslant \mathrm{e}^{2at}\int_\Omega [(u_0 - v_0)^+]^2 \mathrm{d}x = 0.$$

The proof of the result for the subsolution case is analogous.

Next we show some uniform bounds in p.

Lemma 2.2 *Given $T > 0$, there exists an $M = M(T) > 0$, such that $\|u_p\|_{L^\infty(Q_T)} \leqslant M$ for all $p > 1$.*

Proof Take $\psi \geqslant 0$ as the unique solution of

$$\begin{cases}\partial_t\psi - \Delta\psi = a\psi & \text{in } Q_T,\\ \psi(0) = u_0, \quad u(t)|_{\partial\Omega} = 0.\end{cases}$$

Then

$$\partial_t\psi - \Delta\psi - a\psi + b(x)\psi^p \geqslant \partial_t\psi - \Delta\psi - a\psi = 0.$$

Hence, ψ is a supersolution, and from Lemma 2.1, we have that $0 \leqslant u_p \leqslant \psi$. In particular,

$$\|u_p\|_{L^\infty(Q_T)} \leqslant \|\psi\|_{L^\infty(Q_T)} < +\infty, \quad \text{as } u_0 \in L^\infty(\Omega),$$

which proves the result.

Lemma 2.3 *Given $T>0$, the sequence $\{u_p\}_p$ is bounded in $H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;$ $H_0^1(\Omega))$. Thus, there exists a $u\in H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;H_0^1(\Omega))$, such that*

$$\begin{aligned} u_p &\to u \qquad \text{strongly in } L^2(Q_T),\ \text{weakly in } L^2(0,T;H_0^1(\Omega)),\\ \partial_t u_p &\rightharpoonup \partial_t u \quad \text{weakly in } L^2(Q_T). \end{aligned}$$

Moreover, there exists a $C=C(T)>0$, such that

$$\iint_{Q_T} b(x)u_p^{p+1}\mathrm{d}x\mathrm{d}t \leqslant C, \quad \forall p>1. \tag{2.1}$$

Proof Multiplying (1.1) by u_p, and integrating it in Ω, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_p^2(t)\mathrm{d}x + \int_\Omega |\nabla u_p(t)|^2\mathrm{d}x = a\int_\Omega u_p^2(t)\,\mathrm{d}x - \int_\Omega b(x)u_p^{p+1}(t)\mathrm{d}x.$$

Integrating the above equation between 0 and t, we have

$$\begin{aligned} &\frac{1}{2}\int_\Omega u_p^2(t)\mathrm{d}x + \int_0^t \|\nabla u_p(\xi)\|_2^2\mathrm{d}\xi + \iint_{Q_t} b(x)u_p^{p+1}\mathrm{d}x\mathrm{d}t\\ \leqslant{}& \frac{1}{2}\int_\Omega u_0^2\mathrm{d}x + a\iint_{Q_t} u_p^2\mathrm{d}x\mathrm{d}t\\ \leqslant{}& \frac{1}{2}\|u_0\|_2^2 + at|\Omega|(M(t))^2. \end{aligned}$$

Hence, for every $T>0$, $\{u_p\}_p$ is bounded in $L^2(Q_T)$, and (2.1) holds.

Now using $\partial_t u_p$ as a test function ($u_p=0$ on $\partial\Omega$ for all $t>0$, thus $\partial_t u_p(t)\in H_0^1(\Omega)$ for a.e. $t>0$), we have

$$\int_\Omega (\partial_t u_p)^2\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u_p(t)|^2\mathrm{d}x = \frac{a}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_p^2(t)\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega b(x)\frac{u_p^{p+1}(t)}{p+1}\mathrm{d}x.$$

Again after an integration, we have

$$\begin{aligned} &\iint_{Q_t} (\partial_t u_p)^2\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_\Omega |\nabla u_p(t)|^2\mathrm{d}x + \frac{a}{2}\int_\Omega u_0^2\mathrm{d}x + \int_\Omega b(x)\frac{u_p^{p+1}(t)}{p+1}\mathrm{d}x\\ ={}& \frac{1}{2}\int_\Omega |\nabla u_0|^2\mathrm{d}x + \int_\Omega b(x)\frac{u_0^{p+1}}{p+1}\mathrm{d}x + \frac{a}{2}\int_\Omega u_p^2(t)\mathrm{d}x\\ \leqslant{}& \frac{1}{2}\|\nabla u_0\|_2^2 + \frac{b_\infty|\Omega|}{p+1} + \frac{a}{2}M^2|\Omega|, \end{aligned} \tag{2.2}$$

where we have used the fact that $0\leqslant u_0\leqslant 1$ whenever $b(x)\neq 0$, together with the previous lemma.

The proofs of the following two results are inspired by similar computations made in [8–9].

Lemma 2.4 *We have $u(t)\in\mathbb{K}_0$ for a.e. $t>0$.*

Proof Let $\Omega' \Subset \Omega \setminus \Omega_0$ and take $Q'_T := \Omega' \times (0,T)$. Given $m > 1$, we will show that $|\{(x,t) \in Q'_T : u > m\}| = 0$. Denote by $\underline{\mathrm{b}}$ the infimum of $b(x)$ over $\overline{\Omega'}$, which is positive by (b2). Recalling (2.1), we deduce the existence of $C > 0$, such that

$$\begin{aligned} 0 &\leqslant \iint_{\{u_p>m\}\cap Q'_T} \underline{b} u_p \mathrm{d}x\mathrm{d}t \\ &\leqslant \frac{1}{m^p} \iint_{\{u_p>m\}\cap Q'_T} b(x) u_p^{p+1} \mathrm{d}x\mathrm{d}t \\ &\leqslant \frac{1}{m^p} \iint_{Q_T} b(x) u_p^{p+1} \mathrm{d}x\mathrm{d}t \leqslant \frac{C}{m^p}. \end{aligned}$$

Hence, as $m > 1$ and $\underline{b} > 0$,

$$\lim_{p\to+\infty} \iint_{\{u_p>m\}\cap Q'_T} u_p \mathrm{d}x\mathrm{d}t = 0.$$

Now observe that

$$\begin{aligned} 0 &= \lim_{p\to+\infty} \iint_{\{u_p>m\}\cap Q'_T} u_p \mathrm{d}x\mathrm{d}t \\ &= \lim_{p\to+\infty} \Big(\int_0^T \int_{\Omega'} u_p \chi_{\{u_p>m\}} \chi_{\{u>m\}} \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Omega'} u_p \chi_{\{u_p>m\}} \chi_{\{u\leqslant m\}} \mathrm{d}x\mathrm{d}t \Big) \\ &\geqslant \lim_{p\to+\infty} \int_0^T \int_{\Omega'} u_p \chi_{\{u_p>m\}} \chi_{\{u>m\}} \mathrm{d}x\mathrm{d}t. \end{aligned}$$

As $u_p \chi_{\{u_p>m\}} \chi_{\{u>m\}} \to u \chi_{\{u>m\}}$ a.e. and $|u_p \chi_{\{u_p>m\}} \chi_{\{u>m\}}| \leqslant L$ on Q_T, then by the Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} &\lim_{p\to+\infty} \int_0^T \int_{\Omega'} u_p \chi_{\{u_p>m\}} \chi_{\{u>m\}} \mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_{\Omega'} u \chi_{\{u>m\}} \mathrm{d}x\mathrm{d}t \\ &\geqslant m |\{(t,x) \in Q'_T : u(t,x) > m\}| \geqslant 0. \end{aligned}$$

Hence $|\{(x,t) \in Q'_T : \ u(x,t) > m\}| = 0$ whenever $m > 1$.

Lemma 2.5 *Let u be the limit provided by Lemma 2.3. Then, up to a subsequence,*

$$u_p \to u \quad \textit{strongly in } L^2(0,T; H_0^1(\Omega)).$$

Proof Multiply (1.1) by $u_p - u$, and integrate it in Q_T, we have

$$\begin{aligned} &\iint_{Q_T} \partial_t u_p (u_p - u) \mathrm{d}x\mathrm{d}t + \iint_{Q_T} \nabla u_p \cdot \nabla (u_p - u) \mathrm{d}x\mathrm{d}t \\ &+ \iint_{Q_T} b(x) u_p^p (u_p - u) \mathrm{d}x\mathrm{d}t \\ &= \iint_{Q_T} a u_p (u_p - u) \mathrm{d}x\mathrm{d}t, \end{aligned}$$

which, after adding and subtracting $\iint_{Q_T} \nabla u \cdot \nabla(u_p - u)\mathrm{d}x\mathrm{d}t$, is equivalent to

$$\begin{aligned}&\iint_{Q_T} \partial_t u_p(v-u_p)\mathrm{d}x\mathrm{d}t + \iint_{Q_T} |\nabla(u_p-u)|^2\mathrm{d}x\mathrm{d}t \\ &+ \iint_{Q_T} \nabla u \cdot \nabla(u_p-u)\mathrm{d}x\mathrm{d}t + \iint_{Q_T} b(x)u_p^p(u_p-u)\mathrm{d}x\mathrm{d}t \\ =& \iint_{Q_T} au_p(u_p-u)\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

By the convergence shown in Lemma 2.3, we have that the terms $\iint_{Q_T} \partial_t u_p(u_p - u)\mathrm{d}x\mathrm{d}t$, $\iint_{Q_T} \nabla u \cdot \nabla(u_p - u)\mathrm{d}x\mathrm{d}t$ and $\iint_{Q_T} au_p(u_p - u)\mathrm{d}x\mathrm{d}t$ tend to zero as $p \to +\infty$. Finally, observe that

$$\begin{aligned}&\iint_{Q_T} b(x)u_p^p(u_p-u)\mathrm{d}x\mathrm{d}t \\ =& \iint_{\{u_p\leqslant u\}} b(x)u_p^p(u_p-u)\mathrm{d}x\mathrm{d}t + \iint_{\{u<u_p\}} b(x)u_p^p(u_p-u)\mathrm{d}x\mathrm{d}t \\ \geqslant& \iint_{\{0\leqslant u_p\leqslant u\}} b(x)u_p^p(u_p-u)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

As $u \leqslant 1$ a.e. in $Q_T' = (0,T) \times \Omega \setminus \Omega_0$ (see Lemma 2.4), we have

$$\begin{aligned}&\Big|\iint_{\{0\leqslant u_p\leqslant u\}} b(x)u_p^p(u_p-u)\mathrm{d}x\mathrm{d}t\Big| \\ \leqslant& \iint_{\{0\leqslant u_p\leqslant u\}\cap Q_T'} b(x)u^p|u_p-u|\mathrm{d}x\mathrm{d}t \\ \leqslant& \iint_{Q_T} b_\infty|u_p-u|\mathrm{d}x\mathrm{d}t \to 0,\end{aligned}$$

whence $\liminf \iint_{Q_T} b(x)u_p^p(u_p - u)\mathrm{d}x\mathrm{d}t \geqslant 0$. Thus

$$\iint_{Q_T} |\nabla(u_p-u)|^2\mathrm{d}x\mathrm{d}t \to 0, \qquad \text{as } p \to +\infty,$$

and the result follows.

Proof of Theorem 1.1 (1) The convergence of u_p to u are the consequences of Lemmas 2.3 and 2.5. Let us then prove first of all that

$$\iint_{Q_T} \partial_t u(v-u)\mathrm{d}x\mathrm{d}t + \iint_{Q_T} \nabla u \cdot \nabla(v-u)\mathrm{d}x\mathrm{d}t \geqslant \iint_{Q_T} au(v-u)\mathrm{d}x\mathrm{d}t \tag{2.3}$$

for every $v \in \widetilde{\mathbb{K}}_0$, where $\widetilde{\mathbb{K}}_0 := \{v \in L^2(0,T; H_0^1(\Omega)) :\ v(t) \in \mathbb{K}_0 \text{ for a.e. } t \in (0,T)\}$. Fix $v \in \widetilde{\mathbb{K}}_0$ and take $0 < \theta < 1$. Multiplying (1.1) by $\theta v - u_p$ and integrating it, we have

$$\begin{aligned}&\iint_{Q_T} \partial_t u_p(\theta v-u_p)\mathrm{d}x\mathrm{d}t + \iint_{Q_T} \nabla u_p \cdot \nabla(\theta v-u_p)\mathrm{d}x\mathrm{d}t \\ &+ \iint_{Q_T} b(x)u_p^p(\theta v-u_p)\mathrm{d}x\mathrm{d}t \\ =& \iint_{Q_T} au_p(\theta v-u_p)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

By Lemmas 2.3 and 2.5, we have

$$\iint_{Q_T} \partial_t u_p(\theta v - u_p)\mathrm{d}x\mathrm{d}t \to \iint_{Q_T} \partial_t u(\theta v - u)\mathrm{d}x\mathrm{d}t,$$
$$\iint_{Q_T} \nabla u_p \cdot \nabla(\theta v - u_p)\mathrm{d}x\mathrm{d}t \to \iint_{Q_T} \nabla u \cdot \nabla(\theta v - u)\mathrm{d}x\mathrm{d}t,$$
$$\iint_{Q_T} u_p(\theta v - u_p)\mathrm{d}x\mathrm{d}t \to \iint_{Q_T} u(\theta v - u)\mathrm{d}x\mathrm{d}t.$$

For the remaining term, as $b(x) = 0$ a.e. in Ω_0 and $v \leqslant 1$ a.e in $\Omega \setminus \Omega_0 \times (0, T)$, we have

$$\begin{aligned}
&\iint_{Q_T} b(x)u_p^p(\theta v - u_p)\mathrm{d}x\mathrm{d}t \\
&= \iint_{0\leqslant u_p \leqslant \theta v} b(x)u_p^p(\theta v - u_p)\mathrm{d}x\mathrm{d}t + \iint_{\theta v < u_p} b(x)u_p^p(\theta v - u_p)\mathrm{d}x\mathrm{d}t \\
&\leqslant \iint_{Q_T'} b(x)\theta^p|\theta v - u_p|\mathrm{d}x\mathrm{d}t \to 0,
\end{aligned}$$

as $p \to +\infty$, because $\theta < 1$. Thus

$$\iint_{Q_T} \partial_t u(\theta v - u)\mathrm{d}x\mathrm{d}t + \iint_{Q_T} \nabla u \cdot \nabla(\theta v - u)\mathrm{d}x\mathrm{d}t \geqslant \iint_{Q_T} au(\theta v - u)\mathrm{d}x\mathrm{d}t,$$

and now we just have to make $\theta \to 1$.

(2) Given $v \in \mathbb{K}_0$, $\xi \in (0, T)$ and $h > 0$, take

$$\widetilde{v}(t) = \begin{cases} v, & t \in [\xi, \xi + h], \\ u(t), & t \notin [\xi, \xi + h]. \end{cases}$$

Then, $\widetilde{v} \in \widetilde{\mathbb{K}}_0$, and from (2.3), we have

$$\int_\xi^{\xi+h}\int_\Omega \partial_t u(v - u)\mathrm{d}x\mathrm{d}t + \int_\xi^{\xi+h}\int_\Omega \nabla u \cdot \nabla(v - u)\,\mathrm{d}x\mathrm{d}t \geqslant \int_\xi^{\xi+h}\int_\Omega au(v - u)\mathrm{d}x\mathrm{d}t.$$

Multiplying this inequality by $\frac{1}{h}$ and making $h \to 0$, we get (1.6), as required.

(3) Finally, it is easy to show that the problem (1.6) has a unique solution. In fact, taking u_1 and u_2 as solutions to (1.6) with the same initial data, we have

$$\begin{aligned}
&\int_\Omega \partial_t(u_1(t) - u_2(t))(u_2(t) - u_1(t)) + \nabla(u_1(t) - u_2(t)) \cdot \nabla(u_2(t) - u_1(t))\mathrm{d}x \\
&\geqslant \int_\Omega a(u_1(t) - u_2(t))(u_2(t) - u_1(t))\mathrm{d}x,
\end{aligned}$$

which is equivalent to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u_1(t) - u_2(t))^2\mathrm{d}x + \int_\Omega |\nabla(u_1(t) - u_2(t))|^2\mathrm{d}x \leqslant \int_\Omega a(u_1(t) - u_2(t))^2\mathrm{d}x.$$

The fact that u_1 and u_2 have the same initial data now implies that

$$\int_\Omega (u_1(t) - u_2(t))^2(t)\mathrm{d}x \leqslant \mathrm{e}^{2at}\int_\Omega (u_0 - u_0)\mathrm{d}x = 0.$$

Hence, $u_p \to u$ for the whole sequence $\{u_p\}_p$, not only for a subsequence.

3 The Nondegenerate Case: Proof of Theorem 1.3

As stated, the results of the previous section are true even in the case of $\Omega_0 = \emptyset$. Let us check that in the nondegenerate case (b2′), we have a stronger convergence as well as a more detailed characterization for the limit u (see (1.8)). This is mainly due to the following powerful estimate.

Lemma 3.1 *There exists a constant $M > 0$ (independent of p), such that $\|u_p\|^{p-1}_{L^\infty(Q)} \leqslant M$ for all $p > 1$.*

Proof Let $b_0 = \inf\limits_{\Omega} b > 0$ and take $M_p > 0$, such that $aM_p - b_0 M_p^p = 0$. Observe that $as - b_0 s^p \leqslant 0$ for $s \geqslant M_p$. Take $N_p := \max\{1, M_p\}$. Multiplying (1.1) by $(u_p(t) - N_p)^+$ (recall that $u_p = 0$ on $\partial\Omega$, whence $(u_p - N_p)^+ = 0$ on the boundary as well), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega ((u_p - N_p)^+)^2 \mathrm{d}x + \int_\Omega |\nabla(u_p - N_p)^+|^2 \mathrm{d}x\\ &= \int_\Omega (au_p - b(x)u_p^p)(u_p - N_p)^+ \mathrm{d}x\\ &\leqslant \int_\Omega (au_p - b_0 u_p^p)(u_p - N_p)^+ \mathrm{d}x\\ &= \int_{u_p \geqslant N_p} (au_p - b_0 u_p^p)(u_p - N_p) \mathrm{d}x \leqslant 0.\end{aligned}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega ((u_p - N_p)^+)^2 \leqslant 0, \quad \int_\Omega ((u_p(t) - N_p)^+)^2 \mathrm{d}x \leqslant \int_\Omega ((u_0 - N_p)^+)^2 \mathrm{d}x,$$

which is zero because $N_p \geqslant 1$. Then

$$0 \leqslant u_p(t, x) \leqslant \max\{1, M_p\},$$

and the result now follows from the fact that $M_p = \left(\frac{a}{b_0}\right)^{\frac{1}{p-1}}$.

Lemma 3.2 *For each $t_2 > t_1 > 0$, $q > 1$ and $\alpha \in (0, 1)$, the sequence $\{u_p\}_p$ is bounded in $W_q^{2,1}(Q_{t_1,t_2})$ and $C_\alpha^{1,0}(\overline{Q}_{t_1,t_2})$. Thus*

$$u_p \rightharpoonup u \quad \text{weakly in } W_q^{2,1}(Q_{t_1,t_2}), \quad u_p \to u \quad \text{strongly in } C_\alpha^{1,0}(\overline{Q}_{t_1,t_2}), \quad \forall \alpha \in (0, 1).$$

Proof From Lemma 3.1 we get that

$$\|au_p\|_{L^\infty(Q)} \leqslant C' M^{\frac{1}{p-1}} \leqslant C'', \quad \|b(x)u_p^p\|_{L^\infty(Q)} \leqslant b_\infty M^{\frac{p}{p-1}} \leqslant C'''.$$

Hence

$$\|\partial_t u_p - \Delta u_p\|_{L^\infty(Q)} \leqslant C, \quad \forall p > 1,$$

which, together with [10, IV. Theorems 9.1 and 10.1] (see also [15, Theorems 7.22 and 7.32]), implies that for every $q > 1$, the sequence $\{u_p\}_p$ is bounded in $W_q^{2,1}(Q_{t_1,t_2})$ independently of p.

Thus, we can use the embedding (1.7) to show that $\{u_p\}_p$ is bounded in $C^{1,0}_\alpha(\overline{Q}_{t_1,t_2})$. As the embedding $C^{1,0}_\alpha \hookrightarrow C^{1,0}_{\alpha'}$ is compact for all $\alpha > \alpha'$, we have the conclusion.

Observe that by Theorem 1.1 the whole sequence u_p already converges to u in some spaces, and hence the convergence obtained in this lemma is also for the whole sequence, not only for a subsequence.

Remark 3.1 It is important to assume Ω smooth (say $\partial\Omega$ of class C^2) to get regularity up to $\partial\Omega$. This will be of crucial importance in the next section. Without such a regularity assumption, we would obtain convergence in each set of the type $\Omega' \times (t_1, t_2)$ with $\Omega' \Subset \Omega$, $0 < t_1 < t_2$.

Now, in view of Theorem 1.3, we want to prove that in this case u solves (1.8). By Lemma 3.1, we know that $\|u_p^{p-1}\|_{L^\infty(Q)} \leqslant M$ for all $p > 1$. This implies the existence of $\psi \geqslant 0$, such that, for every $T > 0$,

$$u_p^{p-1} \rightharpoonup \psi \quad \text{weak-}* \text{ in } L^\infty(Q_T) \text{ and weak in } L^2(Q_T).$$

Thus when we make $p \to +\infty$ in (1.1), we obtain that the limit u satisfies

$$\partial_t u - \Delta u = (a - \psi)u.$$

Moreover,

$$\|u_p\|_\infty \leqslant M^{\frac{1}{p-1}} \to 1, \quad \text{as } p \to \infty,$$

which implies, together with Lemma 3.2, that $0 \leqslant u \leqslant 1$. The proof of Theorem 1.3 will be complete after the following lemmas.

Lemma 3.3 *$\psi = 0$ a.e. in the set $\{(t,x) \in Q : u(x,t) < 1\}$. In particular, this implies that*

$$\partial_t u - \Delta u = au\chi_{\{u<1\}} \quad \textit{a.e. } (x,t) \in Q.$$

Proof Take (x,t) such that $u(x,t) < 1$. As $u_p \to u$ in $C^{1,0}_\alpha$, we can take $\delta > 0$ such that $u_p \leqslant 1 - \delta$ for large p. Then,

$$0 \leqslant u_p^{p-1} \leqslant (1-\delta)^{p-1} \to 0, \quad \text{as } p \to +\infty,$$

whence $\psi(x,t) = 0$. Thus $\psi = 0$ a.e. on $\{(x,t) : u(t,x) < 1\}$.

Finally, as $u \in W^{2,1}_q$ for every $q \geqslant 1$, we have that

$$\partial_t u - \Delta u = 0 \quad \text{a.e. on } \{(x,t) : u(x,t) = 1\},$$

and the proof is complete.

Lemma 3.4 *Let w be a solution of* (1.8)*. Then w solves* (1.6)*.*

Proof Multiply (1.8) by $v - w$ with $v \in \mathbb{K}$. Then we have

$$\begin{aligned}&\int_\Omega \partial_t w(v-w)\mathrm{d}x + \int_\Omega \nabla w \cdot \nabla(v-w)\mathrm{d}x\\ &= a\int_\Omega w\chi_{\{w<1\}}(v-w)\mathrm{d}x\\ &= a\int_\Omega w(v-w)\mathrm{d}x - a\int_\Omega (v-1)\mathrm{d}x\\ &\geqslant a\int_\Omega w(v-w)\mathrm{d}x,\end{aligned}$$

since $v \leqslant 1$ in Ω.

Proof of Theorem 1.3 The convergence $u_p \to u$ strongly in $C_\alpha^{1,0}(\overline{Q}_{t_1,t_2})$ and weakly in $W_q^{2,1}(Q_{t_1,t_2})$ for every $T > 0$ is a consequence of Lemma 3.2. By Lemma 3.3, u satisfies (1.8). Finally, Lemma 3.4 and the uniqueness shown for (1.6) imply the uniqueness of solution of (1.8).

4 Asymptotic Behavior as $t \to \infty$: Proof of Theorem 1.4

In this section, we will study the asymptotic behavior of (1.6) as $t \to +\infty$. First we need to understand what happens in the nondegenerate case (b2′), and prove Theorem 1.4. Then, as we will see, the convergence up to the boundary proved in Lemma 3.2 will be crucial. Only afterwards will we be able to prove Theorem 1.2.

4.1 Proof of Theorem 1.4

We start by showing that the time derivative of u vanishes as $t \to +\infty$.

Proposition 4.1 $\|\partial_t u(t)\|_{L^2(\Omega)} \to 0$ *as* $t \to +\infty$.

In order to prove this proposition, we will show that $\|\partial u_p(t)\|_{L^2(\Omega)} \to 0$ as $t \to +\infty$, uniformly in $p > 1$. To do so, we will use the following result from [2, Lemma 6.2.1].

Lemma 4.1 *Suppose that $y(t)$ and $h(t)$ are nonnegative continuous functions defined on $[0, \infty)$ and satisfy the following conditions*:

$$y'(t) \leqslant A_1 y^2 + A_2 + h(t), \quad \int_0^\infty y(t)\mathrm{d}t \leqslant A_3, \quad \int_0^\infty h(t)\mathrm{d}t \leqslant A_4 \tag{4.1}$$

for some constants $A_1, A_2, A_3, A_4 > 0$. Then

$$\lim_{t\to+\infty} y(t) = 0.$$

Moreover, this convergence is uniform[1] *for all y satisfying* (4.1) *with the same constants A_1, A_2, A_3, A_4.*

[1]This uniformity is not stated in the original lemma, but a close look at the proof allows us to easily obtain that conclusion.

With this in mind, we have the following lemma.

Lemma 4.2 *Let u_p be the solution of* (1.1) *and $a>0$. Then*

$$\|\partial_t u_p(t)\|_2 \to 0, \qquad \text{as } t\to+\infty, \text{ uniformly in } p>1.$$

Proof Let us check that $y(t) := \|\partial_t u_p(t)\|_2^2$ satisfies the assumptions of Lemma 4.1. First of all, (2.2) implies that

$$\int_0^\infty \|\partial_t u_p(t)\|_2^2 \mathrm{d}x \leqslant \|\nabla u_0\|_2^2 + \frac{|\Omega|}{2} + \frac{a}{2}M^2|\Omega|$$

(recall that in the nondegenerate case, $\|u_p\|_{L^\infty(\Omega\times\mathbb{R}^+)}$ is bounded uniformly in p, by Lemma 3.1). Differentiate equation (1.1) with respect to t

$$\partial_t^2 u_p - \Delta\partial_t u_p + p u_p^{p-1}\partial_t u_p = a\partial_t u_p,$$

multiply the above equation by $\partial_t u_p$ and integrate it in Ω at each time t. Then, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\partial_t u_p(t))^2\mathrm{d}x + \int_\Omega |\nabla(\partial_t u_p(t))|^2\mathrm{d}x + p\int_\Omega u_p^{p-1}(t)(\partial_t u_p(t))^2\mathrm{d}x\\ &= a\int_\Omega (\partial_t u_p(t))^2\mathrm{d}x \leqslant \frac{a}{2}\Big(\int_\Omega (\partial_t u_p(t))^2\mathrm{d}x\Big)^2 + \frac{a}{2}.\end{aligned}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t u_p(t)\|_2^2 \leqslant a\|\partial_t u_p\|_2^4 + a.$$

So we can apply the previous lemma with $A_1=a$, $A_2=a$, $A_3=\|\nabla u_0\|_2^2+\frac{|\Omega|}{2}+\frac{a}{2}M^2|\Omega|$, and $h(t)\equiv 0$, $A_4=0$.

Proof of Proposition 4.1 From the previous lemma, we know that, given $\varepsilon>0$, there exist $\bar{t}$ and p_0, such that

$$\|\partial_t u_p(t)\|_2^2 \leqslant \varepsilon, \quad \forall t\geqslant \bar{t},\ \forall p>p_0.$$

Thus for every $\bar{t}\leqslant t_1<t_2$,

$$\int_{t_1}^{t_2} \|\partial_t u_p(t)\|_2^2\mathrm{d}t \leqslant \varepsilon(t_2-t_1), \quad \forall t\geqslant \bar{t},\ \forall p>p_0.$$

As $\partial_t u_p \rightharpoonup \partial_t u$ weakly in $L^2(Q_T)$ for every $T>0$ (see Theorem 1.1), then taking the $\liminf$ as $p\to+\infty$, we get

$$\int_{t_1}^{t_2} \|\partial_t u(t)\|_2^2\mathrm{d}t \leqslant \varepsilon(t_2-t_1).$$

Hence

$$\|\partial_t u(t)\|_2^2 \leqslant \varepsilon, \quad \forall t\geqslant \bar{t},$$

which gives the statement.

Proof of Theorem 1.4 Fix $a>\lambda_1(\Omega)$. By taking $v=0$ in (1.6), we obtain

$$\int_\Omega |\nabla u(t)|^2\mathrm{d}x \leqslant \int_\Omega (-\partial_t u(t)u(t) + au^2)\mathrm{d}x,$$

which implies that $\|u(t)\|_{H_0^1(\Omega)}$ is bounded for $t > 0$. Therefore, up to a subsequence, we have $u(t) \rightharpoonup \overline{u}$ in $H_0^1(\Omega)$ as $t \to +\infty$. Given a subsequence $t_n \to +\infty$ such that $u(t_n) \rightharpoonup \overline{u}$, we know that

$$\begin{aligned}&\int_\Omega \partial_t u(t_n)(v - u(t_n))\mathrm{d}x + \int_\Omega \nabla u(t_n) \cdot \nabla(v - u(t_n))\mathrm{d}x\\ \geqslant\ & a\int_\Omega u(t_n)(v - u(t_n))\mathrm{d}x\end{aligned}$$

for all $v \in \mathbb{K}$, which, together with Proposition 4.1, implies that, as $p \to +\infty$,

$$\int_\Omega \nabla\overline{u} \cdot \nabla(v - \overline{u})\mathrm{d}x \geqslant \int_\Omega a\overline{u}(v - \overline{u})\mathrm{d}x, \quad \forall v \in \mathbb{K}$$

or, equivalently,

$$-\Delta\overline{u} = a\overline{u}\chi_{\{\overline{u}<1\}}$$

(here we are using the equivalence between these two problems, which was shown in [3] and stated in Section 1). Since $\|\overline{u}\|_\infty \leqslant 1$ and $a > \lambda_1(\Omega)$, in order to prove that $\overline{u} = w$ (the unique nontrivial solution of (1.3)), the only thing left to prove is that $\overline{u} \not\equiv 0$.

(2) Let us then check that, for $a > \lambda_1$, $\overline{u} \not\equiv 0$. Fix any $\overline{t} > 0$. By the maximum principle, we have that $u(\overline{t}, x) > 0$ in Ω and $\partial_\nu u(\overline{t}, x) < 0$ on $\partial\Omega$. By the convergence in $C_\alpha^{1,0}$-spaces up to the boundary of Ω (see Theorem 1.3), we have that for $p \geqslant \overline{p}$, $u_p(\overline{t}, x) > 0$ in Ω and $\partial_\nu u_p(\overline{t}, x) < 0$ on $\partial\Omega$. Let φ_1 be the first eigenfunction of the Laplacian in $H_0^1(\Omega)$ with $\varphi_1 > 0$ and $\|\varphi_1\|_\infty = 1$. Then

$$c\varphi_1 \leqslant u_p(\overline{t}, x), \quad \forall x \in \Omega,\ \forall p \geqslant \overline{p} \tag{4.2}$$

for sufficiently small c (independent of p). Moreover, observe that

$$\partial_t(c\varphi_1) - \Delta(c\varphi_1) \leqslant a(c\varphi_1) - b(x)(c\varphi_1)^p$$

if and only if

$$b(x)c^{p-1}\varphi_1^{p-1} \leqslant a - \lambda_1. \tag{4.3}$$

Take $\overline{c} > 0$ such that (4.2)–(4.3) hold. Then, $\overline{c}\varphi_1$ is a subsolution of (1.1) for sufficiently small $\overline{c}$ and for each $p \geqslant \overline{p}$. Then, by Lemma 2.1, we have that $u_p(t, x) \geqslant \overline{c}\varphi_1$ for every $t \geqslant \overline{t}$ and $p \geqslant \overline{p}$. Hence as $p \to \infty$, we also have $u(t, x) \geqslant \overline{c}\varphi_1(x)$ for every $x \in \Omega$, $t \geqslant \overline{t}$. Thus $\overline{u} \not\equiv 0$ and $\overline{u} = w$, the unique solution of (1.3). From the uniqueness, we deduce in particular that $u(t) \rightharpoonup w$ in $H_0^1(\Omega)$ as $t \to \infty$, not only for some subsequence. As for the strong convergence, this is now easy to show, since by taking the difference

$$\partial_t u - \Delta(u(t) - w) = au(t)\chi_{\{u<1\}} - aw\chi_{\{w<1\}}$$

and multiplying it by $u(t) - w$, we get

$$\begin{aligned}&\int_\Omega |\nabla(u(t) - w)|^2\mathrm{d}x\\ =& -\int_\Omega \partial_t u(t)(u(t) - w)\mathrm{d}x + (au(t)\chi_{\{u<1\}} - aw\chi_{\{w<1\}})(u(t) - w) \to 0,\end{aligned}$$

as $t \to \infty$ (recall that both $u(t)$ and w are less than or equal to 1). Thus $u(t) \to w$ strongly in $H_0^1(\Omega)$.

(3) The convergence of the coincidence sets follows as in [16]. As $0 \leqslant \chi_{\{u=1\}}(t) \leqslant 1$, then there exists a function $0 \leqslant \chi^* \leqslant 1$ such that, up to a subsequence,

$$\chi_{\{u=1\}}(t) \rightharpoonup \chi^* \quad \text{weak-}* \text{ in } L^\infty(\Omega), \quad \text{as } t \to +\infty.$$

Since $\chi_{\{u=1\}}(1-u) = 0$ a.e., we have $\chi^*(1-w) = 0$ a.e. Hence $\chi^* = 0$ whenever $w < 1$. Moreover, from the fact that $\partial_t u - \Delta u = au(1 - \chi_{\{u=1\}})$ a.e. in Q, we deduce that $-\Delta w = aw(1-\chi^*)$. As $\Delta w = 0$ a.e. on $\{w = 1\}$ (in fact, $u \in W^{2,q}(\Omega)$ for every $q \geqslant 1$), we conclude that $\chi^* = 1$ on $\{w = 1\}$, whence $\chi^* = \chi_{\{w=1\}}$. Since in general, the $L^\infty(\Omega)$ weak-$*$ convergence of characteristic functions implies the strong convergence in $L^q(\Omega)$ for every $q \geqslant 1$, we have proved (1.9). As a consequence, actually $u(t) \to w$ in H^2-norm.

(4) For $a < \lambda_1(\Omega)$, the function 0 attracts all the solutions of (1.6) with nonnegative initial data. In fact, by taking $v = 0$ in (1.6), we obtain

$$\int_\Omega |\nabla u(t)|^2 \mathrm{d}x \leqslant a \int_\Omega u(t)^2 \,\mathrm{d}x - \int_\Omega \partial_t u(t) u(t) \mathrm{d}x \leqslant \frac{a}{\lambda_1(\Omega)} \int_\Omega |\nabla u(t)|^2 \,\mathrm{d}x + \mathrm{o}(1),$$

as $t \to +\infty$. Thus $\|u(t)\|_{H_0^1(\Omega)} \to 0$.

4.2 Proof of Theorem 1.2

Fix $a \in (\lambda_1(\Omega), \lambda_1(\Omega_0))$. In this case, we have a result stronger than that in Lemma 2.2, with a uniform L^∞ bound in $Q = \Omega \times \mathbb{R}^+$.

Lemma 4.3 *For $a \in (\lambda_1(\Omega), \lambda_1(\Omega_0))$, there exists $C > 0$, such that $\|u_p\|_{L^\infty(Q)} \leqslant C$ for all $p > 1$.*

Proof Here we follow the line of the proof of Claim 1 in [4, p. 224], to which we refer for more details. Define $\Omega_\delta = \{x \in \mathbb{R}^N : \mathrm{dist}(x, \Omega) < \delta\}$. Since $a < \lambda_1(\Omega_0)$, there exists a small δ such that $a < \lambda_1(\Omega_\delta)$ (by continuity of the map $\Omega \mapsto \lambda_1(\Omega)$). Denoting by ϕ_δ the first eigenfunction of $-\Delta$ in $H_0^1(\Omega_\delta)$ and ψ any extension of $\phi|_{\Omega_{\frac{\delta}{2}}}$ to $\overline{\Omega}$ such that $\min_{\overline{\Omega}} \psi > 0$, there exists a $Q > 0$ large enough, such that

$$-\Delta(Q\psi) - aQ\psi + b(x)(Q\psi) \geqslant 0 \quad \text{in } \Omega,$$

and $u_0 \leqslant Q\psi$ in Ω. Thus, $Q\psi$ is a supersolution of (1.1) for all $p > 1$. By Lemma 2.1, we have

$$u_p \leqslant Q\psi \leqslant M \quad \text{for all } (x,t) \in Q.$$

Proof of Theorem 1.2 (1) Fix $a \in (\lambda_1(\Omega), \lambda_1(\Omega_0))$. Having proved Lemma 4.3, we can repeat the proof of Proposition 4.1 word by word and show that

$$\|\partial_t u(t)\|_{L^2(\Omega)} \to 0, \quad \text{as } t \to +\infty.$$

By making $v = 0$ in (1.6), we obtain once again by Lemma 4.3 that $\|u(t)\|_{H_0^1(\Omega)}$ is bounded for $t > 0$. Take $t_n \to +\infty$ such that $u(t_n) \rightharpoonup \overline{u}$ in $H_0^1(\Omega)$ for some $\overline{u} \in H_0^1(\Omega)$. Then $\overline{u} \in \mathbb{K}_0$ and

$$\int_\Omega \nabla\overline{u} \cdot \nabla(v - \overline{u})\mathrm{d}x \geqslant a \int_\Omega \overline{u}(v - \overline{u})\mathrm{d}x, \quad \forall v \in \mathbb{K}_0. \tag{4.4}$$

In [3], Dancer and Du shown that (4.4) has a unique nontrivial nonnegative solution w. In order to prove that $\overline{u} = w$ and conclude the proof for this case, we just have to show that $\overline{u} \not\equiv 0$. This will be a consequence of Theorem 1.4. In fact, considering ϕ_p as the solution of

$$\begin{cases} \partial_t \phi_p - \Delta\phi_p = a\phi_p - \|b\|_\infty \phi_p^p \quad \text{in } Q_T, \\ \varphi_p(0) = v_0, \quad \varphi_p(t)|_{\partial\Omega} = 0 \end{cases}$$

with $v_0 := \inf\{u_0, 1\}$, it is straightforward to see that ϕ_p is a subsolution of (1.1), and

$$u_p \geqslant \phi_p \to w, \quad \text{as } p \to +\infty,$$

where $w \neq 0$ is the unique nontrivial solution of (1.3). This last statement is a consequence of Theorem 1.4, as $0 \leqslant v_0 \leqslant 1$ a.e. in Ω. Thus $\overline{u} \geqslant w \not\equiv 0$, which concludes the proof in this case.

(2) If $a < \lambda_1(\Omega)$, the same reasoning as in the proof of Theorem 1.4 yields that $\|u(t)\|_{H_0^1(\Omega)} \to 0$. As for the case $a \geqslant \lambda_1(\Omega_0)$, if either $\|u(t)\|_\infty$ or $\|u(t)\|_{H_0^1(\Omega)}$ bounded, it is clear from the proof of Proposition 4.1 that $\|\partial_t u(t)\|_{L^2(\Omega)} \to 0$. Repeating the reasoning of the previous step, we would obtain a nontrivial solution of (1.6) for $a \geqslant \lambda_1(\Omega_0)$, contradicting [3, Theorem 1.1].

Remark 4.1 As for the case $a = \lambda_1(\Omega)$, observe that $c\varphi_1$ is always a steady state solution of (1.8) for all $0 < c < 1$, where φ_1 denotes the first eigenfunction of $(-\Delta, H_0^1(\Omega))$ with $\|\varphi_1\|_\infty = 1$. Hence, the long time limit of (1.6) in this case will depend on the initial condition u_0, and we are only able to conclude that, given $t_n \to +\infty$, there exists a subsequence $\{t_{n_k}\}$, such that $u(t_{n_k})$ converges to $c\varphi_1$ for some $c > 0$.

Acknowledgement The second author would like to thank Jesús Hernández and Pedro Freitas for the useful discussions related with this paper.

References

[1] Lions, J. L., Quelques Méthodes de Résolution des Problémes aux Limites Nonlinéaires, Gauthier-Villars, Paris, 1969.

[2] Zheng, S., Nonlinear evolutions equations, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 133, Chapmam & Hall/CRC, Boca Raton, FL, 2004.

[3] Dancer, E. and Du, Y., On a free boundary problem arising from population biology, *Indiana Univ. Math. J.*, **52**, 2003, 51–67.

[4] Dancer, E., Du, Y. and Ma, L., Asymptotic behavior of positive solutions of some elliptic problems, *Pacific Journal of Mathematics*, **210**, 2003, 215–228.

[5] Dancer, E., Du, Y. and Ma, L., A uniqueness theorem for a free boundary problem, *Proc. Amer. Math. Soc.*, **134**, 2006, 3223–3230.

[6] Fraile, J. M., Koch Medina, P., López-Gómez J., et al., Elliptic eigenvalue problems and unbounded continua of positive solutions of a semilinear elliptic equation, *J. Differential Equations*, **127**, 1996, 295–319.

[7] Du, Y. and Guo, Z., The degenerate logistic model and a singularly mixed boundary blow-up problem, *Discrete and Continuous Dyn. Syst.*, **14**, 2006, 1–29.

[8] Boccardo, L. and Murat, F., Increase of power leads to bilateral problems, Composite Media and Homogenization Theory, G. Dal Maso and G. F. Dell'Antonio (eds.), World Scientific, Singapore, 1995, 113–123.

[9] Dall'Aglio, A. and Orsina, L., On the limit of some nonlinear elliptic equations involving increasing powers, *Asymptotic Analysis*, **14**, 1997, 49–71.

[10] Ladyzenskaja, O., Solonnikov, V. and Uralceva, N., Linear and Quasi-linear Equations of Parabolic Type, A. M. S., Providence, RI, 1988.

[11] Guo, Z. and Ma, L., Asymptotic behavior of positive solutions of some quasilinear elliptic problems, *J. London Math. Soc.*, **76**, 2007, 419–437.

[12] Grossi, M., Asymptotic behaviour of the Kazdan-Warner solution in the annulus, *J. Differential Equations*, **223**, 2006, 96–111.

[13] Grossi, M. and Noris, B., Positive constrained minimizers for supercritical problems in the ball, *Proc. Amer. Math. Soc.*, **140**, 2012, 2141–2154.

[14] Bonheure, D. and Serra, E., Multiple positive radial solutions on annuli for nonlinear Neumann problems with large growth, *NoDEA*, **18**, 2011, 217–235.

[15] Lieberman, G., Second Order Parabolic Differential Equations, World Scientific, Singapore, 1996.

[16] Rodrigues, J. F., On a class of parabolic unilateral problems, *Nonlinear Anal.*, **10**, 1986, 1357–1366.