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Global Null Controllability of the 1-Dimensional Nonlinear Slow Diffusion Equation*

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(*Dedicated to Jacques-Louis Lions with the souvenir of his mastery and creativity*)

Abstract The authors prove the global null controllability for the 1-dimensional nonlinear slow diffusion equation by using both a boundary and an internal control. They assume that the internal control is only time dependent. The proof relies on the return method in combination with some local controllability results for nondegenerate equations and rescaling techniques.

Keywords Nonlinear control, Nonlinear slow diffusion equation, Porous medium equation
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1 Introduction

We study the null controllability of the 1-dimensional nonlinear slow diffusion equation, sometimes referred to as the Porous Media Equation (or PME for short), using both internal and boundary controls. The methods we used need such a combination of controls due to the degenerate nature of this quasilinear parabolic equation.

The PME belongs to the more general family of nonlinear diffusion equations of the form

$$y_t - \Delta\phi(y) = f, \tag{1.1}$$

where ϕ is a continuous nondecreasing function with $\phi(0) = 0$. For the PME, the constitutive

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law is precisely given by

$$\phi(y) = |y|^{m-1} y \tag{1.2}$$

with $m \geq 1$.

This family of equations arises in many different frameworks and, depending on the nature of ϕ, it models different diffusion processes, mainly grouped into three categories: "slow diffusion", "fast diffusion" and linear processes.

The "slow diffusion" case is characterized by a finite speed of propagation and the formation of free boundaries, while the "fast diffusion" one is characterized by a finite extinction time, which means that the solution becomes identically zero after a finite time.

If one neglects the source term, i.e., $f \equiv 0$, and imposes the constraint of nonnegativeness to the solutions (which is fundamental in all the applications where y represents for example a density), then one can precisely characterize these phenomena. In fact, it was shown in [12] that the solution of the homogeneous Dirichlet problem associated to (1.1) on a bounded open set Ω of $\mathbb{R}^N$ satisfies a finite extinction time if and only if

$$\int_0^1 \frac{\mathrm{d}s}{\phi(s)} < +\infty,$$

which corresponds to the case $m \in (0,1)$ for constitutive laws given by (1.2). On the contrary, if

$$\int_0^1 \frac{\mathrm{d}s}{\phi(s)} = +\infty, \tag{1.3}$$

(which is the case for $m \geq 1$) then, for any initial datum $y_0 \in H^{-1}(\Omega) \cap L^1(\Omega)$ with $(-\Delta)^{-1} y_0 \in L^\infty(\Omega)$, there is a kind of "retention property". This means that, if $y_0(x) > 0$ on a positively measured subset $\Omega' \subset \Omega$, then $y(\cdot, t) > 0$ on Ω' for any $t > 0$. In addition to (1.3), if ϕ satisfies

$$\int_0^1 \frac{\phi'(s)\mathrm{d}s}{s} < +\infty,$$

(i.e., $m > 1$ in the case of (1.2)) then the solution enjoys a finite speed of propagation and generates a free boundary given by that of its support ($\partial\{y > 0\}$).

Most typical applications of "slow diffusion" are as follows: Nonlinear heat propagation, groundwater filtration and the flow of an ideal gas in a homogeneous porous medium. With regard to the "fast diffusion", it rather finds a paradigmatic application to the flow in plasma physics. Many results and references can by found in the monographs [2, 23].

As already said, the aim of this paper is to show how a combined action of boundary controls and a spatially homogeneous internal control may allow the global extinction of the solution (the so-called global null controllability) in any prescribed temporal horizon $T > 0$. We shall prove the global null controllability for the following two control problems:

$$\mathrm{P}_{DD} \begin{cases} y_t - (y^m)_{xx} = u(t)\chi_I(t), & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = v_0(t)\chi_I(t), & t \in (0,T), \\ y(1,t) = v_1(t)\chi_I(t), & t \in (0,T), \\ y(x,0) = y_0(x), & x \in (0,1), \end{cases} \tag{1.4}$$

and

$$\mathrm{P}_{DN}\begin{cases} y_t-(y^m)_{xx}=u(t)\chi_I(t), & (x,t)\in(0,1)\times(0,T),\\ (y^m)_x(0,t)=0, & t\in(0,T),\\ y(1,t)=v_1(t)\chi_I(t), & t\in(0,T),\\ y(x,0)=y_0(x), & x\in(0,1), \end{cases} \tag{1.5}$$

where $I:=(t_1,T)$ with $t_1\in(0,T)$, $m\geq 1$ and χ_I is the characteristic function of I. In both problems, y represents the state variable and $U_{DN}:=(u\chi_I,0,v_1\chi_I)$, respectively $U_{DD}:=(u\chi_I,v_0\chi_I,v_1\chi_I)$, is the control variable. The function y^m should be more properly written in form (1.2), but as we shall impose the constraint $y\geq 0$, it makes no real difference.

We emphasize the fact that the internal control $u(t)$ has the property to be independent of the space variable x and that all the controls are active only on a part of the time interval. Moreover, as we shall show later, the systems are null controllable in arbitrarily fixed time, and then the localized form of the control $u(t)\chi_I(t)$ (the same for the boundary controls) on a subinterval of $[0,T]$ is more an emphatic difficulty than a real difficulty. It serves mostly to underline that the controls are not active in the first time lapse. In the same way, it could be possible to take a control interval $(\underline{t},\overline{t})$ with $\underline{t},\overline{t}\in(0,T)$ or, even more generally, three different intervals, one for each control v_0,v_1,u, such that the intersection of the three is not empty.

The main results of this paper are contained in the following statement.

Theorem 1.1 *Let $m\in[1,+\infty)$.*

(i) *For any initial data $y_0\in H^{-1}(0,1)$ such that $y_0\geq 0$ and any time $T>0$, there exist controls $v_0(t),v_1(t)$ and $u(t)$ with $v_0(t)\chi_I(t),v_1(t)\chi_I(t)\in H^1(0,T)$, $v_0,v_1\geq 0$ and $u\in L^\infty(0,T)$ such that the solution y of P_{DD} satisfies $y\geq 0$ on $(0,1)\times(0,T)$, and $y(\cdot,T)\equiv 0$ on $(0,1)$.*

(ii) *For any initial data $y_0\in H^{-1}(0,1)$ such that $y_0\geq 0$ and any time $T>0$, there exist controls $v_1(t)$ and $u(t)$ with $v_1(t)\chi_I(t)\in H^1(0,T)$, $v_1\geq 0$ and $u\in L^\infty(0,T)$ such that the solution y of P_{DN} satisfies $y\geq 0$ on $(0,1)\times(0,T)$, and $y(\cdot,T)\equiv 0$ on $(0,1)$.*

Notice that since $H^{-1}(0,1)=(H_0^1(0,1))'$ and $H_0^1(0,1)\subset C([0,1])$, we have $H^{-1}(0,1)\supset\mathcal{M}(0,1)$, where $\mathcal{M}(0,1)$ is the set of bounded Borel measures on $(0,1)$; for instance, the initial datum can be a Dirac mass distribution at a point in $(0,1)$. As said before in the case of "slow diffusion" ($m>1$), the solution may present a free boundary given by the boundary of its support (whenever the support of y_0 is strictly smaller than $[0,1]$). Nevertheless, our strategy is built in order to prevent such a situation. Indeed, on the set of points (x,t) where y vanishes (i.e., on the points $(x,t)\in(0,1)\times(0,T)\setminus\mathrm{supp}(y)$), the diffusion operator is not differentiable at $y\equiv 0$, and so some linearization methods which work quite well for second order semilinear parabolic problems (see, e.g., [13, 17, 19–20]) can not be applied directly. Moreover, the evanescent viscosity perturbation with some higher order terms only gives some controllability results for suitable functions ϕ, as the ones of the Stefan problem (see [13–15]).

Here we follow a different approach which is mainly based on the so-called return method introduced in [9–10] (see [11, Chapter 6] for information on this method). More precisely, we shall prove first the null controllability of problem (1.4) by applying an idea appeared in [8] (for the controllability of the Burgers equation). In the second step, we shall show, using some symmetry arguments, that the same result holds for (1.5).

Our version of the return method consists in choosing a suitable parametrized family of trajectories $\frac{a(t)}{\varepsilon}$, which is independent of the space variable, going from the initial state $y \equiv 0$ to the final state $y \equiv 0$. We shall use the controls to reach one of such trajectories, no matter which one, in some positive time smaller than the final T. Once we fix a partition of the form $0 < t_1 < t_2 < t_3 < T$, we shall choose a function $a(t)$ satisfying the following properties:

(i) $a \in C^2([0,T])$;

(ii) $a(t) = 0$, $0 \le t \le t_1$ and $t = T$;

(iii) $a(t) > 0$, $t \in (t_1, T)$;

(iv) $a(t) = 1$, $t_2 \le t \le t_3$.

Then, the solution y of problem P_{DD} can be written as a perturbation of the explicit solution $\frac{a(t)}{\varepsilon}$ of the same equation with the control $U := \big(\frac{a(t)}{\varepsilon}, \frac{a(t)}{\varepsilon}, \frac{a(t)}{\varepsilon}\big)$ in the following way:

$$y(x,t) = \Big(\frac{a(t)}{\varepsilon} + z(x,t)\Big). \tag{1.6}$$

Now, our aim is to find controls such that $z(\cdot, t_3) \equiv 0$, which means that we have controlled our solution $y(\cdot, t)$ to the state $\frac{1}{\varepsilon}$ at time $t = t_3$; this will be done by using a slight modification of a result in [4]. On the final time interval (t_3, T), we shall use the same trajectory $y(\cdot, t) \equiv \frac{a(t)}{\varepsilon}$ to reach the final state $y(\cdot, T) \equiv 0$. An ideal representation of the trajectory can be seen in Figure 1.

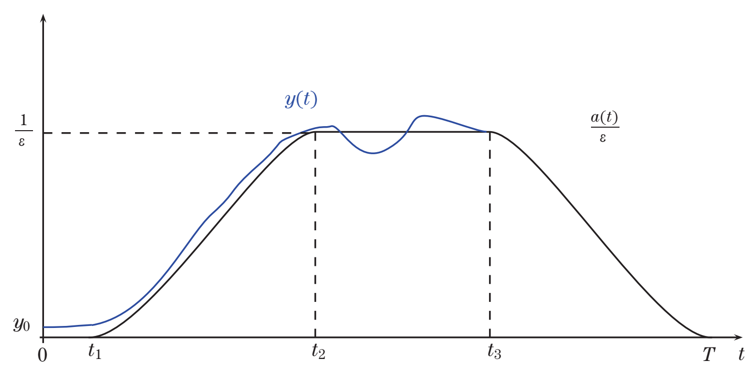

Figure 1 Solution profile

One can see that the central core of our procedure is to drive the initial state to a constant state in a finite time thanks to the use of a boundary and internal control which only depends on the time variable.

On the first interval $(0, t_1)$, we shall not make any use of the controls. So we let the solution $y(t) := y(\cdot, t)$ regularize itself from an initial state in $H^{-1}(0,1)$ to a smoother one in $H_0^1(0,1)$ for $t = t_1$. Then, as the degenerate character of the diffusion operator neglects the diffusion effects outside the support of the state, we move $y(t)$ away from the zero state by asking $z(t) := z(\cdot, t)$ to be nonnegative at least on the interval (t_1, t_2). With this trick, the solution $y(t)$ will be far enough from zero. On the interval (t_2, t_3) the states $y(t)$ will be kept strictly positive even if the internal control $u(t)$ will be allowed to take negative values.

As already mentioned concerning the local retention property, we point out that the presence of the control $u(t)$ is fundamental for the global null controllability. To be more precise, notice that if we assume $u(t) \equiv 0$ then we can find initial states which can not be steered to zero at time T just with some nonnegative boundary controls. As a matter of fact, one can use the well-known family of Barenblatt solutions (see [3, 23]) (also known as ZKB solutions) to show it. Indeed, if we introduce the parameters

$$\alpha = \frac{1}{m+1}, \quad k = \frac{m-1}{2m(m+1)}, \quad \tau \ll 1,$$

and choose C such that $\left(\frac{C}{k}\right)^{\frac{1}{2}}(T+\tau)^{\alpha} < \frac{1}{2}$, then the function

$$y_m(x,t) = (t+\tau)^{-\alpha}\Big(C - k\Big|x - \frac{1}{2}\Big|^2 (t+\tau)^{-2\alpha}\Big)_+^{\frac{1}{m-1}}$$

is a solution of system (1.4) with $u = 0, v_0 = v_1 = 0$ and $y_m(\cdot,T) \neq 0$. Any other solution of system (1.4) with the same initial datum and $v_0, v_1 \geq 0$ would be a supersolution of y_m, which implies that $y_m(\cdot,0)$ can not be connected with $y(\cdot,T) \equiv 0$.

Remark 1.1 It would be very interesting to know if, in the case of the problem P_{DD}, one could take $v_1 = 0$ in Theorem 1.1 as it has been done in [22] for a viscous Burgers' control system.

2 Well-Posedness of the Cauchy Problem

For the existence theory of problem (1.4), we refer to [1, 5–7, 21, 23]; in particular, we shall use a frame similar to the ones in [1, 6]. More precisely, we adopt the following definition.

Definition 2.1 *Let $(v_0, v_1) \in L^\infty(0,T)^2$ and $v_D = (1-x)v_0(t) + xv_1(t)$ and let $u \in L^\infty(0,T)$. Assume that $y_0 \in H^{-1}(0,1)$. We say that y is a weak solution of*

$$\mathrm{P}_{DD} \begin{cases} y_t - (|y|^{m-1}y)_{xx} = u(t), & (x,t) \in (0,1)\times(0,T), \\ y(0,t) = v_0(t), & t \in (0,T), \\ y(1,t) = v_1(t), & t \in (0,T), \\ y(x,0) = y_0(x), & x \in (0,1), \end{cases} \tag{2.1}$$

if

$$y \in C^0([0,T];H^{-1}(0,1)) \quad \textit{and} \quad y(0) = y_0, \quad \textit{in } H^{-1}(0,1), \tag{2.2}$$

$$y \in L^\infty(\tau,T;L^1(0,1)), \quad \forall \tau \in (0,T], \tag{2.3}$$

$$\partial_t y \in L^2(\tau,T;H^{-1}(0,1)), \quad \forall \tau \in (0,T], \tag{2.4}$$

$$|y|^{m-1}y \in |v_D|^{m-1}v_D + L^2(\tau,T;H_0^1(0,1)), \quad \forall \tau \in (0,T], \tag{2.5}$$

and for every $\tau \in (0,T]$, $\xi \in L^2(0,T;H_0^1(0,1))$,

$$\int_\tau^T \langle \partial_t y, \xi\rangle \mathrm{d}t + \int_\tau^T \int_0^1 (|y|^{m-1}y)_x \xi_x \mathrm{d}x\mathrm{d}t = \int_\tau^T \int_0^1 u\xi \mathrm{d}x\mathrm{d}t, \tag{2.6}$$

where the symbol $\langle\cdot,\cdot\rangle$ *stands for the dual pairing between* $H^{-1}(0,1)$ *and* $H_0^1(0,1)$.

Remark 2.1 We have changed the definition of weak solution given in [1] in order to handle the case where y_0 is only in $H^{-1}(0,1)$, instead of $y_0 \in L^{m+1}(0,1)$ as assumed in [1].

The modifications to extend the previous definition to the case of problem P_{ND} are straightforward (see [1]). For instance, the extension to the interior of the boundary datum can be taken now as $v_D = (c_1 + c_2 x^2) v_1(t)$.

With this definition, one has the following proposition.

Proposition 2.1 *The boundary-value problem* (1.4) *has at most one weak solution.*

The proof of Proposition 2.1 is the same as in [1, Theorem 2.4] due to the regularizing effect required in Definition 2.1 (see also [5]).

The next two propositions follow from results which can be found in [1, Theorems 1.7 and 2.4] and [7].

Proposition 2.2 *Suppose that* $(v_0, v_1) \in H^1(0,T)^2$ *and vanishes in a neighbourhood of* $t = 0$*, then there exists one and only one weak solution of problem* (1.4).

Proposition 2.3 *Suppose that* $(v_0, v_1) \in H^1(0,T)^2$ *and that* $y_0 \in L^{m+1}$*, then there exists one and only one weak solution* y *of problem* (1.4). *Moreover, this solution satisfies*

$$y \in L^\infty(0,T;L^1(0,1)), \tag{2.7}$$

$$\partial_t y \in L^2(0,T;H^{-1}(0,1)), \tag{2.8}$$

$$|y|^{m-1}\, y \in |v_D|^{m-1}\, v_D + L^2(0,T;H_0^1(0,1)). \tag{2.9}$$

Now, we emphasize that the solution of problem P_{DD} enjoys an additional semigroup property (we will need it to construct the final trajectory), which directly follows from Definition 2.1, Propositions 2.2 and 2.3.

Lemma 2.1 (Matching) *Suppose that* y_1*, respectively* y_2*, is a weak solution of* (1.4) *on the interval* $(0,T_1)$*, respectively* (T_1,T)*, with* $y_2(T_1) = y_1(T_1) \in L^2(0,1)$*. If we denote*

$$y(t) = \begin{cases} y_1(t), & t \in (0,T_1), \\ y_2(t), & t \in (T_1,T), \end{cases}$$

then y *is a weak solution of* (1.4) *in the interval* $(0,T)$.

3 Proof of the Main Theorem: First Step

In the interval $(0,t_1]$ the solution with no control evolves as in [7], hence $0 \le y^m(t) \in H_0^1(0,1)$ for all $t \in (0,t_1]$. Due to the inclusion $H_0^1(0,1) \subset L^\infty(0,1)$, we get that $y_1(x) := y(x,t_1)$ is a bounded function. We call the solution on the first interval y^0, i.e.,

$$y_{|(0,t_1)} = y^0. \tag{3.1}$$

In order to be able to apply the null controllability result in [4] to the function $z(x,t)$, given in the decomposition (1.6), on the interval (t_2, t_3) we need the H^1-norm of $z(t_2)$ to be small enough. We want to find some estimates of the solution z of

$$\begin{cases} z_t - \Big(m\Big(\dfrac{a(t)}{\varepsilon} + z\Big)^{m-1} z_x\Big)_x = 0, & (x,t) \in (0,1) \times (t_1, t_2), \\ z_x(t,0) = z_x(t,1) = 0, & t \in (t_1, t_2), \\ z(x,0) = y_1(x), & x \in (0,1). \end{cases} \tag{3.2}$$

For the existence, regularity and comparison results for this problem, we refer to [18], where the equation is recast in the form $(|Y|^{\frac{1}{m}} \operatorname{sign}(Y))_t - Y_{xx} = \frac{a'}{\varepsilon}$. From the maximum principle, we deduce that $y_1 \in L^\infty(0,1)$ and $y_1 \geq 0$ imply that $z \in L^\infty((0,1) \times (t_1,t_2))$ and $z \geq 0$. In fact, we have $0 \leq z \leq M$, where $M := \|y_1\|_{L^\infty(0,1)}$ is a solution of the state equation of (3.2), and in particular a super solution of (3.2).

To study the behaviour of z, we will actually make use of rescaling.

3.1 Small initial data and a priori estimates

For $\delta > 0$, we define $\widetilde{z} := \delta z$. Then $\widetilde{z}$ satisfies

$$\begin{cases} \widetilde{z}_t - \Big(m\Big(\dfrac{a(t)}{\varepsilon} + \dfrac{1}{\delta}\widetilde{z}\Big)^{m-1} \widetilde{z}_x\Big)_x = 0, & (x,t) \in (0,1) \times (t_1, t_2), \\ \widetilde{z}_x(t,0) = \widetilde{z}_x(t,1) = 0, & t \in (t_1, t_2), \\ \widetilde{z}(x,0) = \delta y_1, & x \in (0,1). \end{cases} \tag{3.3}$$

After collecting the factor $\frac{1}{\varepsilon}$ and rescaling the time $\tau := \frac{t}{\varepsilon^{m-1}}$, we get

$$\widetilde{z}_t - \Big(m\Big(a(\tau) + \frac{\varepsilon}{\delta}\widetilde{z}\Big)^{m-1} \widetilde{z}_x\Big)_x = 0.$$

Choosing $\delta := \varepsilon^{1-\alpha}$ with $0 < \alpha < 1$, the system can be written in the following form:

$$\begin{cases} \widetilde{z}_\tau - (m(a(\tau) + \varepsilon^\alpha \widetilde{z})^{m-1} \widetilde{z}_x)_x = 0, & (x,\tau) \in (0,1) \times (\tau_1, \tau_2), \\ \widetilde{z}_x(\tau,0) = \widetilde{z}_x(\tau,1) = 0, & \tau \in (\tau_1, \tau_2), \\ \widetilde{z}(x,0) = \varepsilon^{1-\alpha} y_1, & x \in (0,1), \end{cases} \tag{3.4}$$

where $\tau := \frac{t}{\varepsilon^{m-1}}$. For simplicity, we take $\alpha = \frac{1}{2}$.

Thus, the null controllability of system (3.2) is reduced to the null controllability of system (3.4). As we can see, the initial datum in (3.4) are now depending on ε and tend to 0 as $\varepsilon \to 0$.

3.2 H^1-estimate

We recall that, according to regularity theory for linear parabolic equations with bounded coefficients, $\widetilde{z}(t) \in H^2(0,1)$ for $t > 0$ (see, e.g., [16, pp. 360–364]). Multiplying by $\widetilde{z}_{xx}$ the first equation of (3.4) and integrating on $x \in (0,1)$, we get

$$\int_0^1 \widetilde{z}_\tau \widetilde{z}_{xx} \mathrm{d}x = \int_0^1 (m(a(\tau) + \sqrt{\varepsilon}\widetilde{z})^{m-1} \widetilde{z}_x)_x \widetilde{z}_{xx} \mathrm{d}x.$$

Then, integrating by parts and using the boundary condition in (3.4), we are led to

$$\begin{aligned}\frac{1}{2m}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^1 \widetilde{z}_x^2\mathrm{d}x &= -\int_0^1 (a(\tau)+\sqrt{\varepsilon}\widetilde{z})^{m-1}\widetilde{z}_{xx}^2\mathrm{d}x \\ &\quad -\frac{(m-1)}{3}\sqrt{\varepsilon}\int_0^1 (a(\tau)+\sqrt{\varepsilon}\widetilde{z})^{m-2}(\widetilde{z}_x^3)_x\mathrm{d}x \\ &= -\int_0^1 (a(\tau)+\sqrt{\varepsilon}\widetilde{z})^{m-1}\widetilde{z}_{xx}^2\mathrm{d}x \\ &\quad +\frac{(m-1)(m-2)}{3}\varepsilon\int_0^1 (a(\tau)+\sqrt{\varepsilon}\widetilde{z})^{m-3}\widetilde{z}_x^4\mathrm{d}x.\end{aligned}$$

We denote by

$$\begin{aligned}IT_1 &:= -\int_0^1 (a(\tau)+\sqrt{\varepsilon}\widetilde{z})^{m-1}\widetilde{z}_{xx}^2\mathrm{d}x, \\ IT_2 &:= \frac{(m-1)(m-2)}{3}\varepsilon\int_0^1 (a(\tau)+\sqrt{\varepsilon}\widetilde{z})^{m-3}\widetilde{z}_x^4\mathrm{d}x.\end{aligned}$$

We observe that $IT_1 \le 0$. Let us look at the term IT_2. For $m \in (1,2)$, we have that $IT_2 \le 0$. Otherwise,

$$IT_2 \le \frac{(m-1)(m-2)}{3}(a(\tau)+\sqrt{\varepsilon}\|\widetilde{z}\|_\infty)^{m-3}\varepsilon\int_0^1 \widetilde{z}_x^4\mathrm{d}x.$$

The fact that the L^∞-norm of $\widetilde{z}$ is finite comes from that $\widetilde{z} = \delta z$ and that the supremum of z is bounded, as already pointed out. We now use a well-known Gagliardo-Nirenberg's inequality in the case of a bounded interval.

Lemma 3.1 *Suppose* $z \in L^\infty(0,1)$ *with* $z_{xx} \in L^2(0,1)$ *and either* $z(0) = z(1) = 0$ *or* $z_x(0) = z_x(1) = 0$. *Then*

$$\|z_x\|_{L^4} \le \sqrt{3}\|z_{xx}\|_{L^2}^{\frac{1}{2}}\|z\|_{L^\infty}^{\frac{1}{2}}.$$

Proof Integrating by parts and using the boundary conditions, we obtain

$$\int_0^1 z_x^4\mathrm{d}x = \int_0^1 z_x^3 z_x\mathrm{d}x = -3\int_0^1 z_x^2 z_{xx} z\mathrm{d}x.$$

Then, using Cauchy-Schwarz's inequality, we get

$$\|z_x\|_{L^4}^4 \le 3\|z_x\|_{L^4}^2\|z\|_{L^\infty}\|z_{xx}\|_{L^2},$$

and the result follows immediately.

Setting $C' := C\|\widetilde{z}\|_{L^\infty}^2$ and considering that $\|\widetilde{z}_x\|_{L^4}^4 \le C'\|\widetilde{z}_{xx}\|_{L^2}^2$, we have

$$\begin{aligned}\frac{1}{2m}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^1 \widetilde{z}_x^2\mathrm{d}x &\le -\int_0^1 (a(\tau)+\sqrt{\varepsilon}\widetilde{z})^{m-1}\widetilde{z}_{xx}^2\mathrm{d}x \\ &\quad +\frac{(m-1)(m-2)}{3}(a(\tau)+\sqrt{\varepsilon}\|\widetilde{z}\|_\infty)^{m-3}\varepsilon\int_0^1 \widetilde{z}_x^4\mathrm{d}x\end{aligned}$$

$$\begin{aligned}&\le -(a(\tau))^{m-1}\int_0^1 \widetilde{z}_{xx}^2\mathrm{d}x\\&\quad + C'\frac{(m-1)(m-2)}{3}(a(\tau)+\sqrt{\varepsilon}\|\widetilde{z}\|_\infty)^{m-3}\varepsilon\int_0^1 \widetilde{z}_{xx}^2\mathrm{d}x\\&= C''(m,\tau,\varepsilon)\int_0^1 \widetilde{z}_{xx}^2\mathrm{d}x,\end{aligned}$$

where

$$C''(m,\tau,\varepsilon) := \Big(C'\frac{(m-1)(m-2)}{3}(a(\tau)+\sqrt{\varepsilon}\|\widetilde{z}\|_\infty)^{m-3}\varepsilon - (a(\tau))^{m-1}\Big).$$

For $\tau > 0$, we have

$$C''(m,\tau,\varepsilon) < 0,$$

if ε is small enough.

From these estimates, we deduce that the H^1-norm is non-increasing in the interval (τ_1,τ_2). Hence, for all $\rho \ge 0$, we can choose ε small enough to get $\|\widetilde{z}(\tau_2)\|_{H^1(0,1)} \le \varepsilon\|y_1\|_{H^1(0,1)} \le \rho$.

4 The End of the Proof of the Main Theorem

Now, we go back to problem (3.4) but with Dirichlet boundary conditions and initial data $\widetilde{z}(\tau_2)$. We apply an extension method that can be found for instance in [19, Chapter 2]. It consists in extending the space domain from $(0,1)$ to $E := (-d, 1+d)$ and inserting a sparse control in ω, a nonempty open interval whose closure in $\mathbb{R}$ is included in $(-d,0)$. We look at the following system:

$$\begin{cases} w_t - (m(1+\sqrt{\varepsilon}w)^{m-1}w_x)_x = \chi_\omega\widetilde{u}, & (x,\tau)\in Q',\\ w(-d,\tau) = w(1+d,\tau) = 0, & \tau\in(\tau_2,\tau_3),\\ w(x,\tau_2) = w_2(x), & x\in E,\end{cases} \tag{4.1}$$

where $Q' := E\times(\tau_2,\tau_3)$ and $\tau_3 := \frac{t_3}{\varepsilon^{m-1}}$. The function $w_2 \in H_0^1(E)\cap H^2(E)$ is an extension of $\widetilde{z}(\tau_2)$ to E which does not increase the H^1-norm, i.e., $\|w_2\|_{H^1(E)} \le k\|\widetilde{z}(\tau_2)\|_{H^1(0,1)} \le \sqrt{\varepsilon}k\|y_1\|_{H^1(0,1)}$, for some $k>0$ independent of $\widetilde{z}(\tau_2)$.

Proposition 4.1 *There exists a $\rho > 0$ such that, for any initial datum w_2 with $\|w_2\|_{H^1} \le \rho$ and for any ε sufficiently small, system (4.1) is null controllable, i.e., there exists a $\widetilde{u} \in L^2(Q')$ such that $w(\tau_3) = 0$.*

Sketch of the proof It is substantially the same as in [4]. We just have to choose ρ sufficiently small such that the solution of the control problem satisfies, for suitable value of ε, $\|w\|_{L^\infty} < \frac{1}{\sqrt{\varepsilon}}$.

Remark 4.1 Note that, combining the results in [4] and [16, pp. 360–364], the solution of (4.1) satisfies $w(0,\cdot), w(1,\cdot) \in H^1(\tau_2,\tau_3)$.

Proof of Theorem 1.1 We consider the function

$$y(\cdot,t)=\begin{cases} y^0(\cdot,t), & t\in(0,t_1),\\ \dfrac{a(t)}{\varepsilon}+z(\cdot,t)=\dfrac{a(t)}{\varepsilon}+\dfrac{\widetilde{z}(\cdot,t)}{\sqrt{\varepsilon}}, & t\in(t_1,t_2),\\ \dfrac{a(t)}{\varepsilon}+\dfrac{w(\cdot,t)}{\sqrt{\varepsilon}}, & t\in(t_2,t_3),\\ \dfrac{a(t)}{\varepsilon}, & t\in(t_3,T), \end{cases} \tag{4.2}$$

which is a solution of system (1.4) with controls given by

$$u(t):=\frac{a'(t)}{\varepsilon},\quad t\in(0,T), \tag{4.3}$$

$$v_0(t):=\begin{cases} 0, & t\in(0,t_1),\\ \dfrac{a(t)}{\varepsilon}+\dfrac{\widetilde{z}(0,t)}{\sqrt{\varepsilon}}, & t\in(t_1,t_2),\\ \dfrac{a(t)}{\varepsilon}+\dfrac{w(0,t)}{\sqrt{\varepsilon}}, & t\in(t_2,t_3),\\ \dfrac{a(t)}{\varepsilon}, & t\in(t_3,T), \end{cases} \tag{4.4}$$

$$v_1(t):=\begin{cases} 0, & t\in(0,t_1),\\ \dfrac{a(t)}{\varepsilon}+\dfrac{\widetilde{z}(1,t)}{\sqrt{\varepsilon}}, & t\in(t_1,t_2),\\ \dfrac{a(t)}{\varepsilon}+\dfrac{w(1,t)}{\sqrt{\varepsilon}}, & t\in(t_2,t_3),\\ \dfrac{a(t)}{\varepsilon}, & t\in(t_3,T). \end{cases} \tag{4.5}$$

The function satisfies $y\in C([0,T];H^{-1}(0,1))$, and, as one can check using the improved regularity of the solution when it is strictly positive, $(v_1,v_2)\in H^1(0,T)^2$. Combining Propositions 2.2–2.3 and Lemma 2.1, it is easy to see that the function given by (4.2) is the solution in the interval $(0,T)$ of problem (1.4) with nonhomogeneous term (4.3) and boundary conditions given by (4.4)–(4.5).

To conclude, we have from construction that $y(\cdot,T)\equiv 0$.

The proof of part (ii) follows the common argument of extension by symmetry. First, one notices that, using the smoothing property of (1.5) when $u\equiv 0$ and $v_1\equiv 0$, we may assume that y_0 is in $L^2(0,1)$. Then, we consider the auxiliary problem

$$\mathrm{P}^s_{DD}\begin{cases} y_t-(y^m)_{xx}=\widetilde{u}(t)\chi_I(t), & (x,t)\in(-1,1)\times(0,T),\\ y(-1,t)=v_0(t)\chi_I(t), & t\in(0,T),\\ y(1,t)=v_1(t)\chi_I(t), & t\in(0,T),\\ y(x,0)=\widetilde{y}_0(x), & x\in(-1,1) \end{cases} \tag{4.6}$$

with $\widetilde{y}_0\in L^2(-1,1)$ defined by

$$\widetilde{y}_0(x)=y_0(x),\quad \widetilde{y}_0(-x)=y_0(x),\quad \forall x\in(0,1), \tag{4.7}$$

and with $v_0(t) = v_1(t)$. We apply the arguments of part (i) to P^s_{DD} with $(0,1)$ replaced by $(-1,1)$ and adjusting the formulation of (4.1) in such a way that the control region ω is now symmetric with respect to $x = 0$. Then, as we will show later, the restriction of the solution of P^s_{DD} to the space interval $(0,1)$ is the sought trajectory for system P_{DN}.

Lemma 4.1 *Let ω be a nonempty open subset of $[-1-d, 1+d] \setminus [-1,1]$ which is symmetric with respect to (w.r.t.) $x = 0$. Then, if w_2 is symmetric w.r.t. $x = 0$, we can find a control u_s, symmetric w.r.t. $x = 0$, such that the solution w of system (4.1) satisfies*

(1) *w is symmetric w.r.t. $x = 0$,*

(2) *$w(\cdot, \tau_3) = 0$.*

Proof The proof follows almost straightforwardly from [4, Theorems 4.1–4.2]. We just have to minimize the functional which appears in [4, Theorems 4.1] in the space of L^2 functions which are symmetric w.r.t. $x = 0$.

The symmetry of the initial value implies, as a consequence, the symmetry of the solution w.

To conclude the proof of part (ii) of Theorem 1.1, we note that as the solution $y(\cdot, t)$ of (4.6) belongs to $H^2(-1,1)$ for all $t \in (0,T)$, we see that $y_x(0,t) = 0$ for all $t \in (0,T)$ and so, the conclusion is a direct consequence of part (i).

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