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Semi-linear Wave Equations with Effective Damping*

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Abstract The authors study the Cauchy problem for the semi-linear damped wave equation

$$u_{tt} - \triangle u + b(t)u_t = f(u), \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x)$$

in any space dimension $n \geq 1$. It is assumed that the time-dependent damping term $b(t) > 0$ is effective, and in particular $tb(t) \to \infty$ as $t \to \infty$. The global existence of small energy data solutions for $|f(u)| \approx |u|^p$ in the supercritical case of $p > 1 + \frac{2}{n}$ and $p \leq \frac{n}{n-2}$ for $n \geq 3$ is proved.

Keywords Semi-linear equations, Damped wave equations, Critical exponent, Global existence
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1 Introduction

We consider the Cauchy problem for the dissipative semi-linear equation

$$\begin{cases} u_{tt} - \triangle u + b(t)u_t = f(u), \quad t \geq 0, \ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x), \end{cases} \tag{1.1}$$

where the time-dependent damping term $b(t) > 0$ is effective, in particular $tb(t) \to \infty$ as $t \to \infty$, and the nonlinear term satisfies

$$f(0) = 0, \quad |f(u) - f(v)| \lesssim |u - v|(|u| + |v|)^{p-1} \tag{1.2}$$

for a given $p > 1$. Our aim is to establish the existence of $\mathcal{C}([0,\infty), H^1) \cap \mathcal{C}^1([0,\infty), L^2)$ solutions to (1.1) assuming small initial data in the energy space $H^1 \times L^2$ or in some weighted energy spaces. Clearly this will require suitable assumptions on $b(t)$ in (1.1) and on the exponent p in (1.2). In Section 2 , we first present some results related to the semi-linear wave equation with a constant damping term. We refer the interested reader to [10, 13] and to the quoted references for the damped wave equation with the x-dependent damping term $b(x)u_t$. In Section 3, we state our main theorems and some auxiliary results, which we prove in Sections 4–8.

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2 The Classical Semi-linear Damped Wave Equation

Many papers are concerned with the classical semi-linear damped wave equation, i.e., with the case of $b \equiv 1$ given by

$$\begin{cases} u_{tt} - \triangle u + u_t = f(u), \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x). \end{cases} \tag{2.1}$$

For the sake of clarity, we put

$$p_{\mathrm{GN}}(n) = 1 + \frac{2}{n-2} = \frac{n}{n-2} \quad \text{for } n \geq 3,$$
$$p_{\mathrm{Fuj}}(n) = 1 + \frac{2}{n} \quad \text{for } n \geq 1.$$

As stated in [15], for initial data $(u_0, u_1) \in H^1 \times L^2$ with compact support in $B_K(0)$, and $p \leq p_{\mathrm{GN}}(n)$ if $n \geq 3$, the problem (2.1) admits a unique local solution $u \in \mathcal{C}([0,T_m), H^1) \cap \mathcal{C}^1([0,T_m), L^2)$ for some maximal existence time $T_m \in (0, +\infty]$, and for any $t < T_m$ it holds that $\operatorname{supp} u(t,\cdot) \subset B_{K+t}(0)$.

One of the first results on the global existence theory was given in [15] establishing global existence for small data by using the technique of potential well and modified potential well. Let $\widetilde{W} \subset H^1$ be the interior of the set

$$\{u \in H^1 : \ \|\nabla u\|_{L^2}^2 \geq \|u\|_{L^{p+1}}^{p+1}\}.$$

In particular, by assuming $(u_0, u_1) \in \widetilde{W} \times L^2$, the authors removed the compactness assumption on the support of the data, and they proved the local existence of the solution, provided that $p < \frac{n+2}{n-2}$ if $n \geq 3$ (see [15, Theorem 1]). In Theorem 3 of the same paper, they proved the global existence, provided that the data in $\widetilde{W} \times L^2$ satisfy energy smallness assumptions and the exponent satisfies $p \geq 1 + \frac{4}{n}$ with $p < \frac{n+2}{n-2}$ if $n \geq 3$ (we remark that this set is not empty). In such a case, the energy of the solution to (1.1) satisfies the same decay estimates of the linear equation, i.e., $\|u_t(t,\cdot)\|_{L^2}^2 + \|\nabla u(t,\cdot)\|_{L^2}^2 \leq C(1+t)^{-1}$.

Assuming compactly supported data $(u_0, u_1) \in H^1 \times L^2$ to be sufficiently small, a global existence result for $p > p_{\mathrm{Fuj}}(n)$ and $p \leq p_{\mathrm{GN}}(n)$ if $n \geq 3$, was proved in [16] (we remark that this set is never empty). The approach followed in [16] makes use of the Matsumura estimates in [12] for the solution to the Cauchy problem for the classical damped linear wave equation

$$u_{tt} - \triangle u + u_t = 0, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x). \tag{2.2}$$

In order to state these estimates, we define

$$\mathcal{A}_{m,k} := (L^m \cap H^k) \times (L^m \cap H^{k-1}), \tag{2.3}$$
$$\|(u,v)\|_{\mathcal{A}_{m,k}} := \|u\|_{L^m} + \|u\|_{H^k} + \|v\|_{L^m} + \|v\|_{H^{k-1}} \tag{2.4}$$

for $m \in [1,2)$ and $k \in \mathbb{N}$. If $(u_0, u_1) \in \mathcal{A}_{m,1}$ for some $m \in [1,2)$, then the solution to (2.2) satisfies

$$\begin{aligned} \|u(t,\cdot)\|_{L^2} &\leq C(1+t)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}\|(u_0,u_1)\|_{\mathcal{A}_{m,0}}, \\ \|\nabla u(t,\cdot)\|_{L^2} &\leq C(1+t)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{1}{2}}\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}, \\ \|u_t(t,\cdot)\|_{L^2} &\leq C(1+t)^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1}\|(u_0,u_1)\|_{\mathcal{A}_{m,1}}. \end{aligned} \tag{2.5}$$

Since in [16] the data $(u_0, u_1) \in H^1 \times L^2$ has compact support, the authors applied Matsumura's estimates for $m = 1$. Moreover, they found that the energy of the solution to (2.1) satisfies (2.5) for $m = 1$, and they proved a blow-up result in finite time if $p < p_{\rm Fuj}(n)$, provided that $f(u) = |u|^p$ and that $\int_{\mathbb{R}^n} u_j(x)\mathrm{d}x > 0$ for $j = 0, 1$. The same result was obtained in [19] for the case $p = p_{\rm Fuj}(n)$.

We remark that the exponent $p_{\rm Fuj}(n)$ is the Fujita's one, the same which guarantees the existence of a non-negative classical global solution to the semi-linear heat equation

$$u_t - \triangle u = u^p, \quad u(0, x) = u_0(x),$$

provided that $u_0 \geq 0$ is sufficiently smooth. The Fujita exponent is sharp, that is, if $p \leq p_{\rm Fuj}(n)$, the semi-linear heat equation does not admit any global regular solution (see [6]).

Coming back to the global existence theory for the semi-linear classical damped wave equation, the condition on the compact support of the data was relaxed in [9] by assuming small data in a suitable weighted Sobolev space with norm given by

$$I^2 := \int_{\mathbb{R}^n} \mathrm{e}^{\frac{|x|^2}{2}} (|u_1|^2 + |\nabla u_0|^2 + |u_0|^2)\mathrm{d}x \leq \epsilon^2. \tag{2.6}$$

Condition (2.6) implies that $(u_0, u_1) \in (W^{1,1} \cap H^1) \times (L^1 \cap L^2) \subset \mathcal{A}_{1,1}$, and therefore, in [9], the authors used Matsumura's estimates (2.5) for $m = 1$.

Furthermore, in [7], the authors showed that the smallness in weighted Sobolev spaces or compactly supported data can be avoided assuming smallness in $\mathcal{A}_{1,1}$ and the critical exponent remains $p_{\rm Fuj}(n)$ for $n = 1, 2$. Since their technique requires $p > 2$, the authors obtained global existence only for $2 < p \leq 3 = p_{\rm GN}(3)$ if $n = 3$ (we remark that $p_{\rm Fuj}(3) = 1 + \frac{2}{3} < 2$). In [8], this result was extended to the case of initial data in $\mathcal{A}_{m,1}$ for $m \in (1, 2)$.

In this paper, we are going to follow the approach in [7–9]. In particular, we use some Matsumura-type estimates for the linear wave equation with time-dependent effective damping, derived by Wirth [18]. In order to do this, we extend these estimates to a family of Cauchy problems with the initial time as a parameter.

We remark that the Cauchy problem for the classical wave equation (i.e., $b \equiv 1$) is independent of translation in time, since the coefficients of the equation do not depend on t and hence Duhamel's principle is easily applied, whereas for a non-constant $b = b(t)$ the situation is more complicated.

3 Main Results

In order to present our results, we fix the class of effective damping terms $b(t)$ which are of interest in the further discussions.

Hypothesis 3.1 We make the following assumptions on the damping term $b(t)$:
(i) $b(t) > 0$ for any $t \geq 0$.
(ii) $b(t)$ is monotone, and $tb(t) \to \infty$ as $t \to \infty$.
(iii) $((1 + t)^2 b(t))^{-1} \in L^1([0, \infty))$.
(iv) $b \in \mathcal{C}^3$ and

$$\frac{|b^{(k)}(t)|}{b(t)} \lesssim \frac{1}{(1 + t)^k} \tag{3.1}$$

for any $k = 1, 2, 3$.

(v) $\frac{1}{b} \notin L^1$.

The damping term $b(t)$ is said to be effective according to [17–18].

Definition 3.1 *We denote by $B(t, 0)$ the primitive of $\frac{1}{b}(t)$ which vanishes at $t = 0$, that is,*

$$B(t,0) = \int_0^t \frac{1}{b(\tau)} \mathrm{d}\tau. \tag{3.2}$$

Thanks to conditions (i) and (v) in Hypothesis 3.1, $B(t, 0)$ is a positive, strictly increasing function, and $B(t, 0) \to +\infty$ as $t \to \infty$.

Let us consider the Cauchy problem for the linear damped wave equation given by

$$\begin{cases} u_{tt} - \triangle u + b(t) u_t = 0, \\ u(0, x) = u_0(x), \\ u_t(0, x) = u_1(x). \end{cases} \tag{3.3}$$

In 2005, Wirth derived Matsumura-type estimates for the solution to (3.3) (see [17, Theorem 5.5] and [18, Theorem 26]).

Theorem 3.1 *If Hypothesis* 3.1 *is satisfied and* $(u_0, u_1) \in \mathcal{A}_{m,1}$ *for some* $m \in [1, 2]$*, then the solution to the Cauchy problem* (3.3) *satisfies the following decay estimates*:

$$\|u(t,\cdot)\|_{L^2} \le C(1 + B(t,0))^{-\frac{n}{2}(\frac{1}{m} - \frac{1}{2})} \|(u_0, u_1)\|_{\mathcal{A}_{m,0}}, \tag{3.4}$$

$$\|\nabla u(t,\cdot)\|_{L^2} \le C(1 + B(t,0))^{-\frac{n}{2}(\frac{1}{m} - \frac{1}{2}) - \frac{1}{2}} \|(u_0, u_1)\|_{\mathcal{A}_{m,1}}, \tag{3.5}$$

$$\|u_t(t,\cdot)\|_{L^2} \le C(b(t))^{-1}(1 + B(t,0))^{-\frac{n}{2}(\frac{1}{m} - \frac{1}{2}) - 1} \|(u_0, u_1)\|_{\mathcal{A}_{m,1}}. \tag{3.6}$$

In order to prove our results for semi-linear damped wave equations, we need a further assumption on $b(t)$ in the case of increasing $b(t)$.

Hypothesis 3.2 Let $b \in \mathcal{C}^1([0, \infty))$, $b(t) > 0$. We assume that there exists a constant $a \in [0, 1)$, such that

$$t b'(t) \le a b(t), \quad t \ge 0. \tag{3.7}$$

Remark 3.1 We recall that if $b(t)$ is as in Hypothesis 3.1, then it is either increasing or decreasing. If $b(t)$ is decreasing, then (3.7) holds for $a = 0$. On the other hand, if $b(t)$ is increasing, condition (3.7) is stronger than the upper bound of (3.1) for $k = 1$.

Our first result is based on a generalization of the ideas in [9].

Notation 3.1 Given $\rho : \mathbb{R}^n \to [0, \infty)$, we say that $f \in L^q(\rho)$ for some $q \in [1, \infty]$ if $\rho f \in L^q$. Similarly, for any $f \in L^2(\rho)$ such that $\nabla f \in L^2(\rho)$, we write $f \in H^1(\rho)$.

It is easy to see that $H^1(\rho) \hookrightarrow H^1$ if $\rho > 0$ and $\frac{1}{\rho} \in L^\infty$.

Since in this paper we will work with exponential weight functions, for the sake of brevity we will denote $L^q(e^g)$ as L^q_g and $H^1(e^g)$ as H^1_g for any $g : \mathbb{R}^n \to \mathbb{R}$.

We assume that the initial data of (1.1) are small in $H^1_{\alpha|x|^2} \times L^2_{\alpha|x|^2}$ for some $\alpha \in \big(0, \frac{1}{4}\big]$. We put

$$I_\alpha^2 := \int_{\mathbb{R}^n} \mathrm{e}^{2\alpha|x|^2}(|u_0(x)|^2 + |\nabla u_0(x)|^2 + |u_1(x)|^2)\mathrm{d}x. \tag{3.8}$$

Theorem 3.2 *Let $n \geq 1$ and $p > p_{\mathrm{Fuj}}(n)$. Moreover, let $p \leq p_{\mathrm{GN}}(n)$ if $n \geq 3$. Let $\alpha \in \big(0, \frac{1}{4}\big]$. Then there exists an $\epsilon_0 > 0$, such that if $I_\alpha \leq \epsilon_0$, where I_α is introduced in (3.8), then there exists a unique solution to (1.1) in $\mathcal{C}([0,\infty), H^1) \cap \mathcal{C}^1([0,\infty), L^2)$.*

Moreover, there exists a constant $C > 0$, such that the solution satisfies the decay estimates

$$\|u(t,\cdot)\|_{L^2} \leq C\, I_\alpha\, (1 + B(t,0))^{-\frac{n}{4}}, \tag{3.9}$$

$$\|\nabla u(t,\cdot)\|_{L^2} \leq C\, I_\alpha\, (1 + B(t,0))^{-\frac{n}{4}-\frac{1}{2}}, \tag{3.10}$$

$$\|u_t(t,\cdot)\|_{L^2} \leq C\, I_\alpha\, (1 + B(t,0))^{-\frac{n}{4}}(1+t)^{-1}. \tag{3.11}$$

Finally, the wave energy is uniformly bounded in the family of weighted spaces $L^2_{\psi(t,\cdot)}$, where

$$\psi(t,x) = \frac{\alpha|x|^2}{(1 + B(t,0))}, \tag{3.12}$$

namely,

$$\int_{\mathbb{R}^n} \mathrm{e}^{\frac{2\alpha|x|^2}{(1+B(t,0))}}(|\nabla u(t,x)|^2 + |u_t(t,x)|^2)\mathrm{d}x \leq C I_\alpha^2, \quad t \geq 0.$$

We notice that $\psi(0,x) = \alpha|x|^2$ gives the weight at $t = 0$.

The decay estimates (3.9)–(3.11) for the solution to the semi-linear problem (1.1) correspond to the decay estimates (3.4)–(3.6) with $m = 1$, for the solution to the linear problem (3.3). In particular, the decay factor $(1+t)^{-1}$ in (3.11) is equivalent to $(b(t))^{-1}(1 + B(t,0))^{-1}$ in (3.6), as we shall see in Remark 4.9.

Now let us assume $(u_0, u_1) \in \mathcal{A}_{1,1}$ (see (2.3)). We follow the approach in [8] to gain a global existence result for this larger class of data. This goal will restrict our range of admissible n and p.

Theorem 3.3 *Let $n \leq 4$, and let*

$$\begin{cases} p > p_{\mathrm{Fuj}}(n), & \text{if } n = 1, 2, \\ 2 \leq p \leq 3 = p_{\mathrm{GN}}(3), & \text{if } n = 3, \\ p = 2 = p_{\mathrm{GN}}(4), & \text{if } n = 4. \end{cases} \tag{3.13}$$

Let $(u_0, u_1) \in \mathcal{A}_{1,1}$. Then, there exists an $\epsilon_0 > 0$, such that, if

$$\|(u_0, u_1)\|_{\mathcal{A}_{1,1}} \leq \epsilon_0,$$

then there exists a unique solution to (1.1) in $\mathcal{C}([0,\infty), H^1) \cap \mathcal{C}^1([0,\infty), L^2)$. Moreover, there exists a constant $C > 0$, such that the solution satisfies the decay estimates

$$\|u(t,\cdot)\|_{L^2} \leq C\, \|(u_0,u_1)\|_{\mathcal{A}_{1,1}}\, (1 + B(t,0))^{-\frac{n}{4}}, \tag{3.14}$$

$$\|\nabla u(t,\cdot)\|_{L^2} \leq C\, \|(u_0,u_1)\|_{\mathcal{A}_{1,1}}\, (1 + B(t,0))^{-\frac{n}{4}-\frac{1}{2}}, \tag{3.15}$$

$$\|u_t(t,\cdot)\|_{L^2} \leq C\, \|(u_0,u_1)\|_{\mathcal{A}_{1,1}}\, (1 + B(t,0))^{-\frac{n}{4}}(1+t)^{-1}. \tag{3.16}$$

As in Theorem 3.2, the solutions to the semi-linear Cauchy problem (1.1) and to the linear one (3.3) have the same decay rate.

Remark 3.2 Since we are interested in energy solutions in Theorem 3.3 the restriction $p \geq 2$ appears in a natural way. In both Theorems 3.2 and 3.3 the Fujita exponent $p_{\mathrm{Fuj}}(n)$ appears as a lower bound of admissible exponents p. The optimality of this bound is discussed in Section 3.3.

3.1 Examples

Example 3.1 Let us choose

$$b(t) = \frac{\mu}{(1+t)^\kappa} \quad \text{for some } \mu > 0 \text{ and } \kappa \in (-1,1). \tag{3.17}$$

With $\kappa \in (-1,1)$, Hypothesis 3.1 holds. Indeed $tb(t) \approx (1+t)^{1-\kappa}$ and $(1+t)^2 b(t) \approx (1+t)^{2-\kappa}$ as $t \to \infty$, so that $\frac{1}{b} \notin L^1$ and $((1+t)^2 b(t))^{-1} \in L^1$.

Hypothesis 3.2 holds since (3.7) is satisfied for $a = \max\{-\kappa, 0\}$.

We observe that $1+B(t,0) \approx (1+t)^{1+\kappa}$. Therefore we can apply Theorems 3.2–3.3 with $I_\alpha = \epsilon$ and $\|(u_0,u_1)\|_{\mathcal{A}_{1,1}} = \epsilon$, respectively. The decay in (3.9)–(3.11) or in (3.14)–(3.16) can be rewritten as

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2} &\leq C\,\epsilon\,(1+t)^{-(1+\kappa)\frac{n}{4}},\\
\|\nabla u(t,\cdot)\|_{L^2} &\leq C\,\epsilon\,(1+t)^{-(1+\kappa)(\frac{n}{4}+\frac{1}{2})},\\
\|u_t(t,\cdot)\|_{L^2} &\leq C\,\epsilon\,(1+t)^{-(1+\kappa)\frac{n}{4}-1}.
\end{aligned}$$

In particular, for $\kappa = 0$ we have a constant coefficient in the damping term, and we cover the results described in Section 2.

The case of $\kappa = 1$ has recently been studied in [1]. In particular, $p_{\mathrm{Fuj}}(n)$ still remains the critical exponent of the small data solution, provided that $\mu \geq n+2$.

Example 3.2 Let us multiply the function $b(t)$ in (3.17) by a logarithmic positive power. We consider the following coefficient $b(t)$ in the damping term:

$$b(t) = \frac{\mu}{(1+t)^\kappa}(\log(c+t))^\gamma \quad \text{for some } \mu > 0, \gamma > 0 \text{ and } \kappa \in (-1,1], \tag{3.18}$$

where $c = c(\kappa,\gamma) > 1$ is a suitably large positive constant.

It is easy to check that conditions (i) and (iv)–(v) in Hypothesis 3.1 hold and that $tb(t) \to +\infty$ as $t \to \infty$. Moreover, condition (iii) in Hypothesis 3.1 holds for any $\gamma > 0$ if $\kappa \in (-1,1)$ and for any $\gamma > 1$ if $\kappa = 1$, since

$$((1+t)^2 b(t))^{-1} = \frac{1}{\mu(1+t)^{2-\kappa}(\log(c+t))^\gamma}.$$

For $\kappa = 0$ the assumption (ii) in Hypothesis 3.1 is satisfied. Let $\kappa \in (-1,1]$, $\kappa \neq 0$. If we explicitly compute $b'(t)$, then we derive

$$\begin{aligned}
b'(t) &= -\frac{\mu\kappa}{(1+t)^{\kappa+1}}(\log(c+t))^\gamma + \frac{\mu\gamma}{(1+t)^\kappa(c+t)}(\log(c+t))^{\gamma-1}\\
&= \frac{\mu}{(1+t)^{\kappa+1}}(\log(c+t))^\gamma\Big(-\kappa + \frac{\gamma(1+t)}{(c+t)\log(c+t)}\Big),
\end{aligned}$$

and therefore, we get

$$b'(t) \approx \frac{1}{(1+t)^{\kappa+1}}(\log(c+t))^{\gamma} \approx \frac{b(t)}{1+t},$$

provided that $c = c(\kappa,\gamma) > e^{\frac{\gamma}{|\kappa|}}$. We proved that $b(t)$ is monotone and this concludes the proof of Hypothesis 3.1.

If $\kappa \in (0,1]$, then Hypothesis 3.2 holds since $b(t)$ is decreasing. If $\kappa \in (-1,0]$, then (3.7) is satisfied for $c > e^{\frac{\gamma}{1+\kappa}}$. In facts,

$$\frac{tb'(t)}{b(t)} = \frac{t}{1+t}\Big(-\kappa + \frac{\gamma(1+t)}{(c+t)\log(c+t)}\Big) < -\kappa + \frac{\gamma}{\log c} < 1.$$

In particular, in correspondence with $\kappa = 0$, we have

$$\frac{tb'(t)}{b(t)} = \frac{t\gamma}{(c+t)\log(c+t)} < \frac{\gamma}{\log c} < 1.$$

Example 3.3 Analogous to Example 3.2, we can multiply the function $b(t)$ in (3.17) by a logarithmic negative power, that is, we can consider the coefficient

$$b(t) = \frac{\mu}{(1+t)^{\kappa}(\log(c+t))^{\gamma}} \quad \text{for some } \mu > 0,\ \gamma > 0 \text{ and } \kappa \in (-1,1), \tag{3.19}$$

where $c = c(\kappa,\gamma) > 1$ is a suitably large positive constant. It is easy to check that Hypotheses 3.1 and 3.2 are satisfied if $c = c(\kappa,\gamma) > 1$ is sufficiently large.

Example 3.4 We can also consider an iteration of logarithmic functions, eventually with different powers, like

$$b(t) = \frac{\mu}{(1+t)^{\kappa}}(\log(c_1 + (\log(c_2+t))^{\gamma_2}))^{\gamma_1},$$
$$b(t) = \frac{\mu}{(1+t)^{\kappa}}(\log(c_1 + (\log(c_2 + (\log(c_3+\cdots)))^{\gamma_3}))^{\gamma_2})^{\gamma_1}.$$

3.2 A special class of effective damping

In [11, 14], the authors studied damping terms with the time-dependent coefficient (3.17). They obtained the following results.

Theorem 3.4 *Let $p > p_{\mathrm{Fuj}}(n)$ and $p < \frac{n+2}{n-2}$ if $n \ge 3$. Let $b(t) = \mu(1+t)^{-\kappa}$ for $\kappa \in (-1,1)$ and $\mu > 0$. Let $(u_0,u_1) \in H^1 \times L^2$ be compactly supported. Then, there exists an $\epsilon_0 > 0$, such that, if*

$$\int_{\mathbb{R}^n} e^{\frac{(1+\kappa)|x|^2}{2(2+\delta)}}(|u_0(x)|^{p+1} + |\nabla u_0(x)|^2 + |u_1(x)|^2)dx \le \epsilon^2 \tag{3.20}$$

for an arbitrarily small $\delta > 0$ and for some $\epsilon \in (0,\epsilon_0]$, then there exists a unique solution $u \in \mathcal{C}([0,\infty),H^1) \cap \mathcal{C}^1([0,\infty),L^2)$ to (1.1) which satisfies

$$\|u(t,\cdot)\|_{L^2} \le C(\delta)\epsilon\,(1+t)^{-\frac{(1+\kappa)n}{4}+\frac{\varepsilon}{2}}, \tag{3.21}$$

$$\|\nabla u(t,\cdot)\|_{L^2} + \|u_t(t,\cdot)\|_{L^2} \le C(\delta)\epsilon\,(1+t)^{-\frac{(1+\kappa)(n+2)}{4}+\frac{\varepsilon}{2}} \tag{3.22}$$

for a small constant $\varepsilon = \varepsilon(\delta) > 0$ and a large constant $C(\delta)$ with $\varepsilon(\delta) \to 0$ and $C(\delta) \to \infty$ as $\delta \to 0$.

Moreover, in [11], the authors established that there does not exist any global solution $u \in \mathcal{C}([0,\infty), H^1) \cap \mathcal{C}^1([0,\infty), L^2)$ in the case of $f(u) = |u|^p$ with $1 < p \le p_{\mathrm{Fuj}}(n)$ and initial data, such that

$$\int_{\mathbb{R}^n} u_1(x) + \widehat{b}_1 u_0(x) \mathrm{d}x > 0 \quad \text{with } \widehat{b}_1^{-1} = \int_0^\infty \exp\Big(- \int_0^t b(s)\mathrm{d}s \Big) \mathrm{d}t.$$

We remark that in (3.20) the exponents p and κ come into play.

Recalling Notation 3.1, for some $\beta > 0$, $q \ge 1$ and $K > 0$, we put

$$\begin{aligned} D_{\beta,q,K} &= \{(u_0,u_1) \in (\dot{H}^1_{\frac{\beta|x|^2}{2}} \cap L^q_{\frac{\beta|x|^2}{q}}) \times L^2_{\frac{\beta|x|^2}{2}} \mid \mathrm{supp}(u_0,u_1) \subset B_K(0)\}, \\ D_\beta &= H^1_{\frac{\beta|x|^2}{2}} \times L^2_{\frac{\beta|x|^2}{2}}. \end{aligned}$$

Let $\beta(\kappa,\delta) := \frac{1+\kappa}{2(2+\delta)}$. After fixing a small $\delta > 0$, the space of initial data in Theorem 3.4 is given by

$$\bigcup_{K>0} D_{\beta(\kappa,\delta),p+1,K},$$

whereas the space of initial data in Theorem 3.2 is $D_{2\alpha}$ for some $\alpha \in \big(0, \frac{1}{4}\big]$.

Since $\beta(\kappa,\delta) < \frac{1}{2}$, we observe that for any $\delta > 0$, $p > 1$ and $\kappa \le 1$, we have

$$D_{\beta(\kappa,\delta),p+1,K} \subset D_{\beta(\kappa,\delta),2,K} \subset D_{\frac{1}{2},2,K} \subsetneq D_{\frac{1}{2}} \subset D_{2\alpha} \subsetneq \mathcal{A}_{1,1} \tag{3.23}$$

for any $K > 0$ and $\alpha \in \big(0, \frac{1}{4}\big]$. Hence the class of admissible small data in [11] is strictly contained in the class of admissible small data in Theorem 3.2. In particular,

(1) we do not assume compactly supported initial data;

(2) in Theorem 3.2 we do not choose u_0 from a weighted L^{p+1} space but from a weighted L^2 space;

(3) the space with weight $\mathrm{e}^{\beta(\kappa,\delta)|x|^2}$ is properly contained in $D_{\frac{1}{2}}$, the space in Theorem 3.2 corresponding to $\alpha = \frac{1}{4}$;

(4) in Theorem 3.3 we enlarge the class of initial data to $\mathcal{A}_{1,1}$.

We can enlarge the class of initial data, since we use Matsumura's type estimates which are avoided in [11]. This technique has other advantages. First of all, we can consider more general $b(t)$, not only the ones that grow like t^κ (see Examples 3.2–3.3, and Hypothesis 8.1 in Section 8).

Moreover, if $(u_0,u_1) \in D_{\beta(\kappa,\delta),p+1,K}$ for some $K > 0$, then applying Theorem 3.4 we know that there exists an $\epsilon_0 > 0$, such that for any $\epsilon \in (0,\epsilon_0)$ the solution corresponding to data $(\epsilon u_0, \epsilon u_1)$ exists globally in time. Here $\epsilon_0 > 0$ depends on u_0, u_1 and K. Due to (3.23), these data can be used in Theorems 3.2–3.3, but the corresponding $\epsilon_0 > 0$ depends only on (u_0,u_1). Finally, in the decay estimates for the solution u and the energy $(\nabla u, u_t)$, an $\frac{\varepsilon}{2}$ loss of decay appears in Theorem 3.4, while on the contrary, in Theorems 3.2–3.3, we have optimal decay rates.

3.3 Optimality

The sharpness of the Fujita exponent p_{Fuj} in Theorems 3.2 and 3.3 is of special interest. This question was first discussed by Wakasugi from Osaka University and the first author during his scientific stays at TU Bergakademie Freiberg. The following result corresponds to Example 2 in [4], and it is based on a modification of the test function method developed by Zhang [19].

We make the following assumptions on $b(t)$.

Hypothesis 3.3 We assume that $b \in \mathcal{C}^1$ satisfies $b(t) > 0$ and that

$$\liminf_{t\to\infty} \frac{b'(t)}{b(t)^2} > -1,$$

$$\limsup_{t\to\infty} \frac{tb'(t)}{b(t)} < 1.$$

We remark that Hypothesis 3.3 is weaker than Hypothesis 3.1.

Theorem 3.5 *Assume Hypotheses* 3.2–3.3 *and let* $p \le p_{\mathrm{Fuj}}(n)$. *Then the function*

$$\beta(t) := \exp\Big(- \int_0^t b(\tau)\mathrm{d}\tau\Big)$$

is in $L^1(0,\infty)$ *and there exists no global solution* $u \in \mathcal{C}^2([0,\infty)\times\mathbb{R}^n)$ *to* (1.1) *with* $f(u) = |u|^p$ *for initial data* $(u_0, u_1) \in L^1$ *satisfying*

$$\int_{\mathbb{R}^n} (u_0(x) + \widehat{b}_1 u_1(x))\mathrm{d}x > 0, \tag{3.24}$$

where $\widehat{b}_1 := \|\beta\|^{-1}_{L^1(0,\infty)}$.

Example 3.5 Let us consider $b(t) = \mu(1+t)^{-\kappa}$ as in (3.17) in Example 3.1 for some $\kappa \in (-1,1]$ and $\mu > 0$. Then Hypothesis 3.3 holds, provided that $\mu > 1$ if $\kappa = 1$.

Let b be as in (3.18) in Example 3.2, that is, $b(t) = \mu(1+t)^{-\kappa}(\log(c+t))^{\gamma}$. Then Hypothesis 3.3 holds for any $\mu > 0, \kappa \in (-1,1], \gamma > 0$ with a suitable constant c. Analogously, we can prove Hypothesis 3.3, if b is chosen as in Examples 3.3–3.4.

4 Linear Decay Estimates

In order to prove Theorems 3.2–3.3, we have to extend the decay estimates (3.4)–(3.6) given by Wirth for the Cauchy problem (3.3) to a family of parameter-dependent Cauchy problems with initial data $(0, g(s,x))$ for some function g.

Let $s \ge 0$ be a parameter. We consider the following Cauchy problem in $[s,\infty)\times\mathbb{R}^n$:

$$\begin{cases} v_{tt} - \triangle v + b(t)v_t = 0, \quad t \in [s,\infty), \\ v(s,x) = 0, \\ v_t(s,x) = g(s,x). \end{cases} \tag{4.1}$$

It is clear that we have to extend Definition 3.1.

Definition 4.1 *We denote by* $B(t,s)$ *the primitive of* $\frac{1}{b(t)}$ *which vanishes at* $t = s$, *that is,*

$$B(t,s) = \int_s^t \frac{1}{b(\tau)}\mathrm{d}\tau = B(t,0) - B(s,0). \tag{4.2}$$

Then we have the following result.

Theorem 4.1 *Let $b(t)$ satisfy Hypothesis 3.1 and let $g(s,\cdot) \in L^m \cap L^2$ for some $m \in [1,2]$. Then the solution $v(t,x)$ to (4.1) satisfies the following Matsumura-type decay estimates:*

$$\|v(t,\cdot)\|_{L^2} \le C(b(s))^{-1}(1+B(t,s))^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}\|g(s,\cdot)\|_{L^m\cap L^2}, \tag{4.3}$$

$$\|\nabla v(t,\cdot)\|_{L^2} \le C(b(s))^{-1}(1+B(t,s))^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-\frac{1}{2}}\|g(s,\cdot)\|_{L^m\cap L^2}, \tag{4.4}$$

$$\|v_t(t,\cdot)\|_{L^2} \le C(b(s))^{-1}(b(t))^{-1}(1+B(t,s))^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})-1}\|g(s,\cdot)\|_{L^m\cap L^2}. \tag{4.5}$$

We remark that the constant $C>0$ does not depend on s.

We remark that Hypothesis 3.2 does not come into play in Theorem 4.1.

4.1 Application of Duhamel's principle to the semi-linear problem

Let us denote by $E_1(t,s,x)$ the fundamental solution to the linear homogeneous problem (4.1), in particular,

$$E_1(s,s,x) = 0 \quad \text{and} \quad \partial_t E_1(s,s,x) = \delta_x,$$

where δ_x is the Dirac distribution in the x variable. Here the symbol $*_{(x)}$ denotes the convolution with respect to the x variable. By Duhamel's principle, we get

$$u^{\mathrm{nl}}(t,x) = \int_0^t E_1(t,s,x) *_{(x)} f(u(s,x))\mathrm{d}s \tag{4.6}$$

as the solution to the inhomogeneous problem

$$\begin{cases} u^{\mathrm{nl}}_{tt} - \triangle u^{\mathrm{nl}} + b(t)u^{\mathrm{nl}}_t = f(u(t,x)), \quad t\in[0,\infty), \\ u^{\mathrm{nl}}(0,x) = 0, \\ u^{\mathrm{nl}}_t(0,x) = 0. \end{cases} \tag{4.7}$$

Let $u^{\mathrm{lin}}(t,x)$ be the solution to (3.3). Then

$$u^{\mathrm{lin}}(t,x) = E_0(t,0,x) *_{(x)} u_0(x) + E_1(t,0,x) *_{(x)} u_1(x), \tag{4.8}$$

where $E_1(t,0,x)$ is as above, and by $E_0(t,0,x)$ we denote the fundamental solution to the homogeneous Cauchy problem (3.3) with initial data $(\delta_x, 0)$, that is

$$E_0(0,0,x) = \delta_x \quad \text{and} \quad \partial_t E_0(0,0,x) = 0.$$

Now the solution to (1.1) can be written in the form

$$\begin{aligned} u(t,x) &= u^{\mathrm{lin}}(t,x) + u^{\mathrm{nl}}(t,x) \\ &= E_0(t,0,x) *_{(x)} u_0(x) + E_1(t,0,x) *_{(x)} u_1(x) + \int_0^t E_1(t,s,x) *_{(x)} f(u(s,x))\mathrm{d}s. \end{aligned} \tag{4.9}$$

4.2 Properties of $B(t, s)$

In the proofs of Theorems 3.2–3.3, we will make use of some properties of the function $B(t, s)$ which follow from Hypothesis 3.2 for the coefficient $b(t)$.

Remark 4.1 If (3.7) holds, then it follows that the function $\frac{t}{b(t)}$ is increasing and

$$\Big(\frac{t}{b(t)}\Big)' = \frac{b(t) - tb'(t)}{b^2(t)} \geq (1-a)\,\frac{1}{b(t)}.$$

Moreover, since $\frac{|b'(t)|}{b(t)} \leq \frac{M}{1+t}$ for some $M > 0$ (see (3.1)), we derive

$$\Big(\frac{t}{b(t)}\Big)' = \frac{b(t) - tb'(t)}{b^2(t)} \leq \frac{1+M}{b(t)}.$$

In particular, for any $s \in [0, t]$, we can derive

$$B(t, s) = \int_s^t \frac{1}{b(\tau)}\mathrm{d}\tau \approx \frac{t}{b(t)} - \frac{s}{b(s)}. \tag{4.10}$$

Remark 4.2 By integrating (3.7) over $[s, t]$, we derive

$$\frac{b(t)}{b(s)} \leq \Big(\frac{t}{s}\Big)^a \quad \text{for any } s > 0 \text{ and } t \geq s,$$

that is, for any $\lambda \in (0, 1]$ and for any $t \in [0, \infty)$, it holds that

$$b(\lambda t) \geq \lambda^a b(t). \tag{4.11}$$

We remark that, in particular, $b(t) \leq t^a b(1)$ for $t \geq 1$. Therefore, Hypothesis 3.2 implies Hypothesis 3.1(v), since $a \in [0, 1)$.

Remark 4.3 Thanks to (3.1) for $k = 1$, there exists a constant $M \geq 0$, such that

$$\frac{b'(t)}{b(t)} \geq -\frac{M}{1+t} \geq -\frac{M}{t}, \quad t > 0. \tag{4.12}$$

It is clear that if $b(t)$ is increasing, then we can take $M = 0$.

By integrating (4.12) over $[s, t]$, we derive

$$\frac{b(t)}{b(s)} \geq \Big(\frac{t}{s}\Big)^{-M} \quad \text{for any } s > 0 \text{ and } t \geq s,$$

that is, for any $\lambda \in (0, 1]$ and for any $t \in [0, \infty)$, it holds that

$$b(\lambda t) \leq \lambda^{-M} b(t). \tag{4.13}$$

Properties (4.11)–(4.13) play a fundamental role in the next estimates.

Remark 4.4 Conditions (4.11)–(4.13) guarantee that for any fixed $\lambda \in (0, 1)$, we have

$$b(s) \approx b(t), \quad s \in [\lambda t, t]. \tag{4.14}$$

Indeed, let $\lambda_1 := \frac{s}{t}$. Then $\lambda_1 \in [\lambda, 1]$. Hence, we get

$$\lambda^a b(t) \le \lambda_1^a b(t) \le b(s) \le \lambda_1^{-M} b(t) \le \lambda^{-M} b(t)$$

from (4.11)–(4.13).

Remark 4.5 By using (3.7) and its consequences (4.10)–(4.11), we can prove that for any fixed $\lambda \in (0,1)$, it holds that

$$B(t,0) \ge B(t,\lambda t) \approx \frac{t}{b(t)} - \frac{\lambda t}{b(\lambda t)} \ge \frac{t}{b(t)} - \frac{\lambda^{1-a} t}{b(t)} = \delta \frac{t}{b(t)} \approx B(t,0),$$

where we put $\delta = 1 - \lambda^{1-a} > 0$ since $\lambda \in (0,1)$ and $a \in [0,1)$. Therefore,

$$C_{\lambda,a} B(t,0) \le B(t,\lambda t) \le B(t,0) \quad \text{for } \lambda \in (0,1). \tag{4.15}$$

Remark 4.6 By using (4.10) and (4.13), we can prove that for any fixed $\lambda \in (0,1)$, it holds that

$$B(\lambda t, 0) \approx \frac{\lambda t}{b(\lambda t)} \ge \lambda^{1+M} \frac{t}{b(t)},$$

and, consequently,

$$C_{\lambda,M} B(t,0) \le B(\lambda t, 0) \le B(t,0) \quad \text{for } \lambda \in (0,1). \tag{4.16}$$

Remark 4.7 By splitting the interval $[0,t]$ into $[0, \frac{t}{2}]$ and $[\frac{t}{2}, t]$ and using (4.16), we can derive

$$B(s,0) \approx B(t,0), \quad s \in \Big[\frac{t}{2}, t\Big], \tag{4.17}$$

whereas by using (4.15), we get

$$B(t,s) \approx B(t,0), \quad s \in \Big[0, \frac{t}{2}\Big]. \tag{4.18}$$

Remark 4.8 By using Taylor-Lagrange's theorem (with center t) and (4.14) with $\lambda = \frac{1}{2}$, we obtain

$$B(t,s) \approx \frac{t-s}{b(t)} \approx \frac{t-s}{b(s)}, \quad s \in \Big[\frac{t}{2}, t\Big]. \tag{4.19}$$

Indeed, $b(s) \approx b(r) \approx b(t)$ for any $r \in [s,t] \subset \big[\frac{t}{2}, t\big]$, thanks to (4.14) and

$$B(t,s) = B(t,t) + (s-t)\partial_s B(t,r) = 0 + \frac{t-s}{b(r)} \quad \text{for some } r \in [s,t].$$

Remark 4.9 We observe that

$$b(t)(1 + B(t,0)) \approx 1 + b(t)B(t,0) \approx 1 + t.$$

Thanks to (4.10), it suffices to prove only the first equivalence.

Since $b(t) > 0$ for any $t > 0$, the equivalence holds on compact intervals. It remains to observe that the behavior of the two objects is described in both cases by $b(t)B(t,0)$ for $t \to \infty$.

Indeed, since $B(t,0) \to \infty$ (we recall that $\frac{1}{b} \notin L^1$), it follows that $1 + B(t,0) \approx B(t,0)$, therefore $b(t)(1 + B(t,0)) \approx b(t)B(t,0)$. On the other hand, applying once more (4.10), it follows that $b(t)B(t,0) \ge C\, t \to \infty$. Therefore, $1 + b(t)B(t,0) \approx b(t)B(t,0)$.

5 Proof of Theorem 3.2

5.1 Local existence in weighted energy spaces

We have the following local existence result in weighted energy spaces.

Lemma 5.1 *Let $b(t) > 0$ and $1 < p \leq p_{\mathrm{GN}}(n)$. Let $\psi \in \mathcal{C}^1([0,\infty)\times\mathbb{R}^n)$, such that for any $t \geq 0$ and a.e. $x \in \mathbb{R}^n$, one has*

$$\begin{aligned}
&\psi(t,x) \geq 0,\\
&\psi_t(t,x) \leq 0,\\
&b(t)\psi_t(t,x) + |\nabla\psi(t,x)|^2 \leq 0,\\
&\Delta\psi(t,x) > 0,\\
&\inf_{x\in\mathbb{R}^n} \Delta\psi(t,x) = C(t) > 0.
\end{aligned} \tag{5.1}$$

For any $(u_0,u_1) \in H^1(\mathrm{e}^{\psi(0,x)}) \times L^2(\mathrm{e}^{\psi(0,x)})$, there exists a maximal existence time $T_m \in (0,\infty]$, such that (1.1) has a unique solution $u \in \mathcal{C}([0,T_m),H^1) \cap \mathcal{C}^1([0,T_m),L^2)$. Moreover, for any $T < T_m$, it holds that

$$\sup_{[0,T]} \|\mathrm{e}^{\psi(t,\cdot)}u(t,\cdot)\|_{L^2} + \|\mathrm{e}^{\psi(t,\cdot)}\nabla u(t,\cdot)\|_{L^2} + \|\mathrm{e}^{\psi(t,\cdot)}u_t(t,\cdot)\|_{L^2} < \infty.$$

Finally, if $T_m < \infty$, then

$$\limsup_{t\to T_m} \|\mathrm{e}^{\psi(t,\cdot)}u(t,\cdot)\|_{L^2} + \|\mathrm{e}^{\psi(t,\cdot)}\nabla u(t,\cdot)\|_{L^2} + \|\mathrm{e}^{\psi(t,\cdot)}u_t(t,\cdot)\|_{L^2} = \infty. \tag{5.2}$$

The proof follows the same lines of the Appendix of [9]. We underline that the local existence result does not require Hypothesis 3.1 or 3.2.

5.2 Energy estimates in weighted energy spaces

Let us observe that the function $\psi(t,x)$ given in (3.12) satisfies (5.1), since $\alpha \in \big(0,\frac{1}{4}\big]$. Therefore, the local existence result is applicable. Indeed,

$$\psi(t,x) = \frac{\alpha|x|^2}{1+B(t,0)}$$

verifies

$$\psi_t = -\frac{\alpha|x|^2}{(1+B(t,0))^2 b(t)}, \quad \nabla\psi = \frac{2\alpha x}{1+B(t,0)}, \quad \triangle\psi = \frac{2n\alpha}{1+B(t,0)},$$

together with the fundamental property

$$b(t)\psi_t + |\nabla\psi|^2 = -\frac{\alpha(1-4\alpha)|x|^2}{(1+B(t,0))^2} \leq 0, \tag{5.3}$$

since $\alpha \in \big(0,\frac{1}{4}\big]$. We underline that for $\alpha = \frac{1}{4}$ the equation $b(t)\psi_t + |\nabla\psi|^2 = 0$ is related to the symbol of the linear parabolic equation $b(t)u_t - \triangle u = 0$, that is, we have in mind the parabolic effect when we introduce the weight $\mathrm{e}^{\psi(t,x)}$.

Lemma 5.2 *Let us assume that* $(u_0, u_1) \in H^1_{\psi(0,x)} \times L^2_{\psi(0,x)}$, *and let* $\gamma = \frac{2}{p+1} + \varepsilon$ *for some* $\varepsilon > 0$. *If* $u = u(t,x)$ *is a local solution to* (1.1) *in* $[0,T)$, *then for any* $t \in [0,T)$ *the following energy estimate holds*:

$$E(t) \leq CI_\alpha^2 + CI_\alpha^{p+1} + C_\varepsilon \Big(\sup_{[0,t]} (1+B(s,0))^\varepsilon \| \mathrm{e}^{\gamma\psi(s,\cdot)} u(s,\cdot) \|_{L^{p+1}} \Big)^{p+1} \tag{5.4}$$

with I_α *given by* (3.8) *and*

$$E(t) := \frac{1}{2} \int_{\mathbb{R}^n} \mathrm{e}^{2\psi(t,x)} (|u_t(t,x)|^2 + |\nabla u(t,x)|^2) \mathrm{d}x.$$

Proof First we prove that

$$E(t) \lesssim I_\alpha^2 + I_\alpha^{p+1} + \| \mathrm{e}^{\frac{2}{p+1}\psi(t,\cdot)} u(t,\cdot) \|_{L^{p+1}}^{p+1} + \int_0^t \int_{\mathbb{R}^n} |\psi_t(s,x)| \mathrm{e}^{2\psi(s,x)} |u(s,x)|^{p+1} \mathrm{d}x\mathrm{d}s. \tag{5.5}$$

Straight forward calculations give the following relation:

$$\begin{aligned} &\partial_t \Big(\frac{\mathrm{e}^{2\psi}}{2} (|u_t|^2 + |\nabla u|^2 - F(u)) \Big) \\ &= \nabla \cdot (\mathrm{e}^{2\psi} u_t \nabla u) + \psi_t \mathrm{e}^{2\psi} |u_t|^2 + \frac{\mathrm{e}^{2\psi}}{\psi_t} |u_t \nabla \psi - \psi_t \nabla u|^2 - \frac{\mathrm{e}^{2\psi}}{\psi_t} u_t^2 (b(t)\psi_t + |\nabla\psi|^2) - \psi_t \mathrm{e}^{2\psi} F(u), \end{aligned}$$

where $F(u) := \int_0^u f(\tau)\mathrm{d}\tau$ is a primitive of the nonlinear term $|f(\tau)| \simeq |\tau|^p$, hence, $|F(u)| \leq C|u|^{p+1}$. After integration over $[0,t] \times \mathbb{R}^n$, by taking into consideration $\psi_t \leq 0$ and (5.3), we can estimate

$$G(t) \leq G(0) - 2 \int_0^t \int_{\mathbb{R}^n} \psi_t(s,x) \mathrm{e}^{2\psi(s,x)} F(u(s,x)) \mathrm{d}x\mathrm{d}s,$$

where we put

$$G(t) := E(t) - \int_{\mathbb{R}^n} \frac{\mathrm{e}^{2\psi(t,x)}}{2} F(u(t,x)) \mathrm{d}x = \int_{\mathbb{R}^n} \frac{\mathrm{e}^{2\psi(t,x)}}{2} (|u_t(t,x)|^2 + |\nabla u(t,x)|^2 - F(u(t,x))) \mathrm{d}x.$$

We remark that the divergence theorem can be applied with

$$\mathrm{e}^{2\psi(s,\cdot)} u_t(s,\cdot) \nabla u(s,\cdot) \in L^1(\mathbb{R}^n).$$

This follows from Lemma 5.1. Therefore,

$$E(t) \lesssim G(0) + \| \mathrm{e}^{\frac{2}{p+1}\psi(t,\cdot)} u(t,\cdot) \|_{L^{p+1}}^{p+1} + \int_0^t \int_{\mathbb{R}^n} |\psi_t(s,x)| \mathrm{e}^{2\psi(s,x)} |u(s,x)|^{p+1} \mathrm{d}x\mathrm{d}s.$$

In order to obtain (5.5), it remains to show that $G(0) \lesssim I_\alpha^2 + I_\alpha^{p+1}$. This reduces to prove that

$$\int_{\mathbb{R}^n} \mathrm{e}^{2\alpha|x|^2} |u_0|^{p+1} \mathrm{d}x \lesssim I_\alpha^{p+1}.$$

Since $p + 1 < p_{GN}(n) + 1 \leq \frac{2n}{n-2}$ for $n \geq 3$ (no requirement for $n = 1,2$), from Sobolev embedding theorem, it follows that

$$\int_{\mathbb{R}^n} \mathrm{e}^{2\alpha|x|^2} |u_0|^{p+1} \mathrm{d}x \lesssim \Big[\int_{\mathbb{R}^n} \mathrm{e}^{\frac{4\alpha}{p+1}|x|^2} (|u_0|^2 + |\nabla u_0|^2) \mathrm{d}x + \int_{\mathbb{R}^n} \mathrm{e}^{\frac{4\alpha}{p+1}|x|^2} |x|^2 |u_0|^2 \mathrm{d}x \Big]^{\frac{p+1}{2}}.$$

The assumption $p>1$ gives $(1+|x|^2)\mathrm{e}^{\frac{4\alpha}{p+1}|x|^2}\leq C\mathrm{e}^{2\alpha|x|^2}$. This concludes the proof of (5.5).

Now, by virtue of

$$|\psi_t(s,x)|\mathrm{e}^{(2-\gamma(p+1))\psi(s,x)}=\frac{\psi(s,x)}{(1+B(s,0))b(s)}\mathrm{e}^{-(p+1)\varepsilon\psi(s,x)}\leq\frac{C_\varepsilon}{(1+B(s,0))b(s)},$$

from (5.5), we derive

$$E(t)\leq CI_\alpha^2+CI_\alpha^{p+1}+C\|\mathrm{e}^{\frac{2}{p+1}\psi(t,\cdot)}u(t,\cdot)\|_{L^{p+1}}^{p+1}+C_\varepsilon\int_0^t\frac{1}{(1+B(s,0))b(s)}\|\mathrm{e}^{\gamma\psi(s,x)}u(s,x)\|_{L^{p+1}}^{p+1}\mathrm{d}s.$$

For any $\varepsilon>0$, it holds that

$$\int_0^t\frac{1}{(1+B(s,0))^{1+\varepsilon}b(s)}\mathrm{d}s=\int_1^{1+B(t,0)}\frac{1}{\tau^{1+\varepsilon}}\mathrm{d}\tau\leq\frac{1}{\varepsilon},$$

therefore,

$$E(t)\leq CI_\alpha^2+CI_\alpha^{p+1}+C\|\mathrm{e}^{\frac{2}{p+1}\psi(t,\cdot)}u(t,\cdot)\|_{L^{p+1}}^{p+1}+C'_\varepsilon\Big(\sup_{[0,t]}(1+B(s,0))^\varepsilon\|\mathrm{e}^{\gamma\psi(s,\cdot)}u(s,\cdot)\|_{L^{p+1}}\Big)^{p+1}.$$

To complete the proof, it is sufficient to notice that the third term is estimated by the fourth one, since $\gamma>\frac{2}{p+1}$ and $B(s,0)\geq 0$.

5.3 Decay estimates for the semi-linear problem

Let us observe that we can apply the estimates in Theorem 4.1 for $m=1$ if $(u_0,u_1)\in H^1_{\alpha|x|^2}\times L^2_{\alpha|x|^2}$. Indeed, for any $v\in L^2_{\alpha|x|^2}$, it holds that

$$\int_{\mathbb{R}^n}|v(x)|\mathrm{d}x\leq\Big(\int_{\mathbb{R}^n}\mathrm{e}^{2\alpha|x|^2}|v(x)|^2\mathrm{d}x\Big)^{\frac12}\Big(\int_{\mathbb{R}^n}\mathrm{e}^{-2\alpha|x|^2}\mathrm{d}x\Big)^{\frac12}.$$

Hence,

$$H^1_{\alpha|x|^2}\times L^2_{\alpha|x|^2}\subset(W^{1,1}\cap H^1)\times(L^1\cap L^2)\subset\mathcal{A}_{1,1}\,. \tag{5.6}$$

Having in mind the application of Theorem 4.1 for $m=1$, we need to estimate $f(u(s,\cdot))$ in $L^1\cap L^2$ by using the weighted energy spaces.

In a way similar to Lemma 2.5 in [9], after a change of variables, one has for any $\beta\geq 0$,

$$\int_{\mathbb{R}^n}\mathrm{e}^{-\frac{\beta|x|^2}{1+B(t,0)}}\mathrm{d}x=\Big(\frac{1+B(t,0)}{\beta}\Big)^{\frac n2}\int_{\mathbb{R}^n}\mathrm{e}^{-|y|^2}\mathrm{d}y\leq C_\beta(1+B(t,0))^{\frac n2}.$$

Applying Hölder's inequality, this implies that for any $\varepsilon>0$,

$$\|f(u(s,\cdot))\|_{L^1}\leq C\|u(s,\cdot)\|_{L^p}^p\leq C_{\varepsilon,p}(1+B(s,0))^{\frac n4}\|\mathrm{e}^{\varepsilon\psi(s,\cdot)}u(s,\cdot)\|_{L^{2p}}^p. \tag{5.7}$$

On the other hand, by using the trivial estimate $\|\mathrm{e}^{-2\varepsilon p\psi(t,\cdot)}\|_{L^\infty}\leq C$, we get

$$\|f(u(s,\cdot))\|_{L^2}\leq C\|\mathrm{e}^{\varepsilon\psi(s,\cdot)}u(s,\cdot)\|_{L^{2p}}^p. \tag{5.8}$$

Thanks to Theorem 4.1 combined with the estimates (5.7)–(5.8), we are able to prove the following fundamental statement, which is completely analogous to Lemma 2.4 in [9] for $b\equiv 1$.

Lemma 5.3 *For $j+l=0,1$, it holds that*

$$(b(t))^l(1+B(t,0))^{(\frac{n}{4}+\frac{j}{2})+l}\|\nabla^j\partial_t^l u(t,\cdot)\|_{L^2}\leq CI_\alpha+C_\varepsilon\Big(\sup_{[0,t]}h(s)\|\mathrm{e}^{\varepsilon\psi(s,\cdot)}u(s,\cdot)\|_{L^{2p}}\Big)^p,$$

where we put

$$h(s):=(1+B(s,0))^{\frac{\frac{n}{4}+1+\varepsilon}{p}}. \tag{5.9}$$

Proof We come back to the representation of the solution to (1.1) given in (4.9). Recalling (5.6), it holds that $\|(u_0,u_1)\|_{\mathcal{A}_{1,1}}\leq CI_\alpha$. Thanks to (3.4)–(3.5) for $m=1$ and $j=0,1$, we get

$$\|\nabla^j u^{\mathrm{lin}}(t,\cdot)\|_{L^2}\leq CI_\alpha(1+B(t,0))^{-(\frac{n}{4}+\frac{j}{2})},$$

and thanks to (3.6) for $m=1$, we derive

$$\|\partial_t u^{\mathrm{lin}}(t,\cdot)\|_{L^2}\leq C(b(t))^{-1}I_\alpha(1+B(t,0))^{-\frac{n}{4}-1}.$$

Therefore, we can focus our attention on the nonlinear contribution

$$u^{\mathrm{nl}}(t,x)=\int_0^t E_1(t,s,x)*f(u(s,x))\mathrm{d}s.$$

We first consider $s\in\big[0,\frac{t}{2}\big]$. If $s\in\big[0,\frac{t}{2}\big]$, then property (4.18) gives us $B(t,s)\approx B(t,0)$. Therefore, thanks to (4.3)–(4.4), by using (5.7)–(5.8), we estimate

$$\begin{aligned}&\Big\|\nabla^j\int_0^{\frac{t}{2}}E_1(t,s,x)*f(u(s,x))\mathrm{d}s\Big\|_{L^2}\\
&\leq C\int_0^{\frac{t}{2}}(b(s))^{-1}(1+B(t,s))^{-(\frac{n}{4}+\frac{j}{2})}(1+B(s,0))^{\frac{n}{4}}\|\mathrm{e}^{\varepsilon\psi(s,\cdot)}u(s,\cdot)\|_{L^{2p}}^p\mathrm{d}s\\
&\leq C(1+B(t,0))^{-(\frac{n}{4}+\frac{j}{2})}\Big(\sup_{[0,t]}h(s)\|\mathrm{e}^{\varepsilon\psi(s,\cdot)}u(s,\cdot)\|_{L^{2p}}\Big)^p\int_0^{\frac{t}{2}}(b(s))^{-1}(1+B(s,0))^{-(1+\varepsilon)}\mathrm{d}s.\end{aligned}$$

After the change of variables $r=B(s,0)$, we derive

$$\int_0^{\frac{t}{2}}\frac{1}{b(s)}(1+B(s,0))^{-(1+\varepsilon)}\mathrm{d}s=\int_0^{B(\frac{t}{2},0)}(1+r)^{-(1+\varepsilon)}\mathrm{d}r\leq C_\varepsilon. \tag{5.10}$$

Since $E_1(t,t,x)=0$, for any $t\in[0,\infty)$, we remark that

$$\partial_t u^{\mathrm{nl}}(t,x)=\int_0^t\partial_t E_1(t,s,x)*f(u(s,x))\mathrm{d}s.$$

Taking into consideration (4.5), (5.7)–(5.8) and (5.10), we have

$$\begin{aligned}&\Big\|\int_0^{\frac{t}{2}}\partial_t E_1(t,s,x)*f(u(s,x))\mathrm{d}s\Big\|_{L^2}\\
&\leq C\int_0^{\frac{t}{2}}(b(s)b(t))^{-1}(1+B(t,s))^{-\frac{n}{4}-1}(1+B(s,0))^{\frac{n}{4}}\|\mathrm{e}^{\varepsilon\psi(s,\cdot)}u(s,\cdot)\|_{L^{2p}}^p\mathrm{d}s\\
&\leq C(b(t))^{-1}(1+B(t,0))^{-\frac{n}{4}-1}\Big(\sup_{[0,t]}h(s)\|\mathrm{e}^{\varepsilon\psi(s,\cdot)}u(s,\cdot)\|_{L^{2p}}\Big)^p.\end{aligned}$$

Now we consider $s \in \big[\frac{t}{2}, t\big]$. Formula (4.17) gives us $B(s,0) \approx B(t,0)$. On the other hand, (4.19) gives us $B(t,s) \approx \frac{t-s}{b(t)}$. It is sufficient to use the energy estimate (that is, the $L^2 - L^2$ theory for the linear Cauchy problem given by (4.3)–(4.5) with $m = 2$) as follows:

$$\|\nabla^j \partial_t^l E_1(t,s,x) * f(u(s,x))\|_{L^2} \lesssim (b(s))^{-1}(b(t))^{-l}(1+B(t,s))^{-\frac{j}{2}-l}\|u(s)\|_{L^{2p}}^p$$

for $j + l = 0, 1$. Therefore, it follows that

$$\begin{aligned}&\Big\|\int_{\frac{t}{2}}^t \nabla^j \partial_t^l E_1(t,s,x) * f(u(s,x))\mathrm{d}s\Big\|_{L^2}\\ &\le C\Big(\sup_{[0,t]} h(s)\|\mathrm{e}^{\varepsilon\psi(s,\cdot)}u(s,\cdot)\|_{L^{2p}}\Big)^p\Big(h\Big(\frac{t}{2}\Big)\Big)^{-p}\frac{1}{(b(t))^l}\int_{\frac{t}{2}}^t \frac{1}{b(s)}(1+B(t,s))^{-\frac{j}{2}-l}\mathrm{d}s.\end{aligned}$$

For $j = 0$ and $l = 0$, we derive

$$\int_{\frac{t}{2}}^t \frac{1}{b(s)}\mathrm{d}s = B\Big(t, \frac{t}{2}\Big) \le 1 + B(t,0), \tag{5.11}$$

whereas for $j = 1$ and $l = 0$, after putting $r = B(t,s)$, we conclude

$$\begin{aligned}\int_{\frac{t}{2}}^t \frac{1}{b(s)}(1+B(t,s))^{-\frac{1}{2}}\mathrm{d}s &= \int_0^{B(t,\frac{t}{2})}(1+r)^{-\frac{1}{2}}\mathrm{d}r\\ &= 2\Big(1+B\Big(t,\frac{t}{2}\Big)\Big)^{\frac{1}{2}} - 2 \lesssim (1+B(t,0))^{\frac{1}{2}},\end{aligned} \tag{5.12}$$

and, similarly, for $j = 0$ and $l = 1$, we obtain

$$\begin{aligned}\int_{\frac{t}{2}}^t \frac{1}{b(s)}(1+B(t,s))^{-1}\mathrm{d}s &= \int_0^{B(t,\frac{t}{2})}(1+r)^{-1}\mathrm{d}r\\ &= \log\Big(1+B\Big(t,\frac{t}{2}\Big)\Big) \le \log(1+B(t,0)).\end{aligned} \tag{5.13}$$

To conclude the proof, it is sufficient to notice that

$$\Big(h\Big(\frac{t}{2}\Big)\Big)^{-p}(b(t))^{-l}(1+B(t,0))^{1-\frac{j}{2}-l}(\log(1+B(t,0))^l \lesssim (b(t))^{-l}(1+B(t,0))^{-\frac{n}{4}-\frac{j}{2}-l}$$

for $j + l = 0, 1$.

5.4 Conclusion of the proof of Theorem 3.2

Let us define

$$\begin{aligned}W(\tau) := {}&\|\mathrm{e}^{\psi(\tau,\cdot)}(\partial_t, \nabla)u(\tau,\cdot)\|_{L^2} + (1+B(\tau,0))^{(\frac{n}{4}+\frac{1}{2})}\|\nabla u(\tau,\cdot)\|_{L^2}\\ &+ b(\tau)(1+B(\tau,0))^{\frac{n}{4}+1}\|u_t(\tau,\cdot)\|_{L^2} + (1+B(\tau,0))^{\frac{n}{4}}\|u(\tau,\cdot)\|_{L^2}.\end{aligned}$$

Thanks to Lemmas 5.2–5.3, we can estimate

$$\begin{aligned}\sup_{[0,t]} W(\tau) \lesssim {}& I_\alpha + I_\alpha^{\frac{p+1}{2}} + \Big(\sup_{\tau\in[0,t]}(1+B(\tau,0))^{\varepsilon}\|\mathrm{e}^{\gamma\psi(\tau,\cdot)}u(\tau,\cdot)\|_{L^{p+1}}\Big)^{\frac{p+1}{2}}\\ &+ \Big(\sup_{\tau\in[0,t]} h(\tau)\|\mathrm{e}^{\varepsilon\psi(\tau,\cdot)}u(\tau,\cdot)\|_{L^{2p}}\Big)^p.\end{aligned}$$

In order to manage the last two terms, we use a Gagliardo-Nirenberg type inequality (see Lemma A.2 in Appendix) and get

$$\|e^{\sigma\psi(t,\cdot)}v\|_{L^q} \leq C_\sigma(1+B(t,0))^{\frac{1-\theta(q)}{2}}\,\|\nabla v\|_{L^2}^{1-\sigma}\,\|e^{\psi(t,\cdot)}\nabla v\|_{L^2}^{\sigma} \tag{5.14}$$

for any $\sigma \in [0,1]$ and $v \in H^1_{\sigma\psi(t,\cdot)}$, where

$$\theta(q) := \frac{n}{2} - \frac{n}{q} = n\Big(\frac{1}{2} - \frac{1}{q}\Big) \tag{5.15}$$

for $q \geq 2$, together with $q \leq 2^*$ if $n \geq 3$, where $2^* := \frac{2n}{n-2} = 2p_{\mathrm{GN}}(n)$.

By using (5.14), since $\gamma = \frac{2}{p+1} + \varepsilon$, it follows that

$$\|e^{\gamma\psi(\tau,\cdot)}u(\tau,\cdot)\|_{L^{p+1}} \leq W(\tau)\,(1+B(\tau,0))^{\frac{1-\theta(p+1)}{2}-(1-\frac{2}{p+1}-\varepsilon)(\frac{n}{4}+\frac{1}{2})}, \tag{5.16}$$

$$\|e^{\varepsilon\psi(\tau,\cdot)}u(\tau,\cdot)\|_{L^{2p}} \leq W(\tau)\,(1+B(\tau,0))^{\frac{1-\theta(2p)}{2}-(1-\varepsilon)(\frac{n}{4}+\frac{1}{2})}. \tag{5.17}$$

Recalling (5.9), we observe that the quantities

$$\max_{\tau\in[0,t]}(1+B(\tau,0))^{\frac{1-\theta(p+1)}{2}-(1-\frac{2}{p+1}-\varepsilon)(\frac{n}{4}+\frac{1}{2})+\varepsilon}, \tag{5.18}$$

$$\max_{\tau\in[0,t]}(1+B(\tau,0))^{\frac{\frac{n}{4}+1+\varepsilon}{p}+\frac{1-\theta(2p)}{2}-(1-\varepsilon)(\frac{n}{4}+\frac{1}{2})} \tag{5.19}$$

are uniformly bounded in $[0,\infty)$, provided that $\varepsilon > 0$ is sufficiently small, since $p > p_{\mathrm{Fuj}}(n)$. Indeed,

$$\frac{1-\theta(p+1)}{2} - \Big(1-\frac{2}{p+1}\Big)\Big(\frac{n}{4}+\frac{1}{2}\Big) = \frac{\frac{n}{4}+1}{p} + \frac{1-\theta(2p)}{2} - \Big(\frac{n}{4}+\frac{1}{2}\Big) = \frac{1-\frac{(p-1)n}{2}}{p} < 0.$$

Let us define

$$M(t) := \max_{[0,t]} W(\tau),$$

and let $\epsilon = I_\alpha$. We remark that $M(0) = W(0) \leq (2+b(0))\epsilon$. We have proved that

$$M(t) \leq c_0(\epsilon + \epsilon^{p+1}) + c_1(M(t))^{\frac{p+1}{2}} + c_2(M(t))^p \tag{5.20}$$

for some $c_0, c_1, c_2 > 0$. We claim that there exists a constant $\epsilon_0 > 0$ such that for any $\epsilon \in (0,\epsilon_0]$ it holds that

$$M(t) \leq C\epsilon, \tag{5.21}$$

in particular $E(t) \leq C^2\epsilon^2$, uniformly with respect to $t \in [0,\infty)$. Straightforward calculations (see [9]) also give

$$\|e^{\psi(t,\cdot)}u(t,\cdot)\|_{L^2} \lesssim \epsilon(1+t), \quad t \in [0,T). \tag{5.22}$$

Thanks to (5.21)–(5.22), the global existence of the solution follows by contradiction with the condition (5.2) of Lemma 5.1. Let us prove our claim (5.21). We define

$$\phi(x) = x - c_1 x^{\frac{p+1}{2}} - c_2 x^p$$

for some fixed constants $c_1, c_2 > 0$. We notice that $\phi(0) = 0$ and $\phi'(0) = 1$. Moreover, $\phi(x) \leq x$ for any $x \geq 0$, and we take $\overline{x} > 0$ such that $\phi'(x) \geq \frac{1}{2}$ on $[0, \overline{x}]$. Therefore, ϕ is strictly increasing and $\phi(x) \leq x \leq 2\phi(x)$ for any $x \in [0, \overline{x}]$. Let

$$\epsilon_0 := \min\Big\{1, \frac{\overline{x}}{2 + b(0)}, \frac{\overline{x}}{4c_0}\Big\}.$$

If $I_\alpha = \epsilon$ for some $\epsilon \in (0, \epsilon_0]$, then

$$M(0) = W(0) \leq (2 + b(0))\epsilon < \overline{x}. \tag{5.23}$$

Since $\phi(x)$ is strictly increasing on $[0, \overline{x}]$, it follows from (5.23) that

$$\phi(M(0)) \leq \phi(\overline{x}). \tag{5.24}$$

Thanks to (5.20), we get

$$\phi(M(t)) \leq c_0(\epsilon + \epsilon^p) \leq 2c_0\epsilon \tag{5.25}$$

for any $t \geq 0$. Since $M(t)$ is a continuous function and

$$2c_0\epsilon < 2c_0\epsilon_0 \leq \frac{\overline{x}}{2} \leq \phi(\overline{x}),$$

it follows from (5.24)–(5.25) that $M(t) \in (0, \overline{x})$ for any $t \geq 0$. Therefore, since $x \leq 2\phi(x)$ in $[0, \overline{x}]$, we also derive from (5.25) that

$$M(t) \leq 2\phi(M(t)) \leq 4c_0\epsilon.$$

This concludes the proof of (5.21) and, as a consequence, the global existence result. The relation (5.21) implies directly the decay estimates (3.9)–(3.11) for the semi-linear problem (1.1) (see Remark 4.9).

6 Proof of Theorem 3.3

In order to prove the global existence of a solution in $\mathcal{C}([0,\infty), H^1) \cap \mathcal{C}^1([0,\infty), L^2)$ such that the estimates (3.14)–(3.16) are satisfied for any $t \geq 0$, we introduce the space

$$X(t) = \big\{u \in \mathcal{C}([0,t], H^1) \cap \mathcal{C}^1([0,t], L^2)\big\}$$

with the norm

$$\begin{aligned}\|u\|_{X(t)} := \sup_{0\leq\tau\leq t} \big[&(1 + B(\tau,0))^{\frac{n}{4}} \|u(\tau,\cdot)\|_{L^2} + (1 + B(\tau,0))^{\frac{n}{4}+\frac{1}{2}} \|\nabla u(\tau,\cdot)\|_{L^2} \\ &+ (1 + B(\tau,0))^{\frac{n}{4}} (1+\tau)\|u_t(\tau,\cdot)\|_{L^2}\big].\end{aligned}$$

We remark that if $u \in X(t)$, then $\|u\|_{X(s)} \leq \|u\|_{X(t)}$ for any $s \leq t$.

We shall prove that for any data $(u_0, u_1) \in \mathcal{A}_{1,1}$ the operator N which is defined by

$$Nu(t,x) = E_0(t,0,x) *_{(x)} u_0(x) + E_1(t,0,x) *_{(x)} u_1(x) + \int_0^t E_1(t,s,x) *_{(x)} f(u(s,x))\mathrm{d}s$$

satisfies the following two estimates:

$$\|Nu\|_{X(t)} \leq C\,\|(u_0,u_1)\|_{\mathcal{A}_{1,1}} + C\|u\|^p_{X(t)}, \tag{6.1}$$
$$\|Nu - Nv\|_{X(t)} \leq C\|u-v\|_{X(t)}(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}) \tag{6.2}$$

uniformly with respect to $t \in [0,\infty)$. Arguing as we did at the end of the proof of Theorem 3.2 from (6.1) it follows that N maps $X(t)$ into itself for small data. These estimates lead to the existence of a unique solution of $u = Nu$. In fact, taking the recurrence sequence $u_{-1} = 0,\ u_j = N(u_{j-1})$ for $j = 0,1,2,\cdots$, we apply (6.1) with $\|(u_0,u_1)\|_{\mathcal{A}_{1,1}} = \epsilon$ and see inductively that

$$\|u_j\|_{X(t)} \leq C_1\epsilon, \tag{6.3}$$

where $C_1 = 2C$ for any $\epsilon \in [0,\epsilon_0]$ with $\epsilon_0 = \epsilon_0(C_1)$ sufficiently small.

Once the uniform estimate (6.3) is checked, we use (6.2) once more and find

$$\|u_{j+1} - u_j\|_{X(t)} \leq C\epsilon^{p-1}\|u_j - u_{j-1}\|_{X(t)} \leq 2^{-1}\|u_j - u_{j-1}\|_{X(t)} \tag{6.4}$$

for $\epsilon \leq \epsilon_0$ sufficiently small. From (6.4), we get inductively $\|u_j - u_{j-1}\|_{X(t)} \leq C2^{-j}$ so that $\{u_j\}$ is a Cauchy sequence in the Banach space $X(t)$ converging to the unique solution of $N(u) = u$. Since all of the constants are independent of t, we can take $t \to \infty$ and gain the global existence result. Finally, we see that the definition of $\|u\|_{X(t)}$ leads to the decay estimates (3.14)–(3.16).

Therefore, to complete the proof, it remains only to establish (6.1)–(6.2). More precisely, we put

$$\|v\|_{X_0(t)} := \sup_{0\leq\tau\leq t}[(1+B(\tau,0))^{\frac{n}{4}}\|v(\tau,\cdot)\|_{L^2} + (1+B(\tau,0))^{\frac{n}{4}+\frac{1}{2}}\|\nabla v(\tau,\cdot)\|_{L^2}], \tag{6.5}$$

and prove two slightly stronger inequalities than (6.1)–(6.2), namely,

$$\|Nu\|_{X(t)} \leq C\,\|(u_0,u_1)\|_{\mathcal{A}_{1,1}} + C\|u\|^p_{X_0(t)}, \tag{6.6}$$
$$\|Nu - Nv\|_{X(t)} \leq C\|u-v\|_{X_0(t)}(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)}). \tag{6.7}$$

These conditions will follow from the next proposition in which the restriction on the power p and on the dimension n will appear.

Proposition 6.1 *Let us assume* (3.13). *Let* $(u_0,u_1) \in \mathcal{A}_{1,1}$ *and* $u \in X(t)$. *For* $j + l = 0,1$ *it holds that*

$$\begin{aligned}&(1+t)^l(1+B(t,0))^{(\frac{n}{4}+\frac{j}{2})}\|\nabla^j\partial_t^l Nu(t,\cdot)\|_{L^2}\\ &\leq C\,\|(u_0,u_1)\|_{\mathcal{A}_{1,1}} + C\|u\|^p_{X_0(t)},\end{aligned} \tag{6.8}$$
$$\begin{aligned}&(1+t)^l(1+B(t,0))^{(\frac{n}{4}+\frac{j}{2})}\|\nabla^j\partial_t^l (Nu(t,\cdot) - Nv(t,\cdot))\|_{L^2}\\ &\leq C\|u-v\|_{X_0(t)}(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)}).\end{aligned} \tag{6.9}$$

Proof We first prove (6.8). As in the proof of Theorem 3.2, we use two different strategies for $s \in \left[0,\frac{t}{2}\right]$ and $s \in \left[\frac{t}{2},t\right]$ to control the integral term in Nu. In particular, we use Matsumura's type estimate (4.3)–(4.5) for $m = 1$ if $s \in \left[0,\frac{t}{2}\right]$ and for $m = 2$ (i.e., energy estimates) if $s \in \left[\frac{t}{2}, t\right]$. Together with (3.4)–(3.6) and Remark 4.9, we get

$$\begin{aligned}\|\nabla^j \partial_t^l N u(t,\cdot)\|_{L^2} \leq C(1+t)^{-l}(1+B(t,0))^{-(\frac{n}{4}+\frac{j}{2})}\|(u_0,u_1)\|_{\mathcal{A}_{1,1}}\\ + C\int_0^{\frac{t}{2}} (b(s))^{-1}(b(t))^{-l}(1+B(t,s))^{-(\frac{n}{4}+\frac{j}{2}+l)}\|f(u(s,\cdot))\|_{L^1\cap L^2}\mathrm{d}s\\ + C\int_{\frac{t}{2}}^t (b(s))^{-1}(b(t))^{-l}(1+B(t,s))^{-\frac{j}{2}-l}\|f(u(s,\cdot))\|_{L^2}\mathrm{d}s \qquad (6.10)\end{aligned}$$

for $j+l=0,1$. By (1.2), we can estimate $|f(u)| \lesssim |u|^p$, so that

$$\|f(u(s,\cdot))\|_{L^1\cap L^2} \lesssim \|u(s,\cdot)\|_{L^p}^p + \|u(s,\cdot)\|_{L^{2p}}^p,$$

and, similarly,

$$\|f(u(s,\cdot))\|_{L^2} \lesssim \|u(s,\cdot)\|_{L^{2p}}^p.$$

We apply Gagliardo-Nirenberg inequality (see Remark A.1 in Appendix):

$$\|u(s,\cdot)\|_{L^p}^p \lesssim \|u(s,\cdot)\|_{L^2}^{p(1-\theta(p))}\,\|\nabla u(s,\cdot)\|_{L^2}^{p\theta(p)}, \qquad (6.11)$$

$$\|u(s,\cdot)\|_{L^{2p}}^p \lesssim \|u(s,\cdot)\|_{L^2}^{p(1-\theta(2p))}\,\|\nabla u(s,\cdot)\|_{L^2}^{p\theta(2p)}, \qquad (6.12)$$

where

$$\theta(p) = \frac{n}{2}\,\frac{p-2}{p}, \quad \theta(2p) = \frac{n}{2}\,\frac{p-1}{p}.$$

We remark that the requisite $\theta(p) \geq 0$ implies that $p \geq 2$, whereas the requisite $\theta(2p) \leq 1$ implies that $p \leq p_{\mathrm{GN}}(n)$ if $n \geq 3$. The main difference with respect to the proof of Theorem 3.2 is that to apply Gagliardo-Nirenberg inequality, we need $p \geq 2$, since we use the $L^p \cap L^{2p}$ norm of u, not its L^{p+1} norm.

We estimate $\|f(u(s,\cdot))\|_{L^1\cap L^2}$ and $\|f(u(s,\cdot))\|_{L^2}$ by using (6.11)–(6.12) and $\|u\|_{X_0(t)}$:

$$\|f(u(s,\cdot))\|_{L^1\cap L^2} \lesssim \|u\|_{X_0(s)}^p(1+B(s,0))^{-p(\frac{n}{4}+\frac{\theta(p)}{2})} = \|u\|_{X_0(s)}^p(1+B(s,0))^{-\frac{(p-1)n}{2}}, \qquad (6.13)$$

since $\theta(p) < \theta(2p)$, whereas

$$\|f(u(s,\cdot))\|_{L^2} \lesssim \|u\|_{X_0(s)}^p(1+B(s,0))^{-p(\frac{n}{4}+\frac{\theta(2p)}{2})} = \|u\|_{X_0(s)}^p(1+B(s,0))^{-\frac{(2p-1)n}{4}}. \qquad (6.14)$$

By summarizing, we find

$$\begin{aligned}&\|\nabla^j \partial_l N u(t,\cdot)\|_{L^2}\\ &\leq C(1+t)^{-l}(1+B(t,0))^{-(\frac{n}{4}+\frac{j}{2})}\epsilon\\ &\quad + C\|u\|_{X_0(t)}^p \int_0^{\frac{t}{2}} (b(s))^{-1}(b(t))^{-l}(1+B(t,s))^{-(\frac{n}{4}+\frac{j}{2}+l)}(1+B(s,0))^{-\frac{(p-1)n}{2}}\mathrm{d}s\\ &\quad + C\|u\|_{X_0(t)}^p \int_{\frac{t}{2}}^t (b(s))^{-1}(b(t))^{-l}(1+B(t,s))^{-\frac{j}{2}-l}(1+B(s,0))^{-\frac{(2p-1)n}{4}}\mathrm{d}s\end{aligned}$$

for $j+l=0,1$. First, let $s \in \left[0, \frac{t}{2}\right]$. Due to (4.18) and (4.10), we can estimate

$$\begin{aligned}&\int_0^{\frac{t}{2}} (b(s))^{-1}(b(t))^{-l}(1+B(t,s))^{-(\frac{n}{4}+\frac{j}{2}+l)}(1+B(s,0))^{-\frac{(p-1)n}{2}}\mathrm{d}s\\ &\lesssim (1+B(t,0))^{-(\frac{n}{4}+\frac{j}{2})}(1+t)^{-l}.\end{aligned}$$

Indeed, since $p > p_{\mathrm{Fuj}}(n)$, after the change of variables $r = B(s,0)$, we get

$$\int_0^{\frac{t}{2}} \frac{1}{b(s)}(1+B(s,0))^{-\frac{(p-1)n}{2}}\mathrm{d}s = \int_0^{B(\frac{t}{2},0)}(1+r)^{-\frac{(p-1)n}{2}}\mathrm{d}r \le C.$$

Analogously, for $s \in \left[\frac{t}{2}, t\right]$, by using (4.17), we have

$$\begin{aligned}&\int_{\frac{t}{2}}^t \frac{1}{b(s)}\frac{1}{(b(t))^l}(1+B(t,s))^{-\frac{j}{2}-l}(1+B(t,0))^{-\frac{(2p-1)n}{4}}\mathrm{d}s\\ &\le C(1+B(t,0))^{-\frac{(2p-1)n}{4}}\frac{1}{(b(t))^l}\int_{\frac{t}{2}}^t \frac{1}{b(s)}(1+B(t,s))^{-\frac{j}{2}-l}\mathrm{d}s.\end{aligned}$$

Thanks to (5.11)–(5.13) in the proof of Theorem 3.2, we get

$$\begin{aligned}&\frac{1}{(b(t))^l}(1+B(t,0))^{-\frac{(2p-1)n}{4}}\int_{\frac{t}{2}}^t \frac{1}{b(s)}(1+B(t,s))^{-\frac{j}{2}-l}\mathrm{d}s\\ &\le C(1+B(t,0))^{-\frac{(2p-1)n}{4}+1-\frac{j}{2}-l}(b(t))^{-l}(\log(1+B(t,0)))^l\\ &\lesssim (1+B(t,0))^{-\frac{n}{4}-\frac{j}{2}}(1+t)^{-l}.\end{aligned}$$

By using Remark 4.9, we prove (6.8), once we get

$$(1+B(t,0))^{1-\frac{(p-1)n}{2}}(\log(1+B(t,0)))^l \le C, \quad l = 0,1$$

as follows with $p > p_{\mathrm{Fuj}}(n)$.

Now we prove (6.9). We remark that

$$\|Nu - Nv\|_{X(t)} = \Big\|\int_0^t E_1(t,s,x) *_{(x)} (f(u(s,x)) - f(v(s,x)))\mathrm{d}s\Big\|_{X(t)}.$$

Thanks to (4.3)–(4.5) we can estimate

$$\begin{aligned}&\|\nabla^j\partial_t^l E_1(t,s,x) *_{(x)} (f(u(s,x)) - f(v(s,x)))\|_{L^2}\\ &\lesssim \begin{cases}(b(s))^{-1}(b(t))^{-l}(1+B(t,s))^{-\frac{j}{2}-l-\frac{n}{4}}\|f(u(s,\cdot)) - f(v(s,\cdot))\|_{L^1\cap L^2}, & s \in \left[0, \dfrac{t}{2}\right],\\ (b(s))^{-1}(b(t))^{-l}(1+B(t,s))^{-\frac{j}{2}-l}\|f(u(s,\cdot)) - f(v(s,\cdot))\|_{L^2}, & s \in \left[\dfrac{t}{2}, t\right]\end{cases}\end{aligned}$$

for $j + l = 0, 1$. By using (1.2) and Hölder's inequality, we can now estimate

$$\begin{aligned}\|f(u(s,\cdot)) - f(v(s,\cdot))\|_{L^1} &\lesssim \|u(s,\cdot) - v(s,\cdot)\|_{L^p}(\|u(s,\cdot)\|_{L^p}^{p-1} + \|v(s,\cdot)\|_{L^p}^{p-1}),\\ \|f(u(s,\cdot)) - f(v(s,\cdot))\|_{L^2} &\lesssim \|u(s,\cdot) - v(s,\cdot)\|_{L^{2p}}(\|u(s,\cdot)\|_{L^{2p}}^{p-1} + \|v(s,\cdot)\|_{L^{2p}}^{p-1}).\end{aligned}$$

In a similar way to the proof of (6.6), we apply Gagliardo-Nirenberg inequality to the terms

$$\|u(s,\cdot) - v(s,\cdot)\|_{L^q}, \quad \|u(s,\cdot)\|_{L^q}, \quad \|v(s,\cdot)\|_{L^q}$$

with $q = p$ and $q = 2p$, and we conclude the proof of (6.7) by using the assumption $p > p_{\mathrm{Fuj}}(n)$ and the convergence of the integrals in (5.11)–(5.13).

7 Proof of Theorem 4.1

In order to prove Theorem 4.1, we follow the strategy in [18]. The main goal is to show how the strategy can be extended to a parameter-dependent family of Cauchy problems. For additional details, we refer the reader to that paper. We also address the interested reader to [17], where linear estimates were also obtained in the case of noneffective and scale-invariant dissipation. Energy estimates for the last two cases have recently been developed in the case of a time-dependent propagation speed and higher order equations (see [2–3]). It remains open the chance to use these estimates to study semilinear waves with noneffective damping.

We will prove a statement more general than (4.3)–(4.5), namely,

$$\begin{aligned} &\|\partial_t^l \partial_x^\alpha v(t,\cdot)\|_{L^2} \\ &\leq C(b(s))^{-1}(1+B(t,s))^{-\frac{|\alpha|}{2}-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}(b(t))^{-l}(1+B(t,s))^{-l}\|g(s,\cdot)\|_{L^m\cap H^{[|\alpha|+l-1]^+}} \end{aligned} \tag{7.1}$$

for $l=0,1$ and for any $\alpha\in\mathbb{N}^n$. The inequality (7.1) for $|\alpha|\leq 1-l$ gives us (4.3)–(4.5).

We perform the Fourier transform of (4.1) and make the change of variables

$$y(t,\xi) := \frac{\lambda(t)}{\lambda(s)}\widehat{v}(t,\xi), \quad \text{where } \lambda(t) := \exp\Big(\frac{1}{2}\int_0^t b(\tau)\mathrm{d}\tau\Big), \tag{7.2}$$

so that we derive the Cauchy problem

$$y'' + m(t,\xi)y = 0, \quad y(s,\xi) = 0, \quad y'(s,\xi) = \widehat{g}(s,\xi), \tag{7.3}$$

where we put

$$m(t,\xi) := |\xi|^2 - \Big(\frac{1}{4}b^2(t) + \frac{1}{2}b'(t)\Big).$$

Let us define $\eta(t) := \frac{b(t)}{2}$ and

$$\langle\xi\rangle_{\eta(t)} := \sqrt{||\xi|^2 - \eta^2(t)|}.$$

We divide the extended phase space $[s,\infty)\times\mathbb{R}^n$ into four zones. We define the following hyperbolic, pseudo-differential, reduced and elliptic zones in correspondence with sufficiently small $\varepsilon>0$ and sufficiently large $N>0$:

$$\begin{aligned} Z_{\mathrm{hyp}}(N) &= \Big\{t\geq s,\ |\xi|\geq\eta(t),\ \frac{\langle\xi\rangle_{\eta(t)}}{\eta(t)}\geq N\Big\}, \\ Z_{\mathrm{pd}}(N,\varepsilon) &= \Big\{t\geq s,\ |\xi|\geq\eta(t),\ \varepsilon\leq\frac{\langle\xi\rangle_{\eta(t)}}{\eta(t)}\leq N\Big\}, \\ Z_{\mathrm{red}}(\varepsilon) &= \Big\{t\geq s,\ \frac{\langle\xi\rangle_{\eta(t)}}{\eta(t)}\leq\varepsilon\Big\}, \\ Z_{\mathrm{ell}}(\varepsilon) &= \Big\{t\geq s,\ |\xi|\leq\eta(t),\ \frac{\langle\xi\rangle_{\eta(t)}}{\eta(t)}\geq\varepsilon\Big\}. \end{aligned}$$

Remark 7.1 Since $\eta(t)$ is monotone there exists the limit

$$\eta_\infty := \lim_{t\to\infty}\eta(t)\in[0,\infty].$$

We distinguish the following four cases:

(1) If $\eta(t) \searrow 0$, then for any $\xi \neq 0$, there exists a $T_{|\xi|} \geq s$ such that $(t,\xi) \in Z_{\rm hyp}(N)$ for any $t \geq T_{|\xi|}$.

(2) If $\eta(t) \searrow \eta_\infty > 0$, then for any $|\xi| > \eta_\infty\sqrt{N^2+1}$, there exists a $T_{|\xi|} \geq s$ such that $(t,\xi) \in Z_{\rm hyp}(N)$ for any $t \geq T_{|\xi|}$. Moreover, $(t,\xi) \in Z_{\rm ell}(\varepsilon)$ for any $|\xi| \leq \eta_\infty\sqrt{1-\varepsilon^2}$ and $(t,\xi) \in Z_{\rm hyp}(N)$ for any $|\xi| \geq \eta(s)\sqrt{N^2+1}$.

(3) If $\eta(t) \nearrow \eta_\infty > 0$, then for any $|\xi| < \eta_\infty\sqrt{1-\varepsilon^2}$, there exists a $T_{|\xi|} \geq s$ such that $(t,\xi) \in Z_{\rm ell}(N)$ for any $t \geq T_{|\xi|}$. Moreover, $(t,\xi) \in Z_{\rm ell}(\varepsilon)$ for any $|\xi| \leq \eta(s)\sqrt{1-\varepsilon^2}$ and $(t,\xi) \in Z_{\rm hyp}(N)$ for any $|\xi| \geq \eta_\infty\sqrt{N^2+1}$.

(4) If $\eta(t) \nearrow \infty$, then for any $\xi \in \mathbb{R}^n$, there exists a $T_{|\xi|} \geq s$ such that $(t,\xi) \in Z_{\rm ell}(N)$ for any $t \geq T_{|\xi|}$.

We define
$$h(t,\xi) = \chi\Big(\frac{\langle\xi\rangle_{\eta(t)}}{\varepsilon\eta(t)}\Big)\varepsilon\eta(t) + \Big(1-\chi\Big(\frac{\langle\xi\rangle_{\eta(t)}}{\varepsilon\eta(t)}\Big)\Big)\sqrt{|m(t,\xi)|},$$
where $\chi \in \mathcal{C}^\infty[0,+\infty)$ localizes: $\chi(\zeta) = 1$ if $0 \leq \zeta \leq \frac{1}{2}$ and $\chi(\zeta) = 0$ if $\zeta \geq 1$. For any $(t,\xi) \notin Z_{\rm red}(\varepsilon)$, it holds that $|m(t,\xi)| \geq C\varepsilon^2\eta^2(t)$. Therefore, $h(t,\xi) \geq C_1\varepsilon\eta(t)$.

Let $V(t,\xi) = ({\rm i}h(t,\xi)y(t,\xi), y'(t,\xi))^{\rm T}$. From (7.3), we obtain
$$V' = \begin{pmatrix} \dfrac{h'(t,\xi)}{h(t,\xi)} & {\rm i}h(t,\xi) \\ \dfrac{{\rm i}m(t,\xi)}{h(t,\xi)} & 0 \end{pmatrix} V, \quad V(s,\xi) = (0,\widehat{g}(s,\xi))^{\rm T}. \tag{7.4}$$

For any $t \geq t_1 \geq s$, we denote by $\mathcal{E}(t,t_1,\xi)$ the fundamental solution of (7.4), that is, the matrix which solves
$$\partial_t\mathcal{E}(t,t_1,\xi) = \begin{pmatrix} \dfrac{h'(t,\xi)}{h(t,\xi)} & {\rm i}h(t,\xi) \\ \dfrac{{\rm i}m(t,\xi)}{h(t,\xi)} & 0 \end{pmatrix} \mathcal{E}(t,t_1,\xi)\,, \quad \mathcal{E}(t_1,t_1,\xi) = I \tag{7.5}$$
for any $t \geq t_1$. It is clear that
$$V(t,\xi) = \mathcal{E}(t,s,\xi)(0,\widehat{g}(s,\xi))^{\rm T}, \quad \mathcal{E}(t,t_2,\xi) = \mathcal{E}(t,t_2,\xi)\,\mathcal{E}(t_2,t_1,\xi) \quad \text{for any } t \geq t_2 \geq t_1 \geq s.$$

For $t_2 \geq t_1$ and $(t_2,\xi),(t_1,\xi) \in Z_{\rm hyp}(N,\varepsilon)$, we will write $\mathcal{E}(t_2,t_1,\xi) = \mathcal{E}_{\rm hyp}(t_2,t_1,\xi)$. It is similar for the other zones.

7.1 Diagonalization in the hyperbolic zone

Recalling the definition of χ, in $Z_{\rm hyp}(N)$ it holds that $h(t,\xi) = \sqrt{m(t,\xi)}$. Therefore, we can write the system in (7.4) as
$$\partial_t V = {\rm i}\sqrt{m(t,\xi)}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} V + \frac{\partial_t\sqrt{m(t,\xi)}}{\sqrt{m(t,\xi)}}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V. \tag{7.6}$$

The constant matrix
$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

is the diagonalizer of the principal part of (7.6), that is,

$$P\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}P^{-1}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}P^{-1}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

If we put $W(t,\xi)=PV(t,\xi)$, then (7.6) becomes

$$\partial_t W=\mathrm{i}\sqrt{m(t,\xi)}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}W+\frac{\partial_t\sqrt{m(t,\xi)}}{2\sqrt{m(t,\xi)}}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}W. \tag{7.7}$$

Then we apply a step of refined diagonalization to (7.7). The second diagonalizer depends on $\sqrt{m(t,\xi)}$ and $\partial_t\sqrt{m(t,\xi)}$. For this reason, there will appear terms in which $\partial_t^2\sqrt{m(t,\xi)}$ also comes into play. By using (3.1) for $k=1,2,3$ (we recall that both $b(t)$ and $b'(t)$ appear in the definition of $m(t,\xi)$), we derive suitable estimates for the entries of the new system.

By summarizing, for any $(t_1,\xi),(t_2,\xi)\in Z_{\mathrm{hyp}}(N)$ with $t_1\le t_2$, the fundamental solution in (7.5) can be written as

$$\mathcal{E}_{\mathrm{hyp}}(t_2,t_1,\xi)=\widetilde{\mathcal{E}}_{\mathrm{hyp},0}(t_2,t_1,\xi)Q_{\mathrm{hyp}}(t_2,t_1,\xi),$$

where

$$\widetilde{\mathcal{E}}_{\mathrm{hyp},0}(t_2,t_1,\xi)=\mathrm{diag}\Big(\exp\Big(-\mathrm{i}\int_{t_1}^{t_2}\sqrt{m(\tau,\xi)}\mathrm{d}\tau\Big),\exp\Big(\mathrm{i}\int_{t_1}^{t_2}\sqrt{m(\tau,\xi)}\mathrm{d}\tau\Big)\Big),$$

and $\|Q_{\mathrm{hyp}}(t_2,t_1,\xi)\|\le C$ uniformly. We remark that in the last estimate we used the property

$$m(t_2,\xi)\approx|\xi|\approx m(t_1,\xi)\,,$$

which holds in $Z_{\mathrm{hyp}}(N)$, to control the term

$$\exp\Big(\frac{1}{2}\int_{t_1}^{t_2}\frac{\partial_\tau\sqrt{m(\tau,\xi)}}{\sqrt{m(\tau,\xi)}}\mathrm{d}\tau\Big)=\Big(\frac{m(t_2,\xi)}{m(t_1,\xi)}\Big)^{\frac{1}{4}},$$

which appears after the refined diagonalization step.

7.2 Diagonalization in the elliptic zone

In $Z_{\mathrm{ell}}(\varepsilon)$ it holds that $h(t,\xi)=\sqrt{-m(t,\xi)}$, therefore we can write the system in (7.4) as

$$\partial_t V=\mathrm{i}\sqrt{-m(t,\xi)}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}V+\frac{\partial_t\sqrt{-m(t,\xi)}}{\sqrt{-m(t,\xi)}}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}V. \tag{7.8}$$

The constant matrix

$$\widetilde{P}=\frac{1}{\sqrt{2}}\begin{pmatrix}-\mathrm{i} & 1\\ \mathrm{i} & 1\end{pmatrix}$$

is the diagonalizer of the principal part of (7.8). If we put $W(t,\xi)=\widetilde{P}V(t,\xi)$, then (7.8) becomes

$$\partial_t W=\sqrt{-m(t,\xi)}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}W+\frac{\partial_t\sqrt{-m(t,\xi)}}{2\sqrt{-m(t,\xi)}}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}W. \tag{7.9}$$

If $t_1 \geq \overline{t}$ with a sufficiently large $\overline{t} \geq s$, then we can perform a step of refined diagonalization. On the other hand, since the subzone

$$Z_{\mathrm{comp}}(\varepsilon, \overline{t}) = \{t \leq \overline{t}\} \cap Z_{\mathrm{ell}}(\varepsilon) \subset [s, \overline{t}] \times \{|\xi| \leq \max\{\eta(s), \eta(\overline{t})\}\}$$

is compact, the fundamental solution is bounded there. So we may assume $t_1 \geq \overline{t}$. For any $(t_1, \xi), (t_2, \xi) \in Z_{\mathrm{ell}}(\varepsilon)$ with $t_1 \leq t_2$ the fundamental solution in (7.5) can be written as

$$\mathcal{E}_{\mathrm{ell}}(t_2, t_1, \xi) = \widetilde{\mathcal{E}}_{\mathrm{ell},0}(t_2, t_1, \xi) Q_{\mathrm{ell}}(t_2, t_1, \xi),$$

where

$$\widetilde{\mathcal{E}}_{\mathrm{ell},0}(t_2, t_1, \xi) = \Big(\frac{m(t_2,\xi)}{m(t_1,\xi)}\Big)^{\frac{1}{4}} \operatorname{diag}\Big(\exp\Big(\int_{t_1}^{t_2} \sqrt{-m(\tau,\xi)}\mathrm{d}\tau\Big), \exp\Big(-\int_{t_1}^{t_2} \sqrt{-m(\tau,\xi)}\mathrm{d}\tau\Big)\Big)$$

and $\|Q_{\mathrm{ell}}(t_2, t_1, \xi)\| \leq C$ uniformly. We remark that the term

$$\exp\Big(\frac{1}{2}\int_{t_1}^{t_2} \frac{\partial_\tau \sqrt{m(\tau,\xi)}}{\sqrt{m(\tau,\xi)}}\mathrm{d}\tau\Big) = \Big(\frac{m(t_2,\xi)}{m(t_1,\xi)}\Big)^{\frac{1}{4}},$$

which appears in $\widetilde{\mathcal{E}}_{\mathrm{ell},0}(t_2, t_1, \xi)$ is not bounded. Consequently, it can not be included in $Q_{\mathrm{ell}}(t_2, t_1, \xi)$ as we did during the diagonalization procedure in $Z_{\mathrm{hyp}}(N)$.

7.3 Estimates in the reduced and pseudo-differential zones

In $Z_{\mathrm{red}}(\varepsilon)$ we can estimate $\sqrt{|m(t,\xi)|} \leq C\varepsilon\eta(t)$ and therefore also $h(t,\xi) \leq C\varepsilon\eta(t)$. By rough estimates this implies

$$\|\mathcal{E}_{\mathrm{red}}(t_2, t_1, \xi)\| \leq \exp\Big(C\varepsilon \int_{t_1}^{t_2} \eta(\tau)\mathrm{d}\tau\Big).$$

Since C is independent of ε we can take $\varepsilon < \frac{1}{2C}$, so that the exponential growth is slower than the growth of $\frac{\lambda(t_2)}{\lambda(t_1)}$.

In $Z_{\mathrm{pd}}(N,\varepsilon)$ it holds that $h(t,\xi) = \sqrt{m(t,\xi)}$. We can roughly estimate by using the symbol class of $\frac{\partial_t \sqrt{m(t,\xi)}}{\sqrt{m(t,\xi)}}$:

$$\|\mathcal{E}_{\mathrm{pd}}(t_2, t_1, \xi)\| \leq \exp\Big(c\int_{t_1}^{t_2} (1+\tau)^{-1}\mathrm{d}\tau\Big) = \Big(\frac{1+t_2}{1+t_1}\Big)^c \leq C'_\varepsilon \exp\Big(C\varepsilon\int_{t_1}^{t_2} \eta(\tau)\mathrm{d}\tau\Big)$$

for any $\varepsilon > 0$, since $t\eta(t) \to \infty$.

7.4 Representation of the solution

We come back to our original problem (4.1). Let

$$y(t, s, \xi) = \Psi(t, s, \xi)\widehat{g}(s, \xi)$$

be the solution to (7.3). Then, thanks to our representation for the fundamental solution $\mathcal{E}(t_2, t_1, \xi)$ given in (7.5), we derive

$$\begin{pmatrix} 0 & \mathrm{i}|\xi|\Psi \\ 0 & \Psi' \end{pmatrix} (0, \widehat{g}(s,\xi))^{\mathrm{T}} = \operatorname{diag}\Big(\frac{|\xi|}{h(t,\xi)}, 1\Big)\mathcal{E}(t, s, \xi) \operatorname{diag}(0,1)(0, \widehat{g}(s,\xi))^{\mathrm{T}},$$

that is,

$$\Psi(t,s,\xi) = -\frac{\mathrm{i}\mathcal{E}_{12}(t,s,\xi)}{h(t,\xi)}, \quad \Psi'(t,s,\xi) = \mathcal{E}_{22}(t,s,\xi).$$

We write the Fourier transform of the solution to (4.1) as $\widehat{v}(t,\xi) = \widehat{\Phi}(t,s,\xi)\widehat{g}(s,\xi)$. Recalling (7.2), we obtain

$$\widehat{\Phi}(t,s,\xi) = \frac{\lambda(s)}{\lambda(t)}\Psi(t,s,\xi) = -\mathrm{i}\frac{\lambda(s)}{\lambda(t)}\frac{1}{h(t,\xi)}\mathcal{E}_{12}(t,s,\xi), \tag{7.10}$$

$$\begin{aligned}\widehat{\Phi}'(t,s,\xi) &= \frac{\lambda(s)}{\lambda(t)}\Big(\Psi'(t,s,\xi) - \frac{1}{2}b(t)\Psi(t,s,\xi)\Big) \\ &= \frac{\lambda(s)}{\lambda(t)}\Big(\mathcal{E}_{22}(t,s,\xi) + \frac{\mathrm{i}b(t)}{2h(t,\xi)}\mathcal{E}_{12}(t,s,\xi)\Big).\end{aligned} \tag{7.11}$$

According to Remark 7.1, for any frequency $\xi \neq 0$ and initial time $s \geq 0$ (without loss of generality, we can assume $s \geq \bar{t}$) we can distinguish various cases. We first consider the case of $\eta(t)$ decreasing, $\eta(t) \searrow \eta_\infty$ with $\eta_\infty \in [0,+\infty)$, and $(s,\xi) \in Z_{\mathrm{ell}}$, that is, $|\xi| \leq \eta(s)\sqrt{1-\varepsilon^2}$.

(1) If $|\xi| > \eta_\infty\sqrt{N^2+1}$, then there exist $t_{\mathrm{pd}} > t_{\mathrm{red}} > t_{\mathrm{ell}} \geq s$, such that for any $t \geq t_{\mathrm{pd}}$ it follows that

$$\mathcal{E}(t,s,\xi) = \mathcal{E}_{\mathrm{hyp}}(t,t_{\mathrm{pd}},\xi)\mathcal{E}_{\mathrm{pd}}(t_{\mathrm{pd}},t_{\mathrm{red}},\xi)\mathcal{E}_{\mathrm{red}}(t_{\mathrm{red}},t_{\mathrm{ell}},\xi)\mathcal{E}_{\mathrm{ell}}(t_{\mathrm{ell}},s,\xi).$$

In particular, this happens for any frequency $\xi \neq 0$ if $\eta_\infty = 0$.

(2) If $\eta_\infty\sqrt{1+\varepsilon^2} < |\xi| \leq \eta_\infty\sqrt{N^2+1}$, then there exist $t_{\mathrm{red}} > t_{\mathrm{ell}} \geq s$, such that for any $t \geq t_{\mathrm{red}}$ it follows that

$$\mathcal{E}(t,s,\xi) = \mathcal{E}_{\mathrm{pd}}(t,t_{\mathrm{red}},\xi)\mathcal{E}_{\mathrm{red}}(t_{\mathrm{red}},t_{\mathrm{ell}},\xi)\mathcal{E}_{\mathrm{ell}}(t_{\mathrm{ell}},s,\xi).$$

(3) If $\eta_\infty\sqrt{1-\varepsilon^2} < |\xi| \leq \eta_\infty\sqrt{1+\varepsilon^2}$, then there exists a time $t_{\mathrm{ell}} \geq s$ such that for any $t \geq t_{\mathrm{ell}}$ it follows that

$$\mathcal{E}(t,s,\xi) = \mathcal{E}_{\mathrm{red}}(t,t_{\mathrm{ell}},\xi)\mathcal{E}_{\mathrm{ell}}(t_{\mathrm{ell}},s,\xi).$$

(4) If $|\xi| \leq \eta_\infty\sqrt{1-\varepsilon^2}$, then $\mathcal{E}(t,s,\xi) = \mathcal{E}_{\mathrm{ell}}(t,s,\xi)$.

On the other hand, if $|\xi| \geq \eta(s)\sqrt{N^2+1}$, then $\mathcal{E}(t,s,\xi) = \mathcal{E}_{\mathrm{hyp}}(t,s,\xi)$ for any $t \in [s,\infty)$. The intermediate cases are clear.

If we consider the case of $\eta(t)$ increasing, $\eta(t) \nearrow \eta_\infty$ with $\eta_\infty \in (0,+\infty]$, then the situation is reversed. In particular, for any frequency $|\xi| \in [\eta(s)\sqrt{N^2+1}, \eta_\infty\sqrt{1-\varepsilon^2})$ (if this set is not empty), there exist $t_{\mathrm{red}} > t_{\mathrm{pd}} > t_{\mathrm{hyp}} \geq s$ such that for any $t \geq t_{\mathrm{red}}$ it follows that

$$\mathcal{E}(t,s,\xi) = \mathcal{E}_{\mathrm{ell}}(t,t_{\mathrm{red}},\xi)\mathcal{E}_{\mathrm{red}}(t_{\mathrm{red}},t_{\mathrm{pd}},\xi)\mathcal{E}_{\mathrm{pd}}(t_{\mathrm{pd}},t_{\mathrm{hyp}},\xi)\mathcal{E}_{\mathrm{hyp}}(t_{\mathrm{hyp}},s,\xi).$$

7.5 Estimates for the multipliers

We have to derive estimates for $|\widehat{\Phi}(t,s,\xi)|$ in each zone of the extended phase space. The estimates for $|\widehat{\Phi}'(t,s,\xi)|$ will be obtained by a more refined approach. Since $\mathcal{E}_{12}(t,s,\xi)$ is multiplied by

$$\frac{\lambda(s)}{\lambda(t)}\frac{1}{h(t,\xi)},$$

we look in each zone for an estimate of the scalar and non-negative term

$$a(t_2,t_1,\xi) := \frac{\lambda(t_1)}{\lambda(t_2)}\frac{h(t_1,\xi)}{h(t_2,\xi)}\|\mathcal{E}(t_2,t_1,\xi)\|$$

for any $(t_1,\xi),(t_2,\xi)$ in that zone with $t_1 \le t_2$. Indeed, from (7.10), it follows that

$$|\widehat{\Phi}(t,s,\xi)| \lesssim \frac{1}{h(s,\xi)}\, a(t,s,\xi).$$

Following the ideas from the proof of Theorem 17 in [18], we can easily check that the desired estimate in $Z_{\mathrm{ell}}(\varepsilon)$ is

$$a_{\mathrm{ell}}(t_2,t_1,\xi) \lesssim \exp\Big(- C\,|\xi|^2 \int_{t_1}^{t_2} \frac{1}{b(\tau)}\mathrm{d}\tau\Big) = \exp(-C\,|\xi|^2\, B(t_2,t_1)). \tag{7.12}$$

We remark that the estimate

$$\frac{h(t_1,\xi)}{h(t_2,\xi)}\,\frac{(m(t_2,\xi))^{\frac14}}{(m(t_1,\xi))^{\frac14}} \approx \frac{(m(t_1,\xi))^{\frac14}}{(m(t_2,\xi))^{\frac14}}$$

plays a fundamental role.

In $Z_{\mathrm{red}}(\varepsilon)$ it holds that $h(t,\xi)\approx\eta(t)\approx|\xi|$ while $h(t,\xi)\approx|\xi|$ in $Z_{\mathrm{pd}}(N,\varepsilon)$ and in $Z_{\mathrm{hyp}}(N)$. Therefore, we can assume $\frac{h(t_1,\xi)}{h(t_2,\xi)} \approx 1$ in all these zones. The best estimate is obtained in $Z_{\mathrm{hyp}}(N)$. Since $\mathcal{E}(t_2,t_1,\xi)$ is bounded, we conclude

$$a_{\mathrm{hyp}}(t_2,t_1,\xi) \lesssim \frac{\lambda(t_1)}{\lambda(t_2)}\,; \tag{7.13}$$

on the other hand, in $Z_{\mathrm{pd}}(N,\varepsilon)$, we have

$$a_{\mathrm{pd}}(t_2,t_1,\xi) \le \frac{(1+t_2)^c}{(1+t_1)^c}\frac{\lambda(t_1)}{\lambda(t_2)}, \tag{7.14}$$

whereas in $Z_{\mathrm{red}}(\varepsilon)$, we have

$$a_{\mathrm{red}}(t_2,t_1,\xi) \le \exp\Big(C\varepsilon \int_{t_1}^{t_2} b(\tau)\mathrm{d}\tau\Big)\frac{\lambda(t_1)}{\lambda(t_2)} \equiv \Big(\frac{\lambda(t_1)}{\lambda(t_2)}\Big)^{1-2\delta}, \tag{7.15}$$

where we choose $\varepsilon>0$ such that $\delta := C\varepsilon < \frac12$. It is clear that in the zones $Z_{\mathrm{hyp}}(N)$, $Z_{\mathrm{pd}}(N,\varepsilon)$ and $Z_{\mathrm{red}}(\varepsilon)$ we can uniformly estimate $a(t_2,t_1,\xi)$ by the upper bound from (7.15), which is the worst among (7.13)–(7.15). Moreover, we remark that the parameter $|\xi|$ does not come into play in these estimates. Nevertheless, we should be careful when we compare with the estimate (7.12), which has a completely different structure. Having this in mind, we define

$$\Pi_{\mathrm{hyp}}(\varepsilon) = Z_{\mathrm{red}}(\varepsilon) \cup Z_{\mathrm{pd}}(N,\varepsilon) \cup Z_{\mathrm{hyp}}(N),$$

and we denote by $t_{|\xi|}$ the separating curve among $Z_{\mathrm{ell}}(\varepsilon)$ and $\Pi_{\mathrm{hyp}}(\varepsilon)$, that is, the separating curve between $Z_{\mathrm{ell}}(\varepsilon)$ and $Z_{\mathrm{red}}(\varepsilon)$. This curve is given by

$$\eta^2(t_{|\xi|}) - |\xi|^2 = \varepsilon^2\eta^2(t_{|\xi|}), \quad \text{i.e.,} \quad t_{|\xi|} = \eta^{-1}\Big(\frac{|\xi|}{\sqrt{1-\varepsilon^2}}\Big).$$

We distinguish two cases:

(1) For small frequencies $|\xi| \le \eta(s)\sqrt{1-\varepsilon^2}$, since $h(s,\xi) \approx \eta(s) \approx b(s)$, it holds that

$$|\widehat{\Phi}(t,s,\xi)| \lesssim \frac{1}{b(s)} \exp(-C|\xi|^2 B(t,s)) \quad \text{for } t \le t_{|\xi|}, \tag{7.16}$$

$$|\widehat{\Phi}(t,s,\xi)| \lesssim \frac{1}{b(s)} \exp(-C|\xi|^2 B(t_{|\xi|},s)) \Big(\frac{\lambda(t_{|\xi|})}{\lambda(t)}\Big)^{1-2\delta} \quad \text{for } t \ge t_{|\xi|}. \tag{7.17}$$

We assume that $t_{|\xi|} = \infty$ if $|\xi| \le \eta_\infty\sqrt{1-\varepsilon^2}$ (in particular, this is trivially true if $\eta(t)$ is increasing).

(2) For large frequencies $|\xi| \ge \eta(s)\sqrt{1-\varepsilon^2}$, since $h(s,\xi) \approx |\xi|$, it holds that

$$|\widehat{\Phi}(t,s,\xi)| \lesssim \frac{1}{|\xi|} \Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{1-2\delta} \quad \text{for } t \le t_{|\xi|}, \tag{7.18}$$

$$|\widehat{\Phi}(t,s,\xi)| \lesssim \frac{1}{|\xi|} \Big(\frac{\lambda(s)}{\lambda(t_{|\xi|})}\Big)^{1-2\delta} \exp(-C|\xi|^2 B(t,t_{|\xi|})) \quad \text{for } t \ge t_{|\xi|}. \tag{7.19}$$

We assume that $t_{|\xi|} = \infty$ if $|\xi| \ge \eta_\infty\sqrt{1-\varepsilon^2}$ (in particular, this is trivially true if $\eta(t)$ is decreasing).

7.6 Estimates for the time derivative of the multipliers

We consider $\widehat{\Phi}'(t,s,\xi)$. In $\Pi_{\mathrm{hyp}}(\varepsilon)$, we directly use the representation (7.11) together with $b(t) \lesssim h(t,\xi)$ and $h(s,\xi) \approx |\xi| \approx h(t,\xi)$. Therefore, for large frequencies $|\xi| \ge \eta(s)\sqrt{1-\varepsilon^2}$ and for $t \le t_{|\xi|}$, we can estimate

$$|\widehat{\Phi}'(t,s,\xi)| \lesssim \Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{1-2\delta}, \tag{7.20}$$

whereas for small frequencies $|\xi| \le \eta(s)\sqrt{1-\varepsilon^2}$ and $t \ge t_{|\xi|}$, we get

$$|\widehat{\Phi}'(t,s,\xi)| \lesssim |\widehat{\Phi}'(t_{|\xi|},s,\xi)| \Big(\frac{\lambda(t_{|\xi|})}{\lambda(t)}\Big)^{1-2\delta}. \tag{7.21}$$

It remains to estimate two objects:

(1) $\widehat{\Phi}'(t,s,\xi)$ in the case of small frequencies $|\xi| \le \eta(s)\sqrt{1-\varepsilon^2}$ for any $t \le t_{|\xi|}$,

(2) $\widehat{\Phi}'(t,s,\xi)$ for large frequencies $|\xi| \ge \eta(s)\sqrt{1-\varepsilon^2}$ and for any $t \ge t_{|\xi|}$ (we remark that this case comes into play only if $\eta(t)$ is decreasing).

A direct estimate for $\widehat{\Phi}'(t,s,\xi)$ is not appropriate for small frequencies $|\xi| \le \eta(s)\sqrt{1-\varepsilon^2}$ and $t \le t_{|\xi|}$. Taking account of

$$\widehat{\Phi}'' + |\xi|^2\widehat{\Phi} + b(t)\widehat{\Phi}' = 0, \quad \widehat{\Phi}(s,s,\xi) = 0, \quad \widehat{\Phi}'(s,s,\xi) = 1$$

and setting $y(t,\xi) = \widehat{\Phi}'(t,s,\xi)$, we get

$$y' + b(t)y = -|\xi|^2\widehat{\Phi}(t,s,\xi), \quad y(s,\xi) = 1. \tag{7.22}$$

This leads to the integral equation

$$y(t,\xi) = \exp\Big(-\int_s^t b(\tau)\mathrm{d}\tau\Big)\Big(y(s,\xi) - \int_s^t \exp\Big(\int_s^\tau b(\sigma)\mathrm{d}\sigma\Big)|\xi|^2\widehat{\Phi}(\tau,s,\xi)\mathrm{d}\tau\Big),$$

that is,

$$\widehat{\Phi}'(t,s,\xi) = \frac{\lambda^2(s)}{\lambda^2(t)} - \int_s^t \frac{\lambda^2(\tau)}{\lambda^2(t)} |\xi|^2 \widehat{\Phi}(\tau,s,\xi) \mathrm{d}\tau.$$

In a similar way to Lemma 20 in [18], we can prove that

$$|\widehat{\Phi}'(t,s,\xi)| \lesssim \frac{|\xi|^2}{b(s)b(t)} \exp(-C|\xi|^2 B(t,s)). \tag{7.23}$$

Indeed, by using (7.16) in the integral and applying integration by parts $\big($we remark that $b(\tau)\frac{\lambda^2(\tau)}{\lambda^2(t)} = \partial_\tau\big(\frac{\lambda^2(\tau)}{\lambda^2(t)}\big)\big)$, we get

$$\begin{aligned}|\widehat{\Phi}'(t,s,\xi)| &\lesssim \frac{\lambda^2(s)}{\lambda^2(t)} + \int_s^t \Big(\frac{\lambda^2(\tau)}{\lambda^2(t)} b(\tau)\Big)\Big(\frac{|\xi|^2}{b(s)b(\tau)} \exp(-C|\xi|^2 B(\tau,s))\Big)\mathrm{d}\tau \\ &= \frac{\lambda^2(s)}{\lambda^2(t)} + \frac{|\xi|^2}{b(s)b(t)} \exp(-C|\xi|^2 B(t,s)) \\ &\quad - \frac{1}{b(s)} \int_s^t \frac{\lambda^2(\tau)}{\lambda^2(t)} \partial_\tau \Big(\frac{|\xi|^2}{b(\tau)} \exp(-C|\xi|^2 B(\tau,s))\Big)\mathrm{d}\tau.\end{aligned}$$

One can show that for $\eta(t)$ increasing or decreasing the second term determines the desired estimate. Therefore, we derive (7.23). Combined with (7.21), this allows us to derive for small frequencies $|\xi| \leq \eta(s)\sqrt{1-\varepsilon^2}$ the following estimates:

$$|\widehat{\Phi}'(t,s,\xi)| \lesssim \frac{|\xi|^2}{b(s)b(t)} \exp(-C|\xi|^2 B(t,s)) \quad \text{for } t \leq t_{|\xi|}, \tag{7.24}$$

$$|\widehat{\Phi}'(t,s,\xi)| \lesssim \frac{|\xi|}{b(s)} \exp(-C|\xi|^2 B(t_{|\xi|},s)) \Big(\frac{\lambda(t_{|\xi|})}{\lambda(t)}\Big)^{1-2\delta} \quad \text{for } t \geq t_{|\xi|}. \tag{7.25}$$

We remark that we used $b(t_{|\xi|}) \approx |\xi|$ in (7.25).

To estimate $\widehat{\Phi}'(t,s,\xi)$ for large frequencies $|\xi| \geq \eta(s)\sqrt{1-\varepsilon^2}$ and for any $t \geq t_{|\xi|}$, we slightly modify this approach. Indeed, we still put $y(t,\xi) = \widehat{\Phi}'(t,s,\xi)$, but now we look for an estimate of the solution to

$$\begin{cases} y' + b(t)y = -|\xi|^2 \widehat{\Phi}(t,s,\xi), & t \geq t_{|\xi|}, \\ y(t_{|\xi|},\xi) = \widehat{\Phi}'(t_{|\xi|},s,\xi). \end{cases} \tag{7.26}$$

By using (7.19) for $\widehat{\Phi}(t,s,\xi)$ and (7.20) for $\widehat{\Phi}'(t_{|\xi|},s,\xi)$, we derive for $t \geq t_{|\xi|}$ the following inequality:

$$\begin{aligned}&|\widehat{\Phi}'(t,s,\xi)| \\ &\lesssim \frac{\lambda^2(t_{|\xi|})}{\lambda^2(t)} \Big[\Big(\frac{\lambda(s)}{\lambda(t_{|\xi|})}\Big)^{1-2\delta} + \int_{t_{|\xi|}}^t \frac{\lambda^2(\tau)}{\lambda^2(t_{|\xi|})} |\xi|^2 \Big(\frac{1}{|\xi|}\Big(\frac{\lambda(s)}{\lambda(t_{|\xi|})}\Big)^{1-2\delta} \exp(-C|\xi|^2 B(\tau,t_{|\xi|}))\Big)\mathrm{d}\tau\Big] \\ &\lesssim \Big(\frac{\lambda(s)}{\lambda(t_{|\xi|})}\Big)^{1-2\delta} \Big[\frac{\lambda^2(t_{|\xi|})}{\lambda^2(t)} + \frac{1}{|\xi|} \int_{t_{|\xi|}}^t \Big(\frac{\lambda^2(\tau)}{\lambda^2(t)} b(\tau)\Big)\Big(\frac{|\xi|^2}{b(\tau)} \exp(-C|\xi|^2 B(\tau,t_{|\xi|}))\Big)\mathrm{d}\tau\Big].\end{aligned}$$

We can now easily follow the previous reasoning. Therefore, we derive for large frequencies $|\xi| \geq \eta(s)\sqrt{1-\varepsilon^2}$ and for any $t \geq t_{|\xi|}$ the estimate

$$|\widehat{\Phi}'(t,s,\xi)| \lesssim \Big(\frac{\lambda(s)}{\lambda(t_{|\xi|})}\Big)^{1-2\delta} \frac{|\xi|}{b(t)} \exp(-C|\xi|^2 B(t,t_{|\xi|})). \tag{7.27}$$

7.7 Small frequencies and large frequencies

We are now in the position to prove the following statement.

Lemma 7.1 *For any $s \in [0,\infty)$ and for any $t \geq s$, let us define*

$$\Theta(t,s) := \max\{\eta(s),\eta(t)\}\sqrt{1-\varepsilon^2}.$$

Then the estimates (7.18)–(7.20) hold for any $|\xi| \geq \Theta(t,s)$, whereas for any $|\xi| \leq \Theta(t,s)$, we have the following:

$$|\widehat{\Phi}(t,s,\xi)| \lesssim \frac{1}{b(s)}\exp(-C'|\xi|^2 B(t,s)), \tag{7.28}$$

$$|\widehat{\Phi}'(t,s,\xi)| \lesssim \frac{|\xi|^2}{b(s)b(t)}\exp(-C'|\xi|^2 B(t,s)). \tag{7.29}$$

Remark 7.2 The small frequencies $|\xi| \leq \Theta(t,s)$ are the ones such that $(s,\xi) \in Z_{\mathrm{ell}}(\varepsilon)$ or $(t,\xi) \in Z_{\mathrm{ell}}(\varepsilon)$, whereas the large frequencies $|\xi| \geq \Theta(t,s)$ are the ones for which (s,ξ), $(t,\xi) \in \Pi_{\mathrm{hyp}}(\varepsilon)$.

Proof of Lemma 7.1 The first part of Lemma 7.1 is trivial, since $|\xi| \geq \Theta(t,s)$ means that $(s,\xi) \in Z_{\mathrm{hyp}}(\varepsilon)$ and $t \leq t_{|\xi|}$. To prove (7.28)–(7.29), for $|\xi| \leq \Theta(t,s)$, we distinguish three cases:

(A) $|\xi| \leq \min\{\eta(s),\eta(t)\}\sqrt{1-\varepsilon^2}$;

(B) η is decreasing and $\eta(t)\sqrt{1-\varepsilon^2} \leq |\xi| \leq \eta(s)\sqrt{1-\varepsilon^2}$;

(C) η is increasing and $\eta(s)\sqrt{1-\varepsilon^2} \leq |\xi| \leq \eta(t)\sqrt{1-\varepsilon^2}$.

In the case (A) the two conditions (7.28)–(7.29) coincide with (7.16) and (7.24).

Now let $\eta(t)$ be a decreasing function. Since $|\xi| \lesssim b(\sigma)$ for any $t_{|\xi|} \leq \sigma$, it holds that

$$\begin{aligned}\exp(-C_1|\xi|^2 B(t_{|\xi|},s))\Big(\frac{\lambda(t_{|\xi|})}{\lambda(t)}\Big)^{2C_2} &= \exp\Big(-C_1|\xi|^2\int_s^{t_{|\xi|}}\frac{1}{b(\tau)}\mathrm{d}\tau - C_2\int_{t_{|\xi|}}^t b(\sigma)\mathrm{d}\sigma\Big)\\ &\leq \exp(-\min\{C_1,C_2\}|\xi|^2 B(t,s)).\end{aligned}$$

So (7.28)–(7.29) immediately follows from (7.17) and (7.25) in the case (B). Let $\eta(t)$ be an increasing function. Since $|\xi| \lesssim b(\tau)$ for any $t_{|\xi|} \leq \sigma$, it holds that

$$\begin{aligned}\Big(\frac{\lambda(s)}{\lambda(t_{|\xi|})}\Big)^{2C_1}\exp(C_2|\xi|^2 B(t_{|\xi|},s)) &= \exp\Big(-C_1|\xi|^2\int_s^{t_{|\xi|}} b(\tau)\mathrm{d}\tau - C_2\int_{t_{|\xi|}}^t \frac{1}{b(\sigma)}\mathrm{d}\sigma\Big)\\ &\leq \exp(-\min\{C_1,C_2\}|\xi|^2 B(t,s)).\end{aligned}$$

Then (7.28)–(7.29) follows from (7.19) and (7.27) by using $b(s) \lesssim |\xi|$ in the case (C).

7.8 Matsumura-type estimates

In order to estimate the L^2 norm of $\partial_t^l\partial_x^\alpha\Phi(t,s,x) *_{(x)} g(s,x)$ for $l = 0,1$ and for any $|\alpha| \geq 0$, we follow the ideas in [12] and we distinguish between small and large frequencies. We fix $t \in [s,\infty)$.

Lemma 7.2 *The following estimate holds for large frequencies* $|\xi| \geq \Theta = \Theta(t,s)$:

$$\||\xi|^{|\alpha|}\partial_t^l \widehat{\Phi}(t,s,\cdot)\widehat{g}(s,\cdot)\|_{L^2_{\{|\xi|\geq\Theta\}}} \lesssim \frac{1}{b(s)}\Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{1-2\delta}\|g(s,\cdot)\|_{H^{[|\alpha|+l-1]^+}} \tag{7.30}$$

for $l=0,1$ *and for any* $|\alpha|\geq 0$, *where* $[x]^+$ *denotes the positive part of* x.

Proof First, let $|\alpha|+l\geq 1$. We can estimate

$$\||\xi|^{|\alpha|}\partial_t^l \widehat{\Phi}(t,s,\cdot)\widehat{g}(s,\cdot)\|_{L^2_{\{|\xi|\geq\Theta\}}} \leq \||\xi|^{1-l}\partial_t^l \widehat{\Phi}(t,s,\cdot)\|_{L^\infty_{\{|\xi|\geq\Theta\}}}\||\xi|^{|\alpha|+l-1}\widehat{g}(s,\cdot)\|_{L^2_{\{|\xi|\geq\Theta\}}}$$

for any $|\alpha|+l\geq 1$ since $|\xi|\leq\langle\xi\rangle$. The second term can be estimated by $\|g(s,\cdot)\|_{H^{|\alpha|+l-1}}$. Thanks to the estimates (7.18) and (7.20), namely,

$$|\partial_t^l\widehat{\Phi}(t,s,\xi)| \lesssim |\xi|^{-1+l}\Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{1-2\delta},$$

we get a decay uniformly in $|\xi|\geq\Theta$ which is given by

$$\Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{1-2\delta} = \exp\Big(-\Big(\frac{1}{2}-\delta\Big)\int_s^t b(\tau)\mathrm{d}\tau\Big).$$

Now let $|\alpha|=l=0$. If $\eta_\infty>0$, then $\Theta(t,s)\geq C=\eta_\infty\sqrt{1-\varepsilon^2}>0$ for any s,t, and we can follow the reasoning above since $|\xi|^{-1}\approx\langle\xi\rangle^{-1}$ uniformly in $|\xi|\geq C$. Otherwise, if $\eta(t)\to 0$, then after recalling that $b(s)\lesssim|\xi|$ for large frequencies, we can estimate

$$\|\widehat{\Phi}(t,s,\cdot)\widehat{g}(s,\cdot)\|_{L^2_{|\xi|\geq\Theta}} \lesssim \frac{1}{b(s)}\Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{1-2\delta}\|g(s,\cdot)\|_{L^2}.$$

This completes the proof.

Remark 7.3 If $\eta(t)\to\eta_\infty>0$, or if we are interested in an estimate for $s\in[0,S]$ and $t\geq s$ for some fixed $S>0$, then $\Theta(t,s)$ is uniformly bounded by a positive constant. Therefore (see the proof of Lemma 7.2), we can replace $\|g(s,\cdot)\|_{H^{[|\alpha|+l-1]^+}}$ in the estimate (7.30) by $\|g(s,\cdot)\|_{H^{|\alpha|+l-1}}$, that is, by $\|g(s,\cdot)\|_{H^{-1}}$ in the case $|\alpha|=l=0$.

In particular, this is possible if we are only interested in estimates for $s=0$. This explains the difference in the regularity of the initial data $(0,g(s,\cdot))$ if we compare (3.4) ($L^m\cap H^{-1}$ regularity) with (4.3) ($L^m\cap L^2$ regularity).

Lemma 7.3 *The following estimate holds for small frequencies* $|\xi|\leq\Theta=\Theta(t,s)$:

$$\||\xi|^{|\alpha|}\partial_t^l\widehat{\Phi}(t,s,\cdot)\widehat{g}(s,\cdot)\|_{L^2_{\{|\xi|\leq\Theta\}}} \lesssim \frac{1}{b(s)}(B(t,s)b(t))^{-l}(B(t,s))^{-\frac{|\alpha|}{2}-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}\|g(s,\cdot)\|_{L^m} \tag{7.31}$$

for $l=0,1$ *and for any* $|\alpha|\geq 0$.

Proof Let m' and p be defined by $\frac{1}{m}+\frac{1}{m'}=1$ and $\frac{1}{p}+\frac{1}{m'}=\frac{1}{2}$, that is, $\frac{1}{p}=\frac{1}{m}-\frac{1}{2}$. We can estimate

$$\||\xi|^{|\alpha|}\partial_t^l\widehat{\Phi}(t,s,\cdot)\widehat{g}(s,\cdot)\|_{L^2_{\{|\xi|\leq\Theta\}}} \leq \||\xi|^{|\alpha|}\partial_t^l\widehat{\Phi}(t,s,\cdot)\|_{L^p_{\{|\xi|\leq\Theta\}}}\|\widehat{g}(s,\cdot)\|_{L^{m'}_{\{|\xi|\leq\Theta\}}}.$$

We can control $\|\widehat{g}(s,\xi)\|_{L^{m'}}$ by $\|g(s,\cdot)\|_{L^m}$. So we have to control the L^p norm of the multiplier. Thanks to (7.28)–(7.29), we can estimate

$$\||\xi|^{|\alpha|}\partial_t^l\widehat{\Phi}(t,s,\cdot)\|_{L^p_{\{|\xi|\leq\Theta\}}} \lesssim \frac{1}{b(s)(b(t))^l}\Big(\int_{\{|\xi|\leq\Theta\}}|\xi|^{p(|\alpha|+2l)}\exp(-Cp|\xi|^2B(t,s))\mathrm{d}\xi\Big)^{\frac{1}{p}}.$$

Let $\rho = Cp|\xi|^2 B(t,s)$. After a change of variables to spherical harmonics (the term ρ^{n-1} appears), we conclude

$$\int_{\{|\xi|\le\Theta\}} |\xi|^{p(|\alpha|+2l)} \exp(-Cp|\xi|^2 B(t,s)) \mathrm{d}\xi \lesssim (B(t,s))^{-\frac{p(|\alpha|+2l)+n}{2}} \int_0^\infty \rho^{p(|\alpha|+2l)+n-1} \mathrm{e}^{-\rho} \mathrm{d}\rho.$$

We remark that the case $\Theta(t,s)\to\infty$ brings no additional difficulties. The integral is bounded and we get a decay given by

$$\frac{1}{b(s)(b(t))^l}(B(t,s))^{-\frac{|\alpha|}{2}-l-\frac{n}{2p}} = \frac{1}{b(s)}(B(t,s)b(t))^{-l}(B(t,s))^{-\frac{|\alpha|}{2}-\frac{n}{2}(\frac{1}{m}-\frac{1}{2})}. \tag{7.32}$$

The proof is finished.

One can easily check that the decay function given in (7.31) is worse than the one in (7.30). Therefore, combining together (7.30)–(7.31), we derive (7.1). This concludes the proof of Theorem 4.1.

8 Generalizations and Improvements

8.1 Admissible damping terms

We may include oscillations in the damping term $b(t)u_t$ if we replace Hypotheses 3.1–3.2 by the following.

Hypothesis 8.1 We assume that $b = b(t)$ satisfies the conditions (i) and (iii)–(v) in Hypothesis 3.1. Moreover, we assume the existence of an admissible shape function $\eta : [0,\infty) \to [0,\infty)$ such that

$$\Big|\frac{b(t)}{\eta(t)} - 2\Big| \lesssim \frac{1}{1+t},$$

and $\eta\in\mathcal{C}^1$, $\eta(t)>0$, monotone, and $t\eta(t)\to\infty$ as $t\to\infty$. Finally, it satisfies (3.7), that is, $t\eta'(t)\le a\eta(t)$ for some $a\in[0,1)$.

Then the statements of Theorem 4.1 and Theorems 3.2–3.3 are still valid.

Remark 8.1 Let us assume that we have a life-span estimate for the local solution to (1.1), which guarantees that $T_m(\epsilon)\to\infty$ as $\epsilon\to 0$, where $T_m = T_m(\epsilon)\in(0,\infty]$ is the maximal existence time (see Lemma 5.1). Then condition (3.7) in Hypothesis 3.2 can be weakened to

$$l := \limsup_{t\to\infty} \frac{t\eta'(t)}{\eta(t)} < 1, \tag{8.1}$$

that is, it holds that

$$\frac{t\eta'(t)}{\eta(t)} \le a < 1, \quad t\ge t_0 \tag{8.2}$$

for some $t_0\ge 0$, where we take $a\in(l,1)$. Indeed, there exists an $\epsilon_1(t_0)>0$ such that $T_m(\epsilon)\ge 2t_0$ for any $\epsilon\in(0,\epsilon_1(t_0)]$, and this allows us to rewrite the proofs of Theorems 3.2–3.3 starting from t_0.

8.2 Semi-linear damped wave equation with small data in $L^m \cap H^1$

An intermediate case between the L^2 framework in [15] and the L^1 context in [7] has been studied in [8]. For initial data in $\mathcal{A}_{m,1}$, the authors find the critical exponent $p(n,m) = 1 + \frac{2m}{n}$ for $n \leq 6$, for any $m \in (1,2)$ if $n = 1,2$ and for suitable $m \in [\overline{m}, \overline{\overline{m}})$ if $3 \leq n \leq 6$.

If we consider data $(u_0, u_1) \in \mathcal{A}_{m,1}$ for some $m \in (1,2)$, then we can follow [8] to extend Theorem 3.3. The range of admissible exponents for the nonlinear term will also depend on the choice of $m \in (1,2)$.

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Appendix Gagliardo-Nirenberg Inequality

Here we state some Gagliardo-Nirenberg type inequalities which come into play in the proofs of Theorems 3.2–3.3.

Lemma A.1 (Gagliardo-Nirenberg Inequality (see [5,Theorem 9.3, Part 1])) *Let* $j, m \in \mathbb{N}$ *with* $j < m$, *and let* $u \in \mathcal{C}_c^m(\mathbb{R}^n)$, *i.e.,* $u \in \mathcal{C}^m$ *with compact support. Let* $a \in \left[\frac{j}{m}, 1\right]$, *and let* p, q, r *in* $[1, \infty]$ *be such that*

$$j - \frac{n}{q} = \Big(m - \frac{n}{r}\Big)a - \frac{n}{p}(1-a).$$

Then

$$\|D^j u\|_{L^q} \le C_{n,m,j,p,r,a} \|D^m u\|_{L^r}^a \ \|u\|_{L^p}^{1-a} \tag{A.1}$$

provided that

$$\Big(m - \frac{n}{r}\Big) - j \notin \mathbb{N}, \tag{A.2}$$

i.e., $\frac{n}{r} > m - j$ *or* $\frac{n}{r} \notin \mathbb{N}$. *If* (A.2) *is not satisfied, then* (A.1) *holds provided that* $a \in \left[\frac{j}{m}, 1\right)$.

Remark A.1 If $j = 0$, $m = 1$ and $r = p = 2$, then (A.1) is reduced to

$$\|u\|_{L^q} \lesssim \|\nabla u\|_{L^2}^{\theta(q)} \ \|u\|_{L^2}^{1-\theta(q)}, \tag{A.3}$$

where $\theta(q)$ is given by

$$-\frac{n}{q} = \Big(1 - \frac{n}{2}\Big)\theta(q) - \frac{n}{2}(1 - \theta(q)) = \theta(q) - \frac{n}{2}, \tag{A.4}$$

that is, $\theta(q)$ is as in (5.15). It is clear that $\theta(q) \ge 0$ if and only if $q \ge 2$. Analogously $\theta(q) \le 1$ if and only if

$$\text{either} \quad n = 1, 2 \quad \text{or} \quad q \le 2^* := \frac{2n}{n-2}. \tag{A.5}$$

Applying a density argument, the inequality (A.3) holds for any $u \in H^1$. Assuming $q < \infty$ the condition (A.2) can be neglected, also for $n = 2$. By summarizing, the estimate (A.3) holds for any finite $q \ge 2$ if $n = 1, 2$ and for any $q \in [2, 2^*]$ if $n \ge 3$.

In weighted spaces $H^1_{\psi(t,\cdot)}$, we can derive the following statements.

Lemma A.2 *Let* $q \ge 2$ *satisfy* (A.5), *and let* $\theta(q)$ *be as in* (A.4). *We have the following properties for any* $\sigma \in [0, 1]$ *and* $t \ge 0$:

(i) *Let* $\psi \ge 0$. *If* $v \in H^1_\psi$, *then* $v \in H^1_{\sigma\psi}$ *and for* $j = 0, 1$ *one has*

$$\|\mathrm{e}^{\sigma\psi(t,\cdot)} \nabla^j v(t,\cdot)\|_2 \le \|\nabla^j v\|_2^{1-\sigma} \|\mathrm{e}^{\psi(t,\cdot)} \nabla^j v(t,\cdot)\|_2^{\sigma}.$$

(ii) *Let* $\Delta\psi \ge 0$. *If* $v \in H^1_{\sigma\psi}$, *then* $\mathrm{e}^{\sigma\psi(t,\cdot)} v \in H^1$ *and*

$$\|\nabla(\mathrm{e}^{\sigma\psi(t,\cdot)} v)\|_2 \le \|\mathrm{e}^{\sigma\psi(t,\cdot)} \nabla v\|_2.$$

(iii) *Let* $\Delta\psi \geq 0$*. If* $v \in H^1_\psi$*, then*

$$\|e^{\sigma\psi(t,\cdot)}v\|_{L^q} \lesssim \|e^{\sigma\psi(t,\cdot)}v\|_{L^2}^{1-\theta(q)} \|e^{\sigma\psi(t,\cdot)}\nabla v\|_{L^2}^{\theta(q)}.$$

(iv) *Let* $\psi \geq 0$ *such that* $\inf_{x\in\mathbb{R}^n} \Delta\psi(t,x) =: C(t) > 0$*. Then*

$$\|e^{\sigma\psi(t,\cdot)}v\|_{L^q} \leq (C(t))^{-\frac{1-\theta(q)}{2}} \|e^{\sigma\psi(t,\cdot)}\nabla v\|_2.$$

Proof The property (i) is trivial for $\sigma = 0$ and requires only Hölder's inequality for $\sigma \in (0,1]$. The property (ii) is obtained by integration by parts (see [9, Lemma 2.3]). For (iii) one combines (ii) with a Gagliardo-Nirenberg inequality (Lemma A.1). For (iv) one combines (iii) with integration by parts used in proving (ii).