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# Asymptotic Behavior of a Structure Made by a Plate and a Straight Rod

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**Abstract** This paper is devoted to describing the asymptotic behavior of a structure made by a thin plate and a thin perpendicular rod in the framework of nonlinear elasticity. The authors scale the applied forces in such a way that the level of the total elastic energy leads to the Von-Kármán's equations (or the linear model for smaller forces) in the plate and to a one-dimensional rod-model at the limit. The junction conditions include in particular the continuity of the bending in the plate and the stretching in the rod at the junction.

**Keywords** Nonlinear elasticity, Junction, Straight rod, Plate
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## 1 Introduction

In this paper we consider the junction problem between a plate and a rod as their thicknesses tend to zero. We denote by $\delta$ and $\varepsilon$ the respective half thickness of the plate $\Omega_\delta$ and the rod $B_\varepsilon$, respectively. The structure is clamped on a part of the lateral boundary of the plate and it is free on the rest of its boundary. We assume that this multi-structure is made of elastic materials (possibly different in the plate and in the rod). In order to simplify the analysis we consider Saint-Venant-Kirchhoff's materials with Lamé's coefficients of order 1 in the plate and of order $q_\varepsilon^2 = \varepsilon^{2\eta}$ in the rod with $\eta > -1$ (see (1.1)). It allows us to deal with a rod made of the same material as the plate, or of a softer ($\eta > 0$) or stiffer ($-1 < \eta < 0$). It is well-known that the limit behaviors in both the two parts of this multi-structure depend on the order of the infimum of the elastic energy with respect to the parameters $\delta$ and $\varepsilon$. Indeed this order is governed by the ones of the applied forces on the structure. In the present paper, we suppose that the orders of the applied forces depend on $\delta$ (for the plate) and $\varepsilon$ (for the rod) and via two new real parameters $\kappa$ and $\kappa'$ (see Subsection 5.1). The parameters $\kappa$, $\kappa'$ and $\eta$ are linked in such a way that the infimum of the total elastic energy is of order $\delta^{2\kappa-1}$. As far as a minimizing sequence $v_\delta$ of the energy is concerned, this leads to the following estimates of the Green-St Venant's strain tensors

$$\|\nabla v_\delta^{\rm T}\nabla v_\delta - \mathbf{I}_3\|_{L^2(\Omega_\delta;\mathbb{R}^{3\times 3})} \le C\delta^{\kappa-\frac{1}{2}}, \quad \|\nabla v_\delta^{\rm T}\nabla v_\delta - \mathbf{I}_3\|_{L^2(B_\varepsilon;\mathbb{R}^{3\times 3})} \le C\frac{\delta^{\kappa-\frac{1}{2}}}{q_\varepsilon}.$$

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The limit model for the plate is the Von Kármán system ($\kappa = 3$) or the classical linear plate model ($\kappa > 3$). Similarly, in order to obtain either a nonlinear model or the classical linear model in the rod, the order of $\|\nabla v_\delta^{\mathrm{T}} \nabla v_\delta - \mathbf{I}_3\|_{L^2(B_\varepsilon;\mathbb{R}^{3\times 3})}$ must be less than $\varepsilon^{\kappa'}$ with $\kappa' \geq 3$. Hence, $\delta$, $\varepsilon$ and $q_\varepsilon$ are linked by the relation

$$\delta^{\kappa-\frac{1}{2}} = q_\varepsilon \varepsilon^{\kappa'}.$$

Moreover, still for the above estimates of the Green-St Venant's strain tensors, the bending in the plate is of order $\delta^{\kappa-2}$ and the stretching in the rod is of order $\varepsilon^{\kappa'-1}$. Since we wish at least these two quantities to match at the junction, it is essential to have

$$\delta^{\kappa-2} = \varepsilon^{\kappa'-1}.$$

Finally, the two relations between the parameters lead to

$$\delta^3 = q_\varepsilon^2 \varepsilon^2 = \varepsilon^{2+2\eta}. \tag{1.1}$$

Under the relation (1.1), we prove that in the limit model, the rotation of the cross-section and the bending of the rod in the junction are null. The limit plate model (nonlinear or linear) is coupled with the limit rod model (nonlinear or linear) via the bending in the plate and the stretching in the rod.

A similar problem, but starting within the framework of the linear elasticity, is investigated in [19]. In this work the rod is also clamped at its bottom. This additional boundary condition makes easier the analysis of the linear system of elasticity. In [19], the authors also assume that

$$\frac{\varepsilon}{\delta^2} \to +\infty. \tag{1.2}$$

With this extra condition they obtained the same linear limit model as we do here in the case $\kappa > 3$ and $\kappa' > 3$, and they wondered if the condition (1.2) is necessary or purely technical in order to obtain the junction conditions. The present article shows that this condition is not necessary to carry out the analysis.

The derivation of the limit behavior of a multi-structure such as the one considered here relies on two main arguments. Firstly, it is convenient to derive "Korn's type inequalities" both in the plate and in the rod. Secondly, one needs estimates of a deformation in the junction (in order to obtain the limit junction conditions). In this paper this is achieved through the use of two main tools given in Lemmas 4.1 and 5.2. For the plate, since it is clamped on a part of its lateral boundary, a "Korn's type inequality" is given in [8]. For the rod the issue is more intricate because the rod is nowhere clamped. In the first step, we derive sharp estimates of a deformation $v$ in the junction with respect to the parameters and to the $L^2$ norm (over the whole structure) of the linearized strain tensor $\nabla v + (\nabla v)^{\mathrm{T}} - 2\mathbf{I}_3$. This is the object of Lemma 4.1. In the second step, in Lemma 5.2, we estimate the $L^2$ norm of the linearized strain tensor of $v$ in the rod with respect to the parameters and to the $L^2$ norms of $\mathrm{dist}(\nabla v, SO(3))$ in the rod and in the plate. The proofs of these two lemmas strongly rely on the decomposition techniques for the displacements and the deformations of the plate and the rod. Once these technical results are established, we are in a position to scale the applied forces and in the case

$\kappa = 3$ or $\kappa' = 3$ to state an adequate assumption on these forces in order to finally obtain a total elastic energy of an order less than $\delta^5$.

In Section 2 we introduce a few general notations. Section 3 is devoted to recalling a main tool that we use in the whole paper, namely, the decomposition technique of the deformation of thin structures. In Section 4, the estimates provided by this method allow us to derive sharp estimates on the bending and the cross-section rotation of the rod at the junction together with the difference between the bending of the plate and the stretching of the rod at the junction. In Section 5, we introduce the elastic energy and we specify the scaling with respect to $\delta$ and $\kappa$ on the applied forces in order to obtain a total elastic energy of order $\delta^{2\kappa-1}$. In Section 6, we give the asymptotic behavior of the Green-St Venant's strain tensors in the plate and in the rod. In Section 7, we characterize the limit of the sequence of the rescaled infimum of the elastic energy in terms of the minimum of a limit energy.

As general references on the theory of elasticity we refer to [2] and [13]. The reader is referred to [1, 20, 29] for an introduction of rod models and to [13–15, 18] for plate models. As for junction problems in multi-structures we refer to [3–6, 9, 11–12, 14, 16, 19, 21–22, 24–28]. For the decomposition method in thin structures we refer to [7–8, 10, 23].

## 2 Notations and Definitions of the Structure

Let us introduce a few notations and definitions concerning the geometry of the plate and the rod. We denote by $I_d$ the identity map of $\mathbb{R}^3$.

Let $\omega$ be a bounded domain in $\mathbb{R}^2$ with the Lipschitzian boundary included in the plane $(O;\mathbf{e}_1,\mathbf{e}_2)$ such that $O\in\omega$ and let $\delta>0$. The plate is the domain

$$\Omega_\delta = \omega\times]-\delta,\delta[.$$

Let $\gamma_0$ be an open subset of $\partial\omega$ which is made of a finite number of connected components (whose closure is disjoint). The corresponding lateral part of the boundary of $\Omega_\delta$ is

$$\Gamma_{0,\delta} = \gamma_0\times]-\delta,\delta[.$$

The rod is defined by

$$B_{\varepsilon,\delta} = D_\varepsilon\times]-\delta,L[,\quad D_\varepsilon = D(O,\varepsilon),\quad D = D(O,1),$$

where $\varepsilon>0$ and $D_r = D(O,r)$ is the disc of radius $r$ and center at the origin $O$. The whole structure is denoted by

$$\mathcal{S}_{\delta,\varepsilon} = \Omega_\delta\cup B_{\varepsilon,\delta},$$

while the junction is

$$C_{\delta,\varepsilon} = \Omega_\delta\cap B_{\varepsilon,\delta} = D_\varepsilon\times]-\delta,\delta[.$$

The set of admissible deformations of the plate is

$$\mathbb{D}_\delta = \{v\in H^1(\Omega_\delta;\mathbb{R}^3)\mid v = I_d \text{ on } \Gamma_{0,\delta}\}.$$

The set of admissible deformations of the structure is

$$\mathbb{D}_{\delta,\varepsilon} = \{v \in H^1(\mathcal{S}_{\delta,\varepsilon};\mathbb{R}^3) \mid v = I_d \text{ on } \Gamma_{0,\delta}\}.$$

The aim of this paper is to study the asymptotic behavior of the structure $\mathcal{S}_{\delta,\varepsilon}$ in the case that both parameters $\delta$ and $\varepsilon$ go to 0. In order to simplify this study, we link $\delta$ and $\varepsilon$ by assuming that

$$\text{there exists a } \theta \in \mathbb{R}_+^* \text{ such that } \delta = \varepsilon^\theta, \tag{2.1}$$

where $\theta$ is a fixed constant (see Subsection 5.1). Nevertheless, we keep the parameters $\delta$ and $\varepsilon$ in the estimates given in Sections 3–4.

## 3 Some Reviews about the Decompositions in the Plates and the Rods

From now on, in order to simplify the notations, for any open set $\mathcal{O} \subset \mathbb{R}^3$ and any field $u \in H^1(\mathcal{O};\mathbb{R}^3)$, we denote

$$\mathbf{G}_s(u,\mathcal{O}) = \|\nabla u + (\nabla u)^{\mathrm{T}}\|_{L^2(\mathcal{O};\mathbb{R}^{3\times 3})}.$$

We recall Theorem 4.3 established in [23]. Any displacement $u \in H^1(\Omega_\delta;\mathbb{R}^3)$ of the plate is decomposed as

$$u(x) = \mathcal{U}(x_1,x_2) + x_3\mathcal{R}(x_1,x_2)\wedge \mathbf{e}_3 + \overline{u}(x), \quad x \in \Omega_\delta, \tag{3.1}$$

where $\mathcal{U}$ and $\mathcal{R}$ belong to $H^1(\omega;\mathbb{R}^3)$ and $\overline{u}$ belongs to $H^1(\Omega_\delta;\mathbb{R}^3)$. The sum of the first two terms $U_e(x) = \mathcal{U}(x_1,x_2) + x_3\mathcal{R}(x_1,x_2)\wedge\mathbf{e}_3$ is called the elementary displacement associated to $u$.

The following theorem is proved in [21] for displacements in $H^1(\Omega_\delta;\mathbb{R}^3)$ and in [23] for displacements in $W^{1,p}(\Omega_\delta;\mathbb{R}^3)$ $(1 < p < +\infty)$.

**Theorem 3.1** *Let $u \in H^1(\Omega_\delta;\mathbb{R}^3)$. There exists an elementary displacement $U_e(x) = \mathcal{U}(x_1,x_2) + x_3\mathcal{R}(x_1,x_2)\wedge\mathbf{e}_3$ and a warping $\overline{u}$ satisfying* (3.1) *such that*

$$\begin{aligned}
&\|\overline{u}\|_{L^2(\Omega_\delta;\mathbb{R}^3)} \le C\delta\mathbf{G}_s(u,\Omega_\delta),\\
&\|\nabla\overline{u}\|_{L^2(\Omega_\delta;\mathbb{R}^3)} \le C\mathbf{G}_s(u,\Omega_\delta),\\
&\Big\|\frac{\partial\mathcal{R}}{\partial x_\alpha}\Big\|_{L^2(\omega;\mathbb{R}^3)} \le \frac{C}{\delta^{\frac{3}{2}}}\mathbf{G}_s(u,\Omega_\delta),\\
&\Big\|\frac{\partial\mathcal{U}}{\partial x_\alpha} - \mathcal{R}\wedge\mathbf{e}_\alpha\Big\|_{L^2(\omega;\mathbb{R}^3)} \le \frac{C}{\delta^{\frac{1}{2}}}\mathbf{G}_s(u,\Omega_\delta),
\end{aligned} \tag{3.2}$$

*where the constant $C$ does not depend on $\delta$.*

The warping $\overline{u}$ satisfies the following relations:

$$\int_{-\delta}^{\delta}\overline{u}(x_1,x_2,x_3)\mathrm{d}x_3 = 0, \quad \int_{-\delta}^{\delta}x_3\overline{u}_\alpha(x_1,x_2,x_3)\mathrm{d}x_3 = 0 \quad \text{for a.e. } (x_1,x_2)\in\omega. \tag{3.3}$$

If a deformation $v$ belongs to $\mathbb{D}_\delta$, then the displacement $u = v - I_d$ is equal to 0 on $\Gamma_{0,\delta}$. In this case the fields $\mathcal{U}$, $\mathcal{R}$ and the warping $\overline{u}$ satisfy

$$\mathcal{U} = \mathcal{R} = 0 \quad \text{on } \gamma_0, \quad \overline{u} = 0 \quad \text{on } \Gamma_{0,\delta}. \tag{3.4}$$

Then, from (3.2), for any deformation $v \in \mathbb{D}_\delta$ the corresponding displacement $u = v - I_d$ verifies the following estimates (see [21]):

$$\begin{aligned} &\|\mathcal{R}\|_{H^1(\omega;\mathbb{R}^3)} + \|\mathcal{U}_3\|_{H^1(\omega)} \le \frac{C}{\delta^{\frac{3}{2}}}\mathbf{G}_s(u,\Omega_\delta),\\ &\|\mathcal{R}_3\|_{L^2(\omega)} + \|\mathcal{U}_\alpha\|_{H^1(\omega)} \le \frac{C}{\delta^{\frac{1}{2}}}\mathbf{G}_s(u,\Omega_\delta). \end{aligned} \tag{3.5}$$

The constants depend only on $\omega$.

From the above estimates we deduce the following Korn's type inequalities for the displacement $u$

$$\begin{aligned} &\|u_\alpha\|_{L^2(\Omega_\delta)} \le C_0\mathbf{G}_s(u,\Omega_\delta), \quad \|u_3\|_{L^2(\Omega_\delta)} \le \frac{C_0}{\delta}\mathbf{G}_s(u,\Omega_\delta),\\ &\|u - \mathcal{U}\|_{L^2(\Omega_\delta;\mathbb{R}^3)} \le \frac{C}{\delta}\mathbf{G}_s(u,\Omega_\delta),\\ &\|\nabla u\|_{L^2(\Omega_\delta;\mathbb{R}^9)} \le \frac{C}{\delta}\mathbf{G}_s(u,\Omega_\delta). \end{aligned} \tag{3.6}$$

Through the use of a different decomposition of the deformation $v$ which is introduced in [8] (see also Appendix), the following estimate also holds:

$$\|\mathcal{U}_3\|_{H^1(\omega)} \le \frac{C}{\delta^{\frac{3}{2}}}\|\text{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)}. \tag{3.7}$$

Now, we consider a displacement $u \in H^1(B_{\varepsilon,\delta};\mathbb{R}^3)$ of the rod $B_{\varepsilon,\delta}$. This displacement can be decomposed as (see Theorem 3.1 of [23])

$$u(x) = \mathcal{W}(x_3) + \mathcal{Q}(x_3) \wedge (x_1\mathbf{e}_1 + x_2\mathbf{e}_2) + \overline{w}(x), \quad x \in B_{\varepsilon,\delta}, \tag{3.8}$$

where $\mathcal{W}$, $\mathcal{Q}$ belong to $H^1(-\delta, L;\mathbb{R}^3)$ and $\overline{w}$ belongs to $H^1(B_{\varepsilon,\delta};\mathbb{R}^3)$. The sum of the first two terms $\mathcal{W}(x_3) + \mathcal{Q}(x_3) \wedge (x_1\mathbf{e}_1 + x_2\mathbf{e}_2)$ is called an elementary displacement of the rod.

The following theorem is established in [20] for displacements in $H^1(B_{\varepsilon,\delta};\mathbb{R}^3)$ and in [23] for displacements in $W^{1,p}(B_{\varepsilon,\delta};\mathbb{R}^3)$ $(1 < p < +\infty)$.

**Theorem 3.2** *Let $u \in H^1(B_{\varepsilon,\delta};\mathbb{R}^3)$. There exists an elementary displacement $\mathcal{W}(x_3) + \mathcal{Q}(x_3) \wedge (x_1\mathbf{e}_1 + x_2\mathbf{e}_2)$ and a warping $\overline{w}$ satisfying* (3.8)*, such that*

$$\begin{aligned} &\|\overline{w}\|_{L^2(B_{\varepsilon,\delta};\mathbb{R}^3)} \le C\varepsilon\mathbf{G}_s(u,B_{\varepsilon,\delta}),\\ &\|\nabla\overline{w}\|_{L^2(B_{\varepsilon,\delta};\mathbb{R}^{3\times3})} \le C\mathbf{G}_s(u,B_{\varepsilon,\delta}),\\ &\Big\|\frac{\mathrm{d}\mathcal{Q}}{\mathrm{d}x_3}\Big\|_{L^2(-\delta,L;\mathbb{R}^3)} \le \frac{C}{\varepsilon^2}\mathbf{G}_s(u,B_{\varepsilon,\delta}),\\ &\Big\|\frac{\mathrm{d}\mathcal{W}}{\mathrm{d}x_3} - \mathcal{Q}\wedge\mathbf{e}_3\Big\|_{L^2(-\delta,L;\mathbb{R}^3)} \le \frac{C}{\varepsilon}\mathbf{G}_s(u,B_{\varepsilon,\delta}), \end{aligned} \tag{3.9}$$

*where the constant $C$ does not depend on $\varepsilon$, $\delta$ and $L$.*

The warping $\overline{w}$ satisfies the following relations:

$$\begin{aligned}&\int_{D_\varepsilon}\overline{w}(x_1,x_2,x_3)\mathrm{d}x_1\mathrm{d}x_2=0,\quad \int_{D_\varepsilon}x_\alpha\overline{w}_3(x_1,x_2,x_3)\mathrm{d}x_1\mathrm{d}x_2=0,\\&\int_{D_\varepsilon}\{x_1\overline{w}_2(x_1,x_2,x_3)-x_2\overline{w}_1(x_1,x_2,x_3)\}\mathrm{d}x_1\mathrm{d}x_2=0\quad\text{for a.e. } x_3\in]-\delta,L[.\end{aligned}\tag{3.10}$$

Then, from (3.9), for any displacement $u\in H^1(B_{\varepsilon,\delta};\mathbb{R}^3)$ the terms of the decomposition of $u$ verify

$$\begin{aligned}&\|\mathcal{Q}-\mathcal{Q}(0)\|_{H^1(-\delta,L;\mathbb{R}^3)}\le\frac{C}{\varepsilon^2}\mathbf{G}_s(u,B_{\varepsilon,\delta}),\\&\|\mathcal{W}_3-\mathcal{W}_3(0)\|_{H^1(-\delta,L)}\le\frac{C}{\varepsilon}\mathbf{G}_s(u,B_{\varepsilon,\delta}),\\&\|\mathcal{W}_\alpha-\mathcal{W}_\alpha(0)\|_{H^1(-\delta,L)}\le\frac{C}{\varepsilon^2}\mathbf{G}_s(u,B_{\varepsilon,\delta})+C\varepsilon\|\mathcal{Q}(0)\|_2.\end{aligned}\tag{3.11}$$

Now, in order to obtain Korn's type inequalities for the displacement $w$, the following section is devoted to giving estimates on $\mathcal{Q}(0)$ and $\mathcal{W}(0)$.

## 4 Estimates at the Junction

Let us set

$$H^1_{\gamma_0}(\omega)=\{\varphi\in H^1(\omega);\ \varphi=0\ \text{ on }\ \gamma_0\}.$$

Let $v\in\mathbb{D}_{\delta,\varepsilon}$ be a deformation whose displacement $u=v-I_d$ is decomposed as in Theorems 3.1–3.2. We define the function $\widetilde{\mathcal{U}}_3$ as the solution of the following variational problem

$$\begin{cases}\widetilde{\mathcal{U}}_3\in H^1_{\gamma_0}(\omega),\\ \displaystyle\int_\omega\nabla\widetilde{\mathcal{U}}_3\nabla\varphi=\int_\omega(\mathcal{R}\wedge\mathbf{e}_\alpha)\cdot\mathbf{e}_3\frac{\partial\varphi}{\partial x_\alpha}\mathrm{d}x,\\ \forall\varphi\in H^1_{\gamma_0}(\omega).\end{cases}\tag{4.1}$$

Indeed, due to the third estimate in (3.5), $\widetilde{\mathcal{U}}_3$ satisfies

$$\|\widetilde{\mathcal{U}}_3\|_{H^1(\omega)}\le\frac{C}{\delta^{\frac32}}\mathbf{G}_s(u,\Omega_\delta).\tag{4.2}$$

The definition (4.1) of $\widetilde{\mathcal{U}}_3$ together with the fourth estimate in (3.2) leads to

$$\|\mathcal{U}_3-\widetilde{\mathcal{U}}_3\|_{H^1(\omega)}\le\frac{C}{\delta^{\frac12}}\mathbf{G}_s(u,\Omega_\delta),\tag{4.3}$$

and moreover

$$\Big\|\frac{\partial\widetilde{\mathcal{U}}_3}{\partial x_\alpha}-(\mathcal{R}\wedge\mathbf{e}_\alpha)\cdot\mathbf{e}_3\Big\|_{L^2(\omega)}\le\frac{C}{\delta^{\frac12}}\mathbf{G}_s(u,\Omega_\delta).\tag{4.4}$$

Now, let $\rho_0>0$ be fixed such that $D(O,\rho_0)\subset\subset\omega$. Since $\mathcal{R}\in H^1(\omega;\mathbb{R}^3)$, the function $\widetilde{\mathcal{U}}_3$ belongs to $H^2(D(O,\rho_0))$ and the third estimate in (3.5) gives

$$\|\widetilde{\mathcal{U}}_3\|_{H^2(D(O,\rho_0))}\le\frac{C}{\delta^{\frac32}}\mathbf{G}_s(u,\Omega_\delta).\tag{4.5}$$

Besides, estimates (3.7) and (4.3) lead to

$$\|\widetilde{\mathcal{U}}_3\|_{L^6(D(O,\rho_0))} \leq \frac{C}{\delta^{\frac{1}{2}}}\mathbf{G}_s(u,\Omega_\delta) + \frac{C}{\delta^{\frac{3}{2}}}\|\mathrm{dist}(\nabla v, SO(3)\|_{L^2(\Omega_\delta)}. \tag{4.6}$$

**Lemma 4.1** *We have the following estimates:*

$$|\mathcal{W}_\alpha(0)|^2 \leq \frac{C}{\varepsilon\delta}[\mathbf{G}_s(u,\Omega_\delta)]^2 + C\Big[1+\frac{\delta^2}{\varepsilon^2}\Big]\frac{\delta}{\varepsilon^2}[\mathbf{G}_s(u,B_{\varepsilon,\delta})]^2, \tag{4.7}$$

$$|\mathcal{W}_3(0)-\widetilde{\mathcal{U}}_3(0,0)|^2 \leq \frac{C}{\delta^2}\Big[1+\frac{\varepsilon^2}{\delta}\Big][\mathbf{G}_s(u,\Omega_\delta)]^2 + C\frac{\delta}{\varepsilon^2}[\mathbf{G}_s(u,B_{\varepsilon,\delta})]^2, \tag{4.8}$$

$$\begin{aligned}|\widetilde{\mathcal{U}}_3(0,0)|^2 \leq{}& \frac{C}{\delta^3}\mathbf{G}_s(u,\Omega_\delta)\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)} + \frac{C}{\delta^2}[\mathbf{G}_s(u,\Omega_\delta)]^2\\ &+\frac{C}{\delta^3}[\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)}]^2.\end{aligned} \tag{4.9}$$

*The vector $\mathcal{Q}(0)$ satisfies the following estimate:*

$$\|\mathcal{Q}(0)\|_2^2 \leq \frac{C}{\varepsilon^2\delta}\Big[1+\frac{\varepsilon}{\delta^2}\Big][\mathbf{G}_s(u,\Omega_\delta)]^2 + C\frac{\delta}{\varepsilon^4}[\mathbf{G}_s(u,B_{\varepsilon,\delta})]^2. \tag{4.10}$$

*The constant $C$ is independent of $\varepsilon$ and $\delta$.*

**Proof** The two decompositions of $u = v - I_d$ give, for a.e. $x$ in the common part of the plate and the rod $C_{\delta,\varepsilon}$,

$$\mathcal{U}(x_1,x_2) + x_3\mathcal{R}(x_1,x_2)\wedge\mathbf{e}_3 + \overline{u}(x) = \mathcal{W}(x_3) + \mathcal{Q}(x_3)\wedge(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) + \overline{w}(x). \tag{4.11}$$

**Step 1** Estimates on $\mathcal{W}(0)$.

In this step we prove (4.7) and (4.8). Taking into account the equalities (3.3) and (3.10) on the warpings $\overline{u}$ and $\overline{w}$, we deduce that the averages on the cylinder $C_{\delta,\varepsilon}$ of both sides of the above equality (4.11) give

$$\mathcal{M}_{D_\varepsilon}(\mathcal{U}) = \mathcal{M}_{I_\delta}(\mathcal{W}), \tag{4.12}$$

where

$$\mathcal{M}_{D_\varepsilon}(\mathcal{U}) = \frac{1}{|D_\varepsilon|}\int_{D_\varepsilon}\mathcal{U}(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2, \quad \mathcal{M}_{I_\delta}(\mathcal{W}) = \frac{1}{2\delta}\int_{-\delta}^{\delta}\mathcal{W}(x_3)\mathrm{d}x_3.$$

Besides, using (3.5) we have

$$\|\mathcal{U}_\alpha\|^2_{L^2(D_\varepsilon)} \leq C\varepsilon\|\mathcal{U}_\alpha\|^2_{L^4(\omega)} \leq C\varepsilon\|\mathcal{U}_\alpha\|^2_{H^1(\omega)} \leq \frac{C\varepsilon}{\delta}[\mathbf{G}_s(u,\Omega_\delta)]^2.$$

From these estimates we get

$$|\mathcal{M}_{I_\delta}(\mathcal{W}_\alpha)|^2 = |\mathcal{M}_{D_\varepsilon}(\mathcal{U}_\alpha)|^2 \leq \frac{C}{\varepsilon\delta}[\mathbf{G}_s(u,\Omega_\delta)]^2. \tag{4.13}$$

Moreover, for any $p\in[2,+\infty[$, using (4.3), we deduce that

$$\begin{aligned}\|\mathcal{U}_3-\widetilde{\mathcal{U}}_3\|_{L^2(D_\varepsilon)} &\leq C\varepsilon^{1-\frac{2}{p}}\|\mathcal{U}_3-\widetilde{\mathcal{U}}_3\|_{L^p(\omega)} \leq C_p\varepsilon^{1-\frac{2}{p}}\|\mathcal{U}_3-\widetilde{\mathcal{U}}_3\|_{H^1(\omega)}\\ &\leq C_p\frac{\varepsilon^{1-\frac{2}{p}}}{\delta^{\frac{1}{2}}}\mathbf{G}_s(u,\Omega_\delta).\end{aligned} \tag{4.14}$$

Then we replace $\mathcal{U}_3$ with $\widetilde{\mathcal{U}}_3$ in (4.12) to obtain

$$|\mathcal{M}_{D_\varepsilon}(\widetilde{\mathcal{U}}_3) - \mathcal{M}_{I_\delta}(\mathcal{W}_3)|^2 \leq \frac{C_p}{\varepsilon^{\frac{4}{p}}\delta}[\mathbf{G}_s(u,\Omega_\delta)]^2. \tag{4.15}$$

We carry on by comparing $\mathcal{M}_{D_\varepsilon}(\widetilde{\mathcal{U}}_3)$ with $\widetilde{\mathcal{U}}_3(0,0)$. Let us set

$$\mathbf{r}_\alpha = \mathcal{M}_{D_\varepsilon}((\mathcal{R}\wedge\mathbf{e}_\alpha)\cdot\mathbf{e}_3) = \frac{1}{|D_\varepsilon|}\int_{D_\varepsilon}(\mathcal{R}(x_1,x_2)\wedge\mathbf{e}_\alpha)\cdot\mathbf{e}_3\mathrm{d}x_1\mathrm{d}x_2 \tag{4.16}$$

and consider the function $\Psi(x_1,x_2) = \widetilde{\mathcal{U}}_3(x_1,x_2) - \mathcal{M}_{D_\varepsilon}(\widetilde{\mathcal{U}}_3) - x_1\mathbf{r}_2 - x_2\mathbf{r}_1$. Due to the estimate (4.5), we first obtain

$$\Big\|\frac{\partial^2\Psi}{\partial x_\alpha\partial x_\beta}\Big\|_{L^2(D_\varepsilon)} \leq \frac{C}{\delta^{\frac{3}{2}}}\mathbf{G}_s(u,\Omega_\delta). \tag{4.17}$$

Secondly, from (3.2) and the Poincaré-Wirtinger's inequality in the disc $D_\varepsilon$, we get

$$\|(\mathcal{R}\wedge\mathbf{e}_\alpha)\cdot\mathbf{e}_3 - \mathcal{M}_{D_\varepsilon}((\mathcal{R}\wedge\mathbf{e}_\alpha)\cdot\mathbf{e}_3)\|_{L^2(D_\varepsilon)} \leq C\frac{\varepsilon}{\delta^{\frac{3}{2}}}\mathbf{G}_s(u,\Omega_\delta).$$

Using the above inequality and (4.4), we deduce that

$$\|\nabla\Psi\|^2_{L^2(D_\varepsilon;\mathbb{R}^2)} \leq C\Big(\frac{1}{\delta} + \frac{\varepsilon^2}{\delta^3}\Big)[\mathbf{G}_s(u,\Omega_\delta)]^2. \tag{4.18}$$

Noting that $\mathcal{M}_{D_\varepsilon}(\Psi) = 0$, the above inequality and the Poincaré-Wirtinger's inequality in the disc $D_\varepsilon$ lead to

$$\|\Psi\|^2_{L^2(D_\varepsilon)} \leq C\frac{\varepsilon^2}{\delta}\Big(1 + \frac{\varepsilon^2}{\delta^2}\Big)[\mathbf{G}_s(u,\Omega_\delta)]^2. \tag{4.19}$$

From inequalities (4.17)–(4.19), we deduce that

$$\|\Psi\|^2_{\mathcal{C}^0(\overline{D_\varepsilon})} \leq C\Big(\frac{1}{\delta} + \frac{\varepsilon^2}{\delta^3}\Big)[\mathbf{G}_s(u,\Omega_\delta)]^2$$

which in turn gives

$$|\Psi(0,0)|^2 = |\widetilde{\mathcal{U}}_3(0,0) - \mathcal{M}_{D_\varepsilon}(\widetilde{\mathcal{U}}_3)|^2 \leq C\Big(\frac{1}{\delta} + \frac{\varepsilon^2}{\delta^3}\Big)[\mathbf{G}_s(u,\Omega_\delta)]^2.$$

This last estimate and (4.15) yield

$$|\widetilde{\mathcal{U}}_3(0,0) - \mathcal{M}_{I_\delta}(\mathcal{W}_3)|^2 \leq \frac{C}{\delta}\Big(\frac{C_p}{\varepsilon^{\frac{4}{p}}} + \frac{\varepsilon^2}{\delta^2}\Big)[\mathbf{G}_s(u,\Omega_\delta)]^2. \tag{4.20}$$

In order to estimate $\mathcal{M}_{I_\delta}(\mathcal{W}_3) - \mathcal{W}_3(0)$, we set $y(x_3) = \mathcal{W}(x_3) - \mathcal{Q}(0)x_3\wedge\mathbf{e}_3$. Estimates in Theorem 3.2 together with the use of Poincaré inequality in order to estimate $\|\mathcal{Q} - \mathcal{Q}(0)\|_{L^2(-\delta,\delta;\mathbb{R}^3)}$ give

$$\begin{aligned}&\Big\|\frac{\mathrm{d}y_\alpha}{\mathrm{d}x_3}\Big\|_{L^2(-\delta,\delta)} \leq C\Big(\frac{1}{\varepsilon} + \frac{\delta}{\varepsilon^2}\Big)\mathbf{G}_s(u,B_{\varepsilon,\delta}),\\ &\Big\|\frac{\mathrm{d}y_3}{\mathrm{d}x_3}\Big\|_{L^2(-\delta,\delta)} \leq \frac{C}{\varepsilon}\mathbf{G}_s(u,B_{\varepsilon,\delta}),\end{aligned}$$

which imply

$$l\|y_\alpha - y_\alpha(0)\|^2_{L^2(-\delta,\delta)} \le C\frac{\delta^2}{\varepsilon^2}\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)[\mathbf{G}_s(u,B_{\varepsilon,\delta})]^2,$$
$$l\|y_3 - y_3(0)\|^2_{L^2(-\delta,\delta)} \le C\frac{\delta^2}{\varepsilon^2}[\mathbf{G}_s(u,B_{\varepsilon,\delta})]^2.$$

Then, taking the averages on $]-\delta,\delta[$ we obtain

$$\begin{aligned}|\mathcal{M}_{I_\delta}(\mathcal{W}_\alpha) - \mathcal{W}_\alpha(0)|^2 &\le C\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)\frac{\delta}{\varepsilon^2}[\mathbf{G}_s(u,B_{\varepsilon,\delta})]^2,\\ |\mathcal{M}_{I_\delta}(\mathcal{W}_3) - \mathcal{W}_3(0)|^2 &\le C\frac{\delta}{\varepsilon^2}[\mathbf{G}_s(u,B_{\varepsilon,\delta})]^2.\end{aligned} \tag{4.21}$$

Finally, from (4.13), (4.20) and the above last inequality, we obtain (4.7) and the following estimate:

$$|\mathcal{W}_3(0) - \widetilde{\mathcal{U}}_3(0,0)|^2 \le \frac{C}{\delta}\Big[\frac{C_p}{\varepsilon^{\frac{4}{p}}} + \frac{\varepsilon^2}{\delta^2}\Big][\mathbf{G}_s(u,\Omega_\delta)]^2 + C\frac{\delta}{\varepsilon^2}[\mathbf{G}_s(u,B_{\varepsilon,\delta})]^2. \tag{4.22}$$

Choosing $p=\max\big(2,\frac{4}{\theta}\big)$ (recall that $\delta=\varepsilon^\theta$) we get (4.8).

**Step 2** We prove the estimate on $\widetilde{\mathcal{U}}_3(0,0)$.

First recall the Gagliardo-Nirenberg's inequality

$$\|\nabla\widetilde{\mathcal{U}}_3\|_{L^3(D(O,\rho_0);\mathbb{R}^2)} \le C\|\widetilde{\mathcal{U}}_3\|^{\frac12}_{H^2(D(O,\rho_0))}\|\widetilde{\mathcal{U}}_3\|^{\frac12}_{L^6(D(O,\rho_0))}.$$

Together with estimates (4.5) and (4.6) we obtain

$$\|\nabla\widetilde{\mathcal{U}}_3\|_{L^3(D(O,\rho_0);\mathbb{R}^2)} \le \frac{C}{\delta^{\frac32}}[\mathbf{G}_s(u,\Omega_\delta)]^{\frac12}[\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)}]^{\frac12} + \frac{C}{\delta}\mathbf{G}_s(u,\Omega_\delta).$$

Due to (4.6) and the above inequality we get

$$\begin{aligned}\|\widetilde{\mathcal{U}}_3\|_{W^{1,3}(D(O,\rho_0))} \le{}& \frac{C}{\delta^{\frac32}}[\mathbf{G}_s(u,\Omega_\delta)]^{\frac12}[\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)}]^{\frac12} + \frac{C}{\delta}\mathbf{G}_s(u,\Omega_\delta)\\ &+ \frac{C}{\delta^{\frac32}}\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)},\end{aligned}$$

which in turn shows that the estimate on $\widetilde{\mathcal{U}}_3(0,0)$ holds.

**Step 3** We prove the estimate on $\mathcal{Q}(0)$.

We recall (see Definition 3 in [23]) that the field $\mathcal{Q}$ is defined by

$$\mathcal{Q}_1(x_3) = \frac{4}{\pi\varepsilon^4}\int_{D_\varepsilon} x_1u_3(x)\mathrm{d}x_1\mathrm{d}x_2, \quad \mathcal{Q}_2(x_3) = -\frac{4}{\pi\varepsilon^4}\int_{D_\varepsilon} x_2u_3(x)\mathrm{d}x_1\mathrm{d}x_2,$$
$$\mathcal{Q}_3(x_3) = \frac{2}{\pi\varepsilon^4}\int_{D_\varepsilon}\{x_1u_2(x) - x_2u_1(x)\}\mathrm{d}x_1\mathrm{d}x_2, \quad \text{for a.e. } x_3\in]-\delta,L[.$$

Now, using again the equalities (3.3) and (3.10) on the warpings $\overline{u}$ and $\overline{w}$, the two decompositions (4.11) of $u$ in the cylinder $C_{\delta,\varepsilon}$ lead to

$$\Big|\frac{\varepsilon^2}{4}\mathcal{M}_{I_\delta}(\mathcal{Q}_\alpha)\Big| = |\mathcal{M}_{D_\varepsilon}(\mathcal{U}_3\,x_\alpha)|, \quad \Big|\frac{\varepsilon^2}{2}\mathcal{M}_{I_\delta}(\mathcal{Q}_3)\Big| = |\mathcal{M}_{D_\varepsilon}(\mathcal{U}_2\,x_1 - \mathcal{U}_1\,x_2)|.$$

Noticing that $\mathcal{M}_{D_\varepsilon}(\mathcal{U}_1\, x_2) = \mathcal{M}_{D_\varepsilon}([\mathcal{U}_1 - \mathcal{M}_{D_\varepsilon}(\mathcal{U}_1)]x_2)$ and applying the Poincaré-Wirtinger's inequality with (3.5), we have

$$|\mathcal{M}_{I_\delta}(\mathcal{Q}_3)|^2 \le \frac{C}{\varepsilon^2\delta}[\mathbf{G}_s(u,\Omega_\delta)]^2. \tag{4.23}$$

From the definition of the function $\Psi$ and the constants $\mathbf{r}_\alpha$ introduced in Step 1, we deduce that

$$|\mathcal{M}_{D_\varepsilon}(\mathcal{U}_3 x_\alpha)| \le |\mathcal{M}_{D_\varepsilon}(\Psi x_\alpha)| + |\mathcal{M}_{D_\varepsilon}([\mathcal{U}_3 - \widetilde{U}_3]x_\alpha)| + C\varepsilon^2|\mathbf{r}_\alpha|. \tag{4.24}$$

Estimate (4.19) gives

$$|\mathcal{M}_{D_\varepsilon}(\Psi x_\alpha)|^2 \le C\frac{\varepsilon^2}{\delta}\Big(1+\frac{\varepsilon^2}{\delta^2}\Big)[\mathbf{G}_s(u,\Omega_\delta)]^2, \tag{4.25}$$

while (3.5) leads to

$$|\mathbf{r}_\alpha|^2 \le \frac{C}{\varepsilon^2}\|\mathcal{R}\|^2_{L^2(D_\varepsilon;\mathbb{R}^3)} \le \frac{C}{\varepsilon}\|\mathcal{R}\|^2_{L^4(D_\varepsilon;\mathbb{R}^3)} \le \frac{C}{\varepsilon}\|\mathcal{R}\|^2_{H^1(\omega;\mathbb{R}^3)} \le \frac{C}{\varepsilon\delta^3}[\mathbf{G}_s(u,\Omega_\delta)]^2 \tag{4.26}$$

and (4.3) with the Poincaré-Wirtinger's inequality yields

$$|\mathcal{M}_{D_\varepsilon}([\mathcal{U}_3 - \widetilde{U}_3]x_\alpha)|^2 \le \frac{C\varepsilon^2}{\delta}[\mathbf{G}_s(u,\Omega_\delta)]^2. \tag{4.27}$$

Finally, from (4.24)–(4.27), we obtain

$$|\mathcal{M}_{I_\delta}(\mathcal{Q}_\alpha)|^2 \le \frac{C}{\varepsilon^2\delta}\Big(1+\frac{\varepsilon}{\delta^2}\Big)[\mathbf{G}_s(u,\Omega_\delta)]^2. \tag{4.28}$$

The third estimate in (3.9) implies

$$\|\mathcal{Q}(0) - \mathcal{M}_{I_\delta}(\mathcal{Q})\|_2^2 \le C\frac{\delta}{\varepsilon^4}[\mathbf{G}_s(u,B_{\varepsilon,\delta})]^2. \tag{4.29}$$

From (4.28)–(4.29), we get (4.10).

## 5 Elastic Structure

In this section we assume that the structure $\mathcal{S}_{\delta,\varepsilon}$ is made of an elastic material. The associated local energy $\widehat{W}_\varepsilon : \mathcal{S}_{\delta,\varepsilon}\times\mathbf{X}_3 \to \mathbb{R}^+$ is the following St Venant-Kirchhoff's law (see [9])

$$\widehat{W}_\varepsilon(x,F) = \begin{cases} Q_\varepsilon(x, F^{\mathrm{T}}F - \mathbf{I}_3), & \text{if } \det(F) > 0,\\ +\infty, & \text{if } \det(F) \le 0,\end{cases} \tag{5.1}$$

where $\mathbf{X}_3$ is the space of $3\times 3$ symmetric matrices and the quadratic form $Q_\varepsilon(x,\cdot)$ is given by

$$Q_\varepsilon(x,E) = \begin{cases} Q_p(E), & \text{if } x\in\ \Omega_\delta\setminus C_{\delta,\varepsilon},\\ q_\varepsilon^2 Q_r(E), & \text{if } x\in B_{\varepsilon,\delta}\setminus C_{\delta,\varepsilon},\\ Q_p(E), & \text{if } x\in\ C_{\delta,\varepsilon}\quad\text{and } q_\varepsilon\le 1,\\ q_\varepsilon^2 Q_r(E), & \text{if } x\in C_{\delta,\varepsilon}\quad\text{and } q_\varepsilon > 1\end{cases}$$

with

$$Q_p(E) = \frac{\lambda_p}{8}(\mathrm{tr}(E))^2 + \frac{\mu_p}{4}\mathrm{tr}(E^2), \quad Q_r(E) = \frac{\lambda_r}{8}(\mathrm{tr}(E))^2 + \frac{\mu_r}{4}\mathrm{tr}(E^2), \tag{5.2}$$

and $(\lambda_p, \mu_p)$ (resp. $(q_\varepsilon^2\lambda_r, q_\varepsilon^2\mu_r)$) are the Lamé's coefficients of the plate (resp. the rod). The constant $q_\varepsilon$ depends only on the rod, and we set $q_\varepsilon = \varepsilon^\eta$, the parameter $\eta$ being such that

(1) $\eta = 0$ for the same order for the Lamé's coefficients in the plate and in the rod;

(2) $\eta > 0$ for a softer material in the rod than in the plate;

(3) $\eta < 0$ for a softer material in the plate than in the rod.

Observe that the definition of $Q_\varepsilon(x, E)$ shows that

$$Q_\varepsilon(x, E) \geq \overline{\mu}(1_{\Omega_\delta}(x) + 1_{B_{\varepsilon,\delta}}(x)q_\varepsilon^2))\mathrm{tr}(E^2) \tag{5.3}$$

for a.e. $x \in \mathcal{S}_{\delta,\varepsilon}$ and $\forall E \in \mathbf{X}_3$, where

$$\overline{\mu} = \frac{\inf\{\mu_p, \mu_r\}}{8}. \tag{5.4}$$

Let us recall (see, e.g. [7] or [18]) that for any $3\times 3$ matrix $F$ such that $\det(F) > 0$ we have

$$\mathrm{tr}([F^{\mathrm{T}}F - \mathbf{I}_3]^2) = |||F^{\mathrm{T}}F - \mathbf{I}_3|||^2 \geq \ \mathrm{dist}\ (F, SO(3))^2. \tag{5.5}$$

We define the total energy $J_\delta(v)$[1] over $\mathbb{D}_{\delta,\varepsilon}$ by

$$J_\delta(v) = \int_{\mathcal{S}_{\delta,\varepsilon}} \widehat{W}_\varepsilon(x, \nabla v)(x)\mathrm{d}x - \int_{\mathcal{S}_{\delta,\varepsilon}} f_\delta(x)\cdot(v(x) - I_d(x))\mathrm{d}x. \tag{5.6}$$

### 5.1 Relations between $\delta$, $\varepsilon$ and $q_\varepsilon$

In Subsection 5.2 we scale the applied forces in order to have the infimum of this total energy of order $\delta^{2\kappa-1}$ with $\kappa \geq 3$. In such a way, the minimizing sequences $(v_\delta)$ satisfy

$$\|\nabla v_\delta^{\mathrm{T}}\nabla v_\delta - \mathbf{I}_3\|_{L^2(\Omega_\delta;\mathbb{R}^{3\times3})} \leq C\delta^{\kappa-\frac{1}{2}}, \quad \|\nabla v_\delta^{\mathrm{T}}\nabla v_\delta - \mathbf{I}_3\|_{L^2(B_{\varepsilon,\delta};\mathbb{R}^{3\times3})} \leq C\frac{\delta^{\kappa-\frac{1}{2}}}{q_\varepsilon}.$$

The above estimate in the plate $\Omega_\delta$ leads to the Von Kármán limit model ($\kappa = 3$) or the classical linear plate model ($\kappa > 3$). Since we wish at least to recover the linear model in the rod which corresponds to a Green-St Venant's strain tensor in the rod of order $\varepsilon^{\kappa'}$ with $\kappa' > 3$, we are led to assume that

$$\delta^{\kappa-\frac{1}{2}} = q_\varepsilon\varepsilon^{\kappa'}. \tag{5.7}$$

Furthermore, still for the above estimates of the Green-St Venant's strain tensors, the bending in the plate is of order $\delta^{\kappa-2}$ and the stretching in the rod is of order $\varepsilon^{\kappa'-1}$. In this paper, we wish these two quantities to match at the junction and it is essential to have

$$\delta^{\kappa-2} = \varepsilon^{\kappa'-1}. \tag{5.8}$$

[1] For later convenience, we have added the term $\int_{\mathcal{S}_{\delta,\varepsilon}} f_\delta(x)\cdot I_d(x)\mathrm{d}x$ to the usual standard energy, and indeed this does not affect the minimizing problem for $J_\delta$.

As a consequence of the above relations (5.7)–(5.8) we deduce that

$$\delta^3 = q_\varepsilon^2 \varepsilon^2 = \varepsilon^{2\eta+2}, \tag{5.9}$$

which implies that $\eta$ must be chosen such that $\eta > -1$.

From now on we assume that (5.9) holds and to recover a slightly general model in the rod we extend the analysis to $\kappa' \geq 3$.

### 5.2 Assumptions on the forces and energy estimates

Let $v \in \mathbb{D}_{\delta,\varepsilon}$ be a deformation. The estimate (4.5) and those in Lemma 4.1 yield

$$\begin{aligned}
|\mathcal{W}_\alpha(0)|^2 &\leq \frac{C}{\varepsilon\delta}[\mathbf{G}_s(u,\Omega_\delta)]^2 + C\Big[1+\frac{\delta^2}{\varepsilon^2}\Big]\frac{\delta}{\varepsilon^2}[\mathbf{G}_s(u,B_{\varepsilon,\delta})]^2,\\
|\mathcal{W}_3(0)|^2 &\leq \frac{C}{\delta^2}\Big[1+\frac{\varepsilon^2}{\delta}\Big][\mathbf{G}_s(u,\Omega_\delta)]^2 + C\frac{\delta}{\varepsilon^2}[\mathbf{G}_s(u,B_{\varepsilon,\delta})]^2\\
&\quad + \frac{C}{\delta^3}\mathbf{G}_s(u,\Omega_\delta)\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)}\\
&\quad + \frac{C}{\delta^3}[\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)}]^2,\\
\|\mathcal{Q}(0)\|_2^2 &\leq \frac{C}{\varepsilon^2\delta}\Big[1+\frac{\varepsilon}{\delta^2}\Big][\mathbf{G}_s(u,\Omega_\delta)]^2 + C\frac{\delta}{\varepsilon^4}[\mathbf{G}_s(u,B_{\varepsilon,\delta})]^2.
\end{aligned} \tag{5.10}$$

The following lemma gives the estimates of the displacement $u = v - I_d$ in the rod $B_{\varepsilon,\delta}$.

**Lemma 5.1** *For any deformation $v$ in $\mathbb{D}_{\delta,\varepsilon}$, the displacement $u = v - I_d$ satisfies the following Korn's type inequality in the rod $B_{\varepsilon,\delta}$:*

$$\begin{aligned}
\|u_\alpha\|^2_{L^2(B_{\varepsilon,\delta})} &\leq \frac{C}{\varepsilon^2}[\mathbf{G}_s(u,B_{\varepsilon,\delta})]^2 + C\frac{\varepsilon+\delta^2}{\delta^3}[\mathbf{G}_s(u,\Omega_\delta)]^2,\\
\|u_3\|^2_{L^2(B_{\varepsilon,\delta})} &\leq C[\mathbf{G}_s(u,B_{\varepsilon,\delta})]^2 + C\frac{\varepsilon^2}{\delta^2}\Big[1+\frac{\varepsilon}{\delta}\Big][\mathbf{G}_s(u,\Omega_\delta)]^2\\
&\quad + \frac{C\varepsilon^2}{\delta^3}\mathbf{G}_s(u,\Omega_\delta)\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)}\\
&\quad + \frac{C\varepsilon^2}{\delta^3}[\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)}]^2,\\
\|\nabla u\|^2_{L^2(B_{\varepsilon,\delta};\mathbb{R}^9)} &\leq \frac{C}{\varepsilon^2}[\mathbf{G}_s(u,B_{\varepsilon,\delta})]^2 + C\frac{\varepsilon+\delta^2}{\delta^3}[\mathbf{G}_s(u,\Omega_\delta)]^2,\\
\|u-\mathcal{W}\|^2_{L^2(B_{\varepsilon,\delta};\mathbb{R}^3)} &\leq C[\mathbf{G}_s(u,B_{\varepsilon,\delta})]^2 + C\frac{(\varepsilon+\delta^2)\varepsilon^2}{\delta^3}[\mathbf{G}_s(u,\Omega_\delta)]^2.
\end{aligned} \tag{5.11}$$

**Proof** We define the rigid displacement $\mathbf{r}$ by $\mathbf{r}(x) = \mathcal{W}(0) + \mathcal{Q}(0) \wedge x$. Hence, we have

$$\begin{aligned}
&\|\mathbf{r}_\alpha\|_{L^2(B_{\varepsilon,\delta};\mathbb{R}^3)} \leq C\varepsilon(|\mathcal{W}_\alpha(0)| + \|\mathcal{Q}(0)\|_2),\\
&\|\mathbf{r}_3\|_{L^2(B_{\varepsilon,\delta};\mathbb{R}^3)} \leq C\varepsilon|\mathcal{W}_3(0)| + C\varepsilon^2\|\mathcal{Q}(0)\|_2,\\
&\|\nabla\mathbf{r}\|_{L^2(B_{\varepsilon,\delta};\mathbb{R}^9)} \leq C\varepsilon\|\mathcal{Q}(0)\|_2.
\end{aligned} \tag{5.12}$$

Besides, from (3.11) we obtain the following inequalities for the displacement $u - r$:

$$\begin{aligned}
&\|u_\alpha - \mathbf{r}_\alpha\|_{L^2(B_{\varepsilon,\delta})} \le \frac{C}{\varepsilon}\mathbf{G}_s(u, B_{\varepsilon,\delta}),\\
&\|u_3 - \mathbf{r}_3\|_{L^2(B_{\varepsilon,\delta})} \le C\mathbf{G}_s(u, B_{\varepsilon,\delta}),\\
&\|\nabla u - \nabla \mathbf{r}\|_{L^2(B_{\varepsilon,\delta};\mathbb{R}^9)} \le \frac{C}{\varepsilon}\mathbf{G}_s(u, B_{\varepsilon,\delta}),
\end{aligned}$$

which lead to the three estimates in (5.11) using (5.12). Before obtaining the estimate of $u - \mathcal{W}$ we write (see (3.8))

$$u(x) - \mathcal{W}(x_3) = (\mathcal{Q}(x_3) - \mathcal{Q}(0)) \wedge (x_1\mathbf{e}_1 + x_2\mathbf{e}_2) + \overline{u}(x) + \mathcal{Q}(0) \wedge (x_1\mathbf{e}_1 + x_2\mathbf{e}_2).$$

Then, due to the estimates (3.9), (3.11) and (5.10) we finally get the last inequality in (5.11).

The following lemma is one of the key point of this article in order to obtain a priori estimates on minimizing sequences of the total energy.

**Lemma 5.2** *Let $v \in \mathbb{D}_{\delta,\varepsilon}$ be a deformation and $u = v - I_d$. We have*

$$\mathbf{G}_s(u, \Omega_\delta) \le C\|\text{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)} + C_1\frac{\|\text{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)}}{\delta^{\frac{5}{2}}} \tag{5.13}$$

*and the following estimate on $\mathbf{G}_s(u, B_{\varepsilon,\delta})$*

$$\begin{aligned}
\mathbf{G}_s(u, B_{\varepsilon,\delta}) \le\ & C\|\text{dist}(\nabla v, SO(3))\|_{L^2(B_{\varepsilon,\delta})} + C_2\frac{\|\text{dist}(\nabla v, SO(3))\|^2_{L^2(B_{\varepsilon,\delta})}}{\varepsilon^3}\\
&+ C[\delta^2 + \varepsilon^{\frac{3}{2}}]\frac{\|\text{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)}}{\varepsilon\delta^3}.
\end{aligned} \tag{5.14}$$

*The constant $C$ does not depend on $\delta$ and $\varepsilon$.*

The proof is postponed in Section 8.

As an immediate consequence of the Lemmas 5.1–5.2, we get the full estimates of the displacement $u = v - I_d$ in the rod.

**Corollary 5.1** *For any deformation $v$ in $\mathbb{D}_{\delta,\varepsilon}$, the displacement $u = v - I_d$ satisfies the following nonlinear Korn's type inequality in the rod $B_{\varepsilon,\delta}$*

$$\begin{aligned}
\|u_\alpha\|_{L^2(B_{\varepsilon,\delta})} \le\ & C\frac{\|\text{dist}(\nabla v, SO(3))\|_{L^2(B_{\varepsilon,\delta})}}{\varepsilon} + 2C_2\frac{\|\text{dist}(\nabla v, SO(3))\|^2_{L^2(B_{\varepsilon,\delta})}}{\varepsilon^4}\\
&+ C\Big[(\delta + \varepsilon^{\frac{1}{2}})\frac{\|\text{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)}}{\delta^{\frac{3}{2}}}\\
&+ \Big(\frac{\varepsilon^{\frac{1}{2}}}{\delta^4} + \frac{\delta^2 + \varepsilon^{\frac{3}{2}}}{\varepsilon^2\delta^3}\Big)\|\text{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)}\Big],\\
\|u_3\|_{L^2(B_{\varepsilon,\delta})} \le\ & C\|\text{dist}(\nabla v, SO(3))\|_{L^2(B_{\varepsilon,\delta})} + 2C_2\frac{\|\text{dist}(\nabla v, SO(3))\|^2_{L^2(B_{\varepsilon,\delta})}}{\varepsilon^3}\\
&+ C\varepsilon\Big[\frac{\|\text{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)}}{\delta^{\frac{3}{2}}} + \frac{\delta^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}}}{\delta^4}\|\text{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)}\\
&+ \frac{\|\text{dist}(\nabla v, SO(3))\|^{\frac{3}{2}}_{L^2(\Omega_\delta)}}{\delta^{\frac{11}{4}}}\Big],
\end{aligned}$$

$$\begin{aligned}\|\nabla u\|_{L^2(B_{\varepsilon,\delta};\mathbb{R}^9)} \le{}& C\frac{\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(B_{\varepsilon,\delta})}}{\varepsilon} + 2C_2\frac{\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(B_{\varepsilon,\delta})}}{\varepsilon^4}\\ &+ C\Big[(\delta+\varepsilon^{\frac12})\frac{\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)}}{\delta^{\frac32}}\\ &+\Big(\frac{\varepsilon^{\frac12}}{\delta^4}+\frac{\delta^2+\varepsilon^{\frac32}}{\varepsilon^2\delta^3}\Big)\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)}\Big],\\ \|u-\mathcal{W}\|_{L^2(B_{\varepsilon,\delta};\mathbb{R}^3)} \le{}& C\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(B_{\varepsilon,\delta})} + 2C_2\frac{\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(B_{\varepsilon,\delta})}}{\varepsilon^3}\\ &+ C\Big[\varepsilon(\delta+\varepsilon^{\frac12})\frac{\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)}}{\delta^{\frac32}}\\ &+\Big(\frac{(\varepsilon^{\frac12}+\delta)\varepsilon}{\delta^4}+\frac{\delta^2+\varepsilon^{\frac32}}{\varepsilon\delta^3}\Big)\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)}\Big].\end{aligned}$$

**First assumptions on the forces** To introduce the scaling on $f_\delta$, let us consider $f_r$, $g_1$, $g_2$ in $L^2(0,L;\mathbb{R}^3)$ and $f_p \in L^2(\omega;\mathbb{R}^3)$, and assume that the force $f_\delta$ is given by

$$\begin{aligned} f_\delta(x) &= q_\varepsilon^2\varepsilon^{\kappa'}\Big[f_{r,1}(x_3)\mathbf{e}_1 + f_{r,2}(x_3)\mathbf{e}_2 + \frac{1}{\varepsilon}f_{r,3}(x_3)\mathbf{e}_3 + \frac{x_1}{\varepsilon^2}g_1(x_3)+\frac{x_2}{\varepsilon^2}g_2(x_3)\Big],\\ &\qquad\qquad x\in B_{\varepsilon,\delta},\ x_3>\delta,\\ f_{\delta,\alpha}(x) &= \delta^{\kappa-1}f_{p,\alpha}(x_1,x_2),\quad f_{\delta,3}(x)=\delta^{\kappa}f_{p,3}(x_1,x_2),\quad x\in\Omega_\delta. \end{aligned} \tag{5.15}$$

We set

$$N(f_p) = \|f_p\|_{L^2(\omega;\mathbb{R}^3)},\quad N(f_r) = \|f_r\|_{L^2(0,L;\mathbb{R}^3)} + \sum_{\alpha=1}^{2}\|g_\alpha\|_{L^2(0,L;\mathbb{R}^3)}. \tag{5.16}$$

We recall that $\overline{\mu}$ is defined in (5.4).

**Lemma 5.3** *Let $v \in \mathbb{D}_{\delta,\varepsilon}$ be such that $J_\delta(v) \le 0$ and $u = v - I_d$. Under the assumption* (5.15) *on the applied forces, we have:*

(1) *If $\kappa > 3$ and $\kappa' > 3$, then*

$$\begin{aligned}&\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)} + q_\varepsilon\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(B_{\varepsilon,\delta})}\\ &\le C\delta^{\kappa-\frac12}(N(f_p)+N(f_r)+[N(f_r)]^2).\end{aligned} \tag{5.17}$$

(2) *If $\kappa = 3$ and $\kappa' > 3$, then there exists a constant $C^*$ which does not depend on $\delta$ and $\varepsilon$ such that, if the forces applied to the plate $\Omega_\delta$ satisfy*

$$N(f_p) < C^*\overline{\mu}, \tag{5.18}$$

*then* (5.17) *still holds true.*

(3) *If $\kappa > 3$ and $\kappa' = 3$, then there exists a constant $C^{**}$ which does not depend on $\delta$ and $\varepsilon$ such that, if the forces applied to the rod $B_{\varepsilon,\delta}$ satisfy*

$$N(f_r) < C^{**}\overline{\mu}, \tag{5.19}$$

*then* (5.17) *still holds.*

(4) *If* $\kappa = 3$ *and* $\kappa' = 3$*, then if the applied forces satisfy* (5.18)–(5.19) *then* (5.17) *still holds.*

*The constants* $C$, $C^*$ *and* $C^{**}$ *depend only on* $\omega$ *and* $L$.

Recall that we want geometric energy in the plate $\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)}$ of an order less than $\delta^{\frac{5}{2}}$ in order to obtain a limit Von Kármán plate model. Lemma 5.3 prompts us to adopt the conditions (5.18) (if $\kappa = 3$) and (5.19) (if $\kappa' = 3$). Let us notice that in the case $\kappa = 3$ under the only assumption (5.15) on the forces (i.e. without assumption (5.18)) the geometric energy is generally of order $\delta^{\frac{3}{2}}$ which corresponds to a limit model allowing large deformations (see [10]).

**Second assumptions on the forces** From now on, in the whole paper, we assume that
(1) If $\kappa = 3$ then

$$N(f_p) < C^* \overline{\mu}. \tag{5.20}$$

(2) If $\kappa' = 3$ then

$$N(f_r) < C^{**} \overline{\mu}. \tag{5.21}$$

**Proof of Lemma 5.3** Notice that $J_\delta(I_d) = 0$. So, in order to minimize $J_\delta$ we only need to consider deformations $v$ of $\mathbb{D}_{\delta,\varepsilon}$ such that $J_\delta(v) \le 0$. From (3.6) and the assumption (5.15) on the body forces, we obtain that for any $v \in \mathbb{D}_{\delta,\varepsilon}$ and for $u = v - I_d$,

$$\begin{aligned}
&\Big| \int_{\mathcal{S}_{\delta,\varepsilon}} f_\delta(x) \cdot u(x) \mathrm{d}x \Big| \\
&\le C_0 \delta^{\kappa - \frac{1}{2}} N(f_p) \mathbf{G}_s(u, \Omega_\delta) + \sqrt{\pi} q_\varepsilon^2 \varepsilon^{\kappa'+1} \Big( \frac{1}{\varepsilon} \|f_{r,3}\|_{L^2(0,L)} \|u_3\|_{L^2(B_{\varepsilon,\delta})} \\
&\quad + \sum_{\alpha=1}^{2} (\|f_{r,\alpha}\|_{L^2(0,L)} \|u_\alpha\|_{L^2(B_{\varepsilon,\delta})} + \|g_\alpha\|_{L^2(0,L;\mathbb{R}^3)} \|u - \mathcal{W}\|_{L^2(B_{\varepsilon,\delta};\mathbb{R}^3)}) \Big).
\end{aligned} \tag{5.22}$$

Now we use (5.13), Corollary 5.1 and the relations (5.7)–(5.9) to obtain

$$\begin{aligned}
\Big| \int_{\mathcal{S}_{\delta,\varepsilon}} f_\delta(x) \cdot u(x) \mathrm{d}x \Big| &\le C_1 C_0 \delta^{\kappa - 3} N(f_p) \|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)} \\
&\quad + C[\delta^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}}] \delta^{\kappa - 3} N(f_r) \|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)} \\
&\quad + C[\delta^2 + \varepsilon^{\frac{3}{2}}] \varepsilon^{\kappa' - 3} N(f_r) \|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)} \\
&\quad + C \delta^{\kappa - \frac{7}{4}} N(f_r) \|\mathrm{dist}(\nabla v, SO(3))\|^{\frac{3}{2}}_{L^2(\Omega_\delta)} \\
&\quad + 2 C_2 \sqrt{\pi} q_\varepsilon^2 \varepsilon^{\kappa' - 3} N(f_r) \|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(B_{\varepsilon,\delta})} \\
&\quad + C \delta^{\kappa - \frac{1}{2}} \{N(f_p) + N(f_r)\} \|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)} \\
&\quad + C q_\varepsilon^2 \varepsilon^{\kappa'} N(f_r) \|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(B_{\varepsilon,\delta})}.
\end{aligned} \tag{5.23}$$

From (5.1), (5.3) and (5.5) we have

$$\begin{aligned}
&\overline{\mu} (\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)} + q_\varepsilon^2 \|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(B_{\varepsilon,\delta})}) \\
&\le \int_{\mathcal{S}_{\delta,\varepsilon}} \widehat{W}_\varepsilon(x, \nabla v)(x) \mathrm{d}x \le \int_{\mathcal{S}_{\delta,\varepsilon}} f_\delta(x) \cdot u(x) \mathrm{d}x.
\end{aligned} \tag{5.24}$$

Then using (5.23) and observing that for any $X \geq 0$, we have

$$C\delta^{\kappa-\frac{7}{4}} N(f_r) X^{\frac{3}{2}} \leq \frac{\overline{\mu}}{2}\delta^{\kappa-3}X^2 + \frac{2C^2[N(f_r)]^2}{\overline{\mu}}\delta^{\kappa-\frac{1}{2}}X,$$

and we get

$$\begin{aligned}
&\Big[\frac{\overline{\mu}}{2} - C_1C_0\delta^{\kappa-3}N(f_p) - C\{[\delta^2+\varepsilon^{\frac{3}{2}}]\varepsilon^{\kappa'-3} \\
&\quad + [\delta^{\frac{1}{2}}+\varepsilon^{\frac{1}{2}}]\delta^{\kappa-3}\}N(f_r)\Big]\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)} \\
&\quad + [\overline{\mu} - 2C_2\sqrt{\pi}\varepsilon^{\kappa'-3}N(f_r)]q_\varepsilon^2\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(B_{\varepsilon,\delta})} \\
&\leq C\delta^{\kappa-\frac{1}{2}}\{N(f_p)+N(f_r)+[N(f_r)]^2\}\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)} \\
&\quad + Cq_\varepsilon^2\varepsilon^{\kappa'}N(f_r)\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(B_{\varepsilon,\delta})} \\
&\leq C\delta^{\kappa-\frac{1}{2}}\{N(f_p)+N(f_r)+[N(f_r)]^2\}(\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)} \\
&\quad + q_\varepsilon\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(B_{\varepsilon,\delta})}).
\end{aligned} \tag{5.25}$$

Now, recall that $\kappa \geq 3$ and $\kappa' \geq 3$. So, first $[\delta^2+\varepsilon^{\frac{3}{2}}]\varepsilon^{\kappa'-3} + [\delta^{\frac{1}{2}}+\varepsilon^{\frac{1}{2}}]\delta^{\kappa-3} \to 0$. Secondly, setting $C^* = \frac{1}{2C_1C_0}$ and $C^{**} = \frac{1}{2C_2\sqrt{\pi}}$ then (5.17) holds in any case of the lemma.

Recalling that $\delta^{\kappa-\frac{1}{2}} = q_\varepsilon\varepsilon^{\kappa'}$, we first deduce from Lemma 5.3

$$\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)} \leq C\delta^{\kappa-\frac{1}{2}}, \quad \|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(B_{\varepsilon,\delta})} \leq C\varepsilon^{\kappa'}. \tag{5.26}$$

Then applying (5.13) of Lemma 5.2, we obtain

$$\mathbf{G}_s(u, \Omega_\delta) \leq C\delta^{\kappa-\frac{1}{2}}, \tag{5.27}$$

while (5.14) gives

$$\mathbf{G}_s(u, B_{\varepsilon,\delta}) \leq C\varepsilon^{\kappa'} + C[\delta^2+\varepsilon^{\frac{3}{2}}]\frac{\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)}}{\varepsilon\delta^3} \leq C\varepsilon^{\kappa'} + C[\delta^2+\varepsilon^{\frac{3}{2}}]\frac{\delta^{2\kappa-4}}{\varepsilon}$$

and (5.8) yields

$$\mathbf{G}_s(u, B_{\varepsilon,\delta}) \leq C\varepsilon^{\kappa'}. \tag{5.28}$$

Finally for any deformation $v \in \mathbb{D}_{\delta,\varepsilon}$ and $u = v - I_d$ such that $J(v) \leq 0$, we have

$$\int_{\mathcal{S}_{\delta,\varepsilon}} f_\delta \cdot u\,\mathrm{d}x \leq C\delta^{2\kappa-1}. \tag{5.29}$$

Moreover, the above inequality together with (5.24) shows that

$$\int_{\mathcal{S}_{\delta,\varepsilon}} \widehat{W}_\varepsilon(x, \nabla v)(x)\,\mathrm{d}x \leq C\delta^{2\kappa-1}, \tag{5.30}$$

which in turn leads to

$$\|\nabla v^{\mathrm{T}}\nabla v - \mathbf{I}_3\|_{L^2(\Omega_\delta;\mathbb{R}^{3\times 3})} \leq C\delta^{\kappa-\frac{1}{2}}, \quad \|\nabla v^{\mathrm{T}}\nabla v - \mathbf{I}_3\|_{L^2(B_{\varepsilon,\delta};\mathbb{R}^{3\times 3})} \leq C\varepsilon^{\kappa'}. \tag{5.31}$$

From (5.29) we also obtain

$$c\delta^{2\kappa-1} \le J_\delta(v) \le 0. \tag{5.32}$$

We set

$$m_\delta = \inf_{v\in \mathbb{D}_{\delta,\varepsilon}} J_\delta(v). \tag{5.33}$$

In general, a minimizer of $J_\delta$ does not exist on $\mathbb{D}_{\delta,\varepsilon}$. As a consequence of (5.32), we have

$$c \le \frac{m_\delta}{\delta^{2\kappa-1}} \le 0.$$

## 6 Limits of the Green-St Venant's Strain Tensors

In this subsection and the following one, we consider a sequence of deformations $(v_\delta)$ belonging to $\mathbb{D}_{\delta,\varepsilon}$ and satisfying $(u_\delta = v_\delta - I_d)$

$$\mathbf{G}_s(u_\delta, \Omega_\delta) \le C\delta^{\kappa-\frac{1}{2}}, \quad \mathbf{G}_s(u_\delta, B_{\varepsilon,\delta}) \le C\varepsilon^{\kappa'}.$$

For any open subset $\mathcal{O} \subset \mathbb{R}^2$ and for any field $\psi \in H^1(\mathcal{O};\mathbb{R}^3)$, we denote

$$\gamma_{\alpha\beta}(\psi) = \frac{1}{2}\Big(\frac{\partial \psi_\alpha}{\partial x_\beta} + \frac{\partial \psi_\beta}{\partial x_\alpha}\Big), \quad (\alpha,\beta) \in \{1,2\}. \tag{6.1}$$

### 6.1 The rescaling operators

Before rescaling the domains, we introduce the reference domain $\Omega$ for the plate and the one $B$ for the rod

$$\Omega = \omega \times ]-1,1[, \quad B = D\times ]0,L[= D(O,1)\times ]0,L[.$$

As usual when dealing with thin structures, we rescale $\Omega_\delta$ and $B_{\varepsilon,\delta}$ using (for the plate) the operator

$$\Pi_\delta(w)(x_1,x_2,X_3) = w(x_1,x_2,\delta X_3) \quad \text{for any } (x_1,x_2,X_3) \in \Omega$$

defined for e.g. $w \in L^2(\Omega_\delta)$ for which $\Pi_\delta(w) \in L^2(\Omega)$ and using (for the rod) the operator

$$P_\varepsilon(w)(X_1,X_2,x_3) = w(\varepsilon X_1, \varepsilon X_2, x_3) \quad \text{for any } (X_1,X_2,x_3) \in B$$

defined for e.g. $w \in L^2(B_{\varepsilon,\delta})$ for which $P_\varepsilon(w) \in L^2(B)$.

### 6.2 Asymptotic behavior in the plate

Following Section 2 we decompose the restriction of $u_\delta = v_\delta - I_d$ to the plate. Theorem 3.1 gives $\mathcal{U}_\delta$, $\mathcal{R}_\delta$ and $\overline{u}_\delta$, and then estimates in (3.5) lead to the following convergences for a subsequence still indexed by $\delta$

$$\begin{aligned}
&\frac{1}{\delta^{\kappa-2}}\mathcal{U}_{\delta,3} \to \mathcal{U}_3 \quad \text{strongly in } H^1(\omega),\\
&\frac{1}{\delta^{\kappa-1}}\mathcal{U}_{\delta,\alpha} \rightharpoonup \mathcal{U}_\alpha \quad \text{weakly in } H^1(\omega),\\
&\frac{1}{\delta^{\kappa-2}}\mathcal{R}_\delta \rightharpoonup \mathcal{R} \quad \text{weakly in } H^1(\omega;\mathbb{R}^3),\\
&\frac{1}{\delta^{\kappa}}\Pi_\delta(\overline{u}_\delta) \rightharpoonup \overline{u} \quad \text{weakly in } L^2(\omega;H^1(-1,1;\mathbb{R}^3)),\\
&\frac{1}{\delta^{\kappa-1}}\Big(\frac{\partial \mathcal{U}_\delta}{\partial x_\alpha} - \mathcal{R}_\delta \wedge \mathbf{e}_\alpha\Big) \rightharpoonup \mathcal{Z}_\alpha \quad \text{weakly in } L^2(\omega;\mathbb{R}^3).
\end{aligned} \tag{6.2}$$

The boundary conditions in (3.4) give here

$$\mathcal{U}_3 = 0, \quad \mathcal{U}_\alpha = 0, \quad \mathcal{R} = 0 \quad \text{on } \gamma_0, \tag{6.3}$$

while (6.2) shows that $\mathcal{U}_3 \in H^2(\omega)$ with

$$\frac{\partial \mathcal{U}_3}{\partial x_1} = -\mathcal{R}_2, \quad \frac{\partial \mathcal{U}_3}{\partial x_2} = \mathcal{R}_1. \tag{6.4}$$

We also have

$$\begin{aligned}
&\frac{1}{\delta^{\kappa-1}}\Pi_\delta(u_{\delta,\alpha}) \rightharpoonup \mathcal{U}_\alpha - X_3\frac{\partial \mathcal{U}_3}{\partial x_\alpha} \quad \text{weakly in } H^1(\Omega),\\
&\frac{1}{\delta^{\kappa-2}}\Pi_\delta(u_{\delta,3}) \to \mathcal{U}_3 \quad \text{strongly in } H^1(\Omega),
\end{aligned} \tag{6.5}$$

which shows that the rescaled limit displacement is a Kirchhoff-Love displacement.

In [8] the limit of the Green-St Venant's strain tensor of the sequence $v_\delta$ is also derived. Let us set

$$\overline{u}_p = \overline{u} + \frac{X_3}{2}(\mathcal{Z}_1\cdot\mathbf{e}_3)\mathbf{e}_1 + \frac{X_3}{2}(\mathcal{Z}_2\cdot\mathbf{e}_3)\mathbf{e}_2 \tag{6.6}$$

and

$$\mathcal{Z}_{\alpha\beta} = \begin{cases} \gamma_{\alpha\beta}(\mathcal{U}) + \dfrac{1}{2}\dfrac{\partial \mathcal{U}_3}{\partial x_\alpha}\dfrac{\partial \mathcal{U}_3}{\partial x_\beta}, & \text{if } \kappa = 3,\\ \gamma_{\alpha\beta}(\mathcal{U}), & \text{if } \kappa > 3. \end{cases} \tag{6.7}$$

Then we have

$$\frac{1}{2\delta^{\kappa-1}}\Pi_\delta((\nabla v_\delta)^{\mathrm{T}}\nabla v_\delta - \mathbf{I}_3) \rightharpoonup \mathbf{E}_p(\mathcal{U},\overline{u}_p) \quad \text{weakly in } L^1(\Omega;\mathbb{R}^9),$$

where the symmetric matrix $\mathbf{E}_p(\mathcal{U},\overline{u}_p)$ is defined by

$$\mathbf{E}_p(\mathcal{U},\overline{u}_p) = \begin{pmatrix} -X_3\dfrac{\partial^2 \mathcal{U}_3}{\partial x_1^2} + \mathcal{Z}_{11} & -X_3\dfrac{\partial^2 \mathcal{U}_3}{\partial x_1\partial x_2} + \mathcal{Z}_{12} & \dfrac{1}{2}\dfrac{\partial \overline{u}_{p,1}}{\partial X_3}\\ * & -X_3\dfrac{\partial^2 \mathcal{U}_3}{\partial x_2^2} + \mathcal{Z}_{22} & \dfrac{1}{2}\dfrac{\partial \overline{u}_{p,2}}{\partial X_3}\\ * & * & \dfrac{\partial \overline{u}_{p,3}}{\partial X_3} \end{pmatrix}. \tag{6.8}$$

### 6.3 Asymptotic behavior in the rod

Now, we decompose the restriction of $u_\delta = v_\delta - I_d$ to the rod. Theorem 3.2 gives $\mathcal{W}_\delta$, $\mathcal{Q}_\delta$ and $\overline{w}_\delta$, and then the estimates in (3.11) and (5.10) allow to claim that

$$\begin{aligned}
&\|\overline{w}_\delta\|_{L^2(B_{\varepsilon,\delta};\mathbb{R}^3)} \le C\varepsilon^{\kappa'+1}, \quad \|\nabla\overline{w}_\delta\|_{L^2(B_{\varepsilon,\delta};\mathbb{R}^3)} \le C\varepsilon^{\kappa'},\\
&\|\mathcal{Q}_\delta - \mathcal{Q}_\delta(0)\|_{H^1(-\delta,L;\mathbb{R}^3)} \le C\varepsilon^{\kappa'-2}, \quad \Big\|\frac{\mathrm{d}\mathcal{W}_\delta}{\mathrm{d}x_3} - \mathcal{Q}_\delta\wedge\mathbf{e}_3\Big\|_{L^2(-\delta,L;\mathbb{R}^3)} \le C\varepsilon^{\kappa'-1},\\
&\|\mathcal{W}_{\delta,3} - \mathcal{W}_{\delta,3}(0)\|_{H^1(-\delta,L)} \le C\varepsilon^{\kappa'-1},\\
&\|\mathcal{W}_\delta - \mathcal{W}_\delta(0) - \mathcal{Q}_\delta(0)x_3\wedge\mathbf{e}_3\|_{H^1(-\delta,L;\mathbb{R}^3)} \le C\varepsilon^{\kappa'-2}.
\end{aligned} \tag{6.9}$$

Moreover, from (4.8) and (5.10), we get

$$\begin{aligned}
&|\mathcal{W}_{\delta,\alpha}(0)| \le C\delta^{\frac12}(\delta+\varepsilon^{\frac12})\varepsilon^{\kappa'-2},\\
&|\mathcal{W}_{\delta,3}(0) - \widetilde{\mathcal{U}}_{\delta,3}(0,0)| \le C(\delta^{\frac12}+\varepsilon)\varepsilon^{\kappa'-1},\\
&\|\mathcal{Q}_\delta(0)\|_2 \le C(\delta^{\frac12}+\varepsilon^{\frac12})\varepsilon^{\kappa'-2}.
\end{aligned} \tag{6.10}$$

Due to the above estimates we are in a position to prove the following lemma.

**Lemma 6.1** *There exists a subsequence still indexed by $\delta$ such that*

$$\begin{aligned}
&\frac{1}{\varepsilon^{\kappa'-2}}\mathcal{W}_{\delta,\alpha} \to \mathcal{W}_\alpha \quad \textit{strongly in } H^1(0,L),\\
&\frac{1}{\varepsilon^{\kappa'-1}}\mathcal{W}_{\delta,3} \rightharpoonup \mathcal{W}_3 \quad \textit{weakly in } H^1(0,L),\\
&\frac{1}{\varepsilon^{\kappa'-2}}\mathcal{Q}_\delta \rightharpoonup \mathcal{Q} \quad \textit{weakly in } H^1(0,L;\mathbb{R}^3),\\
&\frac{1}{\varepsilon^{\kappa'}}P_\varepsilon(\overline{w_\delta}) \rightharpoonup \overline{w} \quad \textit{weakly in } L^2(0,L;H^1(D;\mathbb{R}^3)),\\
&\frac{1}{\varepsilon^{\kappa'-1}}\Big(\frac{\partial\mathcal{W}_{\delta,1}}{\partial x_3} - \mathcal{Q}_{\delta,2}\Big) \rightharpoonup \mathcal{Z}_1 \quad \textit{weakly in } L^2(B),\\
&\frac{1}{\varepsilon^{\kappa'-1}}\Big(\frac{\partial\mathcal{W}_{\delta,2}}{\partial x_3} + \mathcal{Q}_{\delta,1}\Big) \rightharpoonup \mathcal{Z}_2 \quad \textit{weakly in } L^2(B).
\end{aligned} \tag{6.11}$$

*We also have $\mathcal{W}_\alpha \in H^2(0,L)$ and*

$$\frac{\mathrm{d}\mathcal{W}_1}{\mathrm{d}x_3} = \mathcal{Q}_2, \quad \frac{\mathrm{d}\mathcal{W}_2}{\mathrm{d}x_3} = -\mathcal{Q}_1. \tag{6.12}$$

*The junction conditions*

$$\mathcal{W}_\alpha(0) = 0, \quad \mathcal{Q}(0) = 0, \quad \mathcal{W}_3(0) = \mathcal{U}_3(0,0) \tag{6.13}$$

*hold. Setting*

$$\overline{w}_r = \overline{w} + [X_1\mathcal{Z}_1 + X_2\mathcal{Z}_2]\mathbf{e}_3, \tag{6.14}$$

*we have*

$$\frac{1}{2\varepsilon^{\kappa'-1}}P_\varepsilon((\nabla v_\delta)^{\mathrm{T}}\nabla v_\delta - \mathbf{I}_3) \rightharpoonup \mathbf{E}_r(\mathcal{W},\mathcal{Q}_3,\overline{u}_r) \quad \textit{weakly in } L^1(B;\mathbb{R}^{3\times3}), \tag{6.15}$$

*where the symmetric matrices* $\mathbf{E}_r(\mathcal{W},\mathcal{Q}_3,\overline{u}_r)$ *and* $\mathbf{F}(\mathcal{Q})$ *are defined by*

$$\mathbf{E}_r(\mathcal{W},\mathcal{Q}_3,\overline{u}_r)=\begin{pmatrix}\gamma_{11}(\overline{w}_r) & \gamma_{12}(\overline{w}_r) & -\frac{1}{2}X_2\frac{\mathrm{d}\mathcal{Q}_3}{\mathrm{d}x_3}+\frac{1}{2}\frac{\partial\overline{w}_{r,3}}{\partial X_1}\\ * & \gamma_{22}(\overline{w}_r) & \frac{1}{2}X_1\frac{\mathrm{d}\mathcal{Q}_3}{\mathrm{d}x_3}+\frac{1}{2}\frac{\partial\overline{w}_{r,3}}{\partial X_2}\\ * & * & -X_1\frac{d^2\mathcal{W}_1}{\mathrm{d}x_3^2}-X_2\frac{d^2\mathcal{W}_2}{\mathrm{d}x_3^2}+\frac{\mathrm{d}\mathcal{W}_3}{\mathrm{d}x_3}\end{pmatrix}+\mathbf{F}(\mathcal{Q})$$

*with*

$$\mathbf{F}(\mathcal{Q})=\begin{cases}\frac{1}{2}(\|\mathcal{Q}\|_2^2\mathbf{I}_3-\mathcal{Q}.\mathcal{Q}^{\mathrm{T}}), & \text{if } \kappa'=3,\\ 0, & \text{if } \kappa'>3.\end{cases}\tag{6.16}$$

**Proof** First, the estimates (6.9) and (6.10) imply that the sequences $\frac{1}{\varepsilon^{\kappa'-2}}\mathcal{W}_{\delta,\alpha}$, $\frac{1}{\varepsilon^{\kappa'-1}}\mathcal{W}_{\delta,3}$, $\frac{1}{\varepsilon^{\kappa'-2}}\mathcal{Q}_\delta$ are bounded in $H^1(0,L;\mathbb{R}^k)$ for $k=1$ or $k=3$. Taking into account also (6.9) and upon extracting a subsequence it follows that the convergences in (6.11) hold together with (6.12). The first strong convergence in (6.11) is in particular a consequence of (6.9). The junction conditions on $\mathcal{Q}$ and $\mathcal{W}_\alpha$ are immediate consequences of (6.10) and the convergences (6.11).

In order to obtain the junction condition between the bending in the plate and the stretching in the rod, note first that the sequence $\frac{1}{\delta^{\kappa-2}}\widetilde{\mathcal{U}}_{\delta,3}$ converges strongly in $H^1(\omega)$ to $\mathcal{U}_3$ because of (4.3) and the first convergence in (6.2). Besides, this sequence is uniformly bounded in $H^2(D(O,\rho_0))$, and hence it converges strongly to the same limit $\mathcal{U}_3$ in $C^0(D(O,\rho_0))$. Moreover, the weak convergence of the sequence $\frac{1}{\varepsilon^{\kappa'-1}}\mathcal{W}_{\delta,3}$ in $H^1(0,L)$ implies the convergence of $\frac{1}{\varepsilon^{\kappa'-1}}\mathcal{W}_{\delta,3}(0)$ to $\mathcal{W}_3(0)$. Using the third estimate in (6.10) gives the last condition in (6.13).

Once the convergences in (6.11) are established, the limit of the rescaled Green-St Venant strain tensor of the sequence $v_\delta$ is analyzed in [7] and it gives (6.16).

The above lemma and the decomposition (3.8) lead to

$$\begin{aligned}&\frac{1}{\varepsilon^{\kappa'-2}}P_\varepsilon(u_{\delta,\alpha})\to\mathcal{W}_\alpha\quad\text{strongly in } H^1(B),\\ &\frac{1}{\varepsilon^{\kappa'-1}}P_\varepsilon(u_{\delta,1}-\mathcal{W}_{\delta,1})\rightharpoonup -X_2\mathcal{Q}_3\quad\text{weakly in } H^1(B),\\ &\frac{1}{\varepsilon^{\kappa'-1}}P_\varepsilon(u_{\delta,2}-\mathcal{W}_{\delta,2})\rightharpoonup X_1\mathcal{Q}_3\quad\text{weakly in } H^1(B),\\ &\frac{1}{\varepsilon^{\kappa'-1}}P_\varepsilon(u_{\delta,3})\rightharpoonup\mathcal{W}_3-X_1\frac{\mathrm{d}\mathcal{W}_1}{\mathrm{d}x_3}-X_2\frac{\mathrm{d}\mathcal{W}_2}{\mathrm{d}x_3}\quad\text{weakly in } H^1(B),\end{aligned}\tag{6.17}$$

which show that the limit rescaled displacement is a Bernoulli-Navier displacement.

## 7 Asymptotic Behavior of the Sequence $\frac{m_\delta}{\delta^{2\kappa-1}}$

The goal of this section is to establish Theorem 7.1. Let us first introduce a few notations.

We set

$$\begin{aligned}\mathbb{D}_0 = \Big\{&(\mathcal{U}, \mathcal{W}, \mathcal{Q}_3) \in H^1(\omega;\mathbb{R}^3) \times H^1(0,L;\mathbb{R}^3) \times H^1(0,L) \mid \\ &\mathcal{U}_3 \in H^2(\omega),\ \mathcal{W}_\alpha \in H^2(0,L),\ \mathcal{U} = 0,\ \frac{\partial \mathcal{U}_3}{\partial x_\alpha} = 0 \text{ on } \gamma_0, \\ &\mathcal{W}_3(0) = \mathcal{U}_3(0,0),\ \mathcal{W}_\alpha(0) = \frac{\mathrm{d}\mathcal{W}_\alpha}{\mathrm{d}x_3}(0) = \mathcal{Q}_3(0) = 0\Big\}.\end{aligned} \tag{7.1}$$

We introduce below the "limit" rescaled elastic energies for the plate and the rod[2]

$$\begin{aligned}\mathcal{J}_p(\mathcal{U}) &= \frac{E_p}{3(1-\nu_p^2)}\int_\omega \Big[(1-\nu_p)\sum_{\alpha,\beta=1}^{2}\Big|\frac{\partial^2\mathcal{U}_3}{\partial x_\alpha \partial x_\beta}\Big|^2 + \nu_p(\Delta\mathcal{U}_3)^2\Big] \\ &\quad + \frac{E_p}{(1-\nu_p^2)}\int_\omega \Big[(1-\nu_p)\sum_{\alpha,\beta=1}^{2}|\mathcal{Z}_{\alpha\beta}|^2 + \nu_p(\mathcal{Z}_{11}+\mathcal{Z}_{22})^2\Big], \\ \mathcal{J}_r(\mathcal{W},\mathcal{Q}_3) &= \frac{E_r\pi}{8}\int_0^L \Big[\Big|\frac{d^2\mathcal{W}_1}{\mathrm{d}x_3^2}\Big|^2 + \Big|\frac{d^2\mathcal{W}_2}{\mathrm{d}x_3^2}\Big|^2\Big] + \frac{E_r\pi}{2}\Big|\frac{\mathrm{d}\mathcal{W}_3}{\mathrm{d}x_3} + \mathbf{F}_{33}\Big|^2 \\ &\quad + \frac{\mu_r\pi}{8}\int_0^L \Big|\frac{\mathrm{d}\mathcal{Q}_3}{\mathrm{d}x_3}\Big|^2,\end{aligned} \tag{7.2}$$

where the $\mathcal{Z}_{\alpha\beta}$'s are given by

$$\mathcal{Z}_{\alpha\beta} = \begin{cases}\gamma_{\alpha\beta}(\mathcal{U}) + \dfrac{1}{2}\dfrac{\partial\mathcal{U}_3}{\partial x_\alpha}\dfrac{\partial\mathcal{U}_3}{\partial x_\beta}, & \text{if } \kappa = 3, \\ \gamma_{\alpha\beta}(\mathcal{U}), & \text{if } \kappa > 3,\end{cases}$$

and $\mathbf{F}_{33}$ is given by

$$\mathbf{F}_{33} = \begin{cases}\dfrac{1}{2}\Big(\Big|\dfrac{\mathrm{d}\mathcal{W}_1}{\mathrm{d}x_3}\Big|^2 + \Big|\dfrac{\mathrm{d}\mathcal{W}_2}{\mathrm{d}x_3}\Big|^2\Big), & \text{if } \kappa' = 3, \\ 0, & \text{if } \kappa' > 3.\end{cases} \tag{7.3}$$

The total energy of the plate-rod structure is given by the functional $\mathcal{J}$ defined over $\mathbb{D}_0$

$$\mathcal{J}(\mathcal{U},\mathcal{W},\mathcal{Q}_3) = \mathcal{J}_p(\mathcal{U}) + \mathcal{J}_r(\mathcal{W},\mathcal{Q}_3) - \mathcal{L}(\mathcal{U},\mathcal{W},\mathcal{Q}_3) \tag{7.4}$$

with

$$\mathcal{L}(\mathcal{U},\mathcal{W},\mathcal{Q}_3) = 2\int_\omega f_p\cdot\mathcal{U}\mathrm{d}x_3 + \pi\int_0^L f_r\cdot\mathcal{W}\mathrm{d}x_3 + \frac{\pi}{2}\int_0^L g_\alpha\cdot(\mathcal{Q}\wedge\mathbf{e}_\alpha)\mathrm{d}x_3, \tag{7.5}$$

where

$$\mathcal{Q} = -\frac{\mathrm{d}\mathcal{W}_2}{\mathrm{d}x_3}\mathbf{e}_1 + \frac{\mathrm{d}\mathcal{W}_1}{\mathrm{d}x_3}\mathbf{e}_2 + \mathcal{Q}_3\mathbf{e}_3. \tag{7.6}$$

It is worth noting that the functional $\mathcal{J}_p(\mathcal{U})$ corresponds to the elastic energy of a Von Kármán plate model for $\kappa = 3$ (see [17]) and to the classical linear plate model for $\kappa > 3$. Similarly, the functional $\mathcal{J}_r(\mathcal{W},\mathcal{Q}_3)$ corresponds to a nonlinear rod model derived in [7] for $\kappa' = 3$ and to

[2] $E_p$, $\nu_p$ are the Young modulus and the Poisson's ratio of the plate, while $E_r$ is the Young modulus of the rod.

the classical linear rod model for $\kappa' > 3$. Let us also notice that in the space $\mathbb{D}_0$ the bending in the plate is equal to the stretching in the rod at the junction while the bending and the section-rotation of the rod in the junction are equal to 0 (see (7.6)).

In the lemma below we give sufficient conditions on the applied forces in order to insure the existence of at least a minimizer of $\mathcal{J}$ (see [17] for a proof of the result for different boundary conditions for the displacement on $\partial\omega$).

**Lemma 7.1** *We have*

(1) *If $\kappa > 3$ and $\kappa' > 3$, then the minimization problem*

$$\min_{(\mathcal{U},\mathcal{W},\mathcal{Q}_3)\in\mathbb{D}_0} \mathcal{J}(\mathcal{U},\mathcal{W},\mathcal{Q}_3) \tag{7.7}$$

*admits a unique solution.*

(2) *If $\kappa = 3$ and $\kappa' > 3$, then there exists a constant $C_l^*$ such that, if $(f_{p,1}, f_{p,2})$ satisfies*

$$\|f_{p,1}\|^2_{L^2(\omega)} + \|f_{p,2}\|^2_{L^2(\omega)} < C_p^*, \tag{7.8}$$

*then* (7.7) *admits at least a solution.*

(3) *If $\kappa > 3$ and $\kappa' = 3$, then there exists a constant $C_l^{**}$ such that, if $f_{r,3}$ satisfies*

$$\|f_{r,3}\|_{L^2(0,L)} < C_r^{**}, \tag{7.9}$$

*then* (7.7) *admits at least a solution.*

(4) *If $\kappa = 3$ and $\kappa' = 3$, then if the applied forces $(f_{p,1}, f_{p,2})$ and $f_{r,3}$ satisfy* (7.8) *and* (7.9)*, respectively, then* (7.7) *admits at least a solution.*

**Proof** First, in the case $\kappa > 3$ and $\kappa' > 3$, the result is well known.

We prove the lemma in the case $\kappa = 3$ and $\kappa' = 3$. The two other cases are simpler and left to the reader.

Due to the boundary conditions on $\mathcal{U}_3$ in $\mathbb{D}_0$, we immediately have

$$\|\mathcal{U}_3\|^2_{H^2(\omega)} \le C\mathcal{J}_p(\mathcal{U}). \tag{7.10}$$

Then we get

$$\begin{aligned}\sum_{\alpha,\beta=1}^{2} \|\gamma_{\alpha\beta}(\mathcal{U})\|^2_{L^2(\omega)} &\le \mathcal{J}_p(\mathcal{U}) + C\|\nabla\mathcal{U}_3\|^4_{L^4(\omega;\mathbb{R}^2)} \\ &\le \mathcal{J}_p(\mathcal{U}) + C[\mathcal{J}_p(\mathcal{U})]^2.\end{aligned} \tag{7.11}$$

Thanks to the 2D Korn's inequality, we obtain

$$\|\mathcal{U}_1\|^2_{H^1(\omega)} + \|\mathcal{U}_2\|^2_{H^1(\omega)} \le C\mathcal{J}_p(\mathcal{U}) + C_p[\mathcal{J}_p(\mathcal{U})]^2. \tag{7.12}$$

Again, due to the boundary conditions on $\mathcal{W}_\alpha$ and $\mathcal{Q}_3$ in $\mathbb{D}_0$, we immediately have

$$\|\mathcal{W}_1\|^2_{H^2(0,L)} + \|\mathcal{W}_2\|^2_{H^2(0,L)} + \|\mathcal{Q}_3\|^2_{H^1(0,L)} \le \mathcal{J}_r(\mathcal{W},\mathcal{Q}_3). \tag{7.13}$$

Then we get

$$\begin{aligned}\Big\|\frac{\mathrm{d}\mathcal{W}_3}{\mathrm{d}x_3}\Big\|^2_{L^2(0,L)} &\le \mathcal{J}_r(\mathcal{W},\mathcal{Q}_3) + C\Big\{\Big\|\frac{\mathrm{d}\mathcal{W}_1}{\mathrm{d}x_3}\Big\|^4_{L^4(0,L)} + \Big\|\frac{\mathrm{d}\mathcal{W}_2}{\mathrm{d}x_3}\Big\|^4_{L^4(0,L)}\Big\} \\ &\le \mathcal{J}_r(\mathcal{W},\mathcal{Q}_3) + C[\mathcal{J}_r(\mathcal{W},\mathcal{Q}_3)]^2.\end{aligned} \tag{7.14}$$

From the above inequality and (7.10), we obtain

$$\begin{aligned}\|\mathcal{W}_3\|^2_{L^2(0,L)} &\le C|\mathcal{W}_3(0)|^2 + C\Big\|\frac{\mathrm{d}\mathcal{W}_3}{\mathrm{d}x_3}\Big\|^2_{L^2(0,L)} \\ &\le C\mathcal{J}_p(\mathcal{U}) + C\mathcal{J}_r(\mathcal{W},\mathcal{Q}_3) + C_r[\mathcal{J}_r(\mathcal{W},\mathcal{Q}_3)]^2.\end{aligned} \tag{7.15}$$

Since $\mathcal{J}(0,0,0)=0$, let us consider a minimizing sequence $(\mathcal{U}^{(N)},\mathcal{W}^{(N)},\mathcal{Q}_3^{(N)}) \in \mathbb{D}_0$ satisfying $\mathcal{J}(\mathcal{U}^{(N)},\mathcal{W}^{(N)},\mathcal{Q}_3^{(N)}) \le 0$,

$$m = \inf_{(\mathcal{U},\mathcal{W},\mathcal{Q}_3)\in\mathbb{D}_0} \mathcal{J}(\mathcal{U},\mathcal{W},\mathcal{Q}_3) = \lim_{N\to+\infty} \mathcal{J}(\mathcal{U}^{(N)},\mathcal{W}^{(N)},\mathcal{Q}_3^{(N)}),$$

where $m \in [-\infty, 0]$.

With the help of (7.10)–(7.15) we get

$$\begin{aligned}&\mathcal{J}_p(\mathcal{U}^{(N)}) + \mathcal{J}_r(\mathcal{W}^{(N)},\mathcal{Q}_3^{(N)}) \le C\|f_{p3}\|\sqrt{\mathcal{J}_p(\mathcal{U}^{(N)})} \\ &+ (\|f_{p,1}\|^2_{L^2(\omega)} + \|f_{p,2}\|^2_{L^2(\omega)})^{\frac{1}{2}}\Big(C\sqrt{\mathcal{J}_p(\mathcal{U}^{(N)})} + \sqrt{C_p}\mathcal{J}_p(\mathcal{U}^{(N)})\Big) \\ &+ \sum_{\alpha=1}^{2}(\|f_{r\alpha}\|_{L^2(0,L)} + \|g_\alpha\|_{L^2(0,L;\mathbb{R}^3)})\sqrt{\mathcal{J}_r(\mathcal{W}^{(N)},\mathcal{Q}_3^{(N)})} \\ &+ \|f_{r,3}\|_{L^2(0,L)}\Big(C\sqrt{\mathcal{J}_r(\mathcal{W}^{(N)},\mathcal{Q}_3^{(N)})} + C\sqrt{\mathcal{J}_p(\mathcal{U}^{(N)})} + \sqrt{C_r}\mathcal{J}_r(\mathcal{W}^{(N)},\mathcal{Q}_3^{(N)})\Big).\end{aligned} \tag{7.16}$$

Choosing $C_p^* = \frac{1}{C_p}$ and $C_r^{**} = \frac{1}{\sqrt{C_r}}$, if the applied forces satisfy (7.8) and (7.9), then the following estimates hold

$$\begin{aligned}&\|\mathcal{U}_3^{(N)}\|_{H^2(\omega)} + \|\mathcal{U}_1^{(N)}\|_{H^1(\omega)} + \|\mathcal{U}_2^{(N)}\|_{H^1(\omega)} + \|\mathcal{W}_1^{(N)}\|_{H^2(0,L)} \\ &+ \|\mathcal{W}_2^{(N)}\|_{H^2(0,L)} + \|\mathcal{Q}_3^{(N)}\|_{H^1(0,L)} + \|\mathcal{W}_3^{(N)}\|_{H^1(0,L)} \le C,\end{aligned} \tag{7.17}$$

where the constant $C$ does not depend on $N$.

As a consequence, there exists $(\mathcal{U}^{(*)},\mathcal{W}^{(*)},\mathcal{Q}_3^{(*)}) \in \mathbb{D}_0$ such that for a subsequence

$$\begin{aligned}&\mathcal{U}_3^{(N)} \rightharpoonup \mathcal{U}_3^{(*)} \quad \text{weakly in } H^2(\omega) \text{ and strongly in } W^{1,4}(\omega), \\ &\mathcal{U}_\alpha^{(N)} \rightharpoonup \mathcal{U}_\alpha^{(*)} \quad \text{weakly in } H^1(\omega), \\ &\mathcal{W}_\alpha^{(N)} \rightharpoonup \mathcal{W}_\alpha^{(*)} \quad \text{weakly in } H^2(0,L) \text{ and strongly in } W^{1,4}(0,L), \\ &\mathcal{Q}_3^{(N)} \rightharpoonup \mathcal{Q}_3^{(*)} \quad \text{weakly in } H^1(0,L), \\ &\mathcal{W}_3^{(N)} \rightharpoonup \mathcal{W}_3^{(*)} \quad \text{weakly in } H^1(0,L).\end{aligned}$$

Finally, since $\mathcal{J}$ is weakly sequentially continuous in

$$H^2(\omega)\times H^1(\omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^3)\times H^2(0,L;\mathbb{R}^2)\times H^1(0,L;\mathbb{R}^2)\times L^2(0,L)$$

with respect to

$$(\mathcal{U}_3, \mathcal{U}_1, \mathcal{U}_2, \mathcal{Z}_{11}, \mathcal{Z}_{12}, \mathcal{Z}_{22}, \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{Q}_3, \mathbf{F}_{33}),$$

the above weak and strong convergences imply that

$$\mathcal{J}(\mathcal{U}^{(*)}, \mathcal{W}^{(*)}, \mathcal{Q}_3^{(*)}) = m = \min_{(\mathcal{U}, \mathcal{W}, \mathcal{Q}_3) \in \mathbb{D}_0} \mathcal{J}(\mathcal{U}, \mathcal{W}, \mathcal{Q}_3),$$

which ends the proof of the lemma.

The following theorem is the main result of the paper. It characterizes the limit of the rescaled infimum of the total energy $\frac{m_\delta}{\delta^{2\kappa-1}} = \frac{1}{\delta^{2\kappa-1}} \inf_{v \in \mathbb{D}_{\delta,\varepsilon}} J_\delta(v)$ as the minimum of the limit energy $\mathcal{J}$ over the space $\mathbb{D}_0$. Due to the conditions on the fields $\mathcal{U}, \mathcal{W}, \mathcal{Q}_3$ in $\mathbb{D}_0$, this minimization problem modelizes the junction of a 2d plate model with a 1d rod model of the type "plate bending-rod stretching".

**Theorem 7.1** *Under the assumptions* (5.15), (5.20)–(5.21) *and* (7.8)–(7.9) *on the forces, we have*

$$\lim_{\delta \to 0} \frac{m_\delta}{\delta^{2\kappa-1}} = \min_{(\mathcal{U}, \mathcal{W}, \mathcal{Q}_3) \in \mathbb{D}_0} \mathcal{J}(\mathcal{U}, \mathcal{W}, \mathcal{Q}_3), \tag{7.18}$$

*where the functional $\mathcal{J}$ is defined by* (7.4).

**Proof** **Step 1** In this step we show that

$$\min_{(\mathcal{U}, \mathcal{W}, \mathcal{Q}_3) \in \mathbb{D}_0} \mathcal{J}(\mathcal{U}, \mathcal{W}, \mathcal{Q}_3) \leq \liminf_{\delta \to 0} \frac{m_\delta}{\delta^{2\kappa-1}}. \tag{7.19}$$

Let $(v_\delta)_\delta$ be a sequence of deformations belonging to $\mathbb{D}_{\delta,\varepsilon}$ such that

$$\lim_{\delta \to 0} \frac{J_\delta(v_\delta)}{\delta^{2\kappa-1}} = \liminf_{\delta \to 0} \frac{m_\delta}{\delta^{2\kappa-1}}. \tag{7.20}$$

One can always assume that $J_\delta(v_\delta) \leq 0$ without loss of generality. From the analysis of the previous section and in particular from estimates (5.26), the sequence $v_\delta$ satisfies

$$\|\mathrm{dist}(\nabla v_\delta, SO(3))\|_{L^2(\Omega_\delta)} \leq C\delta^{\kappa-\frac{1}{2}}, \quad \|\mathrm{dist}(\nabla v_\delta, SO(3))\|_{L^2(B_{\varepsilon,\delta})} \leq C\varepsilon^{\kappa'}. \tag{7.21}$$

Estimates in (5.31) give

$$\|\nabla v_\delta^{\mathrm{T}} \nabla v_\delta - \mathbf{I}_3\|_{L^2(\Omega_\delta; \mathbb{R}^{3\times 3})} \leq C\delta^{\kappa-\frac{1}{2}}, \quad \|\nabla v_\delta^{\mathrm{T}} \nabla v_\delta - \mathbf{I}_3\|_{L^2(B_{\varepsilon,\delta}; \mathbb{R}^{3\times 3})} \leq C\varepsilon^{\kappa'}. \tag{7.22}$$

Firstly, for any fixed $\delta$, the displacement $u_\delta = v_\delta - I_d$, restricted to $\Omega_\delta$, is decomposed as in Theorem 3.1. Due to the second estimate in (7.21), we can apply the results of Subsection 6.2 to the sequence $(v_\delta)$. As a consequence, there exists a subsequence (still indexed by $\delta$) and $\mathcal{U}^{(0)}, \mathcal{R}^{(0)} \in H^1(\omega; \mathbb{R}^3)$, such that the convergences in (6.2) and (6.5) hold. Due to (6.3) and (6.4), the field $\mathcal{U}_3$ belongs to $H^2(\omega)$, and we have the boundary conditions

$$\mathcal{U}^{(0)} = 0, \quad \nabla U_3^{(0)} = 0 \text{ on } \gamma_0. \tag{7.23}$$

Subsection 6.2 also shows that there exists $\overline{u}_p^{(0)} \in L^2(\omega; H^1(-1, 1; \mathbb{R}^3))$ such that

$$\frac{1}{2\delta^{\kappa-1}} (\nabla v_\delta^{\mathrm{T}} \nabla v_\delta - \mathbf{I}_3) \rightharpoonup \mathbf{E}_p^{(0)} \quad \text{weakly in } L^2(\Omega; \mathbb{R}^9), \tag{7.24}$$

where $\mathbf{E}_p^{(0)} = \mathbf{E}_p(\mathcal{U}^{(0)}, \overline{u}_p^{(0)})$ (see (6.8)).

Moreover, thanks to the first estimate in (7.22), the weak convergence (7.24) actually occurs in $L^2(\Omega; \mathbb{R}^9)$.

Secondly, still for fixed $\delta$, the displacement $u_\delta = v_\delta - I_d$, restricted to $B_{\varepsilon,\delta}$, is decomposed as in Theorem 3.1. Again due to the third estimate in (7.22), we can apply the results of Subsection 6.3 to the sequence $(v_\delta)$. As a consequence, there exists a subsequence (still indexed by $\delta$) and $\mathcal{W}^{(0)}, \mathcal{Q}_3^{(0)} \in H^1(0, L; \mathbb{R}^3)$, such that the convergences in (6.11) hold. As a consequence of (6.12), the field $\mathcal{W}^{(0)}$ belongs to $H^2(0, L)$ and we have

$$\frac{\mathrm{d}\mathcal{W}^{(0)}}{\mathrm{d}x_3} = \mathcal{Q}_3^{(0)} \wedge \mathbf{e}_3.$$

The junction conditions in (6.13) give

$$\mathcal{Q}^{(0)}(0) = 0, \quad \mathcal{W}_\alpha^{(0)}(0) = 0, \quad \mathcal{W}_3^{(0)}(0) = \mathcal{U}_3^{(0)}(0,0). \tag{7.25}$$

The triplet $(\mathcal{U}^{(0)}, \mathcal{W}^{(0)}, \mathcal{Q}_3^{(0)})$ belongs to $\mathbb{D}_0$.

Subsection 6.3 also shows that there exists $\overline{w}_r^{(0)} \in L^2(0, L; H^1(D; \mathbb{R}^3))$ such that

$$\frac{1}{2\varepsilon^{\kappa'-1}} P_\varepsilon((\nabla v_\delta)^{\mathrm{T}} \nabla v_\delta - \mathbf{I}_3) \rightharpoonup \mathbf{E}_r^{(0)} \quad \text{weakly in } L^2(B; \mathbb{R}^{3\times 3}), \tag{7.26}$$

where the symmetric matrix $\mathbf{E}_r^{(0)} = \mathbf{E}_r(\mathcal{W}^{(0)}, \mathcal{Q}_3^{(0)}, \overline{u}_r^{(0)})$ (see (6.16)). Moreover, thanks to the second estimate in (7.22), the weak convergence (7.26) actually occurs in $L^2(B; \mathbb{R}^9)$.

First of all, we have

$$\begin{aligned}
&\frac{1}{\delta^{2\kappa-1}} \int_{\mathcal{S}_{\delta,\varepsilon}} \widehat{W}_\varepsilon(x, \nabla v_\delta) \mathrm{d}x \\
&\geq \frac{1}{\delta^{2\kappa-1}} \int_{\Omega_\delta \setminus C_{\delta,\varepsilon}} \widehat{W}_\varepsilon(x, \nabla v_\delta) \mathrm{d}x + \frac{1}{q_\varepsilon^2 \varepsilon^{2\kappa'}} \int_{B_{\varepsilon,\delta} \setminus C_{\delta,\varepsilon}} \widehat{W}_\varepsilon(x, \nabla v_\delta) \mathrm{d}x \\
&= \int_\Omega Q_p\Big(\chi_{\Omega \setminus D_\varepsilon \times ]-1,1[} \Pi_\delta\Big[\frac{1}{\delta^{\kappa-1}}((\nabla v_\delta)^{\mathrm{T}} \nabla v_\delta - \mathbf{I}_3)\Big]\Big) \\
&\quad + \int_B Q_r\Big(\chi_{B \setminus D \times ]0,\delta[} P_\varepsilon\Big[\frac{1}{\varepsilon^{\kappa'-1}}((\nabla v_\delta)^{\mathrm{T}} \nabla v_\delta - \mathbf{I}_3)\Big]\Big).
\end{aligned}$$

From the weak convergences of the Green-St Venant's tensors in (7.24) and (7.26) (recall that these convergences hold in $L^2$) and the limit of the term involving the forces (7.28), we obtain

$$\liminf_{\delta \to 0} \frac{J_\delta(v_\delta)}{\delta^{2\kappa-1}} \geq \int_\Omega Q(\mathbf{E}_p^{(0)}) + \int_B Q(\mathbf{E}_r^{(0)}) - \lim_{\delta \to 0} \frac{1}{\delta^{2\kappa-1}} \int_{\mathcal{S}_{\delta,\varepsilon}} f_\delta \cdot (v_\delta - I_d). \tag{7.27}$$

In order to derive the last limit in (7.27), we use the assumptions on the forces (5.15) and the convergences (6.2) and (6.11), and this leads to

$$\lim_{\delta \to 0} \frac{1}{\delta^{2\kappa-1}} \int_{\mathcal{S}_{\delta,\varepsilon}} f_\delta \cdot (v_\delta - I_d) = \mathcal{L}(\mathcal{U}^{(0)}, \mathcal{W}^{(0)}, \mathcal{Q}_3^{(0)}), \tag{7.28}$$

where $\mathcal{L}(\mathcal{U}, \mathcal{W}, \mathcal{Q}_3)$ is given by (7.5) for any triplet in $\mathbb{D}_0$. From (7.27)–(7.28), we obtain

$$\liminf_{\delta \to 0} \frac{J_\delta(v_\delta)}{\delta^{2\kappa-1}} \geq \int_\Omega Q(\mathbf{E}_p^{(0)}) + \int_B Q(\mathbf{E}_r^{(0)}) - \mathcal{L}(\mathcal{U}^{(0)}, \mathcal{W}^{(0)}, \mathcal{Q}_3^{(0)}). \tag{7.29}$$

The next step in the derivation of the limit energy consists in minimizing $\int_{-1}^{1} Q_p(\mathbf{E}_p^{(0)})\mathrm{d}X_3$ with respect to $\overline{u}_p^{(0)}$. Explicit calculations show that

$$\begin{aligned}\int_{-1}^{1} Q_p(\mathbf{E}_p^{(0)})\mathrm{d}X_3 \geq & \int_{-1}^{1} Q_p(\mathbf{E}_p(\mathcal{U}^{(0)},\overline{\overline{u}}_p^{(0)}))\mathrm{d}X_3\\ =&\frac{E_p}{3(1-\nu_p^2)}\Big[(1-\nu_p)\sum_{\alpha,\beta=1}^{2}\Big|\frac{\partial^2\mathcal{U}_3^{(0)}}{\partial x_\alpha\partial x_\beta}\Big|^2+\nu_p(\Delta\mathcal{U}_3^{(0)})^2\Big]\\ &+\frac{E_p}{(1-\nu_p^2)}\Big[(1-\nu_p)\sum_{\alpha,\beta=1}^{2}|\mathcal{Z}_{\alpha\beta}^{(0)}|^2+\nu_p(\mathcal{Z}_{11}^{(0)}+\mathcal{Z}_{22}^{(0)})^2\Big],\end{aligned}\tag{7.30}$$

where

$$\begin{aligned}&\overline{\overline{u}}_p^{(0)}(\cdot,\cdot,X_3)=\frac{\nu_p}{1-\nu_p}\Big[\Big(\frac{X_3^2}{2}-\frac{1}{6}\Big)\Delta\mathcal{U}_3^{(0)}-X_3(\mathcal{Z}_{11}^{(0)}+\mathcal{Z}_{22}^{(0)})\Big]\mathbf{e}_3,\\ &\mathcal{Z}_{\alpha\beta}^{(0)}=\begin{cases}\gamma_{\alpha\beta}(\mathcal{U}^{(0)})+\dfrac{1}{2}\dfrac{\partial\mathcal{U}_3^{(0)}}{\partial x_\alpha}\dfrac{\partial\mathcal{U}_3^{(0)}}{\partial x_\beta}, & \text{if } \kappa=3,\\ \gamma_{\alpha\beta}(\mathcal{U}^{(0)}), & \text{if } \kappa>3.\end{cases}\end{aligned}\tag{7.31}$$

Similarly, minimizing $\int_D Q_r(\mathbf{E}_r^{(0)})\mathrm{d}X_1\mathrm{d}X_2$ with respect to $\overline{w}_r^{(0)}$ gives

$$\begin{aligned}\int_D Q_r(\mathbf{E}_r^{(0)})\mathrm{d}X_1\mathrm{d}X_2 \geq &\int_D Q_r(\mathbf{E}_r(\mathcal{W}^{(0)},\mathcal{Q}_3^{(0)},\overline{\overline{u}}_r^{(0)}))\mathrm{d}X_1\mathrm{d}X_2\\ =&\frac{E_r\pi}{8}\Big[\Big|\frac{d^2\mathcal{W}_1^{(0)}}{\mathrm{d}x_3^2}\Big|^2+\Big|\frac{d^2\mathcal{W}_2^{(0)}}{\mathrm{d}x_3^2}\Big|^2\Big]\\ &+\frac{E_r\pi}{2}\Big|\frac{\mathrm{d}\mathcal{W}_3^{(0)}}{\mathrm{d}x_3}+\mathbf{F}_{33}^{(0)}\Big|^2+\frac{\mu_r\pi}{8}\Big|\frac{\mathrm{d}\mathcal{Q}_3^{(0)}}{\mathrm{d}x_3}\Big|^2,\end{aligned}\tag{7.32}$$

where

$$\begin{aligned}\overline{\overline{w}}_{r,1}^{(0)}=&-\nu_r\Big[\frac{X_2^2-X_1^2}{2}\frac{d^2\mathcal{W}_1^{(0)}}{\mathrm{d}x_3^2}-X_1X_2\frac{d^2\mathcal{W}_2^{(0)}}{\mathrm{d}x_3^2}\\ &+X_1\Big(\frac{\mathrm{d}\mathcal{W}_3^{(0)}}{\mathrm{d}x_3}+\mathbf{F}_{33}^{(0)}\Big)\Big]-X_1\mathbf{F}_{11}^{(0)}-\frac{X_2}{2}\mathbf{F}_{12}^{(0)},\\ \overline{\overline{w}}_{r,2}^{(0)}=&-\nu_r\Big[\frac{X_1^2-X_2^2}{2}\frac{d^2\mathcal{W}_2^{(0)}}{\mathrm{d}x_3^2}-X_1X_2\frac{d^2\mathcal{W}_1^{(0)}}{\mathrm{d}x_3^2}\\ &+X_2\Big(\frac{\mathrm{d}\mathcal{W}_3^{(0)}}{\mathrm{d}x_3}+\mathbf{F}_{33}^{(0)}\Big)\Big]-\frac{X_1}{2}\mathbf{F}_{12}^{(0)}-X_2\mathbf{F}_{22}^{(0)},\\ \overline{\overline{w}}_{r,3}^{(0)}=&-X_1\mathbf{F}_{13}^{(0)}-X_2\mathbf{F}_{23}^{(0)}\end{aligned}\tag{7.33}$$

and

$$\mathbf{F}^{(0)}=\begin{cases}\dfrac{1}{2}(\|\mathcal{Q}^{(0)}\|_2^2\mathbf{I}_3-\mathcal{Q}^{(0)}.(\mathcal{Q}^{(0)})^{\mathrm{T}}), & \text{if } \kappa'=3,\\ 0, & \text{if } \kappa'>3.\end{cases}\tag{7.34}$$

In view of (7.29)–(7.30) and (7.32), the proof of (7.19) is achieved.

**Step 2** Under the assumptions (7.8)–(7.9), we know that there exists $(\mathcal{U}^{(1)}, \mathcal{W}^{(1)}, \mathcal{Q}_3^{(1)}) \in \mathbb{D}_0$, such that

$$\min_{(\mathcal{U},\mathcal{W},\mathcal{Q}_3)\in\mathbb{D}_0} \mathcal{J}(\mathcal{U}, \mathcal{W}, \mathcal{Q}_3) = \mathcal{J}(\mathcal{U}^{(1)}, \mathcal{W}^{(1)}, \mathcal{Q}_3^{(1)}).$$

Now, in this step we show that

$$\limsup_{\delta\to 0} \frac{m_\delta}{\delta^{2\kappa-1}} \le \mathcal{J}(\mathcal{U}^{(1)}, \mathcal{W}^{(1)}, \mathcal{Q}_3^{(1)}).$$

Let $\overline{\overline{u}}_p^{(1)}$ be in $L^2(\omega; H^1(-1,1;\mathbb{R}^3))$ obtained through replacing $\mathcal{U}^{(0)}$ by $\mathcal{U}^{(1)}$ in (7.31) and $\overline{\overline{w}}_r^{(1)}$ be in $L^2(0,L;H^1(D;\mathbb{R}^3))$ obtained through replacing $\mathcal{W}^{(0)}$ and $\mathcal{Q}_3^{(0)}$ by $\mathcal{W}^{(1)}$ and $\mathcal{Q}_3^{(1)}$ in (7.33)–(7.34), respectively. Observe that

$$\begin{aligned}\mathcal{J}(\mathcal{U}^{(1)}, \mathcal{W}^{(1)}, \mathcal{Q}_3^{(1)}) = &\int_\Omega Q_p(\mathbf{E}_p(\mathcal{U}^{(1)}, \overline{\overline{u}}_p^{(1)}))\mathrm{d}x + \int_B Q_r(\mathbf{E}_r(\mathcal{W}^{(1)}, \mathcal{Q}_3^{(1)}, \overline{\overline{u}}_r^{(1)}))\mathrm{d}x \\ &- \mathcal{L}(\mathcal{U}^{(1)}, \mathcal{W}^{(1)}, \mathcal{Q}_3^{(1)}).\end{aligned} \tag{7.35}$$

We now consider a sequence $(\mathcal{U}^{(n)}, \mathcal{W}^{(n)}, \mathcal{Q}_3^{(n)}, \overline{u}^{(n)}, \overline{w}^{(n)})_{n\ge 2}$ such that

(1) $\mathcal{U}_\alpha^{(n)} \in W^{2,\infty}(\omega) \cap H^1_{\gamma_0}(\omega)$ and

$$\mathcal{U}_\alpha^{(n)} \to \mathcal{U}_\alpha^{(1)} \quad \text{strongly in } H^1(\omega),$$

(2) $\mathcal{U}_3^{(n)} \in W^{3,\infty}(\omega) \cap H^2_{\gamma_0}(\omega)$ and

$$\mathcal{U}_3^{(n)} \to \mathcal{U}_3^{(1)} \quad \text{strongly in } H^2(\omega),$$

(3) $\mathcal{W}_\alpha^{(n)} \in W^{3,\infty}\big(-\frac{1}{n}, L\big)$ with $\mathcal{W}_\alpha^{(n)} = 0$ in $\big[-\frac{1}{n}, \frac{1}{n}\big]$ and

$$\mathcal{W}_\alpha^{(n)} \to \mathcal{W}_\alpha^{(1)} \quad \text{strongly in } H^2(0,L),$$

(4) $\mathcal{W}_3^{(n)} \in W^{2,\infty}\big(-\frac{1}{n}, L\big)$ with $\mathcal{W}_3^{(n)} = \mathcal{U}_3^{(n)}(0,0)$ in $\big[-\frac{1}{n}, \frac{1}{n}\big]$ and

$$\mathcal{W}_3^{(n)} \to \mathcal{W}_3^{(1)} \quad \text{strongly in } H^1(0,L),$$

(5) $\mathcal{Q}_3^{(n)} \in W^{2,\infty}\big(-\frac{1}{n}, L\big)$ with $\mathcal{Q}_3^{(n)} = 0$ in $\big[-\frac{1}{n}, \frac{1}{n}\big]$ and

$$\mathcal{Q}_3^{(n)} \to \mathcal{Q}_3^{(1)} \quad \text{strongly in } H^1(0,L),$$

(6) $\overline{u}^{(n)} \in W^{1,\infty}(\Omega;\mathbb{R}^3)$ with $\overline{u}^{(n)} = 0$ on $\partial\omega\times]-1,1[$, $\overline{u}^{(n)} = 0$ in the cylinder $D\big(O,\frac{1}{n}\big)\times ]-1,1[$ and

$$\overline{u}^{(n)} \to \overline{\overline{u}}_p^{(1)} \quad \text{strongly in } L^2(\omega; H^1(-1,1;\mathbb{R}^3)),$$

(7) $\overline{w}^{(n)} \in W^{1,\infty}(]-\frac{1}{n}, L[\times D;\mathbb{R}^3)$ with $\overline{w}^{(n)} = 0$ in the cylinder $D\times\big]-\frac{1}{n},\frac{1}{n}\big[$ and

$$\overline{w}^{(n)} \to \overline{\overline{w}}_r^{(1)} \quad \text{strongly in } L^2(0,L;H^1(D;\mathbb{R}^3)).$$

First, the above strong convergences and (7.35) show that

$$\begin{aligned}&\lim_{n\to 0}\Big[\int_\Omega Q_p(\mathbf{E}_p(\mathcal{U}^{(n)}, \overline{u}_p^{(n)}))\mathrm{d}x + \int_B Q_r(\mathbf{E}_r(\mathcal{W}^{(n)}, \mathcal{Q}_3^{(n)}, \overline{u}_r^{(n)}))\mathrm{d}x - \mathcal{L}(\mathcal{U}^{(n)}, \mathcal{W}^{(n)}, \mathcal{Q}_3^{(n)})\Big] \\ &= \int_\Omega Q_p(\mathbf{E}_p(\mathcal{U}^{(1)}, \overline{\overline{u}}_p^{(1)}))\mathrm{d}x + \int_B Q_r(\mathbf{E}_r(\mathcal{W}^{(1)}, \mathcal{Q}_3^{(1)}, \overline{\overline{u}}_r^{(1)}))\mathrm{d}x - \mathcal{L}(\mathcal{U}^{(1)}, \mathcal{W}^{(1)}, \mathcal{Q}_3^{(1)}) \\ &= \mathcal{J}(\mathcal{U}^{(1)}, \mathcal{W}^{(1)}, \mathcal{Q}_3^{(1)}).\end{aligned} \tag{7.36}$$

For fixed $n$, let us consider the following sequence $(v_\delta^{(n)})$ of deformations of the whole structure $\mathcal{S}_{\delta,\varepsilon}$, defined below:

(1) In $\Omega_\delta$ we set

$$\begin{aligned}
v_{\delta,1}^{(n)}(x) &= x_1 + \delta^{\kappa-1}\Big(\mathcal{U}_1^{(n)}(x_1,x_2) - \frac{x_3}{\delta}\frac{\partial \mathcal{U}_3^{(n)}}{\partial x_1}(x_1,x_2) + \delta\overline{u}_1^{(n)}\Big(x_1,x_2,\frac{x_3}{\delta}\Big)\Big),\\
v_{\delta,2}^{(n)}(x) &= x_2 + \delta^{\kappa-1}\Big(\mathcal{U}_2^{(n)}(x_1,x_2) - \frac{x_3}{\delta}\frac{\partial \mathcal{U}_3^{(n)}}{\partial x_2}(x_1,x_2) + \delta\overline{u}_2^{(n)}\Big(x_1,x_2,\frac{x_3}{\delta}\Big)\Big),\\
v_{\delta,3}^{(n)}(x) &= x_3 + \delta^{\kappa-2}\Big(\mathcal{U}_3^{(n)}(x_1,x_2) + \delta^2\overline{u}_3^{(n)}\Big(x_1,x_2,\frac{x_3}{\delta}\Big)\Big).
\end{aligned} \tag{7.37}$$

(2) In $B_{\varepsilon,\delta}$ we set

$$\begin{aligned}
v_{\delta,1}^{(n)}(x) &= x_1 + \delta^{\kappa-1}\Big(\mathcal{U}_1^{(n)}(x_1,x_2) - \frac{x_3}{\delta}\frac{\partial \mathcal{U}_3^{(n)}}{\partial x_1}(x_1,x_2)\Big) + \varepsilon^{\kappa'-2}\Big(\mathcal{W}_1^{(n)}(x_3)\\
&\qquad - x_2\mathcal{Q}_3^{(n)}(x_3) + \varepsilon^2\overline{w}_1^{(n)}\Big(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon},x_3\Big)\Big),\\
v_{\delta,2}^{(n)}(x) &= x_2 + \delta^{\kappa-1}\Big(\mathcal{U}_2^{(n)}(x_1,x_2) - \frac{x_3}{\delta}\frac{\partial \mathcal{U}_3^{(n)}}{\partial x_2}(x_1,x_2)\Big) + \varepsilon^{\kappa'-2}\Big(\mathcal{W}_2^{(n)}(x_3)\\
&\qquad + x_1\mathcal{Q}_3^{(n)}(x_3) + \varepsilon^2\overline{w}_2^{(n)}\Big(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon},x_3\Big)\Big),\\
v_{\delta,3}^{(n)}(x) &= x_3 + \delta^{\kappa-2}\mathcal{U}_3^{(n)}(x_1,x_2) + \varepsilon^{\kappa'-1}\Big([\mathcal{W}_3^{(n)}(x_3) - \mathcal{U}_3^{(n)}(0,0)] - \frac{x_1}{\varepsilon}\frac{\mathrm{d}\mathcal{W}_1^{(n)}}{\mathrm{d}x_3}(x_3)\\
&\qquad - \frac{x_2}{\varepsilon}\frac{\mathrm{d}\mathcal{W}_2^{(n)}}{\mathrm{d}x_3}(x_3) + \varepsilon\overline{w}_3^{(n)}\Big(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon},x_3\Big)\Big).
\end{aligned} \tag{7.38}$$

Obviously, if $\delta$ is small enough $\big($in order to have $\delta \le \frac{1}{n}\big)$ the two expressions of $v_\delta^{(n)}$ match in the cylinder $C_{\delta,\varepsilon}$ and are equal to

$$\begin{aligned}
v_{\delta,1}^{(n)}(x) &= x_1 + \delta^{\kappa-1}\Big(\mathcal{U}_1^{(n)}(x_1,x_2) - \frac{x_3}{\delta}\frac{\partial \mathcal{U}_3^{(n)}}{\partial x_1}(x_1,x_2)\Big),\\
v_{\delta,2}^{(n)}(x) &= x_2 + \delta^{\kappa-1}\Big(\mathcal{U}_2^{(n)}(x_1,x_2) - \frac{x_3}{\delta}\frac{\partial \mathcal{U}_3^{(n)}}{\partial x_2}(x_1,x_2)\Big),\\
v_{\delta,3}^{(n)}(x) &= x_3 + \delta^{\kappa-2}\mathcal{U}_3^{(n)}(x_1,x_2).
\end{aligned} \tag{7.39}$$

By construction, the deformation $v_\delta^{(n)}$ belongs to $\mathbb{D}_{\delta,\varepsilon}$ and satisfies

$$\|\nabla v_\delta^{(n)} - \mathbf{I}_3\|_{L^\infty(\mathcal{S}_{\delta,\varepsilon};\mathbb{R}^9)} \le C(n)\{\delta^{\kappa-2} + \varepsilon^{\kappa'-2}\}.$$

Hence, for a.e. $x \in \mathcal{S}_{\delta,\varepsilon}$ we have $\det(\nabla v_\delta^{(n)}(x)) > 0$ (we recall that $\kappa \ge 3$ and $\kappa' \ge 3$). Then we have

$$m_\delta \le J_\delta(v_\delta^{(n)}). \tag{7.40}$$

In the expression (7.37) of the displacement $v_\delta - I_d$, the explicit dependence with respect to $\delta$ permits deriving directly the limit of the Green-St Venant's strain tensor as $\delta$ tends to 0 ($n$ being fixed)

$$\frac{1}{2\delta^{\kappa-1}}\Pi_\delta((\nabla v_\delta^{(n)})^{\mathrm{T}}\nabla v_\delta^{(n)} - \mathbf{I}_3) \to \mathbf{E}_p^{(n)} \quad \text{strongly in } L^\infty(\Omega;\mathbb{R}^9), \tag{7.41}$$

where the symmetric matrix $\mathbf{E}_p^{(n)}$ is defined by

$$\mathbf{E}_p^{(n)} = \begin{pmatrix} -X_3\dfrac{\partial^2\mathcal{U}_3^{(n)}}{\partial x_1^2} + \mathcal{Z}_{11}^{(n)} & -X_3\dfrac{\partial^2\mathcal{U}_3^{(n)}}{\partial x_1\partial x_2} + \mathcal{Z}_{12}^{(n)} & \dfrac{1}{2}\dfrac{\partial\overline{u}_1^{(n)}}{\partial X_3} \\ * & -X_3\dfrac{\partial^2\mathcal{U}_3^{(n)}}{\partial x_2^2} + \mathcal{Z}_{22}^{(n)} & \dfrac{1}{2}\dfrac{\partial\overline{u}_2^{(n)}}{\partial X_3} \\ * & * & \dfrac{\partial\overline{u}_3^{(n)}}{\partial X_3} \end{pmatrix},$$

where

$$\mathcal{Z}_{\alpha\beta}^{(n)} = \begin{cases} \gamma_{\alpha\beta}(\mathcal{U}^{(n)}) + \dfrac{1}{2}\dfrac{\partial\mathcal{U}_3^{(n)}}{\partial x_\alpha}\dfrac{\partial\mathcal{U}_3^{(n)}}{\partial x_\beta}, & \text{if } \kappa = 3, \\ \gamma_{\alpha\beta}(\mathcal{U}^{(n)}), & \text{if } \kappa > 3. \end{cases} \tag{7.42}$$

Now, in the rod $B_{\varepsilon,\delta}$, we have

$$\begin{aligned} v_{\delta,1}^{(n)}(x) &= x_1 + \varepsilon^{\kappa'-2}\Big[\mathcal{W}_1^{(n)}(x_3) + \delta\varepsilon\mathcal{U}_1^{(n)}(0,0) - \varepsilon x_3\frac{\partial\mathcal{U}_3^{(n)}}{\partial x_1}(0,0) \\ &\qquad - x_2\mathcal{Q}_3^{(n)}(x_3)\Big] + \widetilde{w}_{\varepsilon,1}^{(n)}(x), \\ v_{\delta,2}^{(n)}(x) &= x_2 + \varepsilon^{\kappa'-2}\Big[\mathcal{W}_2^{(n)}(x_3) + \delta\varepsilon\mathcal{U}_2^{(n)}(0,0) - \varepsilon x_3\frac{\partial\mathcal{U}_3^{(n)}}{\partial x_2}(0,0) \\ &\qquad + x_1\mathcal{Q}_3^{(n)}(x_3)\Big] + \widetilde{w}_{\varepsilon,2}^{(n)}(x), \\ v_{\delta,3}^{(n)}(x) &= x_3 + \varepsilon^{\kappa'-1}\Big[\mathcal{W}_3^{(n)}(x_3) - \frac{x_1}{\varepsilon}\frac{\mathrm{d}\mathcal{W}_1^{(n)}}{\mathrm{d}x_3}(x_3) + x_1\frac{\partial\mathcal{U}_3^{(n)}}{\partial x_1}(0,0) \\ &\qquad - \frac{x_2}{\varepsilon}\frac{\mathrm{d}\mathcal{W}_2^{(n)}}{\mathrm{d}x_3}(x_3) + x_2\frac{\partial\mathcal{U}_3^{(n)}}{\partial x_2}(0,0)\Big] + \widetilde{w}_{\varepsilon,3}^{(n)}(x), \end{aligned} \tag{7.43}$$

where

$$\begin{aligned} \widetilde{w}_{\varepsilon,1}^{(n)}(x) &= \varepsilon^{\kappa'}\overline{w}_1^{(n)}\Big(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}, x_3\Big) + \delta\varepsilon^{\kappa'-1}(\mathcal{U}_1^{(n)}(x_1,x_2) - \mathcal{U}_1^{(n)}(0,0)) \\ &\qquad - x_3\varepsilon^{\kappa'-1}\Big(\frac{\partial\mathcal{U}_3^{(n)}}{\partial x_1}(x_1,x_2) - \frac{\partial\mathcal{U}_3^{(n)}}{\partial x_1}(0,0)\Big), \\ \widetilde{w}_{\varepsilon,2}^{(n)}(x) &= \varepsilon^{\kappa'}\overline{w}_2^{(n)}\Big(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}, x_3\Big) + \delta\varepsilon^{\kappa'-1}(\mathcal{U}_2^{(n)}(x_1,x_2) - \mathcal{U}_2^{(n)}(0,0)) \\ &\qquad - x_3\varepsilon^{\kappa'-1}\Big(\frac{\partial\mathcal{U}_3^{(n)}}{\partial x_2}(x_1,x_2) - \frac{\partial\mathcal{U}_3^{(n)}}{\partial x_2}(0,0)\Big), \\ \widetilde{w}_{\varepsilon,3}^{(n)}(x) &= \varepsilon^{\kappa'}\overline{w}_3^{(n)}\Big(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}, x_3\Big) + \varepsilon^{\kappa'-1}\Big(\mathcal{U}_3^{(n)}(x_1,x_2) - \mathcal{U}_3^{(n)}(0,0) \\ &\qquad - x_1\frac{\partial\mathcal{U}_3^{(n)}}{\partial x_1}(0,0) - x_2\frac{\partial\mathcal{U}_3^{(n)}}{\partial x_2}(0,0)\Big). \end{aligned}$$

First notice that

$$\begin{aligned} \frac{1}{\varepsilon^{\kappa'}}P_\varepsilon(\widetilde{w}_\varepsilon^{(n)}) \to \overline{w}_r^{(n)} &= \overline{w}^{(n)} - x_3\Big[X_1\frac{\partial^2\mathcal{U}_3^{(n)}}{\partial x_1^2}(0,0) + X_2\frac{\partial^2\mathcal{U}_3^{(n)}}{\partial x_1\partial x_2}(0,0)\Big]\mathbf{e}_1 \\ - x_3\Big[X_1\frac{\partial^2\mathcal{U}_3^{(n)}}{\partial x_1\partial x_2}(0,0) &+ X_2\frac{\partial^2\mathcal{U}_3^{(n)}}{\partial x_2^2}(0,0)\Big]\mathbf{e}_2 \quad \text{strongly in } W^{1,\infty}(B;\mathbb{R}^3). \end{aligned} \tag{7.44}$$

As above, the expression (7.43) of the displacement $v_\delta^{(n)} - I_d$ being explicit with respect to $\delta$ and $\varepsilon$, a direct calculation gives

$$\frac{1}{2\varepsilon^{\kappa'-1}} P_\varepsilon((\nabla v_\delta^{(n)})^{\mathrm{T}} \nabla v_\delta^{(n)} - \mathbf{I}_3) \to \mathbf{E}_r^{(n)} \quad \text{strongly in } L^\infty(B; \mathbb{R}^{3\times 3}), \tag{7.45}$$

where the symmetric matrix $\mathbf{E}_r^{(n)}$ is defined by

$$\begin{aligned} \mathbf{E}_r^{(n)} &= \begin{pmatrix} \gamma_{11}(\overline{w}_r^{(n)}) & \gamma_{12}(\overline{w}_r^{(n)}) & -\frac{1}{2} X_2 \frac{\mathrm{d}\mathcal{Q}_3^{(n)}}{\mathrm{d}x_3} + \frac{1}{2}\frac{\partial \overline{w}_{r,3}^{(n)}}{\partial X_1} \\ * & \gamma_{22}(\overline{w}_r^{(n)}) & \frac{1}{2} X_1 \frac{\mathrm{d}\mathcal{Q}_3^{(n)}}{\mathrm{d}x_3} + \frac{1}{2}\frac{\partial \overline{w}_{r,3}^{(n)}}{\partial X_2} \\ * & * & -X_1 \frac{\mathrm{d}^2\mathcal{U}_1^{(n)}}{\mathrm{d}x_3^2} - X_2 \frac{\mathrm{d}^2\mathcal{U}_2^{(n)}}{\mathrm{d}x_3^2} + \frac{\mathrm{d}\mathcal{U}_3^{(n)}}{\mathrm{d}x_3} \end{pmatrix} + \mathbf{F}^{(n)}, \\ \mathbf{F}^{(n)} &= \begin{cases} \frac{1}{2}(\|\mathcal{Q}^{(n)}\|_2^2 \mathbf{I}_3 - \mathcal{Q}^{(n)} \cdot (\mathcal{Q}^{(n)})^{\mathrm{T}}), & \text{if } \kappa' = 3, \\ 0, & \text{if } \kappa' > 3. \end{cases} \end{aligned} \tag{7.46}$$

The definition (5.1) of $\widehat{W}_\varepsilon(x, \nabla v_\delta^{(n)})(x)$ shows that

$$\begin{aligned} &\Big| \frac{1}{\delta^{2\kappa-1}} \int_{\mathcal{S}_{\delta,\varepsilon}} \widehat{W}_\varepsilon(x, \nabla v_\delta^{(n)})(x)\mathrm{d}x - \int_\Omega Q_p\Big(\Pi_\delta\Big[\frac{1}{\delta^{\kappa-1}}((\nabla v_\delta^{(n)})^{\mathrm{T}} \nabla v_\delta^{(n)} - \mathbf{I}_3)\Big]\Big) \\ &- \int_B Q_r\Big(P_\varepsilon\Big[\frac{1}{\varepsilon^{\kappa'-1}}((\nabla v_\delta^{(n)})^{\mathrm{T}} \nabla v_\delta^{(n)} - \mathbf{I}_3)\Big]\Big)\Big| \le C \frac{q_\varepsilon^2}{\delta^{2\kappa-1}} \int_{C_{\delta,\varepsilon}} |||(\nabla v_\delta^{(n)})^{\mathrm{T}} \nabla v_\delta^{(n)} - \mathbf{I}_3|||^2, \end{aligned}$$

where the constant $C$ depends only on the Lamé's constants. Taking into account (5.9) and (7.41), we first obtain that

$$\frac{q_\varepsilon^2}{\delta^{2\kappa-1}} \int_{C_{\delta,\varepsilon}} |||(\nabla v_\delta^{(n)})^{\mathrm{T}} \nabla v_\delta^{(n)} - \mathbf{I}_3|||^2 \to 0$$

as $\delta$ and $\varepsilon$ go to 0. Then, due to (7.41) and (7.45), we finally get

$$\lim_{\delta\to 0} \frac{1}{\delta^{2\kappa-1}} \int_{\mathcal{S}_{\delta,\varepsilon}} \widehat{W}_\varepsilon(x, \nabla v_\delta^{(n)})(x)\mathrm{d}x = \int_\Omega Q(\mathbf{E}_p^{(n)})\mathrm{d}x + \int_B Q(\mathbf{E}_r^{(n)})\mathrm{d}x.$$

Furthermore, from the expressions ((7.37) and (7.38)) of $v_\delta$ in the plate and in the rod, we immediately have

$$\lim_{\delta\to 0} \frac{1}{\delta^{2\kappa-1}} \int_{\mathcal{S}_{\delta,\varepsilon}} f_\delta \cdot (v_\delta^{(n)} - I_d) = \mathcal{L}(\mathcal{U}^{(n)}, \mathcal{W}^{(n)}, \mathcal{Q}_3^{(n)}).$$

Then, the above limits and (7.40) lead to

$$\limsup_{\delta\to 0} \frac{m_\delta}{\delta^{2\kappa-1}} \le \int_\Omega Q(\mathbf{E}_p^{(n)})\mathrm{d}x + \int_B Q(\mathbf{E}_r^{(n)})\mathrm{d}x - \mathcal{L}(\mathcal{U}^{(n)}, \mathcal{W}^{(n)}, \mathcal{Q}_3^{(n)}). \tag{7.47}$$

Now, as $n$ goes to infinity, the above inequality and (7.36) give

$$\limsup_{\delta\to 0} \frac{m_\delta}{\delta^{2\kappa-1}} \le \mathcal{J}(\mathcal{U}^{(1)}, \mathcal{W}^{(1)}, \mathcal{Q}_3^{(1)}). \tag{7.48}$$

This concludes the proof of the theorem.

**Corollary 7.1** *Let $v_\delta$ be a sequence of $\mathbb{D}_{\delta,\varepsilon}$ such that*

$$\lim_{\delta\to 0}\frac{J_\delta(v_\delta)}{\delta^{2\kappa-1}}=\lim_{\delta\to 0}\frac{m_\delta}{\delta^{2\kappa-1}}. \tag{7.49}$$

*Then there exists a subsequence still indexed by $\delta$ such that*

$$\begin{aligned}
&\frac{1}{\delta^{\kappa-1}}\Pi_\delta(u_{\delta,\alpha})\rightharpoonup \mathcal{U}_\alpha^{(0)}-X_3\frac{\partial\mathcal{U}_3^{(0)}}{\partial x_\alpha}\quad \textit{weakly in } H^1(\Omega),\\
&\frac{1}{\delta^{\kappa-2}}\Pi_\delta(u_{\delta,3})\to \mathcal{U}_3^{(0)}\quad \textit{strongly in } H^1(\Omega),\\
&\frac{1}{\varepsilon^{\kappa'-2}}P_\varepsilon(u_{\delta,\alpha})\to \mathcal{W}_\alpha^{(0)}\quad \textit{strongly in } H^1(B),\\
&\frac{1}{\varepsilon^{\kappa'-1}}P_\varepsilon(u_{\delta,1}-\mathcal{W}_{\delta,1})\rightharpoonup -X_2\mathcal{Q}_3^{(0)}\quad \textit{weakly in } H^1(B),\\
&\frac{1}{\varepsilon^{\kappa'-1}}P_\varepsilon(u_{\delta,2}-\mathcal{W}_{\delta,2})\rightharpoonup X_1\mathcal{Q}_3^{(0)}\quad \textit{weakly in } H^1(B),\\
&\frac{1}{\varepsilon^{\kappa'-1}}P_\varepsilon(u_{\delta,3})\rightharpoonup \mathcal{W}_3^{(0)}-X_1\frac{\mathrm{d}\mathcal{W}_1^{(0)}}{\mathrm{d}x_3}-X_2\frac{\mathrm{d}\mathcal{W}_2^{(0)}}{\mathrm{d}x_3}\quad \textit{weakly in } H^1(B),
\end{aligned} \tag{7.50}$$

*where $(\mathcal{U}^{(0)},\mathcal{W}^{(0)},\mathcal{Q}_3^{(0)})$ is a minimizer of $\mathcal{J}$ in $\mathbb{D}_0$.*

**Proof** Step 1 of Theorem 7.1 shows that, for a subsequence still indexed by $\delta$, there exists $(\mathcal{U}^{(0)},\mathcal{W}^{(0)},\mathcal{Q}_3^{(0)})\in\mathbb{D}_0$, such that the convergences in (7.50) hold. Moreover, we have

$$\mathcal{J}(\mathcal{U}^{(0)},\mathcal{W}^{(0)},\mathcal{Q}_3^{(0)})\le\lim_{\delta\to 0}\frac{J_\delta(v_\delta)}{\delta^{2\kappa-1}}=\lim_{\delta\to 0}\frac{m_\delta}{\delta^{2\kappa-1}}=\min_{(\mathcal{U},\mathcal{W},\mathcal{Q}_3)\in\mathbb{D}_0}\mathcal{J}(\mathcal{U},\mathcal{W},\mathcal{Q}_3).$$

So $(\mathcal{U}^{(0)},\mathcal{W}^{(0)},\mathcal{Q}_3^{(0)})$ is a minimizer of $\mathcal{J}$ in $\mathbb{D}_0$. The proof of Step 1 in Theorem 7.1 also shows that the convergences (7.24) and (7.26) of the rescaled Green-St Venant's strain tensors are respectively strong in $L^2(\Omega;\mathbb{R}^{3\times 3})$ and in $L^2(B;\mathbb{R}^{3\times 3})$.

## 8 Appendix

**Proof of Lemma 5.2** The first estimate in (5.13) is proved in Lemma 4.3 of [8]. Now we carry on by estimating $\mathbf{G}_s(u,B_{\varepsilon,\delta})$.

**Step 1** In this step we prove the following inequality

$$\begin{aligned}\mathbf{G}_s(u,B_{\varepsilon,\delta})\le{}& C\|\mathrm{dist}(\nabla v,SO(3))\|_{L^2(B_{\varepsilon,\delta})}\\
&+C\frac{\|\mathrm{dist}(\nabla v,SO(3))\|^2_{L^2(B_{\varepsilon,\delta})}}{\varepsilon^3}+C\varepsilon|||\mathbf{Q}(0)-\mathbf{I}_3|||^2.\end{aligned} \tag{8.1}$$

The restriction of the displacement $u=v-I_d$ to the rod $B_{\varepsilon,\delta}$ is decomposed as (see Theorem 2.2.2 of [7])

$$u(x)=\mathcal{W}(x_3)+(\mathbf{Q}(x_3)-\mathbf{I}_3)(x_1\mathbf{e}_1+x_2\mathbf{e}_2)+\overline{w}'(x),\quad x\in B_{\varepsilon,\delta}, \tag{8.2}$$

where $\mathcal{W}\in H^1(-\delta,L;\mathbb{R}^3)$, $\mathbf{Q}\in H^1(-\delta,L;SO(3))$ and $\overline{w}'\in H^1(B_{\varepsilon,\delta};\mathbb{R}^3)$. This displacement is also decomposed as in (3.8). In both decompositions, the field $\mathcal{W}$ is the average of $u$ on the cross-sections of the rod.

We know (see Theorem 2.2.2 established in [7]) that the fields $\mathcal{W}$, $\mathbf{Q}$ and $\overline{w}'$ satisfy

$$
\begin{aligned}
&\|\overline{w}'\|_{L^2(B_{\varepsilon,\delta};\mathbb{R}^3)} \le C\varepsilon\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(B_{\varepsilon,\delta})},\\
&\|\nabla\overline{w}'\|_{L^2(B_{\varepsilon,\delta};\mathbb{R}^{3\times3})} \le C\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(B_{\varepsilon,\delta})},\\
&\Big\|\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}x_3}\Big\|_{L^2(-\delta,L;\mathbb{R}^3)} \le \frac{C}{\varepsilon^2}\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(B_{\varepsilon,\delta})},\\
&\Big\|\frac{\mathrm{d}\mathcal{W}}{\mathrm{d}x_3} - (\mathbf{Q}-\mathbf{I}_3)\mathbf{e}_3\Big\|_{L^2(-\delta,L;\mathbb{R}^3)} \le \frac{C}{\varepsilon}\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(B_{\varepsilon,\delta})},\\
&l\|\nabla v-\mathbf{Q}\|_{L^2(B_{\varepsilon,\delta};\mathbb{R}^{3\times3})} \le C\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(B_{\varepsilon,\delta})},
\end{aligned}
\tag{8.3}
$$

where the constant $C$ does not depend on $\varepsilon$, $\delta$ and $L$.

We set $\mathbf{v} = \mathbf{Q}(0)^{\mathrm{T}}v$ and $\mathbf{u} = \mathbf{v} - I_d$. The deformation $\mathbf{v}$ belongs to $H^1(B_{\varepsilon,\delta};\mathbb{R}^3)$ and satisfies

$$\|\mathrm{dist}(\nabla\mathbf{v}, SO(3))\|_{L^2(B_{\varepsilon,\delta})} = \|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(B_{\varepsilon,\delta})}.$$

The last estimate in (8.3) leads to

$$
\begin{aligned}
l\|\nabla\mathbf{u}+(\nabla\mathbf{u})^{\mathrm{T}}\|_{L^2(B_{\varepsilon,\delta};\mathbb{R}^{3\times3})} &\le C\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(B_{\varepsilon,\delta})}\\
&\quad + C\varepsilon\|\mathbf{Q}(0)^{\mathrm{T}}\mathbf{Q}+\mathbf{Q}^{\mathrm{T}}\mathbf{Q}(0)-2\mathbf{I}_3\|_{L^2(-\delta,L;\mathbb{R}^9)}.
\end{aligned}
\tag{8.4}
$$

First, we observe that for any matrices $\mathbf{R}\in SO(3)$, we get $|||\mathbf{R}-\mathbf{I}_3|||^2 = \sqrt{2}|||\mathbf{R}+\mathbf{R}^{\mathrm{T}}-2\mathbf{I}_3|||$. Hence, we have $\sqrt{2}|||\mathbf{Q}(0)^{\mathrm{T}}\mathbf{Q}+\mathbf{Q}^{\mathrm{T}}\mathbf{Q}(0)-2\mathbf{I}_3||| = |||\mathbf{Q}-\mathbf{Q}(0)|||^2$. Using again (8.3), we obtain

$$\|\mathbf{Q}(0)^{\mathrm{T}}\mathbf{Q}+\mathbf{Q}^{\mathrm{T}}\mathbf{Q}(0)-2\mathbf{I}_3\|_{L^2(-\delta,L;\mathbb{R}^9)} \le C\frac{\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(B_{\varepsilon,\delta})}}{\varepsilon^4},$$

which implies with (8.4)

$$\mathbf{G}_s(\mathbf{u}, B_{\varepsilon,\delta}) \le C\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(B_{\varepsilon,\delta})} + C\frac{\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(B_{\varepsilon,\delta})}}{\varepsilon^3}. \tag{8.5}$$

Observing that $\nabla u+(\nabla u)^{\mathrm{T}} = \nabla\mathbf{u}+(\nabla\mathbf{u})^{\mathrm{T}} + (\mathbf{I}_3-\mathbf{Q}(0))^{\mathrm{T}}(\nabla u-(\mathbf{Q}(0)-\mathbf{I}_3)) + (\nabla u-(\mathbf{Q}(0)-\mathbf{I}_3))^{\mathrm{T}}(\mathbf{I}_3-\mathbf{Q}(0)) + 2(\mathbf{Q}(0)+\mathbf{Q}(0)^{\mathrm{T}}-2\mathbf{I}_3)$, we deduce that

$$
\begin{aligned}
\mathbf{G}_s(u, B_{\varepsilon,\delta}) &\le \mathbf{G}_s(\mathbf{u}, B_{\varepsilon,\delta}) + 2|||\mathbf{Q}(0)-\mathbf{I}_3|||\|\nabla u-(\mathbf{Q}(0)-\mathbf{I}_3)\|_{L^2(B_{\varepsilon,\delta};\mathbb{R}^{3\times3})}\\
&\quad + C\varepsilon|||\mathbf{Q}(0)+\mathbf{Q}(0)^{\mathrm{T}}-2\mathbf{I}_3|||\\
&\le \mathbf{G}_s(\mathbf{u}, B_{\varepsilon,\delta}) + C|||\mathbf{Q}(0)-\mathbf{I}_3|||\frac{\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(B_{\varepsilon,\delta})}}{\varepsilon}\\
&\quad + C\varepsilon|||\mathbf{Q}(0)-\mathbf{I}_3|||^2\\
&\le \mathbf{G}_s(\mathbf{u}, B_{\varepsilon,\delta}) + C\frac{\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(B_{\varepsilon,\delta})}}{\varepsilon^3} + C\varepsilon|||\mathbf{Q}(0)-\mathbf{I}_3|||^2.
\end{aligned}
$$

Thanks to (8.5), we obtain (8.1).

Now we carry on by giving two estimates on $|||\mathbf{Q}(0)-\mathbf{I}_3|||^2$.

**Step 2** First estimate on $|||\mathbf{Q}(0)-\mathbf{I}_3|||^2$.

We deal with the restriction of $v$ to the plate. Due to Theorem 3.3 established in [8], the displacement $u = v - I_d$ is decomposed as

$$u(x) = \mathcal{U}(x_1,x_2) + x_3(\mathbf{R}(x_1,x_2)-\mathbf{I}_3)\mathbf{e}_3 + \overline{v}(x), \quad x\in\Omega_\delta, \tag{8.6}$$

where $\mathcal{U}$ belongs to $H^1(\omega;\mathbb{R}^3)$, $\mathbf{R}$ belongs to $H^1(\omega;\mathbb{R}^{3\times 3})$, $\overline{v}$ belongs to $H^1(\Omega_\delta;\mathbb{R}^3)$, and we have the following estimates

$$\begin{aligned}
&\|\overline{v}\|_{L^2(\Omega_\delta;\mathbb{R}^3)} \le C\delta\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)},\\
&\|\nabla\overline{v}\|_{L^2(\Omega_\delta;\mathbb{R}^9)} \le C\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)},\\
&\Big\|\frac{\partial \mathbf{R}}{\partial x_\alpha}\Big\|_{L^2(\omega;\mathbb{R}^9)} \le \frac{C}{\delta^{\frac{3}{2}}}\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)},\\
&\Big\|\frac{\partial \mathcal{U}}{\partial x_\alpha} - (\mathbf{R}-\mathbf{I}_3)\mathbf{e}_\alpha\Big\|_{L^2(\omega;\mathbb{R}^3)} \le \frac{C}{\delta^{\frac{1}{2}}}\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)},\\
&l\|\nabla v - \mathbf{R}\|_{L^2(\Omega_\delta;\mathbb{R}^9)} \le C\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(\Omega_\delta)},
\end{aligned}\tag{8.7}$$

where the constant $C$ does not depend on $\delta$. The following boundary conditions are satisfied

$$\mathcal{U} = 0, \quad \mathbf{R} = \mathbf{I}_3 \quad \text{on } \gamma_0, \quad \overline{v} = 0 \quad \text{on } \Gamma_{0,\delta}. \tag{8.8}$$

The last estimates in (8.3) and (8.7) allow to compare $\mathbf{Q}-\mathbf{I}_3$ and $\mathbf{R}-\mathbf{I}_3$ in the cylinder $C_{\delta,\varepsilon}$. We obtain

$$\begin{aligned}
\varepsilon^2\|\mathbf{Q}-\mathbf{I}_3\|^2_{L^2(-\delta,\delta;\mathbb{R}^9)} \le{}& C\{\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)} + \|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(B_{\varepsilon,\delta})}\}\\
&+ C\delta\|\mathbf{R}-\mathbf{I}_3\|^2_{L^2(D_\varepsilon;\mathbb{R}^9)}.
\end{aligned}$$

Besides, the third estimate in (8.7) and the boundary condition on $\mathbf{R}$ lead to

$$\begin{aligned}
\|\mathbf{R}-\mathbf{I}_3\|^2_{L^2(D_\varepsilon;\mathbb{R}^9)} &\le C\varepsilon^{\frac{3}{2}}\|\mathbf{R}-\mathbf{I}_3\|^2_{L^8(D_\varepsilon;\mathbb{R}^9)} \le C\varepsilon^{\frac{3}{2}}\|\mathbf{R}-\mathbf{I}_3\|^2_{H^1(D_\varepsilon;\mathbb{R}^9)}\\
&\le C\varepsilon^{\frac{3}{2}}\frac{\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)}}{\delta^3}.
\end{aligned}\tag{8.9}$$

Then, we get

$$\begin{aligned}
\varepsilon^2\|\mathbf{Q}-\mathbf{I}_3\|^2_{L^2(-\delta,\delta;\mathbb{R}^9)} \le{}& C\{\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)} + \|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(B_{\varepsilon,\delta})}\}\\
&+ C\varepsilon^{\frac{3}{2}}\frac{\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)}}{\delta^2}.
\end{aligned}\tag{8.10}$$

Furthermore, the third estimate in (8.3) gives

$$\begin{aligned}
|||\mathbf{Q}(0)-\mathbf{I}_3|||^2 &\le \frac{C}{\delta}\|\mathbf{Q}-\mathbf{I}_3\|^2_{L^2(-\delta,\delta;\mathbb{R}^9)} + C\delta\Big\|\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}x_3}\Big\|^2_{L^2(B_{\varepsilon,\delta};\mathbb{R}^9)}\\
&\le \frac{C}{\delta}\|\mathbf{Q}-\mathbf{I}_3\|^2_{L^2(-\delta,\delta;\mathbb{R}^9)} + C\frac{\delta}{\varepsilon^4}\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(B_{\varepsilon,\delta})},
\end{aligned}$$

which using (8.10) yields

$$\begin{aligned}
\varepsilon|||\mathbf{Q}(0)-\mathbf{I}_3|||^2 \le{}& C\Big[\frac{\delta^2}{\varepsilon} + \varepsilon^{\frac{1}{2}}\Big]\frac{\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)}}{\delta^3}\\
&+ C\Big[\delta + \frac{\varepsilon^2}{\delta}\Big]\frac{\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(B_{\varepsilon,\delta})}}{\varepsilon^3}.
\end{aligned}$$

Finally (8.1) and the above estimate lead to

$$\begin{aligned}
\mathbf{G}_s(u, B_{\varepsilon,\delta}) \le{}& C\|\mathrm{dist}(\nabla v, SO(3))\|_{L^2(B_{\varepsilon,\delta})} + C\Big[1 + \frac{\varepsilon^2}{\delta}\Big]\frac{\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(B_{\varepsilon,\delta})}}{\varepsilon^3}\\
&+ C[\delta^2 + \varepsilon^{\frac{3}{2}}]\frac{\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)}}{\varepsilon\delta^3}.
\end{aligned}\tag{8.11}$$

**Step 3** Second estimate on $|||\mathbf{Q}(0)-\mathbf{I}_3|||^2$.

Now, we consider the traces of the two decompositions (8.2) and (8.6) of the displacement $u = v - I_d$ on $D_\varepsilon \times \{0\}$. From (8.3) and (8.7), we have

$$\begin{aligned}&\int_{D_\varepsilon} \|u(x_1,x_2,0)-\mathcal{W}(0)-(\mathbf{Q}(0)-\mathbf{I}_3)(0)(x_1\mathbf{e}_1+x_2\mathbf{e}_2)\|_2^2\\ &= \int_{D_\varepsilon} \|\overline{w}'(x_1,x_2,0)\|_2^2 \le C\varepsilon\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(B_{\varepsilon,\delta})},\\ &\int_{D_\varepsilon} \|u(x_1,x_2,0)-\mathcal{U}(x_1,x_2)\|_2^2\\ &= \int_{D_\varepsilon} \|\overline{v}(x_1,x_2,0)\|_2^2 \le C\delta\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)}.\end{aligned}$$

The above estimates lead to

$$\begin{aligned}&\int_{D_\varepsilon} \|\mathcal{W}(0)+(\mathbf{Q}(0)-\mathbf{I}_3)(x_1\mathbf{e}_1+x_2\mathbf{e}_2)-\mathcal{U}(x_1,x_2)\|_2^2\\ &\le C\delta\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)} + C\varepsilon\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(B_{\varepsilon,\delta})}\end{aligned}$$

and we take the mean values to

$$|\mathcal{W}(0)-\mathcal{M}_{D_\varepsilon}(\mathcal{U})|^2 \le \frac{C}{\varepsilon^2}\{\delta\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)} + \varepsilon\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(B_{\varepsilon,\delta})}\}.$$

The above two estimates give

$$\begin{aligned}&\int_{D_\varepsilon} \|(\mathbf{Q}(0)-\mathbf{I}_3)(x_1\mathbf{e}_1+x_2\mathbf{e}_2)-(\mathcal{U}(x_1,x_2)-\mathcal{M}_{D_\varepsilon}(\mathcal{U}))\|_2^2\\ &\le C\delta\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)} + C\varepsilon\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(B_{\varepsilon,\delta})}.\end{aligned} \tag{8.12}$$

We carry on by estimating $\mathcal{U}-\mathcal{M}_{D_\varepsilon}(\mathcal{U})$. Let us set

$$\mathbf{R}_\alpha = \mathcal{M}_{D_\varepsilon}((\mathbf{R}-\mathbf{I}_3)\mathbf{e}_\alpha) = \frac{1}{|D_\varepsilon|}\int_{D_\varepsilon}(\mathbf{R}(x_1,x_2)-\mathbf{I}_3)\mathbf{e}_\alpha \mathrm{d}x_1\mathrm{d}x_2$$

and we consider the function $\Phi(x_1,x_2) = \mathcal{U}(x_1,x_2)-\mathcal{M}_{D_\varepsilon}(\mathcal{U})-x_1\mathbf{R}_1-x_2\mathbf{R}_2$. Due to the fourth estimate in (8.7) and the Poincaré-Wirtinger's inequality (in order to estimate $\|(\mathbf{R}-\mathbf{I}_3)\mathbf{e}_\alpha-\mathbf{R}_\alpha\|_{L^2(D_\varepsilon;\mathbb{R}^3)}$), we obtain

$$\|\nabla\Phi\|^2_{L^2(D_\varepsilon;\mathbb{R}^2)} \le C\Big(\frac{1}{\delta}+\frac{\varepsilon^2}{\delta^3}\Big)\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)}. \tag{8.13}$$

Noting that $\mathcal{M}_{D_\varepsilon}(\Psi)=0$, the above inequality and the Poincaré-Wirtinger's inequality in the disc $D_\varepsilon$ lead to

$$\|\Phi\|^2_{L^2(D_\varepsilon)} \le C\frac{\varepsilon^2}{\delta}\Big(1+\frac{\varepsilon^2}{\delta^2}\Big)\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)}. \tag{8.14}$$

Estimate (8.12) gives

$$\begin{aligned}&\int_{D_\varepsilon} \|(\mathbf{Q}(0)-\mathbf{I}_3)(x_1\mathbf{e}_1+x_2\mathbf{e}_2)\|_2^2\\ &\le C(\|\Phi\|^2_{L^2(D_\varepsilon)} + \varepsilon^4\|\mathbf{R}_1\|_2^2 + \varepsilon^4\|\mathbf{R}_2\|_2^2 + \delta\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(\Omega_\delta)}\\ &\quad + \varepsilon\|\mathrm{dist}(\nabla v, SO(3))\|^2_{L^2(B_{\varepsilon,\delta})}),\end{aligned}$$

which in turn with (8.9) and (8.14) yields

$$\begin{aligned}&\varepsilon^4(\|(\mathbf{Q}(0)-\mathbf{I}_3)\mathbf{e}_1\|_2^2+\|(\mathbf{Q}(0)-\mathbf{I}_3)\mathbf{e}_2\|_2^2)\\&\le C\Big(\frac{\varepsilon^2}{\delta}+\frac{\varepsilon^{7/2}}{\delta^3}+\delta\Big)\|\mathrm{dist}(\nabla v,SO(3))\|^2_{L^2(\Omega_\delta)}+C\varepsilon\|\mathrm{dist}(\nabla v,SO(3))\|^2_{L^2(B_{\varepsilon,\delta})}\end{aligned}$$

and finally

$$\begin{aligned}&\varepsilon|\|\mathbf{Q}(0)-\mathbf{I}_3\||^2\\&\le C\Big(\frac{\delta^2}{\varepsilon}+\varepsilon^{\frac12}+\frac{\delta^4}{\varepsilon^3}\Big)\frac{\|\mathrm{dist}(\nabla v,SO(3))\|^2_{L^2(\Omega_\delta)}}{\delta^3}+C\varepsilon\frac{\|\mathrm{dist}(\nabla v,SO(3))\|^2_{L^2(B_{\varepsilon,\delta})}}{\varepsilon^3}.\end{aligned}\tag{8.15}$$

Estimates (8.1) and (8.15) yield

$$\begin{aligned}\mathbf{G}_s(u,B_{\varepsilon,\delta})&\le C\|\mathrm{dist}(\nabla v,SO(3))\|_{L^2(B_{\varepsilon,\delta})}+C\frac{\|\mathrm{dist}(\nabla v,SO(3))\|^2_{L^2(B_{\varepsilon,\delta})}}{\varepsilon^3}\\&\quad+C\Big[\varepsilon^{\frac32}+\delta^2+\frac{\delta^4}{\varepsilon^2}\Big]\frac{\|\mathrm{dist}(\nabla v,SO(3))\|^2_{L^2(\Omega_\delta)}}{\varepsilon\delta^3}.\end{aligned}\tag{8.16}$$

**Step 4** Final estimate on $\mathbf{G}_s(u,B_{\varepsilon,\delta})$.

The two estimates of $\mathbf{G}_s(u,B_{\varepsilon,\delta})$ given by (8.11) and (8.16) lead to

(1) if $\varepsilon^2\le\delta$ then

$$\begin{aligned}\mathbf{G}_s(u,B_{\varepsilon,\delta})&\le C\|\mathrm{dist}(\nabla v,SO(3))\|_{L^2(B_{\varepsilon,\delta})}+C\frac{\|\mathrm{dist}(\nabla v,SO(3))\|^2_{L^2(B_{\varepsilon,\delta})}}{\varepsilon^3}\\&\quad+C[\delta^2+\varepsilon^{\frac32}]\frac{\|\mathrm{dist}(\nabla v,SO(3))\|^2_{L^2(\Omega_\delta)}}{\varepsilon\delta^3};\end{aligned}$$

(2) if $\delta\le\varepsilon^2$ then

$$\begin{aligned}\mathbf{G}_s(u,B_{\varepsilon,\delta})&\le C\|\mathrm{dist}(\nabla v,SO(3))\|_{L^2(B_{\varepsilon,\delta})}+C\frac{\|\mathrm{dist}(\nabla v,SO(3))\|^2_{L^2(B_{\varepsilon,\delta})}}{\varepsilon^3}\\&\quad+C[\delta^2+\varepsilon^{\frac32}]\frac{\|\mathrm{dist}(\nabla v,SO(3))\|^2_{L^2(\Omega_\delta)}}{\varepsilon\delta^3}.\end{aligned}$$

We immediately deduce (5.14).

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