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On the Linearized Darboux Equation Arising in Isometric Embedding of the Alexandrov Positive Annulus*

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Abstract In the present paper, the solvability condition of the linearized Gauss-Codazzi system and the solutions to the homogenous system are given. In the meantime, the solvability of a relevant linearized Darboux equation is given. The equations are arising in a geometric problem which is concerned with the realization of the Alexandrov's positive annulus in $\mathbb{R}^3$.

Keywords Alexandrov's positive annulus, Darboux equation, Gauss-Codazzi system, solvability
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1 Introduction

Definition 1.1 (see [1]) *Let T be an annulus, $T = S^1 \times (0,1)$, i.e. $T = \{(x_1, x_2) \mid x_1 \in [-\pi,\pi],\ x_2 \in (0,1)\}$ and $(\vec{r}, g)$ is a smooth (analytic) nonnegative annulus,*

$$\vec{r}:\ \overline{T} \to \mathbb{R}^3, \quad g = \mathrm{d}\vec{r}^2,$$

where $\vec{r}$, g are defined in T and satisfy Alexandrov's assumption (see [1]):

$$\int_T K \mathrm{d}g = 4\pi \quad \text{and} \quad K = 0, \quad \nabla K \neq 0 \quad \text{on} \quad \partial T, \tag{1.1}$$

where K is the Gaussian curvature and $K > 0$ in T. In what follows, we call such an annulus $\vec{r}$ the Alexandrov's positive annulus. And (1.1) is called the Alexandrov condition.

Choosing the origin such that $\vec{n} \cdot \vec{r} > 0$, where $\vec{n}$ is the unit outward normal, we define the function

$$\rho = \frac{1}{2}\vec{r} \cdot \vec{r}. \tag{1.2}$$

Let us consider the Darboux equation satisfied by ρ (see [2])

$$F(x, \rho, \partial\rho, \partial^2\rho) = \frac{1}{|g|}\det(\nabla^2\rho - gI) - K(2\rho - |\nabla\rho|^2) = 0. \tag{1.3}$$

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Its linearized equation of ρ is

$$G = F'(\rho)\phi = \frac{\mathrm{d}}{\mathrm{d}t}(F(\rho + t\phi))\Big|_{t=0}.$$

A direct computation shows that

$$G = -Kh^{ij}(\vec{r}\cdot\vec{n})\nabla_{ij}\phi - 2K(\phi - g^{ij}\rho_i\phi_j),$$

where $(h^{ij}) = (h_{ij})^{-1}$, $(g^{ij}) = (g_{ij})^{-1}$, g_{ij}, h_{ij} are the coefficients of the first and second fundamental forms and $\vec{n}$ is the outward unit normal.

Furthermore, we can rewrite the linearized equation in the divergence form as follows

$$-\frac{1}{\sqrt{|g|}}\partial_i\Big(\frac{K\sqrt{|g|}h^{ij}\partial_j\phi}{(\vec{r}\cdot\vec{n})^2}\Big) - \frac{2K\phi}{(\vec{r}\cdot\vec{n})^3} = f, \tag{1.4}$$

where

$$f = \frac{G}{(\vec{r}\cdot\vec{n})^3}.$$

The Gauss-Codazzi system is well-known as a basic system satisfied by the coefficients of the second fundamental form L, M and N.

For convenience, set

$$l = \frac{L}{\sqrt{|g|}}, \quad m = \frac{M}{\sqrt{|g|}}, \quad n = \frac{N}{\sqrt{|g|}}.$$

The Gauss-Codazzi system says that

$$ln - m^2 = K, \tag{1.5}$$

$$\partial_2 l - \partial_1 m = -l\Gamma_{22}^2 + 2m\Gamma_{12}^2 - n\Gamma_{11}^2, \tag{1.6}$$

$$\partial_2 m - \partial_1 n = l\Gamma_{22}^1 - 2m\Gamma_{12}^1 + n\Gamma_{11}^1. \tag{1.7}$$

Let

$$a_1^1 = -a_2^2 = \dot{m},$$

$$a_2^1 = -\dot{n}, \quad a_1^2 = \dot{l},$$

and then the perturbation of (1.5)–(1.7) is

$$a_1^1 + a_2^2 = 0, \tag{1.8}$$

$$a_1^j h_{j2} - a_2^j h_{j1} = E \tag{1.9}$$

and

$$\partial_1 a_2^j - \partial_2 a_1^j + a_2^l\Gamma_{l1}^j - a_1^l\Gamma_{l2}^j = C_j, \quad j = 1, 2. \tag{1.10}$$

In the present paper, a new method to solve the linearized Gauss-Codazzi system (1.8)–(1.10) is given. First of all, we transform the linearized system to a partial differential equation of the second order discussed sufficiently in [3] and then obtain the solvability condition as well as the solutions to the homogenous problem for the linearized system (1.8)–(1.10).

For the Alexandrov positive annulus, the boundaries are planar (see [1]). We will discuss two different cases respectively:

Case 1 The two boundary planes are parallel;

Case 2 The two boundary planes are not parallel.

For Case 1, let $\vec{k}$ be a unit vector in $\mathbb{R}^3$ which is parallel to the normals of the two boundary planes, and then we have the following theorem.

Theorem 1.1 *The necessary and sufficient condition that the system* (1.8)–(1.10) *is solvable is*

$$\vec{k}\cdot\int_T E\vec{n}+C_j\partial_j\vec{r}=0. \tag{1.11}$$

For Case 2, let $\vec{k}=(0,0,1)$ be a unit vector in $\mathbb{R}^3$ which is parallel to the normal of the boundary planes P_1 where

$$\sigma_0=(\{x_2=0\}\cap\partial T)$$

lies, and $(0,\sin\theta,\cos\theta)$ be the normal of the boundary planes P_2 where

$$\sigma_1=(\{x_2=1\}\cap\partial T)$$

lies and $\cos\theta\neq\pm1$. Then we have the following theorem.

Theorem 1.2 *The necessary and sufficient condition that the system* (1.8)–(1.10) *is solvable is*

$$\vec{k}\cdot\int_T E\vec{n}+C_j\partial_j\vec{r}=\sin\theta\vec{i}\cdot\int_T(E\vec{n}+C_j\partial_j\vec{r})\times\vec{r}. \tag{1.12}$$

In this paper we will see that for any case, the space of the solution to a homogenous problem for the linearized Gauss-Codazzi system is one-dimensional.

For the Alexandrov positive annulus, the linearized Gauss-Codazzi system is a degenerate elliptic system of the first order studied less than the degenerate elliptic equation of the second order. Moreover, solving the partial differential equation of the second order in [3] is closely related to the geometric aspects of the Alexandrov positive annulus in the study of isometric embedding. At the same time, we will associate the linearized Darboux equation with the linearized Gauss-Codazzi system. Concretely, we will turn the linearized Darboux equation to the form of (1.8)–(1.10) with

$$E\vec{n}+C_j\partial_j\vec{r}=f\sqrt{|g|}\vec{r}. \tag{1.13}$$

Therefore, we obtain the solvability condition as well as the solutions to the homogenous problem for (1.4).

Theorem 1.3 *Let* $\vec{r}\in C^\infty(\overline{T},\mathbb{R}^3)$ *be an Alexandrov positive annulus and* $f\in C^\infty(\overline{T})$*. Then the necessary and sufficient condition that* (1.4) *admits a solution* $\phi\in C^\infty(\overline{T})$ *is*

$$\int_T f\vec{r}\,\mathrm{d}A_g=0, \tag{1.14}$$

where A_g is the area element of the metric g. Moreover, the solution is unique up to $\vec{A}\cdot\vec{r}$ for some constant vector $\vec{A}$.

Remark 1.1 If $\vec{r}\in C^\omega(\overline{T},\mathbb{R}^3)$ and $f\in C^\omega(\overline{T})$, correspondingly $\phi\in C^\omega(\overline{T})$ (see [4]).

2 Geometric Preliminaries

Before solving the equations (1.4), (1.8)–(1.10), we need several lemmas.

Lemma 2.1 (see [1]) *If (1.1) is fulfilled, each component of $\vec{r}(\partial T)$ is a planar curve l_i.*

Lemma 2.2 (see [6]) *Let M be a nonnegative compact surface which is of no planar point, $\partial M=\cup l_i$, and each l_i be a planar curve contained in a plane P_i which is tangential to M. Then M is infinitesimal rigid.*

3 The Fundamental Equation

Having the geometric and analytic preliminaries, we will derive the fundamental equation. The process is partly seen in [5]. In what follows, we will turn the linear system (1.8)–(1.10) to a degenerated elliptic equation (see [3]).

Let $\vec{X}$ be a vector, and then $d\vec{X}=\partial_i\vec{X}\mathrm{d}x^i$. Set $a_i^j\partial_j\vec{r}=\vec{Y}_i$, $i=1,2$, where a_i^j, $i,j=1,2$ satisfy (1.8)–(1.10), and $\vec{Z}_i=\partial_i\vec{X}-\vec{Y}_i$ which is to be fixed.

We claim that

$$\partial_1\vec{Z}_2-\partial_2\vec{Z}_1=-E\vec{n}-C_j\partial_j\vec{r}. \tag{3.1}$$

Remark 3.1 Here $\vec{Y}_i$, $\vec{Z}_i$ are not the derivatives of vectors, but $\partial_i\vec{X}$ are.

First, the Poincare lemma says $\mathrm{d}(\mathrm{d}\vec{X})=0$, and then

$$\partial_1\vec{Z}_2-\partial_2\vec{Z}_1=-(\partial_1\vec{Y}_2-\partial_2\vec{Y}_1).$$

To see this, we compute the exterior derivative

$$\mathrm{d}(\vec{Y}_i\mathrm{d}x^i)=(\partial_k a_i^j\partial_j\vec{r})\mathrm{d}x^k\wedge\mathrm{d}x^i+a_i^j(\Gamma_{jk}^l\partial_l\vec{r}+h_{jk}e_3)\mathrm{d}x^k\wedge\mathrm{d}x^i,$$

which implies

$$\partial_1\vec{Y}_2-\partial_2\vec{Y}_1=(a_1^jh_{j2}-a_2^jh_{j1})\vec{n}+(a_2^j-\partial_2a_1^j+a_2^l\Gamma_{l1}^j-a_1^l\Gamma_{l2}^j)\partial_j\vec{r}.$$

By (1.9)–(1.10), we have

$$\partial_1\vec{Z}_2-\partial_2\vec{Z}_1=-E\vec{n}-C_j\partial_j\vec{r}.$$

Since $a_i^j\partial_j\vec{r}=\vec{Y}_i,\ i=1,2$,

$$a_1^1=g^{11}\partial_1\vec{r}\cdot\vec{Y}_1+g^{12}\partial_2\vec{r}\cdot\vec{Y}_1,$$
$$a_2^2=g^{22}\partial_2\vec{r}\cdot\vec{Y}_2+g^{12}\partial_1\vec{r}\cdot\vec{Y}_2.$$

$a_1^1 + a_2^2 = 0$ means

$$g^{11}\partial_1\vec{r}\cdot\vec{Y_1} + g^{12}(\partial_2\vec{r}\cdot\vec{Y_1} + \partial_1\vec{r}\cdot\vec{Y_2}) + g^{22}\partial_2\vec{r}\cdot\vec{Y_2} = 0.$$

Since $\vec{Z_i} = \partial_i\vec{X} - \vec{Y_i}$,

$$\begin{aligned} &g^{11}\partial_1\vec{r}\cdot\vec{X}_1 + g^{12}(\partial_2\vec{r}\cdot\vec{X}_1 + \partial_1\vec{r}\cdot\vec{X}_2) + g^{22}\partial_2\vec{r}\cdot\vec{X}_2 \\ &= g^{11}\partial_1\vec{r}\cdot\vec{Z_1} + g^{12}(\partial_2\vec{r}\cdot\vec{Z_1} + \partial_1\vec{r}\cdot\vec{Z_2}) + g^{22}\partial_2\vec{r}\cdot\vec{Z_2}. \end{aligned} \tag{3.2}$$

Note that

$$\partial_1\vec{r}\cdot\vec{X}_1 = (\partial_1\vec{r}\cdot\vec{X})_1 - \Gamma_{11}^1\partial_1\vec{r}\cdot\vec{X} - \Gamma_{11}^2\partial_2\vec{r}\cdot\vec{X} - h_{11}\vec{n}\cdot\vec{X}, \tag{3.3}$$

$$\partial_1\vec{r}\cdot\vec{X}_2 + \partial_2\vec{r}\cdot\vec{X}_1 = (\partial_2\vec{r}\cdot\vec{X})_1 + (\partial_1\vec{r}\cdot\vec{X})_2 - 2\Gamma_{12}^1\partial_1\vec{r}\cdot\vec{X} - 2\Gamma_{12}^2\partial_2\vec{r}\cdot\vec{X} - 2h_{12}\vec{n}\cdot\vec{X}, \tag{3.4}$$

$$\partial_2\vec{r}\cdot\vec{X}_2 = (\partial_2\vec{r}\cdot\vec{X})_2 - \Gamma_{22}^1\partial_1\vec{r}\cdot\vec{X} - \Gamma_{22}^2\partial_2\vec{r}\cdot\vec{X} - h_{22}\vec{n}\cdot\vec{X}. \tag{3.5}$$

Set

$$u_2 = \sqrt{|g|}(g^{11}\partial_1\vec{r}\cdot\vec{X} + g^{12}\partial_2\vec{r}\cdot\vec{X}), \tag{3.6}$$

$$-u_1 = \sqrt{|g|}(g^{12}\partial_1\vec{r}\cdot\vec{X} + g^{22}\partial_2\vec{r}\cdot\vec{X}) \tag{3.7}$$

and

$$w = \vec{n}\cdot\vec{X}, \tag{3.8}$$

and hence

$$\begin{aligned} \partial_1\vec{r}\cdot\vec{X} &= \frac{1}{\sqrt{|g|}}(g_{11}u_2 - g_{12}u_1), \\ \partial_2\vec{r}\cdot\vec{X} &= \frac{1}{\sqrt{|g|}}(g_{12}u_2 - g_{22}u_1). \end{aligned}$$

Then by (3.3)–(3.8), (3.2) becomes

$$\frac{1}{\sqrt{|g|}}(\partial_1 u_2 - \partial_2 u_1) = 2Hw + g^{ij}\partial_i\vec{r}\cdot\vec{Z_j}. \tag{3.9}$$

Note that

$$w_i = \partial_i\vec{X}\cdot\vec{n} - h_i^j\partial_j\vec{r}\cdot\vec{X} = \vec{Z_i}\cdot\vec{n} - h_i^j\partial_j\vec{r}\cdot\vec{X}, \tag{3.10}$$

i.e.,

$$\begin{aligned} \partial_1 w &= -K\sqrt{|g|}h^{2i}u_i + \vec{Z_1}\cdot\vec{n}, \\ \partial_2 w &= K\sqrt{|g|}h^{1i}u_i + \vec{Z_2}\cdot\vec{n}. \end{aligned}$$

Inserting (3.10) into (3.9) yields the equation satisfied by w

$$-\frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}h^{ij}\partial_j w) - 2Hw = -\frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}h^{ij}\vec{Z_j}\cdot\vec{n}) + g^{ij}\partial_i\vec{r}\cdot\vec{Z_j}, \tag{3.11}$$

which is discussed sufficiently in [3].

If there exists w which solves (3.11), u_1 and u_2 are generated by (3.10). Thus $\vec{X}$ is generated by

$$\vec{X} = \frac{u_2 \partial_1 \vec{r} - u_1 \partial_2 \vec{r}}{\sqrt{|g|}} + w\vec{n}. \tag{3.12}$$

Hence, once $\vec{Z}_i$ are constructed to satisfy (3.1) such that (3.11) is solvable, $\vec{X}$ is determined. Then $\partial_i \vec{X}$, $\vec{Y}_i = \partial_i \vec{X} - \vec{Z}_i$ are given, so we obtain a_i^j by $a_i^j \partial_j \vec{r} = \vec{Y}_i$.

4 The Solvability

Without loss of generality, suppose that $\vec{Z}_j \cdot \vec{n} = 0$ on σ_k, otherwise replace w with $w - u$, where u is a smooth function on σ_k, $\partial_j u = \vec{Z}_j \cdot \vec{n}$, $j = 1, 2$, $k = 0, 1$.

It is necessary that

$$0 = \oint_{\sigma_k} \partial_j \vec{X} \cdot \vec{n} \mathrm{d}x_j = \oint_{\sigma_k} \vec{Z}_j \cdot \vec{n} \mathrm{d}x_j.$$

First, we introduce the boundary value problem for (3.11) (see [3])

$$\begin{cases} \mathcal{L}w = -\dfrac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} h^{ij} \partial_j w) - 2Hw = \mathcal{F}, \\ \displaystyle\oint_{\sigma_k} \sqrt{|g|} h^{ij} \partial_j w \nu_i = 0, \end{cases} \tag{4.1}$$

where

$$\mathcal{F} = -\frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} h^{ij} (\vec{Z}_j \cdot \vec{n} - \partial_j u)) + g^{ij} \partial_i \vec{r} \cdot \vec{Z}_j - 2Hu,$$

$\vec{\nu} = (\nu_1, \nu_2)$ is the unit outward normal of ∂T.

Lemma 4.1 *If there exist $\vec{Z}_i$, $i = 1, 2$ satisfying* (3.1) *on σ_k, $k = 0, 1$,*

$$\nu_2 \vec{Z}_1 \cdot \vec{n} - \nu_1 \vec{Z}_2 \cdot \vec{n} = 0, \tag{4.2}$$

$$\int_{\partial T} (\nu_2 \vec{Z}_1 \times \vec{r} - \nu_1 \vec{Z}_2 \times \vec{r}) = \int_T (-E\vec{n} - C_j \partial_j \vec{r}) \times \vec{r}, \tag{4.3}$$

then the system (1.8)–(1.10) *is solvable.*

Proof Once we prove if $\vec{z}_i$, $i = 1, 2$ satisfy (3.1) and (4.2)–(4.3), (4.1) is solvable, finally the system (1.8)–(1.10) is solvable.

As shown in [3], the differential operator $\mathcal{L}$ is formally self-adjoint and of Fredholmness. Hence, to prove that the boundary value problem (4.1) is solvable, it suffices to verify that $\mathcal{F}$ is perpendicular to the kernel of its adjoint problem. Since the problem is self-adjoint, we only need to compute the kernel of (4.1).

Recalling that $\vec{Z}_j = 0$ and

$$\oint_{\sigma_k} \sqrt{|g|} h^{ij} \partial_j w \nu_i = \oint_{\sigma_k} u_2 \nu_1 + u_1 \nu_2 = 0, \tag{4.4}$$

then

$$\oint_{\sigma_k} (\vec{X} \times \vec{r_j})\nu_i = 0.$$

It is obvious that $d\vec{r} \cdot \vec{X} \times d\vec{r} = 0$. Noting that $\vec{Z_j} = 0$ means $E = 0$ and $C_j = 0$, therefore by (1.8)–(3.10),

$$\partial_1 \vec{X} \times \partial_2 \vec{r} = \partial_2 \vec{X} \times \partial_1 \vec{r},$$

and then there exists a position vector $\vec{v}$ such that $d\vec{v} = \vec{X} \times d\vec{r}$. By Lemmas 2.1–2.2, $\vec{r}$ is infinitesimal rigid. Such a $\vec{v}$ must come from the rigid motion of $\vec{r}$.

$$\vec{v} = \vec{A} \times \vec{r} + \vec{B},$$

where $\vec{A}$ and $\vec{B}$ are constant vectors. Hence $\vec{X} = \vec{A}$,

$$a_i^j = 0 \tag{4.5}$$

and

$$w = \vec{A} \cdot \vec{n},$$

i.e., the kernel of $\mathcal{L}$ is spanned by $\vec{A} \cdot \vec{n}$ for any constant vector $\vec{A}$.

Thus we only need to verify

$$\int_T \Big(-\frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} h^{ij} (\vec{Z_j} \cdot \vec{n} - \partial_j u)) + g^{ij} \partial_i \vec{r} \cdot \vec{Z_j} - 2Hu \Big) \vec{A} \cdot \vec{n} = 0$$

for any constant $\vec{A}$, or simply

$$\int_T \Big(-\frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} h^{ij} \vec{Z_j} \cdot \vec{n} - \partial_j u) + g^{ij} \partial_i \vec{r} \cdot \vec{Z_j} \Big) \vec{n} = 0. \tag{4.6}$$

Note that the expression in the parenthesis in (4.6) is an invariant scalar function on T. Hence we need to verify

$$\int_T h^{ij} (\vec{Z_j} \cdot \vec{n} - \partial_j u) \partial_i \vec{n} + (g^{ij} \partial_i \vec{r} \cdot \vec{Z_j} - 2Hu) \vec{n} = 0. \tag{4.7}$$

By the Weingarten equation, we have

$$\begin{aligned} h^{ij} \vec{Z_j} \cdot \vec{n} \partial_i \vec{n} &= -h^{ij} \vec{Z_j} \cdot \vec{n} h_i^k \partial_k \vec{r} \\ &= -h^{ij} \vec{Z_j} \cdot \vec{n} h_{il} g^{lk} \partial_k \vec{r} \\ &= -\vec{Z_j} \cdot \vec{n} g^{jk} \partial_k \vec{r} \\ &= -g^{ij} \vec{Z_j} \cdot \partial_i \vec{r}. \end{aligned}$$

Hence (4.7) becomes

$$\int_T -g^{ij} (\vec{Z_j} \cdot \vec{n} - \partial_j u) \partial_i \vec{r} + (g^{ij} \partial_i \vec{r} \cdot \vec{Z_j} - 2Hu) \vec{n} = 0. \tag{4.8}$$

An integration by parts shows

$$\begin{aligned}
&\int_T -g^{ij}\partial_j u\partial_i\vec{r} + 2Hu\vec{n} \\
&= \int_T -\frac{1}{\sqrt{|g|}}\sqrt{|g|}g^{ij}\partial_j u\partial_i\vec{r} + 2Hu\vec{n} \\
&= \int_T -\frac{1}{\sqrt{|g|}}\partial_j(\sqrt{|g|}g^{ij}\partial_i\vec{r})u + 2Hu\vec{n} - \int_{\partial T}\sqrt{|g|}ug^{ij}\partial_i\vec{r}\nu_j.
\end{aligned}$$

With the Gauss equation and the identity

$$\partial_k g^{ij} = -g^{il}\Gamma^j_{kl} - g^{jl}\Gamma^i_{kl},$$

we have

$$-\frac{1}{\sqrt{|g|}}\partial_j(\sqrt{|g|}g^{ij}\partial_i\vec{r})u + 2Hu\vec{n} = 0.$$

Hence

$$\int_T -g^{ij}\partial_j u\partial_i\vec{r} + 2Hu\vec{n} = -\int_{\partial T}\sqrt{|g|}ug^{ij}\partial_i\vec{r}\nu_j, \tag{4.9}$$

$$\begin{aligned}
\int_T -g^{ij}(\vec{Z}_j\cdot\vec{n})\partial_i\vec{r} + g^{ij}(\partial_i\vec{r}\cdot\vec{Z}_j)\vec{n} &= \int_T g^{ij}\vec{Z}_j\times(\vec{n}\times\partial_i\vec{r}) \\
&= \int_T \frac{-1}{\sqrt{|g|}}(\vec{Z}_1\times\partial_2\vec{r} - \vec{Z}_2\times\partial_1\vec{r}).
\end{aligned}$$

Since

$$\vec{Z}_1\times\partial_2\vec{r} - \vec{Z}_2\times\partial_1\vec{r} = \partial_2(\vec{Z}_1\times\vec{r}) - \partial_1(\vec{Z}_2\times\vec{r}) + (\partial_1\vec{Z}_2 - \partial_2\vec{Z}_1)\times\vec{r},$$

by (3.1) and integration by parts, we have

$$\int_{\partial T}(\nu_2\vec{Z}_1\times\vec{r} - \nu_1\vec{Z}_2\times\vec{r} - \sqrt{|g|}ug^{ij}\partial_i\vec{r}\nu_j) = \int_T(-E\vec{n} - C_j\partial_j\vec{r})\times\vec{r}. \tag{4.10}$$

By the definition of u, we have

$$\begin{aligned}
&\int_{\partial T}((\nu_2\vec{Z}_1\cdot\vec{n})\vec{n}\times\vec{r} - (\nu_1\vec{Z}_2\cdot\vec{n})\vec{n}\times\vec{r} - \sqrt{|g|}ug^{ij}\partial_i\vec{r}\nu_j) \\
&= \vec{n}\times\int_{\partial T}(\nu_2u_1\vec{r} - \nu_1u_2\vec{r} + \nu_2u\partial_1\vec{r} - \nu_1u\partial_2\vec{r}) \\
&= \vec{n}\times\int_{\partial T}(\nu_2(u\vec{r})_1 - \nu_1(u\vec{r})_2) \\
&= 0.
\end{aligned}$$

Inserting the equality above into (4.10), we have

$$\int_{\partial T}\nu_2(\vec{Z}_1 - (\vec{Z}_1\cdot\vec{n})\vec{n})\times\vec{r} - \nu_1(\vec{Z}_2 - (\vec{Z}_2\cdot\vec{n})\vec{n})\times\vec{r}) = \int_T(-E\vec{n} - C_j\partial_j\vec{r})\times\vec{r}. \tag{4.11}$$

Hence if on σ_k, $k = 0, 1$,

$$\nu_2\vec{Z}_1\cdot\vec{n} - \nu_1\vec{Z}_2\cdot\vec{n} = 0,$$

then

$$\int_{\partial T}(\nu_2\vec{Z}_1\times\vec{r}-\nu_1\vec{Z}_2\times\vec{r})=\int_T(-E\vec{n}-C_j\partial_j\vec{r})\times\vec{r}.$$

Hence if (4.2) and (4.3) are valid, the system (1.8)–(1.10) is solvable.

In what follows, we will give the necessary and sufficient condition that the system (1.8)–(1.10) is solvable based on Lemma 4.1. In the next section, we will see that the solution to (1.8)–(1.10) is not unique.

We will discuss the condition in two cases respectively:

Case 1 The two boundary planes are parallel;

Case 2 The two boundary planes are not parallel.

For Case 1, we choose the coordinates as follows. Let $\vec{k}$ be a unit vector in $\mathbb{R}^3$ which is parallel to the normals of the two boundary planes, $\vec{a}$ be a unit circle parameterized by the arc length parameter x_1 such that

$$\vec{a}\perp\vec{k},\quad |\vec{a}|=|\vec{a}'|=1,\quad \forall x_1\in[0,2\pi].$$

Without loss of generality, suppose that the surface can be of the form

$$\vec{r}(x_1,x_2)=x_2\vec{k}+S(x_1,x_2)\vec{a}(x_1),\quad x_1\in[0,2\pi],\ x_2\in[0,1]. \tag{4.12}$$

In the case, the unit outward norms of σ_0 and σ_1 are $\vec{\nu}=(0,-1)$ and $\vec{\nu}=(0,1)$, respectively. Thus (4.2)–(4.3) become

$$\vec{Z}_1\cdot\vec{n}=0,\quad \text{on } \sigma_k, \tag{4.13}$$

$$\int_{\sigma_1}\vec{Z}_1\times\vec{r}-\int_{\sigma_0}\vec{Z}_1\times\vec{r}=\int_T(-E\vec{n}-C_j\partial_j\vec{r})\times\vec{r}. \tag{4.14}$$

In what follows, we will prove Theorem 1.1.

Proof of Theorem 1.1 The necessity follows easily. Recalling that

$$\partial_1\vec{Z}_2-\partial_2\vec{Z}_1=-(\partial_1\vec{Y}_2-\partial_2\vec{Y}_1) \tag{4.15}$$

with (3.1), we have

$$\partial_1\vec{Y}_2-\partial_2\vec{Y}_1=E\vec{n}+C_j\partial_j\vec{r}, \tag{4.16}$$

and hence

$$\int_T E\vec{n}+C_j\partial_j\vec{r}=\int_T\partial_1\vec{Y}_2-\partial_2\vec{Y}_1=\int_{\sigma_0}\vec{Y}_1-\int_{\sigma_1}\vec{Y}_1.$$

Noting that on ∂T, $a_i^j\partial_j\vec{r}=\vec{Y}_i$, $\vec{Y}_i\cdot\vec{n}=\vec{Y}_i\cdot\vec{k}=0$, therefore

$$\vec{k}\cdot\int_T E\vec{n}+C_j\partial_j\vec{r}=\int_{\sigma_0}\vec{Y}_1\cdot\vec{n}-\int_{\sigma_1}\vec{Y}_1\cdot\vec{n}=0. \tag{4.17}$$

The sufficiency is proved by Lemma 4.1, once we illustrate the fact that if (1.11) is valid, then there exist $\vec{Z}_i$, $i=1,2$ satisfying (3.1) and (4.2)–(4.3).

Assume that $\vec{i}=(1,0,0)$, $\vec{j}=(0,1,0)$, $\vec{k}=(0,0,1)$, $\vec{a}(x_1)=(\cos(x_1),\ \sin(x_1),0)$, and

$$\begin{aligned} &E\vec{n}+C_j\partial_j\vec{r} \\ &=\Big(\sum_{m=-\infty}^{+\infty} u_m(x_2)\exp(\mathrm{i}mx_1), \sum_{m=-\infty}^{+\infty} v_m(x_2)\exp(\mathrm{i}mx_1), \sum_{m=-\infty}^{+\infty} w_m(x_2)\exp(\mathrm{i}mx_1)\Big), \end{aligned} \tag{4.18}$$

$$\int_T (-E\vec{n}-C_j\partial_j\vec{r})\times\vec{r}=(C^1,C^2,C^3). \tag{4.19}$$

Set

$$\vec{Z}_1=\Big(y_1+\int_0^{x_2}u_0, y_2+\int_0^{x_2}v_0+\cos(x_1)\int_0^{x_2}y_3, \int_0^{x_2}w_0\Big), \tag{4.20}$$

where y_i are constants to be fixed.

By (1.11), we have $\int_0^1 w_0=0$ which means such a $\vec{Z}_1$ satisfies (4.13).

(4.14) can be rewritten as

$$\oint_{\sigma_1}\vec{Z}_1\times\vec{r}-\oint_{\sigma_0}\vec{Z}_1\times\vec{r}=(C^1,C^2,C^3), \tag{4.21}$$

where on σ_1,

$$\vec{Z}_1=\Big(y_1+\int_0^1 u_0, y_2+\int_0^1 v_0+y_3\cos(x_1),0\Big), \tag{4.22}$$

$$\vec{r}=(S(x_1,1)\cos(x_1),S(x_1,1)\sin(x_1),1), \tag{4.23}$$

and on σ_0,

$$\begin{aligned} &\vec{Z}_1=(y_1,y_2,0), \\ &\vec{r}=(S(x_1,0)\cos(x_1),S(x_1,0)\sin(x_1),0). \end{aligned}$$

(4.14) is an algebraic system of y_i

$$\begin{cases} 2\pi\Big(y_2+\int_0^1 v_0\Big)=C^1, \\ -2\pi\Big(y_1+\int_0^1 u_0\Big)=C^2, \\ \int_0^{2\pi}((S(x_1,1)-S(x_1,0))(\sin(x_1)y_1-\cos(x_1)y_2)-S(x_1,1)\cos^2(x_1)y_3)\mathrm{d}x_1 \\ \quad =C^3+C^4, \end{cases} \tag{4.24}$$

where

$$C^4=-\int_0^{2\pi}(S(x_1,1)-S(x_1,0))\Big(\sin(x_1)\int_0^1 u_0-\cos(x_1)\int_0^1 v_0\Big)\mathrm{d}x_1.$$

Noting that $S(x_1,1)>0$, $\cos^2(x_1)\geq 0$ and $\int_0^{2\pi}S(x_1,1)\cos^2(x_1)\mathrm{d}x_1>0$, then the algebraic system is solvable.

By (3.1), (4.18) and (4.20), we have

$$\partial_1 \vec{Z}_2 = \Big(-\sum_{m\neq 0} u_m(x_2)\exp(\mathrm{i}mx_1), y_3\cos(x_1) - \sum_{m\neq 0} v_m(x_2)\exp(\mathrm{i}mx_1), -\sum_{m\neq 0} w_m(x_2)\exp(\mathrm{i}mx_1)\Big),$$

which is solvable, and thus there exist such $\vec{Z}_i$, $i = 1, 2$ as required in Lemma 4.1.

For Case 2, without loss of generality, assume that the equations satisfied by P_1 and P_2 are $z = 0$ and $\sin\theta y + \cos\theta z + 1 = 0$, respectively.

Consider a family of planes with the parameter x_2:

$$(1 - x_2)z + x_2(\sin\theta y + \cos\theta z + 1) = 0 \tag{4.25}$$

with the corresponding normals

$$\vec{n}(x_2) = (0, x_2\sin\theta, 1 - x_2 + x_2\cos\theta).$$

Then the surface is of the form

$$\vec{r}(x_1, x_2) = x_2\frac{\vec{n}(x_2)}{|\vec{n}(x_2)|} + S(x_1, x_2)\vec{a}(x_1)A(x_2), x_1 \in [0, 2\pi], \quad x_2 \in [0, 1], \tag{4.26}$$

where $A(x_2)$ is a matrix defined by

$$A(x_2) = \frac{1}{|\vec{n}(x_2)|}\begin{pmatrix} |\vec{n}(x_2)| & 0 & 0 \\ 0 & 1 - x_2 + x_2\cos\theta & -x_2\sin\theta \\ 0 & x_2\sin\theta & 1 - x_2 + x_2\cos\theta \end{pmatrix}.$$

In what follows we will prove Theorem 1.2.

Proof of Theorem 1.2 The necessity follows easily.

$$\begin{aligned}\int_{\sigma_1}\vec{Y}_1\times\vec{r} - \int_{\sigma_0}\vec{Y}_1\times\vec{r} &= \int_T \partial_2(\vec{Y}_1\times\vec{r}) - \partial_1(\vec{Y}_2\times\vec{r}) \\ &= \int_T (\partial_2\vec{Y}_1 - \partial_1\vec{Y}_2)\times\vec{r}) + \vec{Y}_1\times\partial_2\vec{r} - \vec{Y}_2\times\partial_1\vec{r} \\ &= \int_T (-E\vec{n} - C_j\partial_j\vec{r})\times\vec{r},\end{aligned}$$

where we use (4.16) and

$$\vec{Y}_1\times\partial_2\vec{r} - \vec{Y}_2\times\partial_1\vec{r} = 0,$$

and hence

$$\vec{i}\cdot\Big(\int_{\sigma_1}\vec{Y}_1\times\vec{r} - \int_{\sigma_0}\vec{Y}_1\times\vec{r}\Big) = \vec{i}\cdot\int_T -E\vec{n} - C_j\partial_j\vec{r}. \tag{4.27}$$

Recalling that $\vec{Y}_i\cdot\vec{n} = 0$ means $\vec{Y}_1\cdot\vec{k} = 0$ on σ_0 and $\sin\theta\vec{Y}_1\cdot\vec{j} + \cos\theta\vec{Y}_1\cdot\vec{k} = 0$ on σ_1, we have on σ_1,

$$\int_{\sigma_1}\vec{Y}_1\cdot\vec{j} = \frac{\cos\theta}{\sin\theta}\vec{k}\cdot\int_T E\vec{n} + C_j\partial_j\vec{r} \tag{4.28}$$

and

$$\int_{\sigma_1} \vec{Y}_1 \cdot \vec{k} = -\vec{k} \cdot \int_T E\vec{n} + C_j \partial_j \vec{r}, \tag{4.29}$$

where we use (4.16) again.

Inserting (4.26) and (4.28)–(4.29) into (4.27) yields (1.12).

Similarly to Theorem 1.1, the sufficiency is proved by Lemma 4.1, once we illustrate the fact that if (1.12) is valid, there exist $\vec{Z}_i$, $i = 1, 2$ satisfying (3.1) and (4.2)–(4.3).

Set

$$\vec{Z}_1 = \Big(\int_0^{x_2} u_0 + \sin(x_1)\int_0^{x_2} y_1, -\frac{\cos\theta}{\sin\theta}\int_0^1 w_0 - \int_{x_2}^1 v_0 - \cos(x_1)\int_{x_2}^1 y_2, \int_0^{x_2} w_0\Big), \tag{4.30}$$

where y_i are constants to be fixed.

It is easy to check that $\vec{Z}_1$ satisfies (4.13). (4.14) can be rewritten as

$$\oint_{\sigma_1} \vec{Z}_1 \times \vec{r} - \oint_{\sigma_0} \vec{Z}_1 \times \vec{r} = (C^1, C^2, C^3),$$

where on σ_1,

$$\begin{aligned}
&\vec{Z}_1 = \Big(\int_0^1 u_0 + \sin(x_1) y_1, -\frac{\cos\theta}{\sin\theta}\int_0^1 w_0, \int_0^1 w_0\Big),\\
&\vec{r} = S(x_1, 1)(\cos(x_1), \cos\theta\sin(x_1), -\sin\theta\sin(x_1)) + (0, \sin\theta, \cos\theta),
\end{aligned}$$

and on σ_0,

$$\begin{aligned}
&\vec{Z}_1 = \Big(0, -\frac{\cos\theta}{\sin\theta}\int_0^1 w_0 - \int_0^1 v_0 - \cos(x_1) y_2, 0\Big),\\
&\vec{r} = S(x_1, 0)(\cos(x_1), \sin(x_1), 0).
\end{aligned}$$

We obtain an algebraic system of y_i

$$\begin{cases}
\dfrac{\int_0^1 w_0}{\sin\theta} = C^1,\\
y_1 \sin\theta \displaystyle\int_0^{2\pi} S(x_1, 1)\sin^2(x_1)\mathrm{d}x_1 = C^2 + C^5,\\
y_1 \cos\theta \displaystyle\int_0^{2\pi} S(x_1, 1)\sin^2(x_1)\mathrm{d}x_1 - y_2 \int_0^{2\pi} S(x_1, 0)\cos^2(x_1)\mathrm{d}x_1 = C^3 + C^6,
\end{cases}$$

where

$$\begin{aligned}
C^5 &= -\int_0^{2\pi} S(x_1, 1)\Big(\int_0^1 w_0 \cos(x_1) + \sin\theta \int_0^1 u_0 \sin(x_1)\Big)\mathrm{d}x_1 + 2\pi\cos\theta \int_0^1 u_0,\\
C^6 &= -\int_0^{2\pi} S(x_1, 1)\Big(\cos\theta \int_0^1 u_0 \sin(x_1) + \frac{\cos\theta}{\sin\theta}\int_0^1 w_0 \cos(x_1)\Big)\mathrm{d}x_1\\
&\quad + \int_0^{2\pi} S(x_1, 0)\Big(\frac{\cos\theta}{\sin\theta}\int_0^1 w_0 + \int_0^1 v_0\Big)\cos(x_1)\mathrm{d}x_1 - 2\pi\sin\theta \int_0^1 u_0.
\end{aligned}$$

The first equation of the algebraic system above is nothing but (1.12). Note that $S(x_1, 1) > 0$, $S(x_1, 0) > 0$, $\sin^2(x_1) \geq 0$, $\cos^2(x_1) \geq 0$,

$$\int_0^{2\pi} S(x_1, 1)\sin^2(x_1)\mathrm{d}x_1 > 0, \quad \int_0^{2\pi} S(x_1, 0)\cos^2(x_1)\mathrm{d}x_1 > 0.$$

Thus the algebraic system is solvable.

By (3.1), (4.18) and (4.30), we have that

$$\begin{aligned}\partial_1 \vec{Z}_2 = &\Big(y_1 \sin(x_1) - \sum_{m\neq 0} u_m(x_2)\exp(\mathrm{i}mx_1)\Big)\vec{i}\\ &+ \Big(y_2\cos(x_1) - \sum_{m\neq 0} v_m(x_2)\exp(\mathrm{i}mx_1)\Big)\vec{j} - \sum_{m\neq 0} w_m(x_2)\exp(\mathrm{i}mx_1)\vec{k},\end{aligned}$$

which is solvable, and thus there exist such $\vec{Z}_i$, $i = 1, 2$ as required in Lemma 4.1.

In the remainder of the section, we will turn the linearized Darboux equation (1.4) to the form of (1.8)–(1.10) with (1.13) and then prove Theorem 1.3.

Proof of Theorem 1.3 Choosing the local coordinate (x_1, x_2) on T, we assume

$$\vec{r} = \lambda_i \partial_i \vec{r} + \mu\vec{n},$$

where $\lambda_i = g^{ij}\partial_j\vec{r}\cdot\vec{r}$, $\mu = \vec{r}\cdot\vec{n}$. Since $\mu \neq 0$, $\partial_1\vec{r}$, $\partial_2\vec{r}$ and $\vec{r}$ form another moving framework.

Noting that on ∂T, $(\vec{r}\cdot\vec{n})_1 = \partial_1\vec{r}\cdot\vec{n} + \vec{r}\cdot\vec{n}_1 = 0$, hence on ∂T, $\mu \equiv$ constant.

Letting ϕ solve (1.4), we have

$$\vec{\tau} = g^{ij}\phi_i\partial_j\vec{r} + \frac{\phi - g^{ij}\phi_i\rho_j}{\mu}\vec{n},$$

where ρ is defined in (1.2). Then

$$\begin{aligned}&\phi = \vec{\tau}\cdot\vec{r},\\ &\vec{\tau}\cdot\partial_i\vec{r} = \phi_i,\end{aligned}$$

and hence

$$\partial_i\vec{\tau}\cdot\vec{r} = 0.$$

Then (1.4) becomes

$$N(\vec{\tau}_1\cdot\partial_1\vec{r}) - M(\vec{\tau}_1\cdot\partial_2\vec{r} + \vec{\tau}_2\cdot\partial_1\vec{r}) + L(\vec{\tau}_2\cdot\partial_2\vec{r}) = f|g|(\vec{r}\cdot\vec{n})^2. \tag{4.31}$$

Since $\phi = \vec{\tau}\cdot\vec{r}$ is periodic in x_1, $\partial_i\vec{r}\cdot\vec{\tau} = \phi_i$, $\vec{\tau}$ is periodic in x_1 too.

Note that $\partial_i\vec{\tau}\cdot\vec{r} = 0$. Then

$$\partial_i\vec{\tau} = \vec{r}\times\Big(\frac{\partial_i\vec{\tau}\times\vec{r}}{2\rho}\Big).$$

Set

$$\frac{\partial_i\vec{\tau}\times\vec{r}}{2\rho} = a_i^j\partial_j\vec{r} + b_i\vec{r},$$

and then

$$\partial_i \vec{\tau} = \vec{r} \times a_i^j \partial_j \vec{r}.$$

By (4.31) we have

$$-(a_1^j h_{j2} - a_2^j h_{j1}) = f\sqrt{|g|}\vec{r} \cdot \vec{n}. \tag{4.32}$$

Since $(\vec{\tau}_1 \cdot \vec{r})_2 - (\vec{\tau}_2 \cdot \vec{r})_1 = \phi_{12} - \phi_{21} = 0$, $\vec{\tau}_1 \cdot \partial_2 \vec{r} - \vec{\tau}_2 \cdot \partial_1 \vec{r} = 0$, i.e.,

$$a_1^1 + a_2^2 = 0. \tag{4.33}$$

In what follows, we will derive the other equations satisfied by a_i^j.

The Poincare lemma says $\mathrm{d}^2\vec{\tau} = 0$, and therefore

$$\begin{aligned} &\vec{r} \times (a_1^1 \partial_1 \vec{r} + a_1^2 \partial_2 \vec{r})_2 + \partial_2 \vec{r} \times (a_1^1 \partial_1 \vec{r} + a_1^2 \partial_2 \vec{r}) \\ =\ &\vec{r} \times (a_2^1 \partial_1 \vec{r} + a_2^2 \partial_2 \vec{r})_1 + \partial_1 \vec{r} \times (a_2^1 \partial_1 \vec{r} + a_2^2 \partial_2 \vec{r}). \end{aligned}$$

Since $a_1^1 + a_2^2 = 0$, we have

$$\vec{r} \times (a_1^1 \partial_1 \vec{r} + a_1^2 \partial_2 \vec{r})_2 = \vec{r} \times (a_2^1 \partial_1 \vec{r} + a_2^2 \partial_2 \vec{r})_1.$$

By (4.32), we have

$$(a_1^1 \partial_1 \vec{r} + a_1^2 \partial_2 \vec{r})_2 - (a_2^1 \partial_1 \vec{r} + a_2^2 \partial_2 \vec{r})_1 = -f\sqrt{|g|}\vec{r}. \tag{4.34}$$

(4.33) and (4.34) form a system satisfied by a_i^j which is nothing but the linearized Gauss-Codazzi system.

Since $\oint_{\sigma_k} \partial_1 \vec{\tau} \mathrm{d}x_1 = 0$,

$$\begin{aligned} &\oint_{\sigma_k} \vec{r} \times (a_1^1 \partial_1 \vec{r} + a_1^2 \partial_2 \vec{r}) \mathrm{d}x_1 \\ =\ &\oint_{\sigma_k} (\lambda_i \partial_i \vec{r} + \mu \vec{n}) \times (a_1^1 \partial_1 \vec{r} + a_1^2 \partial_2 \vec{r}) \mathrm{d}x_1 \\ =\ &\vec{n} \oint_{\sigma_k} |g|(\lambda_1 a_1^2 - \lambda_2 a_1^1) \mathrm{d}x_1 + \mu \vec{n} \times \oint_{\sigma_k} (a_1^1 \partial_1 \vec{r} + a_1^2 \partial_2 \vec{r}) \mathrm{d}x_1. \end{aligned}$$

Hence $\oint_{\sigma_k} (a_1^1 \partial_1 \vec{r} + a_1^2 \partial_2 \vec{r}) \mathrm{d}x_1 = 0$.

Integrating both sides of (4.34) by parts yields

$$\int_T f\vec{r} \mathrm{d}A_g = \oint_{\partial T} (a_1^1 \partial_1 \vec{r} + a_1^2 \partial_2 \vec{r}) \mathrm{d}x_1.$$

Thus we have proved the necessity in Theorem 1.3.

As to the sufficiency, first we note that the system (4.33)–(4.34) is nothing but (1.8)–(1.9) with (1.13).

Set

$$\vec{Z}_1 = \Big(\int_0^{x^2} u_0, \int_0^{x^2} v_0, \int_0^{x^2} w_0 \Big). \tag{4.35}$$

By (1.14), we have $\int_0^1 u_0 = 0$, $\int_0^1 v_0 = 0$ and $\int_0^1 w_0 = 0$, which mean that $\vec{Z}_1 = 0$ on σ_k, and such a $\vec{Z}_1$ satisfies (4.13)–(4.14). Set

$$\partial_1 \vec{Z}_2 = \Big(-\sum_{m\neq 0} u_m(x^2)\exp(\mathrm{i}mx^1), -\sum_{m\neq 0} v_m(x^2)\exp(\mathrm{i}mx^1), -\sum_{m\neq 0} w_m(x^2)\exp(\mathrm{i}mx^1)\Big)$$

which is solvable, and thus there exist such $\vec{Z}_i$, $i = 1, 2$ as required in Lemma 4.1. Hence the system (4.33)–(4.34) is solvable.

Furthermore, $0 = \oint_{\sigma_k} \partial_j \vec{X} \cdot \vec{n} \mathrm{d}x_j$ and

$$\oint_{\sigma_k} \sqrt{|g|} h^{ij} \partial_j w \nu_i = \oint_{\sigma_k} u_2\nu_1 + u_1\nu_2 = 0. \tag{4.36}$$

So $\oint_{\sigma_k} (\vec{X} \times \vec{r}_j)\mathrm{d}x_j = 0$.

And

$$\oint_{\sigma_k} (\partial_j \vec{X} \times \vec{r})\mathrm{d}x_j = \oint_{\sigma_k} (\vec{X} \times \vec{r})_j \mathrm{d}x_j - \oint_{\sigma_k} (\vec{X} \times \vec{r}_j)\mathrm{d}x_j = 0. \tag{4.37}$$

By (4.37) we have

$$\begin{aligned}
\oint_{\sigma_k} \vec{\tau}_j \mathrm{d}x_j &= -\oint_{\sigma_k} a_j^i \partial_i \vec{r} \times \vec{r} \\
&= -\oint_{\sigma_k} \vec{Y}_j \times \vec{r} \mathrm{d}x_j \\
&= \oint_{\sigma_k} (\vec{Z}_j - \vec{X}_j) \times \vec{r} \mathrm{d}x_j \\
&= 0.
\end{aligned}$$

Thus there exists a $\vec{\tau}$ such that

$$\partial_i \vec{\tau} = \vec{r} \times a_i^j \partial_j \vec{r},$$

where a_i^j satisfy (4.33)–(4.34), and then the function $\phi = \vec{\tau} \cdot \vec{r}$ is the solution to (1.4).

5 The Solution to the Homogenous Problem

Consider the case of $E = 0$ and $C_j = 0$, i.e., the homogenous problem for (1.8)–(1.10). For convenience, define the operator $\vec{\mathcal{L}}$ of a_i^j by

$$\vec{\mathcal{L}}(a_i^j) = \partial_1 \begin{pmatrix} 0 \\ 0 \\ a_2^1 \\ a_2^2 \end{pmatrix} - \partial_2 \begin{pmatrix} 0 \\ 0 \\ a_1^1 \\ a_1^2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 1 \\ h_{12} & h_{22} & -h_{11} & -h_{21} \\ -\Gamma_{12}^1 & -\Gamma_{22}^1 & \Gamma_{11}^1 & \Gamma_{21}^1 \\ -\Gamma_{12}^2 & -\Gamma_{22}^2 & \Gamma_{11}^2 & \Gamma_{21}^2 \end{pmatrix} \begin{pmatrix} a_1^1 \\ a_1^2 \\ a_2^1 \\ a_2^2 \end{pmatrix}. \tag{5.1}$$

It is easy to follow from $\vec{\mathcal{L}}(a_i^j) = 0$ that

$$\partial_2 \vec{Y}_1 - \partial_1 \vec{Y}_2 = 0 \tag{5.2}$$

and

$$\partial_2 (\vec{r} \times \vec{Y}_1) - \partial_1 (\vec{r} \times \vec{Y}_2) = 0. \tag{5.3}$$

By (5.3), integration by parts yields

$$\begin{aligned}
0 &= \int_T (\partial_2(\vec{r}\times\vec{Y}_1) - \partial_1(\vec{r}\times\vec{Y}_2)) \\
&= \int_{\partial T} (\nu_2\vec{r}\times\vec{Y}_1 - \nu_1\vec{r}\times\vec{Y}_2) \\
&= \int_{\partial T} \vec{n}\sqrt{|g|}(\nu_2((g^{11}\rho_1+g^{21}\rho_2)a_1^2 - (g^{12}\rho_1+g^{22}\rho_2)a_1^1) + \nu_2(\vec{r}\cdot\vec{n})\vec{n}\times(a_1^1\partial_1\vec{r}+a_1^2\partial_2\vec{r})) \\
&\quad - \int_{\partial T} (\vec{n}\sqrt{|g|}\nu_1((g^{11}\rho_1+g^{21}\rho_2)a_2^2 - (g^{12}\rho_1+g^{22}\rho_2)a_2^1) \\
&\quad + \nu_1(\vec{r}\cdot\vec{n})\vec{n}\times(a_2^1\partial_1\vec{r}+a_2^2\partial_2\vec{r})).
\end{aligned} \tag{5.4}$$

By (5.2), integration by parts yields

$$\begin{aligned}
0 &= \int_T (\partial_2\vec{Y}_1 - \partial_1\vec{Y}_2) \\
&= \int_{\partial T} (\nu_2\vec{Y}_1 - \nu_1\vec{Y}_2) \\
&= \int_{\partial T} (\nu_2(a_1^1\partial_1\vec{r}+a_1^2\partial_2\vec{r}) - \nu_1(a_2^1\partial_1\vec{r}+a_2^2\partial_2\vec{r})).
\end{aligned} \tag{5.5}$$

It is worth mentioning that

Lemma 5.1 *If for σ_k, $k=0,1$,*

$$\int_{\sigma_k} (\nu_2\vec{Y}_1 - \nu_1\vec{Y}_2) = 0, \tag{5.6}$$

$$\int_{\sigma_k} (\nu_2\vec{r}\times\vec{Y}_1 - \nu_1\vec{r}\times\vec{Y}_2) = 0, \tag{5.7}$$

then the kernel of $\vec{\mathcal{L}}$ is a zero space.

Proof If (5.6)–(5.7) are valid, there exists a vector $\vec{Y}$ such that $\mathrm{d}\vec{Y} = \vec{Y}_i\mathrm{d}x^i$, and

$$\begin{aligned}
0 &= \int_{\sigma_k} (\nu_2\vec{r}\times\vec{Y}_1 - \nu_1\vec{r}\times\vec{Y}_2) \\
&= -\int_{\sigma_k} (\nu_2\partial_1\vec{r}\times\vec{Y} - \nu_1\partial_2\vec{r}\times\vec{Y}) \\
&= -\int_{\sigma_k} \mathrm{d}\vec{r}\times\vec{Y},
\end{aligned}$$

which means that there exists an isometric deformation $\vec{v}$ such that $\mathrm{d}\vec{v} = \vec{Y}\times d\vec{r}$. By Lemmas 2.1–2.2, $\vec{r}$ is infinitesimal rigid. By (4.5), we have $a_i^j = 0$, i.e. the kernel of $\vec{\mathcal{L}}$ is a zero space if we impose the boundary conditions (5.6)–(5.7).

In Case 1, (5.4)–(5.5) are of the form

$$\begin{aligned}
&\vec{k}\Big(\int_{\sigma_1} + \int_{\sigma_0} \sqrt{|g|}((g^{11}\rho_1+g^{21}\rho_2)a_1^2 - (g^{12}\rho_1+g^{22}\rho_2)a_1^1)\Big) \\
&+ \vec{k}\times\Big(\int_{\sigma_1} + \int_{\sigma_0} (a_1^1\partial_1\vec{r}+a_1^2\partial_2\vec{r})\Big) = 0,
\end{aligned} \tag{5.8}$$

$$\int_{\sigma_1} (a_1^1\partial_1\vec{r}+a_1^2\partial_2\vec{r}) - \int_{\sigma_0} (a_1^1\partial_1\vec{r}+a_1^2\partial_2\vec{r}) = 0. \tag{5.9}$$

By (5.8) and $\partial_i\vec{r}\cdot\vec{k}=0$ on $\sigma_k, k=0,1$, it is automatic that

$$\int_{\sigma_1}+\int_{\sigma_0}(a_1^1\partial_1\vec{r}+a_1^2\partial_2\vec{r})=0.$$

Combining the above equality with (5.9), we have

$$\begin{aligned}0&=\int_{\sigma_0}(a_1^1\partial_1\vec{r}+a_1^2\partial_2\vec{r})\\&=\int_{\sigma_1}(a_1^1\partial_1\vec{r}+a_1^2\partial_2\vec{r}),\end{aligned}$$

i.e., (5.6) is automatically satisfied.

Introduce a boundary value problem for the homogenous equations $\vec{\mathcal{L}}(a_i^j)=0$,

$$\begin{cases}\vec{\mathcal{L}}(a_i^j)=0,\\ \oint_{\sigma_1}\sqrt{|g|}((g^{11}\rho_1+g^{21}\rho_2)a_1^2-(g^{12}\rho_1+g^{22}\rho_2)a_1^1)=1,\\ \oint_{\sigma_0}\sqrt{|g|}((g^{11}\rho_1+g^{21}\rho_2)a_1^2-(g^{12}\rho_1+g^{22}\rho_2)a_1^1)=-1.\end{cases}\tag{5.10}$$

Theorem 5.1 *There exists a solution to* (5.10) *which is unique, and the kernel of the operator* $\vec{\mathcal{L}}$ *is spanned by the solution.*

Proof The uniqueness follows easily from Lemma 5.1. The existence of the solution is due to the existence of such $\vec{Z}_i$ satisfying (3.1), (4.2)–(4.3) and

$$\begin{cases}\vec{k}\cdot\oint_{\sigma_1}\vec{Z}_1\times\vec{r}=1,\\ \vec{k}\cdot\oint_{\sigma_0}\vec{Z}_1\times\vec{r}=-1.\end{cases}\tag{5.11}$$

By Lemma 4.1, if $\vec{Z}_i$ satisfy (3.1) and (4.2)–(4.3), the homogenous equations $\vec{\mathcal{L}}(a_i^j)=0$ are solvable. Recalling (4.1) and the definition of $\vec{X}$,

$$\vec{X}=\frac{u_2\partial_1\vec{r}-u_1\partial_2\vec{r}}{\sqrt{|g|}}+w\vec{n},$$

we have

$$\oint_{\sigma_k}\vec{X}\times\partial_1\vec{r}=0,\quad k=0,1,$$

and then

$$\oint_{\sigma_k}\partial_1\vec{X}\times\vec{r}=0,\quad k=0,1.$$

Hence

$$\vec{k}\cdot\oint_{\sigma_i}\partial_1\vec{X}\times\vec{r}=0.$$

Recalling

$$\partial_1\vec{X} = \vec{Z}_1 + \vec{Y}_1,$$
$$\vec{k}\cdot\oint_{\sigma_i}\vec{Z}_1\times\vec{r} = \vec{k}\cdot\oint_{\sigma_k}\vec{r}\times\vec{Y}_1, \quad k=0,1,$$

$\vec{k}\cdot\oint_{\sigma_i}\vec{r}\times\vec{Y}_1$ is nothing but the boundary integral in (5.10).

Set

$$\vec{Z}_1 = -y_1\Big(x_2\int_0^{2\pi}S(x_1,0)\cos^2(x_1)\mathrm{d}x_1 - (1-x_2)\int_0^{2\pi}S(x_1,1)\cos^2(x_1)\mathrm{d}x_1\Big)\cos(x_1)\vec{j},$$

where y_i are constants to be fixed.

In the similar way to the proof of Theorem 1.1, we have that the algebraic system satisfied by y_i is

$$1 = \int_0^{2\pi}S(x_1,0)\cos^2(x_1)\mathrm{d}x_1\int_0^{2\pi}S(x_1,1)\cos^2(x_1)\mathrm{d}x_1 y_1,$$

which is solvable.

Then let

$$\partial_1\vec{Z}_2 = -y_1\Big(\int_0^{2\pi}S(x_1,0)\cos^2(x_1)\mathrm{d}x_1 + \int_0^{2\pi}S(x_1,1)\cos^2(x_1)\mathrm{d}x_1\Big)\cos(x_1)\vec{j},$$

which is solvable. Thus there exist such $\vec{Z}_i$, $i=1,2$ satisfying (3.1), (4.2)–(4.3) and (5.11).

In Case 2, (5.4)–(5.5) are of the form

$$\begin{aligned}0 = {}&\vec{n}(1)\Big(\int_{\sigma_1}\sqrt{|g|}((g^{11}\rho_1+g^{21}\rho_2)a_1^2-(g^{12}\rho_1+g^{22}\rho_2)a_1^1)\Big)+\vec{n}(1)\times\int_{\sigma_1}(a_1^1\partial_1\vec{r}+a_1^2\partial_2\vec{r})\\&-\vec{n}(0)\Big(\int_{\sigma_0}\sqrt{|g|}((g^{11}\rho_1+g^{21}\rho_2)a_1^2-(g^{12}\rho_1+g^{22}\rho_2)a_1^1)\Big)-\vec{n}(0)\times\int_{\sigma_0}(a_1^1\partial_1\vec{r}+a_1^2\partial_2\vec{r}),\end{aligned}$$
$$\int_{\sigma_1}(a_1^1\partial_1\vec{r}+a_1^2\partial_2\vec{r})-\int_{\sigma_0}(a_1^1\partial_1\vec{r}+a_1^2\partial_2\vec{r})=0. \tag{5.12}$$

Note that

$$\vec{n}(1)\cdot\int_{\sigma_1}(a_1^1\partial_1\vec{r}+a_1^2\partial_2\vec{r})=0,$$
$$\vec{n}(0)\cdot\int_{\sigma_0}(a_1^1\partial_1\vec{r}+a_1^2\partial_2\vec{r})=0.$$

So by (5.12), $\int_{\sigma_k}(a_1^1\partial_1\vec{r}+a_1^2\partial_2\vec{r})$ are parallel to $\vec{n}(0)\times\vec{n}(1)$, $k=0,1$.

Assume that

$$\int_{\sigma_k}(a_1^1\partial_1\vec{r}+a_1^2\partial_2\vec{r}) = C\vec{n}(0)\times\vec{n}(1),$$

where C is a constant determined by $\int_{\sigma_k}(a_1^1\partial_1\vec{r}+a_1^2\partial_2\vec{r})$. Then

$$\begin{cases}\vec{n}(1)\Big(C(1-\vec{n}(1)\cdot\vec{n}(0))+\int_{\sigma_1}\sqrt{|g|}((g^{11}\rho_1+g^{21}\rho_2)a_1^2-(g^{12}\rho_1+g^{22}\rho_2)a_1^1)\Big)=0,\\ \vec{n}(0)\Big(C(1-\vec{n}(1)\cdot\vec{n}(0))-\int_{\sigma_0}\sqrt{|g|}((g^{11}\rho_1+g^{21}\rho_2)a_1^2-(g^{12}\rho_1+g^{22}\rho_2)a_1^1)\Big)=0.\end{cases} \tag{5.13}$$

If $\oint_{\sigma_k}(a_1^1\partial_1\vec{r}+a_1^2\partial_2\vec{r})=0$, $k=0,1$, by (5.13) we have

$$\int_{\sigma_k}\sqrt{|g|}((g^{11}\rho_1+g^{21}\rho_2)a_1^2-(g^{12}\rho_1+g^{22}\rho_2)a_1^1)=0,$$

which means (5.6)–(5.7) are valid.

Introduce a boundary value problem for the homogenous equations $\vec{\mathcal{L}}(a_i^j)=0$,

$$\begin{cases}\vec{\mathcal{L}}(a_i^j)=0\\ \displaystyle\int_{\sigma_k}(a_1^1\partial_1\vec{r}+a_1^2\partial_2\vec{r})=\vec{n}(0)\times\vec{n}(1), \quad k=0,1.\end{cases} \tag{5.14}$$

Theorem 5.2 *There exists a solution to* (5.14) *which is unique, and the kernel of the operator* $\vec{\mathcal{L}}$ *is spanned by the solution.*

Proof The uniqueness follows easily from Lemma 5.1. The existence of the solution is due to the existence of such $\vec{Z}_i$ satisfying (3.1), (4.2)–(4.3) and

$$\oint_{\sigma_k}\vec{Z}_1=\vec{n}(1)\times\vec{n}(0), \quad k=0,1. \tag{5.15}$$

By Lemma 4.1, if $\vec{Z}_i$ satisfy (3.1) and (4.2)–(4.3), the homogenous equations $\vec{\mathcal{L}}(a_i^j)=0$ are solvable.

Recalling

$$\begin{aligned}&\partial_1\vec{X}=\vec{Z}_1+\vec{Y}_1,\\ &-\oint_{\sigma_k}\vec{Z}_1=\oint_{\sigma_k}\vec{Y}_1, \quad k=0,1,\end{aligned}$$

then $\oint_{\sigma_k}\vec{Y}_1$ is nothing but the boundary integral in (5.14).

Set

$$\vec{Z}_1=\Big(\frac{\sin\theta}{2\pi}+\sin(x_1)\int_0^{x_2}y_1,-\cos(x_1)\int_{x_2}^1 y_2,0\Big),$$

where y_i are constants to be fixed.

Similarly to the proof of Theorem 1.2, the algebraic system satisfied by y_i is

$$\begin{cases}\sin\theta\Big(\displaystyle\int_0^{2\pi}S(x_1,1)\sin^2(x_1)\mathrm{d}x_1\Big)y_1=C^7,\\ \cos\theta\Big(\displaystyle\int_0^{2\pi}S(x_1,1)\sin^2(x_1)\mathrm{d}x_1\Big)y_1-\Big(\displaystyle\int_0^{2\pi}S(x_1,0)\cos^2(x_1)\mathrm{d}x_1\Big)y_2=C^8,\end{cases}$$

where

$$\begin{aligned}C^7&=\sin\theta\cos\theta-\frac{\sin^2\theta}{2\pi}\int_0^{2\pi}S(x_1,1)\sin(x_1)\mathrm{d}x_1,\\ C^8&=\frac{\sin(\theta)}{2\pi}\int_0^{2\pi}S(x_1,1)\sin(x_1)\mathrm{d}x_1-\sin^2\theta-\frac{\sin\theta\cos\theta}{2\pi}\int_0^{2\pi}S(x_1,1)\sin(x_1)\mathrm{d}x_1,\end{aligned}$$

which is solvable.

Then

$$\partial_1 \vec{Z}_2 = (y_1 \sin(x_1), y_2 \cos(x_1), 0),$$

which is solvable, and thus there exist such $\vec{Z}_i$, $i = 1, 2$ satisfying (3.1), (4.2)–(4.3) and (5.15).

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