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Composite Waves for a Cell Population System Modeling Tumor Growth and Invasion*

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(*In honor of the scientific heritage of Jacques-Louis Lions*)

Abstract In the recent biomechanical theory of cancer growth, solid tumors are considered as liquid-like materials comprising elastic components. In this fluid mechanical view, the expansion ability of a solid tumor into a host tissue is mainly driven by either the cell diffusion constant or the cell division rate, with the latter depending on the local cell density (contact inhibition) or/and on the mechanical stress in the tumor.

For the two by two degenerate parabolic/elliptic reaction-diffusion system that results from this modeling, the authors prove that there are always traveling waves above a minimal speed, and analyse their shapes. They appear to be complex with composite shapes and discontinuities. Several small parameters allow for analytical solutions, and in particular, the incompressible cells limit is very singular and related to the Hele-Shaw equation. These singular traveling waves are recovered numerically.

Keywords Traveling waves, Reaction-diffusion, Tumor growth, Elastic material
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1 Introduction

Models describing cell multiplication within a tissue are numerous and have been widely studied recently, particularly in relation to cancer invasion. Whereas small-scale phenomena are accurately described by individual-based models (IBM in short, see, e.g., [3, 19, 24]), large scale solid tumors can be described by tools from continuum mechanics (see, e.g., [2, 6, 15–18] and [9] for a comparison between IBM and continuum models). The complexity of the subject has led to a number of different approaches, and many surveys are now available [1, 4–5, 21, 25, 32]. They show that the mathematical analysis of these continuum models raises several challenging issues. One of them, which has attracted little attention, is the existence and the structure of traveling waves (see [12, 15]). This is our main interest here, particularly in the context of fluid mechanical models that have been advocated recently [29, 31]. Traveling wave solutions are of special interest also from the biological point as the diameter of 2D monolayers, 3D multicellular spheroids and xenografts. 3D tumors emerging from cells injected into animals

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are found to increase for many cell lines linearly in time indicating a constant growth speed of the tumor border (see [30]).

In this fluid mechanical view, the expansion ability of tumor cells into a host tissue is mainly driven by the cell division rate which depends on the local cell density (contact inhibition) and by the mechanical pressure in the tumor (see [11, 29, 31]). Tumor cells are considered to be of an elastic material, and then respond to pressure by elastic deformation. Denoting by v the velocity field and by ρ the cell population density, we will make use of the following advection-diffusion model:

$$\partial_t \rho + \mathrm{div}(\rho v) - \mathrm{div}(\epsilon \nabla \rho) = \Phi(\rho, \Sigma).$$

In this equation, the third term in the left-hand side describes the active motion of cells that results in their diffusion with a nonnegative diffusion coefficient ϵ. In the right-hand side, $\Phi(\rho, \Sigma)$ is the growth term, which expresses that cells divide freely. Thus it results in an exponential growth, as long as the elastic pressure Σ is less than a threshold pressure denoted by C_p, where the cell division is stopped by contact inhibition (the term “homeostatic pressure” has been used for C_p). This critical threshold is determined by the compression that a cell can experience (see [9]). A simple mathematical representation is

$$\Phi(\rho) = \rho H(C_p - \Sigma(\rho)),$$

where H denotes the Heaviside function $H(v) = 0$ for $v < 0$ and $H(v) = 1$ for $v > 0$, and $\Sigma(\rho)$ denotes the state equation, linking pressure and local cell density. As long as cells are not in contact, the elastic pressure $\Sigma(\rho)$ vanishes whereas it is an increasing function of the population density for larger value of this contact density. Here, after neglecting cell adhesion, we consider the pressure monotonously depending on cell population, such that

$$\Sigma(\rho) = 0, \ \rho \in [0, 1), \quad \Sigma'(\rho) > 0, \ \rho \geq 1. \tag{1.1}$$

The flat region $\rho \in [0, 1)$ induces a degeneracy that is one of the interests of the model for both mathematics and biophysical effects. This region represents that cells are too far apart and do not touch each other. When elastic deformations are neglected, in the incompressible limit of confined cells, this leads to a jump of the pressure from 0 to $+\infty$ at the reference value $\rho = 1$. This highly singular limit leads to the Hele-Shaw type of models (see [28]). Finally, the balance of forces acting on the cells leads under certain hypotheses to the following relationship between the velocity field v and the elastic pressure (see [14]):

$$-C_S \nabla \Sigma(\rho) = -C_z \Delta v + v.$$

This is Darcy’s law which describes the tendency of cells to move down pressure gradients, extended to a Brinkman model by a dissipative force density resulting from internal cell friction due to cell volume changes. C_S and C_z are parameters, relating respectively to the reference elastic and bulk viscosity cell properties with the friction coefficient. The resulting model is then the coupling of this elliptic equation for the velocity field, a conservation equation for the population density of cells and a state equation for the pressure law.

A similar system of equations describing the biomechanical properties of cells has already been suggested as a conclusion in [9] for the radial growth of tumors. That paper proposes to close the system of equations with an elastic fluid model to generalize their derivation for compact tumors that assume a constant density inside the tumor with a surface tension boundary condition. Many other authors have also considered such an approach (see, e.g., [17–18]). In [8, 10, 13, 15] cell-cell adhesion is also taken into account, in contrast with (1.1). Their linear stability analysis explains instabilities of the tumor front which are also observed numerically in [15, 13]. However, many of these works focus on nutrient-limited growth, whereas we are interested here in stress-regulated growth. Besides, most works deal with a purely elastic fluid model. A viscous fluid model was motivated in [8, 10–11] and studied numerically in [8]. Here we include this case in our mathematical study and numerical results. Moreover, we propose here a rigorous analysis of traveling waves, which furnishes in some case explicit expressions of the traveling profile and the speed of the wave.

From a mathematical point of view, the description of the invasive ability of cells can be considered as the search of traveling waves. Furthermore, the study in several dimensions is also very challenging, and we will restrict ourselves to the 1-dimensional case. For reaction-diffusion-advection equations arising from biology, several works were devoted to the study of traveling waves (see, for instance, [22–23, 26–27, 34] and the book [7]). In particular, our model has some formal similarities with the Keller-Segel system with growth treated in [22, 27], and the main difference is that the effect of pressure is repulsive here while it is attractive for the Keller-Segel system. More generally, the influence of the physical parameters on the traveling speed is an issue of interest for us and is one of the objectives of this work. Also the complexity of the composite waves arising from different physical effects is an interesting feature of the model at hand. In particular, the nonlinear degeneracy of the diffusion term is an interesting part of the complexity of the phenomena studied here. For instance, as in [33], we construct waves which vanish on the right half-line.

The aim of this paper is to prove the existence of traveling waves above a minimal speed in various situations. For the clarity of the paper, we present our main results in the table below. As mentioned earlier, the incompressible cell limit corresponds to the particular case, where the pressure law (1.1) has a jump from 0 to $+\infty$ when $\rho = 1$.

The outline of this paper is as follows (see Table 1). In the next section, we present some preliminary notations and an a priori estimate resulting in a maximum principle. In Section 3, we investigate the existence of traveling waves in the simplified inviscid case $C_z = 0$, for which the model reduces to a single continuity equation for ρ. Finally, Section 4 is devoted to the study of the general case $C_z \neq 0$ in the incompressible cells limit. In both parts, some numerical simulations illustrate the theoretical results.

Table 1 The outline of this paper.

$C_z = 0$	$\epsilon = 0$	Theorem 3.1
		(Incompressible cell limit) Remark 3.1
	$\epsilon > 0$	Theorem 3.2
		(Incompressible cell limit) Remark 3.2
$C_z > 0$	$\epsilon = 0$	(Incompressible cell limit, $C_S C_p > 2C_z$) Theorem 4.1
		(Incompressible cell limit, $C_S C_p < 2C_z$) Remark 4.1
	$\epsilon > 0$	(Incompressible cell limit, $C_S C_p > 2C_z$) Theorem 4.2

2 Preliminaries

In a 1-dimensional framework, the considerations in the introduction lead to the following set of equations:

$$\begin{cases}\partial_t\rho+\partial_x(\rho v)=\Phi(\rho)+\epsilon\partial_{xx}\rho,\\ -C_S\partial_x\Sigma(\rho)=-C_z\partial_{xx}v+v.\end{cases}\tag{2.1}$$

This system is considered on the whole real line $\mathbb{R}$ and is complemented with Dirichlet boundary conditions at infinity for v and Neumann boundary condition for ρ. Here C_p, C_S, C_z stand for nonnegative rescaled constants. It will be useful for the mathematical analysis to introduce the function W that solves the elliptic problem

$$-C_z\partial_{xx}W+W=\Sigma(\rho),\quad \partial_xW(\pm\infty)=0.$$

This allows us to set $v=-C_S\partial_xW$ and rewrite the system (2.1) as

$$\begin{cases}\partial_t\rho-C_S\partial_x(\rho\partial_xW)=\Phi(\rho)+\epsilon\partial_{xx}\rho,\\ -C_z\partial_{xx}W+W=\Sigma(\rho).\end{cases}\tag{2.2}$$

We recall that the elastic pressure satisfies (1.1), and the growth function satisfies

$$\Phi(\rho)\geq 0,\quad \Phi(\rho)=0\quad \text{for } \Sigma(\rho)\geq C_p>0.\tag{2.3}$$

2.1 Maximum principle

The nonlocal aspect of the velocity in terms of ρ makes unobvious the correct way to express the maximum principle. In particular, it does not hold directly on the population density, but on the pressure $\Sigma(\rho)$.

Lemma 2.1 *Assume that Φ satisfies (2.3) and that the state equation for Σ satisfies (1.1). Then, setting $\Sigma_M^0=\max_{x\in\mathbb{R}}\Sigma(x,0)$, any classical solution to (2.2) satisfies the maximum principle*

$$\Sigma(\rho)\leq\max(\Sigma_M^0,C_p)\quad \textit{and}\quad \rho\leq\Sigma^{-1}(C_p)=:\rho_M>1,\quad \textit{if}\ \ \Sigma_M^0\leq C_p.\tag{2.4}$$

However notice that, except in the case when C_z vanishes, this problem is not monotonic, and no BV type estimates are available (see [28] for properties when $C_z=0$).

Proof Only the values on the intervals such that $\rho>1$ need to be considered. When $\rho>1$, multiplying the first equation in (2.2) by $\Sigma'(\rho)$, we find

$$\frac{\partial}{\partial t}\Sigma(\rho)-C_S\partial_x\Sigma(\rho)\partial_xW-C_S\rho\Sigma'(\rho)\partial_{xx}W=\Sigma'(\rho)\Phi(\rho)+\epsilon\partial_{xx}\Sigma(\rho)-\epsilon\Sigma''(\rho)|\partial_x\rho|^2.$$

Fix a time t, and consider a point x_0, where $\max_x\Sigma(\rho(x,t))=\Sigma(\rho(x_0,t))$ (the extension to the case that it is not attained is standard [20]). We have $\partial_x\Sigma(\rho(x_0,t))=0$, $\partial_{xx}\Sigma(\rho(x_0,t))\leq 0$, and thus we obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\max_x\Sigma(\rho(x,t))\leq{}&\Sigma'(\rho(x_0,t))\Phi(\rho(x_0,t))+C_S\rho\Sigma'(\rho(x_0,t))\partial_{xx}W(x_0,t)\\&-\epsilon\Sigma''(\rho(x_0,t))|\partial_x\rho(x_0,t)|^2.\end{aligned}$$

Consider a possible value, such that $\Sigma(\rho(x_0,t))>C_p$. Then we can treat the three terms in the right-hand side as follows:

(i) From assumption (2.3), we have $\Phi(\rho(x_0,t))=0$. Then the first term vanishes.

(ii) Also, by assumption (1.1), since $\Sigma'(\rho(x_0,t)) > 0$ for $\rho(x_0,t) \geq 1$, we have $\partial_x \rho(x_0,t) = 0$. Therefore, the third term vanishes.

(iii) Moreover, since $-C_z \partial_{xx} W(x_0,t) = \max_x \Sigma(\rho(x,t)) - W(x_0,t) \geq 0$ (by the maximum principle $W \leq \max \Sigma$), using (ii), we conclude that the second term is non-positive.

We conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t} \max_x \Sigma(\rho(x,t)) \leq 0,$$

and this proves the result.

2.2 Traveling waves

The end of this paper deals with existence of a traveling wave for model (2.2) with the growth term and definition

$$\Phi(\rho) = \rho H(C_p - \Sigma(\rho)), \quad C_p > 0, \quad \rho_M := \Sigma^{-1}(C_p) > 1. \tag{2.5}$$

There are two constant steady states $\rho = 0$ and $\rho = \rho_M := \Sigma^{-1}(C_p)$, and we look for traveling waves connecting these two stationary states. From Lemma 2.1, we may assume that the initial data satisfy $\max_x \Sigma(\rho(x,t=0)) = C_p$ and $\max_x \rho(x,t=0) = \rho_M$. Then, it is natural to obtain the following definition.

Definition 2.1 *A non-increasing traveling wave solution is a solution to the form $\rho(t,x) = \rho(x-\sigma t)$ for a constant $\sigma \in \mathbb{R}$ called the traveling speed, such that $\rho' \leq 0$, $\rho(-\infty) = \rho_M$ and $\rho(+\infty) = 0$.*

With this definition, we are led to look for (ρ, W) satisfying

$$-\sigma \partial_x \rho - C_S \partial_x(\rho \partial_x W) = \rho H(C_p - \Sigma(\rho)) + \epsilon \partial_{xx} \rho, \tag{2.6}$$

$$-C_z \partial_{xx} W + W = \Sigma(\rho), \tag{2.7}$$

$$\rho(-\infty) = \rho_M, \quad \rho(+\infty) = 0, \quad W(-\infty) = C_p, \quad W(+\infty) = 0. \tag{2.8}$$

When $C_z = 0$, (2.6)–(2.7) reduces to one single equation

$$-\sigma \partial_x \rho - C_S \partial_x(\rho \partial_x \Sigma(\rho)) = \rho H(C_p - \Sigma(\rho)) + \epsilon \partial_{xx} \rho. \tag{2.9}$$

In the sequel and in order to make the mathematical analysis more tractable, as depicted in Figure 1, we assume that Σ has the specific form given by

$$\Sigma(\rho) = \begin{cases} 0 & \text{for } \rho \leq 1, \\ C_\nu \ln \rho & \text{for } \rho \geq 1. \end{cases} \tag{2.10}$$

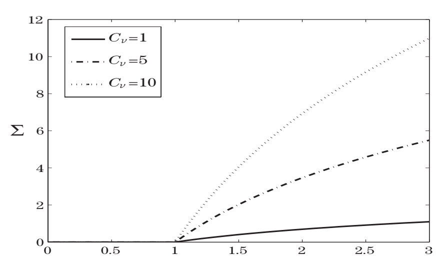


Figure 1 The equation of state as defined by (2.10) for three different values of C_ν.

This form represents logarithmic strain assuming cells of the cuboidal shape (see Appendix). The choice of logarithmic strain conserves the volume of incompressible cells for both small and large deformations. Hence, it is particularly useful as cells, because they are mainly composed of water, and are incompressible on small time scales, such that deformations leave the cell volume invariant.

We will study in particular the case $C_\nu \to +\infty$. We call it the incompressible cell limit, which is both mathematically interesting (see also the derivation of Hele-Shaw equation in [28]) and physically relevant. This limit case boils down to consider the tissue of tumor cells as an incompressible elastic material in a confined environment.

The structure of the problem (2.1) depends deeply on the parameters ϵ and C_z. It is hyperbolic for $\epsilon = C_z = 0$, parabolic when $\epsilon \neq 0$, $C_z = 0$ and coupled parabolic/elliptic in the general case. Therefore, we have to treat the cases separately.

3 Traveling Wave Without Viscosity

When the bulk viscosity is neglected, that is $C_z = 0$, the analysis is much simpler and is closely related to the Fisher/KPP equation (see [7]) with the variant of a complex composite and discontinuous wave. The unknown W can be eliminated. Taking advantage of the state equation for the pression (2.10), we can rewrite (2.9) as a self-contained equation on ρ

$$\begin{cases} -\sigma\partial_x\rho - C_S C_\nu \partial_{xx} Q(\rho) = \rho H(C_p - C_\nu(\ln\rho)_+) + \epsilon\partial_{xx}\rho, \\ \rho(-\infty) = \rho_M, \quad \rho(+\infty) = 0. \end{cases} \tag{3.1}$$

Here f_+ denotes the positive part of f and

$$Q(\rho) = \begin{cases} 0 & \text{for } \rho \le 1, \\ \rho - 1 & \text{for } \rho \ge 1. \end{cases} \tag{3.2}$$

3.1 Traveling waves for $\epsilon = 0$

When the cell motility is neglected, we can find the explicit expression for the traveling waves. More precisely, we establish the following result.

Theorem 3.1 *There exists a $\sigma^* > 0$, such that for all $\sigma \ge \sigma^*$, (3.1)–(3.2) admits a non-negative, non-increasing and discontinuous solution ρ. More precisely, when $\sigma = \sigma^*$ and up to translation, ρ is given by*

$$\rho(x) = \begin{cases} \rho_M := \exp\left(\dfrac{C_p}{C_\nu}\right), & x \le 0, \\ g(x), & x \in (0, x_0),\ x_0 > 0, \\ 0, & x > x_0, \end{cases}$$

where g is a smooth non-increasing function satisfying $g(0) = \rho_M$, $g'(0) = 0$ and $g(x_0) = 1$. Its precise expression is given in the proof.

In other words, when $C_z = 0$ and $\epsilon = 0$, (2.2) admits a nonnegative and non-increasing traveling wave (ρ, W) for $\sigma \ge \sigma^*$.

Notice that, by opposition to the Fisher/KPP equation, we do not have an analytical expression for the minimal speed. Relate that ρ vanishes for large x, a phenomena already known for degenerate diffusion.

Proof Since we are looking for a non-increasing function ρ, we decompose the line to be

$$\mathbb{R} = I_1 \cup I_2 \cup I_3, \quad I_1 = \{\rho(x) = \rho_M\}, \quad I_2 = \{1 < \rho(x) < \rho_M\}, \quad I_3 = \{\rho(x) \le 1\}.$$

Notice that, equivalently $\Sigma(x) = C_p$ in I_1. To fix the notations, we set

$$I_1 = (-\infty, 0], \quad I_2 = (0, x_0), \quad I_3 = [x_0, +\infty).$$

Step 1 (In $I_1 \cup I_2$) ρ satisfies

$$-\sigma\partial_x\rho - C_S C_\nu \partial_{xx}\rho = \rho H(C_p - C_\nu(\ln\rho)_+). \tag{3.3}$$

Therefore, by elliptic regularity, we deduce that the second derivative of ρ is bounded, and therefore $\rho \in C^1(-\infty, x_0)$. On I_1, the function ρ is a constant and by continuity of ρ and $\partial_x\rho$ at $x = 0$, we have the boundary conditions of I_2, such that

$$\rho(0) = \rho_M, \quad \partial_x\rho(0) = 0. \tag{3.4}$$

In I_2, $H(C_p - C_\nu(\ln\rho)_+) = 1$. Solving (3.3) with the boundary conditions in (3.4), we find that if $\sigma > 2\sqrt{C_S C_\nu}$, then

$$\rho(x) = \rho_M \mathrm{e}^{-\frac{\sigma x}{2C_S C_\nu}} \Big(A \exp\Big(\frac{\sqrt{\sigma^2 - 4C_S C_\nu}}{2C_S C_\nu} x\Big) + B \exp\Big(-\frac{\sqrt{\sigma^2 - 4C_S C_\nu}}{2C_S C_\nu} x\Big)\Big)$$

with

$$A = \frac{\sigma + \sqrt{\sigma^2 - 4C_S C_\nu}}{2\sqrt{\sigma^2 - 4C_S C_\nu}}, \quad B = \frac{-\sigma + \sqrt{\sigma^2 - 4C_S C_\nu}}{2\sqrt{\sigma^2 - 4C_S C_\nu}}.$$

In this case, ρ is decreasing for $x > 0$ and vanishes as $x \to +\infty$. Thus there exists a positive x_0, such that $\rho(x_0) = 1$.

When $\sigma < 2\sqrt{C_S C_\nu}$, the solution is

$$\rho(x) = \rho_M \mathrm{e}^{-\frac{\sigma x}{2C_S C_\nu}} \Big(A \cos\Big(\frac{\sqrt{4C_S C_\nu - \sigma^2}}{2C_S C_\nu} x\Big) + B \sin\Big(\frac{\sqrt{4C_S C_\nu - \sigma^2}}{2C_S C_\nu} x\Big)\Big) \tag{3.5}$$

with

$$A = 1, \quad B = \frac{\sigma}{\sqrt{4C_S C_\nu - \sigma^2}}.$$

By a straightforward computation, we deduce

$$\partial_x\rho(x) = -\frac{2\rho_M}{\sqrt{4C_S C_\nu - \sigma^2}} \mathrm{e}^{-\frac{\sigma x}{2C_S C_\nu}} \sin\Big(\frac{\sqrt{4C_S C_\nu - \sigma^2}}{2C_S C_\nu} x\Big).$$

Thus ρ is decreasing in $\big(0, \frac{2C_S C_\nu}{\sqrt{4C_S C_\nu - \sigma^2}}\pi\big)$, and takes negative values at the largest endpoint. There exists an $x_0 > 0$, such that $\rho(x_0) = 1$.

Finally, when $\sigma = 2\sqrt{C_S C_\nu}$, we reach the same conclusion because

$$\rho(x) = \rho_M \Big(\frac{x}{\sqrt{C_S C_\nu}} + 1\Big) \mathrm{e}^{-\frac{x}{\sqrt{C_S C_\nu}}}.$$

Step 2 (On I_3) In $(x_0, +\infty)$, we have $\Sigma = 0$ and $Q(\rho) = 0$ from (3.2). Then equation (3.1) is

$$-\sigma\partial_x\rho = \rho. \tag{3.6}$$

We can write the jump condition at x_0 by integrating (3.1) from x_0^- to x_0^+, which is

$$-\sigma[\rho]_{x_0} - C_S C_\nu [\partial_x Q(\rho)]_{x_0} = 0, \quad \sigma(\rho(x_0^+) - 1) = C_S C_\nu \partial_x \rho(x_0^-).$$

Here $\partial_x \rho(x_0^-) < 0$ can be found, due to the expression of ρ on I_2 as computed above. Thus, we get $\rho(x_0^+)$, which is the boundary condition of (3.6). Then the Cauchy problem (3.6) gives

$$\rho(x) = \Big(1 + \frac{C_S C_\nu}{\sigma} \partial_x \rho(x_0^-)\Big) \mathrm{e}^{-\frac{x}{\sigma}}, \quad x \in I_3.$$

In summary, when $\epsilon = 0$, a nonnegative solution to (3.1) exists under the necessary and sufficient condition

$$\sigma \geq -C_S C_\nu \partial_x \rho(x_0^-). \tag{3.7}$$

The right-hand side also depends on σ. Therefore, it does not obviously imply $\sigma \geq \sigma^*$. To reach this conclusion, and conclude the proof, we shall use Lemma 3.1 below.

Lemma 3.1 *Using the notation in the proof of Theorem* 3.1, *the function* $\sigma \mapsto -C_S C_\nu \partial_x \rho(x_0^-)$ *is nonincreasing. Therefore, there exists a minimal traveling wave velocity* σ^*, *and* (3.7) *is satisfied if and only if* $\sigma \geq \sigma^*$.

Proof We consider (3.3) in $I_2 = (0, x_0)$. We notice that on this interval, $\rho(x)$ is decreasing, and therefore is one to one from $(0, x_0)$ to $(\rho_M, 1)$. We denote by $X(\rho)$ its inverse. Let us define $V = -C_S C_\nu \partial_x \rho$. In I_2, V is nonnegative, and (3.3) can be written as

$$\partial_x V = \sigma \partial_x \rho + \rho = -\frac{V}{C_S C_\nu} \sigma + \rho. \tag{3.8}$$

Setting $\widetilde{V}(\rho) = V(X(\rho))$, by definition of V, we have

$$\partial_\rho \widetilde{V} = \partial_x V \partial_\rho X = \frac{\partial_x V}{\partial_x \rho} = -\partial_x V \frac{C_S C_\nu}{V}.$$

By using (3.8), we finally get the differential equation

$$\begin{cases} \partial_\rho \widetilde{V} = \sigma - \dfrac{C_S C_\nu \rho}{\widetilde{V}} & \text{for } \rho \in (1, \rho_M), \\ \lim\limits_{\rho \to \rho_M} \widetilde{V}(\rho_M) = -C_S C_\nu \partial_x \rho(0) = 0. \end{cases} \tag{3.9}$$

This differential equation has a singularity at ρ_M. We then introduce $z = \rho_m - \rho$ and $Y(z) = \frac{1}{2}\widetilde{V}^2(\rho_M - z)$ for $z \in (0, \rho_M - 1)$. (3.9) becomes

$$\begin{cases} Y'(z) = -\sigma\sqrt{2Y(z)} + C_S C_\nu (\rho_M - z) & \text{for } z \in (0, \rho_M - 1), \\ Y(0) = 0. \end{cases}$$

This ordinary differential equation belongs to the class $Y' = F(z, Y)$ with F one sided Lipschitz in his second variable and $\partial_Y F(z, Y) \leq 0$. Therefore, we can define a unique solution to the above Cauchy problem. Hence there exists a unique nonnegative solution $\widetilde{V}$ to (3.9).

Define $U(\rho) := \frac{\partial \widetilde{V}}{\partial \sigma}$, and our goal is to determine the sign of $U(1)$. We have

$$\frac{\partial^2 \widetilde{V}}{\partial \rho \partial \sigma} = \frac{\partial}{\partial \sigma}\Big(\sigma - \frac{C_S C_\nu}{\widetilde{V}} \rho\Big) = 1 + \frac{C_S C_\nu}{\widetilde{V}^2} \rho \frac{\partial \widetilde{V}}{\partial \sigma}.$$

Then $U(\rho)$ solves on $(1, \rho_M)$,

$$\frac{\partial U}{\partial \rho} = 1 + \frac{C_S C_\nu}{\widetilde{V}^2}\rho U. \tag{3.10}$$

Moreover, we have

$$U(\rho_M) = \frac{\partial \widetilde{V}(\rho_M)}{\partial \sigma} = 0. \tag{3.11}$$

Assume $U(1) > 0$. Let us define $\rho_1 = \sup\{\rho_2 \mid \rho \in (1, \rho_2)$, such that $U(\rho) \geq 0\}$. Then from (3.10), $\frac{\partial U}{\partial \rho}(\rho) \geq 1$ on $(1, \rho_1)$, and thus $U(\rho_1) > U(1) > 0$. By continuity, we should necessarily have $\rho_1 = \rho_M$. However, we then have $\frac{\partial U}{\partial \rho}(\rho) \geq 1$ for all $\rho \in (1, \rho_M)$, which is a contradiction to $U(\rho_M) = 0$. Therefore, $U(1) \leq 0$ and $\widetilde{V}$ is nonincreasing with respect to σ.

Structural Stability Theorem 3.1 shows that there is an infinity of traveling wave solutions. However, as in the Fisher/KPP equation, most of them are unstable. For instance, we can consider some kind of "ignition temperature" approximation to the system (3.1), such that

$$-\sigma \partial_x \rho_\theta - C_S C_\nu \partial_{xx} Q(\rho_\theta) = \xi_\theta(\rho_\theta) H(C_p - C_\nu (\ln \rho_\theta)_+), \tag{3.12}$$

where $\theta \in (0, 1)$ is a small positive parameter and

$$\xi_\theta(\rho) = \begin{cases} \rho & \text{for } \ \rho \in (\theta, \rho_M), \\ 0 & \text{for } \ \rho \in [0, \theta]. \end{cases} \tag{3.13}$$

Then we have the following result.

Lemma 3.2 *Equations* (3.12)–(3.13) *admit a unique couple of solution* $(\sigma_\theta, \rho_\theta)$ *and* $\sigma_\theta \to \sigma^*$ *as* $\theta \to 0$.

Proof As in Theorem 3.1, we solve (3.12) by using the decomposition $\mathbb{R} = I_1 \cup I_2 \cup I_3$. In $I_1 \cup I_2$, $\rho \geq 1 > \theta$, and therefore, ρ is given by the same formula as computed in the proof of Theorem 3.1. On I_3, (3.12) becomes

$$-\sigma_\theta \partial_x \rho_\theta = \xi_\theta(\rho_\theta). \tag{3.14}$$

By contradiction, if $\rho_\theta(x_0^+) \geq \theta$, then (3.14) implies $\rho_\theta(x) = \rho_\theta(x_0^+) \mathrm{e}^{-\frac{x - x_0}{\sigma}}$. Thus, there exists an x_θ, such that $\rho_\theta(x) \leq \theta$ for $x \geq x_\theta$. Then the right-hand side of (3.14) vanishes for $x \geq x_\theta$, and ρ_θ is the constant for $x \geq x_\theta$. This constant has to vanish from the condition at infinity, which contradicts the continuity of ρ_θ. Thus, $\rho_\theta(x_0^+) < \theta$, and (3.14) implies that $\partial_x \rho_\theta = 0$. We conclude that $\rho_\theta = 0$ on I_3. The jump condition at the interface $x = x_0$ gives

$$\sigma_\theta(\rho_\theta(x_0^+) - 1) = C_S C_\nu \partial_x \rho_\theta(x_0^-),$$

which, together with $\rho(x_0^+) = 0$, indicates that

$$\sigma_\theta = -C_S C_\nu \partial_x \rho_\theta(x_0^-).$$

According to Lemma 3.1, there exists a unique σ_θ^*, satisfying the equality above, so does a unique ρ_θ.

Letting $\theta \to 0$ in this formula, we recover the equality case in (3.7) that defines the minimal speed in Theorem 3.1. By continuity of the unique solution, we find $\sigma_\theta \to \sigma^*$.

Remark 3.1 (Incompressible Cells Limit) In the incompressible cells limit $C_\nu \to +\infty$, we can obtain an explicit expression of the traveling wave from Theorem 3.1. Since $\rho_M = \exp\big(\frac{C_p}{C_\nu}\big) \to 1$, we have $\rho(x) \to 1$ in $I_1 \cup I_2$, but Σ carries more structural information. In the first step of the proof, for large C_ν, by using (3.5), we find

$$\Sigma(x) = C_\nu \ln(\rho) \to C_p - \frac{x^2}{2C_S}.$$

We recall that the point x_0 is such that $\rho(x_0) = 1$ or $\Sigma(x_0) = 0$. Therefore, $x_0 = \sqrt{2C_S C_p}$ and

$$C_\nu \partial_x \rho(x_0^-) = \partial_x \Sigma(x_0^-) \to -\sqrt{\frac{2C_p}{C_S}}, \qquad \text{as } C_\nu \to +\infty.$$

Thus $\sigma^* \to \sqrt{2C_p C_S}$. We conclude that, on $I_3 = [x_0, +\infty)$, $\rho(x) \to \big(1 - \frac{\sqrt{2C_pC_S}}{\sigma}\big)\mathrm{e}^{-\frac{x}{\sigma}}$.

3.2 Traveling wave when $\epsilon \neq 0$

We can extend Theorem 3.1 to the case $\epsilon \neq 0$.

Theorem 3.2 *There exists a $\sigma^* > 2\sqrt{\epsilon}$, such that for all $\sigma \geq \sigma^*$, (3.1)–(3.2) admits a nonnegative, non-increasing and continuous solution ρ.*

Thus when $C_z = 0$, system (2.2) admits a nonnegative and non-increasing traveling wave (ρ, W) for $\sigma \geq \sigma^*$.

Proof We follow the proof of Theorem 3.1 and decompose $\mathbb{R} = I_1 \cup I_2 \cup I_3$. Due to the diffusion term in (3.1), $\rho \in C^0(\mathbb{R})$, and we will use the continuity of ρ at the interfaces.

On $I_1 = (-\infty, 0]$, we have $\rho = \rho_M$ and $\Sigma = C_p$.

In $I_2 = (0, x_0)$, the equation (3.1) implies

$$(C_S C_\nu + \epsilon)\partial_{xx}\rho + \sigma \partial_x \rho + \rho = 0, \quad \rho(0) = \rho_M, \quad \partial_x \rho(0) = 0.$$

Therefore, we get the same expressions for ρ on I_2 as in the proof of Theorem 3.1, except that we replace $C_S C_\nu$ by $C_S C_\nu + \epsilon$. Thus, as before, there exists a positive x_0, such that $\rho(x_0) = 1$, and ρ is decreasing in $(0, x_0)$.

On $I_3 = [x_0, +\infty)$, we solve

$$\epsilon \partial_{xx}\rho + \sigma \partial_x \rho + \rho = 0. \tag{3.15}$$

At the interface $x = x_0$, integrating from x_0^- to x_0^+ in (3.1) and using the continuity of ρ, we get

$$C_S C_\nu [\partial_x Q(\rho)]_{x_0} + \epsilon[\partial_x \rho]_{x_0} = 0,$$

that is,

$$\partial_x \rho(x_0^+) = \Big(1 + \frac{C_S C_\nu}{\epsilon}\Big)\partial_x \rho(x_0^-). \tag{3.16}$$

Solving (3.15) with the boundary conditions $\rho(x_0^+) = 1$ and (3.16), we get that if $\sigma < 2\sqrt{\epsilon}$, then ρ is the sum of the trigonometric functions, and therefore will take negative values. Thus $\sigma \geq 2\sqrt{\epsilon}$. In the case $\sigma > 2\sqrt{\epsilon}$,

$$\rho(x) = A \exp\Big(\frac{-\sigma + \sqrt{\sigma^2 - 4\epsilon}}{2\epsilon}(x - x_0)\Big) + B \exp\Big(\frac{-\sigma - \sqrt{\sigma^2 - 4\epsilon}}{2\epsilon}(x - x_0)\Big),$$

where

$$A = \frac{1}{2} + \frac{1}{\sqrt{\sigma^2 - 4\epsilon}}\Big(\frac{\sigma}{2} + (\epsilon + C_S C_\nu)\partial_x \rho(x_0^-)\Big), \quad B = \frac{1}{2} - \frac{1}{\sqrt{\sigma^2 - 4\epsilon}}\Big(\frac{\sigma}{2} + (\epsilon + C_S C_\nu)\partial_x \rho(x_0^-)\Big).$$

After detailed calculation of $\partial_x \rho$ and using $\partial_x \rho(x_0^-) < 0$, we have that ρ is a nonnegative and nonincreasing function if and only if $A \geq 0$, that is,

$$\sqrt{\sigma^2 - 4\epsilon} + \sigma + 2(\epsilon + C_S C_\nu)\partial_x \rho(x_0^-) \geq 0, \quad \sigma > 2\sqrt{\epsilon}. \tag{3.17}$$

In the case $\sigma = 2\sqrt{\epsilon}$, we have

$$\rho(x) = \Big(\Big(\frac{1}{\sqrt{\epsilon}} + \Big(1 + \frac{C_S C_\nu}{\epsilon}\Big)\partial_x \rho(x_0^-)\Big)(x - x_0) + 1\Big)\mathrm{e}^{-\frac{x-x_0}{\sqrt{\epsilon}}}.$$

Thus ρ is a nonnegative and non-increasing function if and only if

$$\frac{1}{\sqrt{\epsilon}} + \Big(1 + \frac{C_S C_\nu}{\epsilon}\Big)\partial_x \rho(x_0^-) \geq 0,$$

which is the same condition as (3.17) by setting $\sigma = 2\sqrt{\epsilon}$. Thus (3.17) is valid for $\sigma \geq 2\sqrt{\epsilon}$. Denoting $U_\epsilon(x) = -(\epsilon + C_S C_\nu)\partial_x \rho(x)$, condition (3.17) can be rewritten as

$$\sigma \geq \mathfrak{F}[\sigma] := \max\Big(2\sqrt{\epsilon}, \min\Big(2U_\epsilon(x_0^-), U_\epsilon(x_0^-) + \frac{\epsilon}{U_\epsilon(x_0^-)}\Big)\Big). \tag{3.18}$$

By a straightforward adaptation of Lemma 3.1, we conclude that $\sigma \mapsto U_\epsilon(x_0^-)$ is nonincreasing with respect to σ. When $U_\epsilon(x_0^-) > \sqrt{\epsilon}$, we have

$$\mathfrak{F}[\sigma] = U_\epsilon(x_0^-) + \frac{\epsilon}{U_\epsilon(x_0^-)}.$$

Then $\mathfrak{F}[\sigma]$ is an increasing function with respect to $U_\epsilon(x_0^-)$ for $U_\epsilon(x_0^-) > \sqrt{\epsilon}$. Together with $\sigma \to U_\epsilon(x_0^-)$ being nonincreasing, $\mathfrak{F}[\sigma]$ is nonincreasing with respect to σ. For the case $U_\epsilon(x_0^-)^2 < \epsilon$, we have

$$\mathfrak{F}[\sigma] = 2\sqrt{\epsilon}.$$

Therefore, for all $\sigma \in (0, +\infty)$, $\mathfrak{F}[\sigma]$ is a nonincreasing function of σ. Hence, there exists a unique σ^*, such that (3.18) is satisfied for every $\sigma \geq \sigma^*$.

Structural Stability We can again select a unique traveling wave when we approximate the growth term by $\xi_\theta(\rho)H(C_p - C_\nu(\ln\rho)_+)$. This can be obtained by considering $\epsilon\partial_{xx}\rho_\theta + \sigma\partial_x\rho_\theta + \xi_\theta(\rho_\theta) = 0$ instead of (3.15) and by matching the values of $\partial_x \rho$ on both sides at the point where $\rho = \theta$. Then, the equality in (3.17) holds, and one unique velocity is selected. As for (3.12), we let $\theta \to 0$, and the minimum traveling velocity σ^* is selected. Then the remark below follows.

Remark 3.2 (Incompressible Cells Limit) In the limit $C_\nu \to +\infty$, we have $\rho(x) \to 1$ in $I_2 = (0, x_0)$ and

$$\Sigma(x) = C_\nu \ln(\rho) \to C_p - \frac{x^2}{2C_S}.$$

Therefore, $x_0 = \sqrt{2C_S C_p}$ and

$$C_\nu \partial_x \rho(x_0^-) = \partial_x \Sigma(x_0^-) \to -\sqrt{\frac{2C_p}{C_S}}, \quad \text{when } C_\nu \to +\infty.$$

Thus (3.17) becomes, for $\sigma \geq 2\sqrt{\epsilon}$,

$$\sqrt{\sigma^2 - 4\epsilon} + \sigma \geq 2\sqrt{2C_pC_S},$$

and we conclude, in this incompressible cells limit, that σ^* is defined by

$$\sigma^* := \max\Big(2\sqrt{\epsilon}, \min\Big(2\sqrt{2C_pC_S}, \sqrt{2C_pC_S} + \frac{\epsilon}{\sqrt{2C_pC_S}}\Big)\Big). \tag{3.19}$$

The kink induced by this formula is a very typical qualitative feature that is recovered in numerical simulations (see Table 2).

Table 2 Numerical values for the traveling speed σ^* with different parameters for $C_\nu = 17.114$ obtained by solving the evolution equation. We observe that the numerical speeds are close to $\sqrt{2C_pC_S} + \frac{\epsilon}{\sqrt{2C_pC_S}}$ or $2\sqrt{\epsilon}$ as computed in (3.19). In the first four lines $\epsilon < 2C_pC_S$, while in the last two $\epsilon > 2C_pC_S$.

C_p	C_S	ϵ	$\sqrt{2C_pC_S} + \frac{\epsilon}{\sqrt{2C_pC_S}}$	$2\sqrt{\epsilon}$	σ^*
0.57	0.001	0.001	0.0634	0.0632	0.0615
0.57	0.01	0.001	0.1161	0.0632	0.1155
1	0.01	0.001	0.1485	0.0632	0.1472
1	0.01	0.01	0.2121	0.200	0.2113
1	0.01	0.1	0.8485	0.632	0.5946
1	0.01	1	7.2125	2.000	1.9069

3.3 Numerical results

In order to perform numerical simulations, we consider a large computational domain $\Omega = [-L, L]$, and we discretize it with a uniform mesh

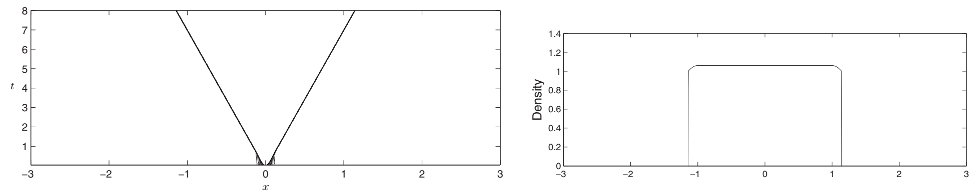
$$\Delta x = \frac{L}{2M}, \quad x_i = i\Delta x, \quad i = -M, \cdots, 0, \cdots, M.$$

We simulate the time evolutionary equation (2.2) with $C_z = 0$ and Neumann boundary conditions. Our algorithm is based on a splitting method. Firstly, we discretize $\partial_t\rho - C_S\partial_{xx}Q(\rho) = 0$ by using the explicit Euler method in time and the second-order centered finite differences in space. After updating ρ^n for one time step, we denote the result by $\rho^{n+\frac{1}{2}}$. Secondly, we solve $\partial_t\rho = \rho H(C_p - \Sigma(\rho))$ by the explicit Euler scheme again, using $\rho^{n+\frac{1}{2}}$ as the initial condition. Then we get ρ^{n+1}.

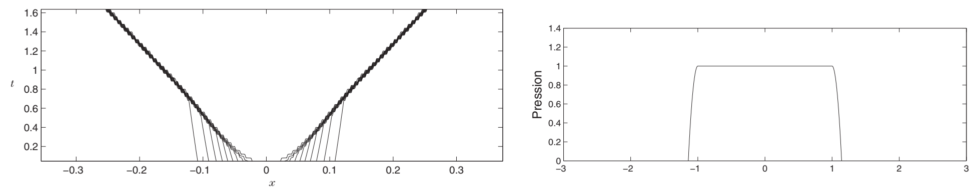
The numerical initial density ρ is a small Gaussian in the center of the computational domain. We take

$$L = 3, \quad C_\nu = 17.114, \quad C_S = 0.01, \quad C_p = 1. \tag{3.20}$$

The numerical traveling wave solution when $C_z = 0$, $\epsilon = 0$ is depicted in Figure 2. We can see that the two fronts propagate in opposite directions with a constant speed. The right propagating front of ρ has a jump from 1 to 0, whereas Σ is continuous, but its derivative $\partial_x\Sigma$ has a jump at the front. Figure 3 presents the numerical results of $C_z = 0$, $\epsilon = 0.02$, where ρ becomes continuous and the front shape of Σ stays the same as for $\epsilon = 0$. Comparing Figures 2 and 3, when there is diffusion, the traveling velocity becomes bigger and the density has a tail.



(a) The solution isolines. (c) Population density.



(b) The zoom in of the subplot in (a). (d) Pressure.

Figure 2 The traveling wave solution for $C_z = 0$, $\epsilon = 0$. The parameters are chosen as in (3.20). In (a) and (b), the horizontal axis is x, and the vertical axis is time. (c) and (d) show the traveling front at $T = 8$.

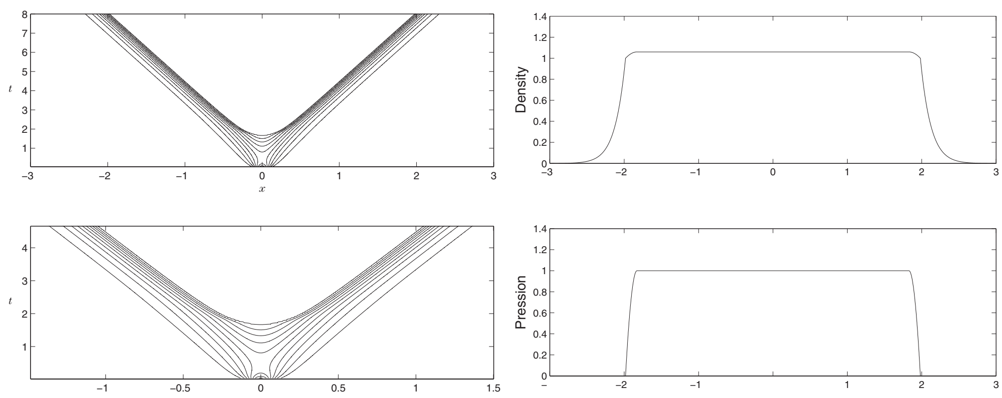


Figure 3 As Figure 2 with $C_z = 0,\ \epsilon = 0.02$.

The numerical traveling velocities for different parameters are given in Table 2, where we can compare them with the analytical formula (3.19) in the incompressible cells limit.

4 Traveling Wave with Viscosity

When $C_z \neq 0$, we can not eliminate the unknown, and we have to deal with the whole system

$$\begin{cases} -\sigma\partial_x\rho - C_S\partial_x\rho\partial_x W - C_S\rho\partial_{xx}W = \rho H(C_p - \Sigma) + \epsilon\partial_{xx}\rho, \\ -C_z\partial_{xx}W + W = \Sigma(\rho), \\ \rho(-\infty) = \rho_M, \quad \rho(+\infty) = 0; \quad W(-\infty) = C_p, \quad W(+\infty) = 0 \end{cases} \tag{4.1}$$

still with the equation of state (2.10). In the interval $\{\rho \geq 1\}$, multiplying (2.6) by $\Sigma'(\rho) = \frac{C_\nu}{\rho}$, we get

$$-\sigma\partial_x\Sigma - C_S\partial_x\Sigma\partial_x W - C_S C_\nu\partial_{xx}W = C_\nu H(C_p - \Sigma) + \epsilon\frac{C_\nu}{\rho}\partial_{xx}\rho \quad \text{for} \quad \rho \geq 1. \tag{4.2}$$

Here the situation is much more complicated, and a new phenomenon appears. We need to clarify the meaning of the discontinuous growth term, when $\Sigma = C_p$, which occurs on an interval and is not well defined in the singular incompressible cells limit as we study here (see (4.5) below). To do so, we use a linear smoothing of the Heaviside function H, such that

$$H_\eta(u) = \min\Big(1, \frac{1}{\eta}u\Big) \quad \text{for } \eta \in (0, C_p). \tag{4.3}$$

There are no explicit or semi-explicit solutions for the traveling waves in general due to the non-local aspect of the field W, and we refer to [27] again for a proof of existence in a related case. Thus we will consider the incompressible cells limit. First, we derive formally the limiting system by letting $C_\nu \to +\infty$. From the state equation, we have $1 \le \rho \le \rho_M \to 1$. Therefore, we need to distinguish the two cases, i.e., $\rho = 1$ and $\rho < 1$. Formally when $\rho < 1$, we find that $\Sigma = 0$, and (4.1) reduces to

$$\begin{cases} -\sigma\partial_x\rho - C_S\partial_x\rho\partial_x W - C_S\rho\partial_{xx}W = \rho + \epsilon\partial_{xx}\rho, & \rho < 1, \\ -C_z\partial_{xx}W + W = 0. \end{cases} \tag{4.4}$$

On the interval, where $\rho = 1$, as $C_\nu \to +\infty$, and the function Σ is not defined in terms of ρ and is left unknown, the formal limit of (4.1) implies a coupled system on W and Σ,

$$\begin{cases} -C_S\partial_{xx}W = H_\eta(C_p - \Sigma), & \rho = 1, \\ -C_z\partial_{xx}W + W = \Sigma. \end{cases} \tag{4.5}$$

Then the existence of traveling waves in the asymptotic case $C_\nu \to +\infty$ boils down to studying the asymptotic system (4.4)–(4.5). As in Section 3, the structure of the problem invites us to distinguish between the two cases, i.e., $\epsilon = 0$ and $\epsilon \neq 0$.

4.1 Case $\epsilon = 0$

Existence of traveling wave in the limit $C_\nu \to +\infty$ In this case, we can establish the following theorem.

Theorem 4.1 *Assume $C_z \neq 0$, $\epsilon = 0$ and $C_SC_p > 2C_z$. Then there exists a $\sigma^* > 0$, such that for all $\sigma \ge \sigma^*$, the asymptotic system* (4.4)–(4.5) *admits a nonnegative and non-increasing solution (ρ, Σ). Furthermore, when $\eta \to 0$, we have $\sigma^* = \sqrt{2C_SC_p} - \sqrt{C_z}$, and the solution is given by*

$$\Sigma(x) = \begin{cases} C_p, & x \le 0, \\ -\dfrac{x^2}{2C_S} - \dfrac{x}{C_S}\sqrt{C_z} + C_p, & 0 < x \le \sqrt{2C_SC_p} - 2\sqrt{C_z} =: x_0, \\ 0, & x > x_0. \end{cases} \tag{4.6}$$

Therefore, Σ has a jump from $\sqrt{\frac{2C_pC_z}{C_S}}$ to 0 at x_0. The population density satisfies

$$\begin{cases} \rho = 1 & \text{for } \ x < x_0, \\ \rho = 0 & \text{for } \ x > x_0, \ \text{ when } \sigma = \sigma^*, \\ \rho = (\sigma - \sigma^* \mathrm{e}^{-\frac{x - x_0}{\sqrt{C_z}}})^{-1-\frac{\sqrt{C_z}}{\sigma}}\mathrm{e}^{-\frac{x-x_0}{\sigma}} & \text{for } \ x > x_0, \ \text{ when } \sigma > \sigma^*. \end{cases}$$

Proof By the maximum principle in Lemma 2.1, and according to Definition 2.1, Σ is bounded by C_p and is nonnegative. Therefore, due to elliptic regularity, $\partial_{xx}W$ is bounded, and

W and $\partial_x W$ are continuous. Following the idea in the proof of Theorem 3.1 or Theorem 3.2, we look for a nonnegative and non-increasing traveling wave defined in $\mathbb{R} = I_1 \cup I_2 \cup I_3$, which has the following form:

(1) On $I_1 = (-\infty, 0]$, we have $\Sigma \in [C_p - \eta, C_p]$, so that the growth term is given by $H_\eta(C_p - \Sigma) = \frac{1}{\eta}(C_p - \Sigma)$.

(2) In $I_2 = (0, x_0)$, we have $\Sigma \in (0, C_p - \eta)$. Thus $H_\eta(C_p - \Sigma) = 1$ and $\rho = 1$.

(3) On $I_3 = [x_0, +\infty)$, we have $\rho < 1$ and $\Sigma = 0$.

On I_1, we have $\rho = 1$, and we solve (4.5). This system can be written as

$$-C_S \partial_{xx} W = \frac{1}{\eta}(C_p - \Sigma), \quad -C_z \partial_{xx} W + W = \Sigma.$$

Eliminating Σ in this system gives

$$-(\eta C_S + C_z)\partial_{xx} W + W = C_p.$$

Together with the boundary conditions of W at $-\infty$, we have

$$W = C_p + A e^{\frac{x}{\sqrt{\eta C_S + C_z}}} \quad \text{and} \quad \Sigma = C_p + \frac{\eta C_S A}{\eta C_S + C_z} e^{\frac{x}{\sqrt{\eta C_S + C_z}}},$$

which is the bounded solution on $I_1 = (-\infty, 0]$. The constant A can be determined as follows. Since Σ depends continuously on ρ and $\rho = 1$ in $I_1 \cup I_2$, Σ is continuous at x_0. Therefore, A is computed by fixing $\Sigma(0) = C_p - \eta$, which gives $A = -\eta - \frac{C_z}{C_S}$.

In I_2, we still have $\rho = 1$. (4.5) can be written as

$$-C_S \partial_{xx} W = 1, \quad -C_z \partial_{xx} W + W = \Sigma.$$

At the interface $x = 0$, W and $\partial_x W$ are continuous and given by their values on I_1. Then we can solve the first equation that gives

$$W(x) = -\frac{x^2}{2C_S} - \frac{x}{C_S}\sqrt{\eta C_S + C_z} + C_p - \eta - \frac{C_z}{C_S}. \tag{4.7}$$

Injecting this expression in the second equation implies

$$\Sigma(x) = -\frac{x^2}{2C_S} - \frac{x}{C_S}\sqrt{\eta C_S + C_z} + C_p - \eta.$$

On I_3, since $\rho < 1$, we have to solve (4.4) with $\epsilon = 0$. The second equation in (4.4) can be solved easily, and the only solution which is bounded on $(x_0, +\infty)$ is

$$W(x) = W(x_0) e^{-\frac{x - x_0}{\sqrt{C_z}}}. \tag{4.8}$$

We fix the value of x_0 by using the continuity of W and the derivative of W at x_0. From (4.8), we have $-\frac{W(x_0)}{\sqrt{C_z}} = \partial_x W(x_0)$. From (4.7), this equality can be rewritten as

$$\frac{1}{\sqrt{C_z}}\left(\frac{x_0^2}{2C_S} + \frac{x_0}{C_S}\sqrt{\eta C_S + C_z} - C_p + \eta + \frac{C_z}{C_S}\right) = -\frac{x_0}{C_S} - \frac{1}{C_S}\sqrt{\eta C_S + C_z}.$$

This is a second order equation for x_0, whose only nonnegative solution (for η small enough) is

$$x_0 = \sqrt{2C_p C_S - \eta C_S} - \sqrt{C_z} - \sqrt{C_z + \eta C_S}. \tag{4.9}$$

Now we determine the expression for ρ on I_3. The jump condition of (4.4) at x_0 in the case $\epsilon = 0$ can be written as $\sigma[\rho]_{x_0} + C_S[\rho\partial_x W]_{x_0} = 0$. The continuity of $\partial_x W$ implies

$$[\rho]_{x_0} = 0 \quad \text{or} \quad \sigma = \sigma^* := -C_S\partial_x W(x_0) = x_0 + \sqrt{\eta C_S + C_z} = \sqrt{2C_pC_S - \eta C_S} - \sqrt{C_z}.$$

From the expression (4.8), the first equation in (4.4) with $\epsilon = 0$ gives

$$\Big(\sigma - \sigma^* \mathrm{e}^{-\frac{x-x_0}{\sqrt{C_z}}}\Big)\partial_x\rho + \Big(1 + \frac{\sigma^*}{\sqrt{C_z}}\mathrm{e}^{-\frac{x-x_0}{\sqrt{C_z}}}\Big)\rho = 0. \tag{4.10}$$

Looking for a non-increasing and nonnegative ρ implies that we should have $\sigma \geq \sigma^*$. After straightforward computation, we get that

$$\partial_x\rho = -\partial_x\Big(\frac{x-x_0}{\sigma} + \Big(1 + \frac{\sqrt{C_z}}{\sigma}\Big)\ln\Big(\sigma - \sigma^*\mathrm{e}^{-\frac{x-x_0}{\sqrt{C_z}}}\Big)\Big)\rho. \tag{4.11}$$

If $[\rho]_{x_0} = 0$ and $\sigma > \sigma^*$, the Cauchy problem (4.11) with $\rho(x_0) = 1$ admits a unique solution, which is given by

$$\rho(x) = \Big(\sigma - \sigma^*\mathrm{e}^{-\frac{x-x_0}{\sqrt{C_z}}}\Big)^{-1-\frac{\sqrt{C_z}}{\sigma}} \mathrm{e}^{-\frac{x-x_0}{\sigma}}.$$

When $\sigma = \sigma^*$, the factor of ρ on the right-hand side of (4.11) has a singularity at $x = x_0$. Therefore, the only solution which does not blow up in $x = x_0$ is $\rho = 0$.

Remark 4.1 When $\sqrt{2C_pC_S} < 2\sqrt{C_z}$, Σ becomes a step function with a jump from C_p to 0 at the point x_0. The corresponding traveling speed is

$$\sigma = -C_S\partial_x W(x_0) = \frac{C_pC_S}{2\sqrt{C_z}}$$

with

$$W(x) = \begin{cases} \frac{C_p}{2}\mathrm{e}^{-\frac{1}{\sqrt{C_z}}(x-x_0)}, & x > x_0, \\ C_p - \frac{C_p}{2}\mathrm{e}^{\frac{1}{\sqrt{C_z}}(x-x_0)}, & x < x_0. \end{cases}$$

The calculations are similar, but simpler than those in Theorem 4.1.

Remark 4.2 (Comparison with the Case $C_z = 0$) In the asymptotic $\eta \to 0$, and when $C_z \to 0$, the expression for σ^* in Theorem 4.1 converges to that obtained for $C_z = 0$. However, we notice that, contrary to the case $C_z = 0$, the growth term does not vanish on I_1 whereas $\Sigma = C_p$. In fact, if the growth term was zero on I_1, then since $\Sigma = C_p$, we would have $\partial_x\Sigma = 0$ and (4.2) gives

$$-C_SC_\nu\partial_{xx}W = 0.$$

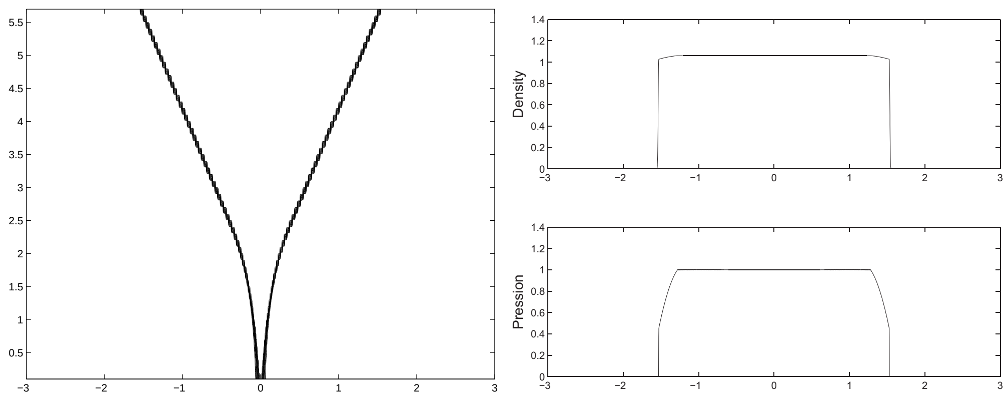
Thus $\partial_{xx}W = 0$ and $W = \Sigma$ on I_1, which can not hold true. That is why we can not use the Heaviside function in the growth term when $\Sigma = C_p$, and the linear approximation in (4.3) allows us to make explicit calculations.

Numerical Results We present some numerical simulations of the full model (2.2) with the growth term $\Phi = \rho H(C_p - \Sigma(\rho))$ and $\epsilon = 0$. As in the previous section, we consider a computational domain $\Omega = [-L, L]$ discretized by a uniform mesh, and use Neumann boundary conditions. System (2.2) is now a coupling of a transport equation for ρ and an elliptic equation for W. We use the following schemes:

(1) The centered three point finite difference method is used to discretize the equation for W.

(2) A splitting method is implemented to update ρ. Firstly, we use a first order upwind discretization of the term $-C_S\partial_x(\rho\partial_x W)$ (i.e., without the right-hand side). Secondly, we solve the growth term $\partial_t\rho = \rho H(C_p - \Sigma(\rho))$ with an explicit Euler scheme.

As before, starting from a Gaussian at the middle of the computational domain, Figure 4 shows the numerical traveling wave solutions for $C_z = 0.01$ and $\epsilon = 0$. We can observe that, at the traveling front, ρ has a jump from 1 to 0, and Σ has a layer and then jumps to zero. These observations are in accordance with our analytical results, and in particular with (4.6) for Σ.



(a) the isolines of the traveling front with respect to time.

(b) the front shapes of the density and pressure.

Figure 4 Numerical results when $C_p = 1$, $C_S = 0.1$, $C_\nu = 17.114$, $C_z = 0.01$ and $\epsilon = 0$.

When $C_z = 0.4$, the relation $C_SC_p > 2C_z$ is no longer satisfied. However, we can perform numerical simulations, and the results are presented in Figure 5. The proof of Theorem 4.1 shows that we can not have a traveling wave which satisfies the continuity relation for W and $\partial_x W$ at the point x_0. In fact, in Figure 5, we notice that the pressure Σ seems to have a jump directly from 1 to 0 at the front position, which is in accordance with Remark 4.1.

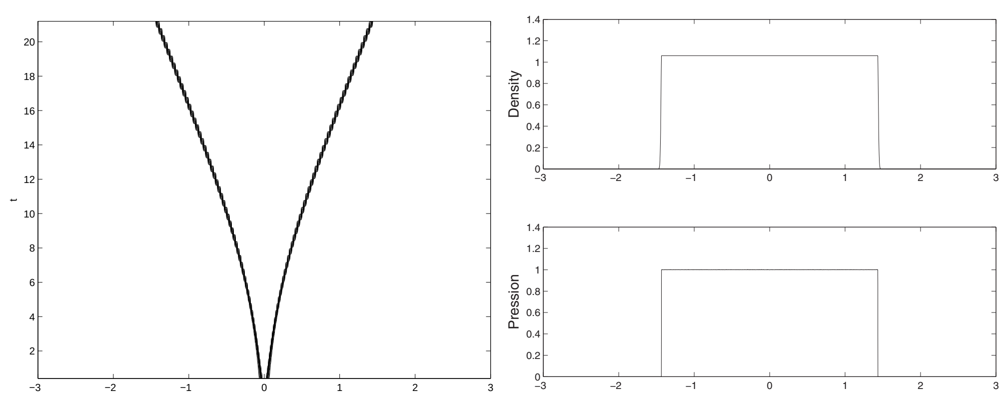


Figure 5 As in Figure 4, but the results violate the condition $C_SC_p > 2C_z$ using $C_p = 1$, $C_S = 0.1$, $C_\nu = 17.114$, $C_z = 0.4$ and $\epsilon = 0$.

With different choices of parameters, the numerical values for the traveling velocities σ and the front jumps of Σ at x_0 are given in Table 3, where we can verify the analytical formula in Theorem 4.1.

Table 3 The traveling speed σ^* for different parameter values satisfying $2C_z < C_pC_S$. The numerical speeds are close to $\sqrt{2C_pC_S} - \sqrt{C_z}$, and the jump of Σ is not far from $\sqrt{\frac{2C_pC_z}{C_S}}$ as calculated in Theorem 4.1.

C_p	C_S	C_z	$\sqrt{2C_pC_S} - \sqrt{C_z}$	$\frac{C_pC_S}{2\sqrt{C_z}}$	σ^*	$\sqrt{\frac{2C_pC_z}{C_S}}$	$\Sigma(x_0)$
0.57	1	0.1	0.7515	0.9012	0.7616	0.3376	0.3342
0.57	1	0.01	0.9677	2.8500	0.9686	0.1068	0.1052
0.57	0.1	0.01	0.2376	0.2850	0.2438	0.3376	0.3362
1	0.1	0.01	0.3472	0.500	0.3507	0.4472	0.4129
1	0.1	0.0	0.4472	-	0.4424	0	0

4.2 Case $\epsilon \neq 0$

Existence of Traveling Waves The case with diffusion such that $\epsilon \neq 0$, can be handled by the same method as above. We have the following result.

Theorem 4.2 *Assume $\epsilon \neq 0$, $C_z \neq 0$ and $C_SC_p > 2C_z$.Then there exists a $\sigma^* > 0$, such that for all $\sigma \geq \sigma^*$, the asymptotic model (4.4)–(4.5) admits a nonnegative and non-increasing solution (ρ, Σ). As $\eta \to 0$, the following bound on the minimal speed holds:*

$$\max\{2\sqrt{\epsilon}, \sqrt{2C_SC_p} - \sqrt{C_z}\} \leq \sigma^* \leq (\sqrt{2C_SC_p} - \sqrt{C_z}) + 2\sqrt{\epsilon\sqrt{\frac{2C_SC_p}{C_z}}},$$

The solution is given by

$$\Sigma(x) = \begin{cases} C_p, & x \leq 0, \\ -\dfrac{x^2}{2C_S} - \dfrac{x}{C_S}\sqrt{C_z} + C_p, & 0 < x \leq \sqrt{2C_SC_p} - 2\sqrt{C_z}, \\ 0, & x > \sqrt{2C_SC_p} - 2\sqrt{C_z}. \end{cases} \tag{4.12}$$

The cell density ρ is a positive, non-increasing $C^1(\mathbb{R})$ function, such that

$$\rho = 1 \quad \text{for } \ x < \sqrt{2C_SC_p} - 2\sqrt{C_z} \quad \text{and} \quad \rho < 1 \quad \text{for } \ x > 2\sqrt{2C_SC_p} - 2\sqrt{C_z}.$$

Proof As above, W and $\partial_x W$ are continuous on $\mathbb{R}$. Moreover, due to the diffusion term, ρ is continuous. Using the same decomposition $\mathbb{R} = I_1 \cup I_2 \cup I_3$ as before, we notice that, in $I_1 \cup I_2$, the problem is independent of ϵ. Thus we have the same conclusion as in Theorem 4.1.

(1) On I_1, we have $\rho = 1$, $\Sigma = C_p - \eta \mathrm{e}^{\frac{x}{\sqrt{\eta C_S + C_z}}}$ and $W = C_p - (\eta + \frac{C_z}{C_S})\mathrm{e}^{\frac{x}{\sqrt{\eta C_S + C_z}}}$.

(2) In I_2, we have $\rho = 1$, $\Sigma = C_p - \eta - \frac{x}{C_S}\sqrt{\eta C_S + C_z} - \frac{x^2}{2C_S}$ and $W = C_p - \eta - \frac{C_z}{C_S} - \frac{x}{C_S}\sqrt{\eta C_S + C_z} - \frac{x^2}{2C_S}$.

(3) On I_3, still from the second equation of (4.4) and the continuity of W and $\partial_x W$, we have

$$\begin{cases} W(x) = \frac{\sqrt{C_z}}{C_S}(\sqrt{C_z + \eta C_S} + x_0)\mathrm{e}^{-\frac{x - x_0}{\sqrt{C_z}}}, \\ x_0 = \sqrt{2C_SC_p - \eta C_s} - \sqrt{C_z} - \sqrt{C_z + \eta C_S}. \end{cases} \tag{4.13}$$

The jump condition at x_0 for the first equation of (4.4) is

$$-\sigma[\rho]_{x_0} - C_S[\rho\partial_x W]_{x_0} = \epsilon[\partial_x \rho]_{x_0},$$

which implies $[\partial_x \rho]_{x_0} = 0$ thanks to the continuity of ρ and $\partial_x W$. Then, from (4.4), when $\rho < 1$, the density satisfies

$$\epsilon\partial_{xx}\rho + \Big(\sigma - \frac{C_S}{\sqrt{C_z}}W\Big)\partial_x\rho + \Big(1 + \frac{C_S}{C_z}W\Big)\rho = 0, \tag{4.14}$$

where W is as in (4.13). This equation is completed with the boundary conditions

$$\rho(x_0) = 1 \quad \text{and} \quad \partial_x\rho(x_0) = 0. \tag{4.15}$$

The Cauchy problem (4.14)–(4.15) admits a unique solution. Moreover, at the point x_0, we deduce from (4.14) that

$$\epsilon\partial_{xx}\rho(x_0) = -1 - \frac{C_S}{C_z}W(x_0) < 0.$$

Therefore, $\partial_x\rho$ is decreasing in the vicinity of x_0. We deduce that $\partial_x\rho \le 0$ for $x \ge x_0$ in the vicinity of x_0. Then if ρ does not have a minimum on $(x_0, +\infty)$, it is a non-increasing function, which necessarily tends to 0 at infinity from (4.14). If ρ admits a minimum at the point $x_m > x_0$, then we have $\partial_{xx}\rho(x_m) > 0$ and $\partial_x\rho(x_m) = 0$. We deduce from (4.14) that

$$\rho(x_m)\Big(1 + \frac{C_S}{C_z}W(x_m)\Big) = -\epsilon\partial_{xx}\rho(x_m) < 0.$$

We conclude that $\rho(x_m) < 0$. Thus there exists a point x_c, such that $\rho(x_c) = 0$. Then on $[x_0, x_c)$, we have that $\rho > 0$, and it is nonincreasing. The question is then to know whether there exists a value of σ for which $x_c = +\infty$. In order to do so, we will compare ρ with $\widetilde{\rho}$ that satisfies

$$\epsilon\partial_{xx}\widetilde{\rho} + \Big(\sigma - \frac{C_S}{\sqrt{C_z}}K\Big)\partial_x\widetilde{\rho} + \Big(1 + \frac{C_S}{C_z}K\Big)\widetilde{\rho} = 0, \quad x \in (x_0, +\infty) \tag{4.16}$$

with the boundary conditions

$$\widetilde{\rho}(x_0) = 1, \quad \partial_x\widetilde{\rho}(x_0) = 0. \tag{4.17}$$

Here K is a given constant which will be defined later.

Lower Bound on $\boldsymbol{\sigma^*}$ Integrating (4.14) from x_0 to $+\infty$, and using $\partial_x W = -\frac{W}{\sqrt{C_z}}$ and the boundary conditions in (4.15), we have

$$\sigma = \sqrt{C_z + \eta C_S} + x_0 + \int_{x_0}^{+\infty} \rho(x)\mathrm{d}x.$$

We deduce that if we had a nonnegative solution ρ, then

$$\sigma \ge \sqrt{C_z + \eta C_S} + x_0 = \sqrt{2C_S C_p - \eta C_s} - \sqrt{C_z}. \tag{4.18}$$

Moreover, from (4.14), we have

$$\epsilon\partial_{xx}\rho + \sigma\partial_x\rho + \rho = \frac{C_S}{\sqrt{C_z}}W\partial_x\rho - \frac{C_S}{C_z}W\rho \le 0.$$

Using the second assertion of Lemma 4.1, we can compare ρ with $\widetilde{\rho}$ that is the solution to (4.16)–(4.17) with $K = 0$. We deduce that $\rho \le \widetilde{\rho}$, since when $\sigma < 2\sqrt{\epsilon}$, $\widetilde{\rho}$ takes negative values on I_3. Thus, ρ is no longer nonnegative, which is a contradiction. Therefore,

$$\sigma \ge 2\sqrt{\epsilon}. \tag{4.19}$$

Upper Bound on σ^* We use the bound $W \le W(x_0)$ to get

$$\epsilon\partial_{xx}\rho + \Big(\sigma - \frac{C_S}{\sqrt{C_z}}W(x_0)\Big)\partial_x\rho + \Big(1 + \frac{C_S}{C_z}W(x_0)\Big)\rho \ge 0. \tag{4.20}$$

Using the assertion (1) of Lemma 4.1, we deduce that ρ is positive on I_3 provided that

$$\sigma \ge \sqrt{2C_SC_p - \eta C_S} - \sqrt{C_z} + 2\sqrt{\epsilon\sqrt{\frac{2C_SC_p}{C_z}}}. \tag{4.21}$$

Thus for all σ satisfying (4.21), there exists a non-increasing and nonnegative solution ρ to (4.14)–(4.15).

However, the bound (4.21) is not satisfactory for small C_z. This is mainly due to the fact that the bound $W(x) \le W(x_0)$ on I_3 is not sharp when C_z is small. We can improve this bound by using the remark that for any $x_z > x_0$, we have $W(x) \le K := W(x_z)$. Let us define $x_z = x_0 + \sqrt{C_z}\,\xi(\sqrt{C_z})$ with a continuous function $\xi : (0,+\infty) \to (0,+\infty)$, such that $\lim\limits_{x\to 0} x\xi(x) = 0$. Let us call $\widehat{\rho}$ a solution to (4.16) on $(x_z,+\infty)$ with $K = W(x_z)$ and the boundary conditions $\widehat{\rho}(x_z) = \rho(x_z) > 0$, $\partial_x\widehat{\rho}(x_z) = \partial_x\rho(x_z) \le 0$. Using the assertion (1) of Lemma 4.1, we deduce that $\rho \ge \widehat{\rho}$, and $\widehat{\rho}$ is positive provided that

$$\sigma \ge \frac{C_S}{\sqrt{C_z}}W(x_z) + 2\sqrt{\epsilon\Big(1 + \frac{C_S}{C_z}W(x_z)\Big)} \tag{4.22}$$

and

$$\alpha + \sqrt{\alpha^2 - 4\beta} \ge -\frac{2\partial_x\rho(x_z)}{\rho(x_z)}, \tag{4.23}$$

where $\epsilon\alpha = \sigma - \frac{C_SW(x_z)}{\sqrt{C_z}}$ and $\epsilon\beta = 1 + \frac{C_SW(x_z)}{C_z}$. When $x_z \to x_0$, we have $\partial_x\rho(x_z) \to 0$, whereas $\alpha > \frac{2}{\sqrt{\epsilon}}$ from (4.22). Thus for $\sqrt{C_z}$ small enough, (4.23) is satisfied provided that (4.22) is satisfied, i.e.,

$$\sigma \ge (\sqrt{2C_SC_p - \eta C_S} - \sqrt{C_z})\mathrm{e}^{-\xi(\sqrt{C_z})} + 2\sqrt{\epsilon}\sqrt{1 + \Big(\sqrt{\frac{2C_SC_p - \eta C_s}{C_z}} - 1\Big)\mathrm{e}^{-\xi(\sqrt{C_z})}}. \tag{4.24}$$

Therefore, choosing the function ξ, such that $\lim\limits_{x\to 0}\frac{\mathrm{e}^{-\xi(x)}}{x} = 0$, we deduce that when $C_z \to 0$, (4.24) becomes $\sigma \ge 2\sqrt{\epsilon}$. One possible choice is $\xi(x) = \ln x^2$.

The proof of Theorem 4.2 uses the following lemma.

Lemma 4.1 *Let α, β, a be positive and $b \le 0$. For $g \in C(\mathbb{R}_+)$, let f and $\widetilde{f}$ be the solutions to the following Cauchy problems on $\mathbb{R}_+$:*

$$f'' + \alpha f' + \beta f = g, \quad f(0) = a, \quad f'(0) = b \tag{4.25}$$

and

$$\widetilde{f}'' + \alpha\widetilde{f}' + \beta\widetilde{f} = 0, \quad \widetilde{f}(0) = a, \quad \widetilde{f}'(0) = b, \tag{4.26}$$

respectively. Then we have

(1) *Assume* $g \geq 0$ *on* $\mathbb{R}^+$*. If* $\alpha^2 \geq 4\beta$ *and* $\alpha + \sqrt{\alpha^2 - 4\beta} \geq -\frac{2b}{a}$*, then* $f(x) \geq \widetilde{f}(x) > 0$ *for* $x \in \mathbb{R}_+$*. Or else, there exists an* $x_c > 0$*, such that* $\widetilde{f}(x_c) = 0$ *and* $\widetilde{f} \geq 0$ *on* $[0, x_c]$*. Moreover, if* $\alpha^2 < 4\beta$*, we have* $f(x) \geq \widetilde{f}(x)$ *for* $x \in \big[0, \frac{2\pi}{\sqrt{4\beta-\alpha^2}}\big]$*; if* $\alpha^2 \geq 4\beta$ *and* $\alpha + \sqrt{\alpha^2 - 4\beta} < \frac{2b}{a}$*, we have* $f(x) \geq \widetilde{f}(x)$ *for* $x \in [0, x_c]$*.*

(2) *Assume* $g \leq 0$ *on* $\mathbb{R}^+$*. If* $\alpha^2 \geq 4\beta$*, then* $f(x) \leq \widetilde{f}(x)$ *for* $x \geq 0$*. If moreover* $\alpha + \sqrt{\alpha^2 - 4\beta} < -\frac{2b}{a}$*, then* f *takes negative values on* $\mathbb{R}_+$*. If* $\alpha^2 < 4\beta$*, then we have* $f(x) \leq \widetilde{f}(x)$ *for* $x \in \big[0, \frac{2\pi}{\sqrt{4\beta-\alpha^2}}\big]$ *and* f *takes negative values on* $\big[0, \frac{2\pi}{\sqrt{4\beta-\alpha^2}}\big]$*.*

Proof Denote by r_1 and r_2 the roots of the characteristic equation $r^2 + \alpha r + \beta = 0$. Then, if $r_1 \neq r_2$, by solving (4.25)–(4.26), we have

$$\begin{aligned} \widetilde{f}(x) &= \frac{r_2 a - b}{r_2 - r_1} \mathrm{e}^{r_1 x} + \frac{r_1 a - b}{r_1 - r_2} \mathrm{e}^{r_2 x}, \\ f(x) &= \widetilde{f}(x) + \int_0^x g(y) \Big(\frac{\mathrm{e}^{r_1(x-y)}}{r_1 - r_2} + \frac{\mathrm{e}^{r_2(x-y)}}{r_2 - r_1} \Big) \mathrm{d}y. \end{aligned} \tag{4.27}$$

First, we assume that $g \geq 0$ on $\mathbb{R}_+$. If $\alpha^2 > 4\beta$, then r_1 and r_2 are real negative. We deduce that

$$\frac{\mathrm{e}^{r_1 x}}{r_1 - r_2} + \frac{\mathrm{e}^{r_2 x}}{r_2 - r_1} > 0,$$

and then $f(x) > \widetilde{f}(x)$ for $x \geq 0$. Moreover, $\widetilde{f}$ vanishes on $\mathbb{R}_+$ if and only if $\min\{r_1, r_2\} \geq \frac{b}{a}$.

If $\alpha^2 < 4\beta$, r_1 and r_2 are complex and $\overline{r}_1 = r_2$. We denote $r_1 = R - \mathrm{i}I$, where $2R = -\alpha$ and $2I = \sqrt{4\beta - \alpha^2}$. We can rewrite then

$$\widetilde{f}(x) = \Big(\frac{R - b}{I} \sin(Ix) + a \cos(Ix) \Big) \mathrm{e}^{Rx}. \tag{4.28}$$

We deduce that there exists an x_c, such that $\widetilde{f}(x_c) = 0$ and $\widetilde{f} \geq 0$ on $[0, x_c]$. Moreover,

$$\frac{\mathrm{e}^{r_1 x}}{r_1 - r_2} + \frac{\mathrm{e}^{r_2 x}}{r_2 - r_1} = \frac{\mathrm{e}^{Rx}}{I} \sin(Ix) \geq 0 \quad \text{for } x \in \Big[0, \frac{\pi}{I}\Big]. \tag{4.29}$$

Thus $f(x) \geq \widetilde{f}(x)$ if $x \in \big[0, \frac{\pi}{I}\big]$.

If $\alpha^2 = 4\beta$, we have $r_1 = r_2 = -\frac{\alpha}{2}$. By straightforward computation, we have $\widetilde{f}(x) = ((b - ar_1)x + a)\mathrm{e}^{rx}$, and

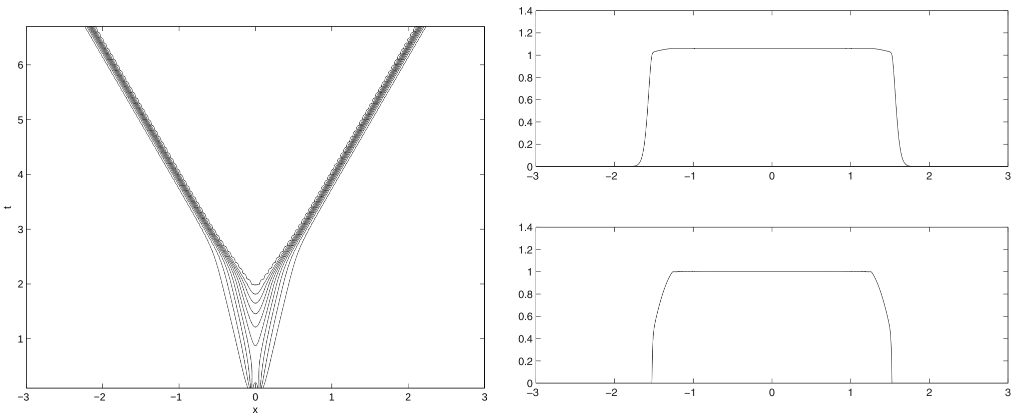
$$f(x) = \widetilde{f}(x) + \int_0^x (x - y) \mathrm{e}^{r_1(x-y)} g(y) \mathrm{d}y. \tag{4.30}$$

For $g \geq 0$, we deduce $f \geq \widetilde{f}$. This concludes the proof of the first point.

Let us consider that $g \leq 0$ on $\mathbb{R}_+$. We deduce the first assertion from (4.27) and (4.30). If $\alpha^2 < 4\beta$, we deduce $f \leq \widetilde{f}$ on $\big[0, \frac{\pi}{I}\big]$ from (4.27) and (4.29). And we have from (4.28) $\widetilde{f}\big(\frac{\pi}{I}\big) = -a\mathrm{e}^{\frac{\pi R}{I}} < 0$, and thus f vanishes on $\big[0, \frac{\pi}{I}\big]$.

Numerical Results We perform numerical simulations of the full system (2.2) by using the same algorithm as in Subsection 4.1 and a centered finite difference scheme for the diffusion term $\epsilon \partial_{xx} \rho$.

We present in Figure 6 the numerical results still with parameters in (3.20) and $C_z = 0.01$, $\epsilon = 0.01$. Comparing Figures 4 and 6, we notice that the profile of ρ has a tail in the latter case.



(a) The isolines of the traveling front with respect to time.

(b) The front shapes of the density and pressure.

Figure 6 The numerical solution when $C_p = 1$, $C_S = 0.01$, $C_\nu = 17.114$, $C_z = 0.01$ and $\epsilon = 0.01$.

Table 4 gives numerical values of the traveling velocity for different parameters. We illustrate numerically the bound on σ^* obtained in the proof of Theorem 4.2.

Table 4 The traveling speed σ^* for (2.2) with different parameter values.

C_p	C_S	C_z	ϵ	$\sqrt{2C_pC_S} - \sqrt{C_z}$	$2\sqrt{\epsilon}$	σ^*
0.57	0.01	0.001	0.01	0.07515	0.20	0.197
0.57	0.1	0.01	0.01	0.2376	0.20	0.321
0.57	1	0.1	0.001	0.7514	0.0632	0.780
0.57	1	0.1	0.01	0.7514	0.2	0.828
0.57	1	0.1	0.1	0.7514	0.632	1.015
0.57	1	0.1	1	0.7514	2	1.974

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Appendix Derivation of the Cuboid State Equation

Cells are modelled as cuboidal elastic bodies of dimensions at rest $L_0 \times l_0 \times h_0$ in x, y, z directions aligned in a row in x direction. At rest, the lineic mass density of the row of cells, in contact but not deformed, is $\rho_0 = \frac{M_{\rm cell}}{L_0}$. We consider the case that the cells are confined in a tube of section $l_0 \times h_0$, where the only possible deformation is along the x axis. This situation can be tested in a direct in-vitro experiment. Moreover, this limit would be expected in case a tumor composed of elastic cells is sufficiently large, such that for the ratio of the cell size L and the radius of curvature R, $\frac{L}{R} \ll 1$ holds, and the cell division is mainly oriented in radial direction as well as the cell-cell tangential friction is sufficiently small, such that a fingering or buckling instability does not occur.

When cells are deformed, we assume that stress and deformation are uniformly distributed, and that the displacements are small. Let L be the size of the cells. The lineic mass density is $\rho = \frac{\rho_0 L_0}{L}$. For $\rho < \rho_0$, the cells are not in contact and $\Sigma(\rho) = 0$; for $\rho \geq \rho_0$, a variation $\mathrm{d}L$ of the size L of the cell corresponds to an infinitesimal strain $\mathrm{d}u = \frac{\mathrm{d}L}{L}$. Therefore, the strain for a cell of size L is $u = \ln\left(\frac{L}{L_0}\right)$. Assuming that a cell is a linear elastic body with Young modulus E and Poisson ratio ν, one finds that the component σ_{xx} of the stress tensor can be written as

$$\sigma_{xx} = -\frac{1-\nu}{(1-2\nu)(1+\nu)} E \ln\left(\frac{\rho}{\rho_0}\right).$$

The state equation is given by

$$\Sigma(\rho) = \begin{cases} 0, & \text{if } \rho \leq \rho_0, \\ \dfrac{1-\nu}{(1-2\nu)(1+\nu)} E \ln\left(\dfrac{\rho}{\rho_0}\right), & \text{otherwise.} \end{cases}$$

Here, $\Sigma(\rho) = -\sigma_{xx}$ is the pressure. Let $\overline{\rho} = \frac{\rho}{\rho_0}$, $\overline{\Sigma} = \frac{\Sigma}{E_0}$ and $\overline{E} = \frac{E}{E_0}$ be the dimensionless density, pressure and Young modulus respectively, with E_0 a reference Young modulus. Then the state equation can be written as

$$\overline{\Sigma}(\overline{\rho}) = \begin{cases} 0, & \text{if } \overline{\rho} \leq 1, \\ C_\nu \ln(\overline{\rho}), & \text{otherwise,} \end{cases}$$

where $C_\nu = \frac{\overline{E}(1-\nu)}{(1-2\nu)(1+\nu)}$. In the article, equations are written in the dimensionless form, and the bars above dimensionless quantities are removed.