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Schwarz Lemma and Hartogs Phenomenon in Complex Finsler Manifold*

Bin SHEN[1] Yibing SHEN[1]

Abstract The authors prove the Schwarz lemma from a compact complex Finsler manifold to another complex Finsler manifold and any complete complex Finsler manifold with a non-positive holomorphic curvature obeying the Hartogs phenomenon.

Keywords Complex Finsler manifold, Schwarz lemma, Hartogs phenomenon
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1 Introduction

The Schwarz lemma is a result of complex analysis of holomorphic functions from the open unit disk to itself. Although the lemma is less celebrated than stronger theorems, such as the Riemann mapping theorem, it is one of the simplest results capturing the rigidity of holomorphic functions. Pick made a new expansion of Schwarz lemma, and Ahlfors first introduced the geometry concept, curvature into the complex analysis (see [1]). In 1978, Yau generalized it into the Kähler manifold (see [2]). Since it is useful in both analysis and geometry, we consider the holomorphic map of the Finsler manifold. In the general case, we need the manifold to be compact because the general maximum principle is not admitted.

The Hartogs phenomenon was first presented as the Hartogs' extension theorem, which is a fundamental result in the theory of functions of several complex variables. Informally, it states that the support of the singularities of such functions cannot be compact. This property of holomorphic functions is also called Hartogs' phenomenon (see [3]). Griffiths and Shiffman generalized it to the Hermitian manifold (see [4–5]). We also consider it in the Finsler case. Given a Finsler curvature condition, the phenomenon is kept.

The method which we use in those problems is based on the definition of the holomorphic curvature in Finsler geometry (see [6]). By this definition, we can also obtain the embedding theorem as a corollary.

2 The Preliminaries

Given a complex manifold M, the real tangent bundle of M is denoted by T_RM. $T^{1,0}M$ denotes the holomorphic tangent bundle of M and $\widetilde{M}$ is $T^{1,0}M\backslash\{0\}$. If the complex dimension of M is n, let $\{z^1,\cdots,z^n\}$ be a set of local complex coordinates, where $z^\alpha = x^\alpha + \mathrm{i}x^{n+\alpha}$. $x^1,\cdots,x^n,x^{n+1},\cdots,x^{2n}$ come to be local real coordinates. A local frame over $\mathbb{R}$ for $T_R\widetilde{M}$ is given by $\{\overset{\circ}{\partial}_1,\cdots,\overset{\circ}{\partial}_{2n},\dot{\overset{\circ}{\partial}}_1,\cdots,\dot{\overset{\circ}{\partial}}_{2n}\}$, where $\overset{\circ}{\partial}_a = \frac{\partial}{\partial x^a}$ and $\dot{\overset{\circ}{\partial}}_a = \frac{\partial}{\partial u^a}$. Analogously, a local frame

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[1]Department of Mathematics, Southeast University, Nanjing 211189, China; Department of Mathematics, Zhejiang University, Hangzhou 310027, China.
E-mail: shenbin_cgs@hotmail.com yibingshen@zju.edu.cn
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over $\mathbb{C}$ for $T^{1,0}\widetilde{M}$ is given by $\{\partial_1, \cdots, \partial_n, \dot{\partial}_1, \cdots, \dot{\partial}_n\}$, where $\partial_\alpha = \frac{\partial}{\partial z^\alpha}$ and $\dot{\partial}_\alpha = \frac{\partial}{\partial v^\alpha}$. In the above, we denote by $z^\alpha = x^\alpha + \mathrm{i}x^{n+\alpha}$ the coordinates on the manifold and $v^\alpha = u^\alpha + \mathrm{i}u^{n+\alpha}$ the vectors on $\widetilde{M}$. Moreover, we use Roman indices to run from 1 to $2n$, whereas Greek indices to run from 1 to n. Then any $(1,0)$-vector can be written as $v = v^\alpha \frac{\partial}{\partial z^\alpha}$. $\{z^\alpha; v^\alpha\}$ is a group of local complex coordinates on $T^{1,0}M$. We define as follows.

Definition 2.1 *A complex Finsler metric on a complex manifold M is a continuous function $F : T^{1,0}M \to \mathbb{R}^+$ satisfying*

(a) $G = F^2$ *is smooth on* $\widetilde{M}$.

(b) $F(v) > 0$ *for any* $v \in \widetilde{M}$.

(c) $F(\lambda v) = |\lambda| F(v)$ *for any* $v \in T^{1,0}M$ *and* $\lambda \in \mathbb{C}$.

We call a complex manifold endowed with such a metric a complex Finsler manifold. Moreover, if the Levi matrix

$$G_{\alpha\overline{\beta}} := (\dot{\partial}_\alpha \dot{\partial}_{\overline{\beta}} G)$$

is positively defined, we call F is strongly pseudoconvex. Generally, the Caratheodory and Kobayashi metrics are strongly pseudoconvex in the strongly convex domain (see [7]). G is smooth on the whole of $T^{1,0}M$, if and only if F is the norm associated to a Hermitian metric. In this case, we shall say that F comes from a Hermitian metric (see [6]).

From the projective map $\pi : TM \to M$, the definition of the real vertical bundle is given by

$$\mathcal{V}_{\mathbb{R}} = \ker\ \pi_* \in T_{\mathbb{R}}\widetilde{M},$$

while the complexified vertical bundle is

$$\mathcal{V}_{\mathbb{C}} = \mathcal{V}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \ker\ \pi_* \in T_{\mathbb{C}}\widetilde{M}.$$

Here the π_* commutes with the complex structure J, for the projection π is holomorphic. $\mathcal{V}_{\mathbb{C}}$ splits into $\mathcal{V}_{\mathbb{C}} = \mathcal{V}^{1,0} \oplus \mathcal{V}^{0,1}$, and we define the complex vertical bundle as

$$\mathcal{V} = \mathcal{V}^{1,0} = \ker\ \pi_* \in T^{1,0}\widetilde{M}.$$

The complex horizontal bundle is a complex subbundle $\mathcal{H}_{\mathbb{C}} \in T_{\mathbb{C}}\widetilde{M}$, which is J-invariant, a conjugation invariant, such that

$$T_{\mathbb{C}}\widetilde{M} = \mathcal{H}_{\mathbb{C}} \oplus \mathcal{V}_{\mathbb{C}}.$$

Since $\mathcal{H}_{\mathbb{C}}$ is also J-invariant, we can write $\mathcal{H}_{\mathbb{C}} = \mathcal{H}^{1,0} \oplus \mathcal{H}^{0,1}$, where $\mathcal{H}^{1,0} = \mathcal{H}_{\mathbb{C}} \cap T^{1,0}\widetilde{M}$. Moreover, $\mathcal{H}^{\overline{1,0}} = \mathcal{H}^{0,1}$, which means that a complex horizontal bundle is completely determined by its $(1,0)$-part $\mathcal{H}^{1,0}$. We simply denote $\mathcal{H}^{1,0}$ as $\mathcal{H}$.

Locally, we shall denote by indices like $\alpha, \overline{\beta}$ and so on the derivatives with respect to the v-coordinates, for example, $G_{\alpha\overline{\beta}} = \frac{\partial^2 G}{\partial v^\alpha \partial \overline{v}^\beta}$. On the other hand, the derivatives with respect to the z-coordinates will be denoted by indices after a semicolon, for instance, $G_{;\mu\nu} = \frac{\partial^2 G}{\partial z^\mu \partial z^\nu}$ or $G_{\alpha;\overline{\nu}} = \frac{\partial^2 G}{\partial \overline{z}^\nu \partial v^\alpha}$. Levi matrix induces the fundamental tensor $G_{\alpha\overline{\beta}} \mathrm{d}z^\alpha \otimes \mathrm{d}\overline{z}^\beta$. Setting

$$\frac{\delta}{\delta z^\alpha} := \frac{\partial}{\partial z^\alpha} - N_\alpha^\beta \frac{\partial}{\partial v^\beta}, \quad \delta v^\alpha := \mathrm{d}v^\alpha + N_\beta^\alpha \mathrm{d}z^\beta, \quad N_\beta^\alpha := G^{\alpha\overline{\gamma}} G_{\overline{\gamma};\beta},$$

we have

$$T_{\mathbb{C}}\widetilde{M} = \mathcal{H} \oplus \overline{\mathcal{H}} \oplus \mathcal{V} \oplus \overline{\mathcal{V}},$$

where $\mathcal{H} = \mathrm{span}\{\frac{\delta}{\delta z^\alpha}\}$, $\mathcal{V} = \mathrm{span}\{\frac{\partial}{\partial v^\alpha}\}$.

To any Hermitian metric, there is an associated unique complex linear connection such that the metric tensor is parallel: the Chern connection. Analogously, to any strongly pseudoconvex Finsler metric, there is a unique good complex vertical connection D making the Hermitian structure parallel. This connection is the Chern-Finsler connection. Precisely, $G_{\alpha\overline{\beta}}\mathrm{d}z^\alpha \otimes \mathrm{d}\overline{z}^\beta$ is a Hermitian metric on the bundle $\mathcal{H}$, i.e., $\langle \delta_\alpha, \delta_\beta \rangle := G_{\alpha\overline{\beta}}$. The Hermitian connection of this metric is called the Chern-Finsler connection of a complex Finsler metric. The connection 1-form is (see [6])

$$\omega^\alpha_\beta := G^{\alpha\overline{\gamma}}\partial G_{\beta\overline{\gamma}} = \Gamma^\alpha_{\beta;\eta}\mathrm{d}z^\eta + C^\alpha_{\beta\eta}\delta v^\eta,$$

where

$$\Gamma^\alpha_{\beta;\eta} := G^{\alpha\overline{\gamma}}\delta_\eta G_{\beta\overline{\gamma}}, \quad C^\alpha_{\beta\eta} := G^{\alpha\overline{\gamma}}G_{\beta\overline{\gamma}\eta}.$$

3 The Holomorphic Curvature

Let us consider the Finsler manifold (M,F) now. The curvature form of the Chern-Finsler connection can be expressed as

$$\Omega^\alpha_\beta := R^\alpha_{\beta;\gamma\overline{\eta}}\mathrm{d}z^\gamma \wedge \mathrm{d}\overline{z^\eta} + S^\alpha_{\beta\gamma;\overline{\eta}}\delta v^\gamma \wedge \mathrm{d}\overline{z^\eta} + P^\alpha_{\beta\overline{\eta};\gamma}\mathrm{d}z^\gamma \wedge \delta\overline{v^\eta} + Q^\alpha_{\beta\gamma\overline{\eta}}\delta v^\gamma \wedge \delta\overline{v^\eta}. \tag{3.1}$$

Moreover, we set $R_{\alpha\overline{\beta};\gamma\overline{\eta}} := G_{\mu\overline{\beta}}R^\mu_{\alpha;\gamma\overline{\eta}}$, where ";" denote the vertical derivative and $G = F^2$.

The holomophic curvature on a Finsler manifold (M,F) is defined as follows.

Definition 3.1 *The holomophic curvature $K_F(v)$ along the direction v is*

$$K_F(v) = K_F(\chi(v)) = \frac{\langle \Omega(\chi,\overline{\chi})\chi,\chi\rangle_v}{G(v)^2}.$$

Here $v \in T^{1,0}M\backslash\{0\}$ and χ is the horizontal lifting from $T^{1,0}M$ to HTM.

Under the local coordinates, we rewrite it as

$$K_F(v) = -\frac{1}{G^2}G_\alpha\delta_{\overline{\beta}}(\Gamma^\alpha_{;\gamma})v^\gamma\overline{v^\beta} = -\frac{1}{G^2}\delta_{\overline{\beta}}(G_{;\gamma})v^\gamma\overline{v^\beta}.$$

When F is Hermitian metric, K_F is the holomorphic curvature of the Hermitian metric. So the first term in the curvature form is a Hermitian quantum, while the other three terms are the non-Hermitian quanta.

Abate and Patrizio gave the following equation:

$$K(\varphi^*G)(0) = K_F(v) - \frac{2}{G(v)^2}\Big\|\nabla_{(\varphi')^H}(\varphi')^H - \frac{\langle\nabla_{(\varphi')^H}(\varphi')^H,\chi\rangle_v}{\langle\chi,\chi\rangle_v}\chi\Big\|^2_v,$$

where $\varphi : \Delta \to M$ is a holomorphic map from the unit disk in $\mathbb{C}$ to the manifold M. Based on it, they gave another analytic expression (see [6])

$$K_F(v) = \sup_c K_{F|_c}([v]). \tag{3.2}$$

The supremum is taken in all the Gauss surfaces that are tangent to v at p.

Remark 3.1 This definition is first given in [6, p. 110, p. 144]. The supremum in their definition is taken in the family of holomorphic maps ϕ from the unit disk in $\mathbb{C}$ to the manifold M with $\phi(0) = p$ and $\phi'(0) = \lambda v$ for $\lambda \in \mathbb{C}^*$. Such families are just the complex 1-dimensional submanifolds passing point p with direction v. The complex 1-dimensional submanifolds of a Finsler manifold are Gauss surfaces. So the definition here is the same as the one given in [6].

When it is on a Riemannian surface, such definition is just the Gauss curvature. Deriving from the definition, we know that the following proposition holds.

Proposition 3.1 *The holomorphic curvature of the submanifold of any Finsler manifold is declined.*

By the curvature declined condition, we see that there must be a section of negative curvature on a manifold of negative holomorphic curvature. We have the following result.

Theorem 3.1 *A compact complex Finsler manifold with negative holomorphic curvature must be a Hodge manifold.*

Proof By Proposition 3.1, there is a line bundle with negative curvature on a negative curved Finsler manifold. The dual bundle is positive. The classical Kodaira embedding theorem implies that a compact complex manifold admitting a positive line bundle can be embedded into $\mathbb{C}P^n$ holomorphically, which is independent of the metric. By this, such a manifold must be a Hodge manifold.

4 The Schwarz Lemma

Using (3.2), we can prove the following Schwarz lemma.

Theorem 4.1 *Let (M_1, F_1), (M_2, F_2) be two Finsler manifolds, and (M_1, F_1) be compact. Their corresponding holomorphic curvatures satisfy $K_{F_1} \geq -B, K_{F_2} \leq -A$ for $A, B > 0$. Let φ be the holomorphic map from M_1 to M_2. Then $\varphi^* F_2^2 \leq \frac{B}{A} F_1^2$.*

Proof For $i = 1, 2$, $K_{F_i} = \sup\limits_C K_{F_i|_c}([v])$. Let $\varphi^* F_2^2 = \mu(p, v) F_1^2$ for a positive function μ depending on p and v. Since the restriction of metric F_1 on the Gauss surface is a Hermitian metric, we set $F_1^2|_c = \lambda \mathrm{d}z \mathrm{d}\overline{z}$, where z is the complex coordinate on c. As the setting,

$$\varphi^* F_2^2|_c = \mu F_1^2|_c = \mu\Big(\xi(z), \xi_*\Big(\frac{\partial}{\partial z}\Big)\Big) \lambda \mathrm{d}z \mathrm{d}\overline{z},$$

where ξ is a holomorphic map from a Gauss surface to M, such that $p = \xi(z)$ and $v = \xi_*(\frac{\partial}{\partial z})$.

By the definition of Gauss curvature on the Gauss surface, we see

$$\sup_c K_{\varphi^* F_2|_c}([v]) = \sup_c \Big(- \frac{\Delta_z \log \mu + \Delta_z \log \lambda}{\mu \lambda} \Big).$$

The Laplacian is taken on c. Since μ takes the maximum on PTM, which is compact since M_1 is compact, we have $\mu(\xi(z), \xi_*(\frac{\partial}{\partial z})) \leq \mu(\xi(0), \xi_*(\frac{\partial}{\partial z}|_0))$ for ξ from the unit disk in $\mathbb{C}$ to M with $\xi(0) = p_0$, $\xi_*(\frac{\partial}{\partial z}|_0) = v_0$ and $\max\limits_{\mathrm{PTM}} \mu = \mu(p_0, v_0)$. It gives $\Delta_z \log \mu(0) \leq 0$.

By the definition, the curvatures can be written as

$$K_{F_1}(v) = \sup_c \{K_{F_1|_c}([v])\} = \sup_c \{K(\psi^* F_1^2)(0)\},$$

where ψ is the holomorphic map from c to M_1 with $\psi(0) = p$, $\psi_* \left(\frac{\partial}{\partial z} = v\right)$.

$$K_{\varphi^* F_2}(v) = \sup_c \{K_{\varphi^* F_2|_c}([v])\} = \sup_c \{K(\psi^* \varphi^* F_2^2)(0)\},$$
$$K_{F_2}(\varphi_* v) = \sup_c \{K_{F_2|_c}([\varphi^* v])\} = \sup_c \{K(\xi^* F_2^2)(0)\},$$

where ξ is the holomorphic map from c to M_2 with $\xi(0) = \varphi(p)$, $\xi_* \left(\frac{\partial}{\partial z} = \varphi_* v\right)$.

It is easy to see $K_{\varphi^* F_2}(v) \leq K_{F_2}(\varphi_* v)$, and it follows immediately that

$$\frac{1}{\mu_0} \sup_c \Big(- \frac{\Delta_z \log \lambda}{\lambda} \Big) \leq -A,$$

where μ_0 is the maximum of μ and is independent of c. By the definition of K_{F_1} and the assumption, it gives

$$\varphi^* F_2^2 \leq \mu_0 F_1^2 \leq \frac{B}{A} F_1^2.$$

It immediately gives the following corollary.

Corollary 4.1 *Let (M_1, g) be a compact Hermitian (or Sasakian) manifold, and (M_2, F_2) be a Finsler manifold. Their corresponding holomorphic curvatures (or transverse holomorphic curvatures) satisfy $K_1 \geq -B, K_{F_2} \leq -A$ for $A, B > 0$. Let φ be the holomorphic (or (Φ, J)-holomorphic) map from M_1 to M_2. Then $\varphi^* F_2^2 \leq \frac{B}{A} g$ (or $\varphi^* F_2^2 \leq \frac{B}{A} g^{\mathrm{T}}$).*

Proof The Hermitian case is obvious, and now we see the Sasakian one. Since φ is (Φ, J)-holomorphic, $(\varphi^* F_2)(X) = F_2(\mathrm{d}\varphi(X))$. Taking X to be ξ, we get $(\varphi^* F_2)(\xi) = 0$, which means that the pull back metric $\varphi^* F_2$ just depends on the transverse metric g^{T}.

Remark 4.1 The Sasakian manifold is an odd dimensional one, which is considered as the twin sister of the Kähler manifold. One can refer to [8–9] for more details.

5 The Hartogs Phenomenon

The Hartogs phenomenon is from the Hartogs extension theorem. The theorem shows that, when $n \geq 2$ and $0 \leq a < b$, any holomorphic function defined in a spherical shell $D^n_{a,b} = \{z \in \mathbb{C}^n \mid a < |z|^2 < b\}$ can be extended to the ball B^n_b (of radius centered at the origin). In other words, there exists a holomorphic function on B^n_b whose restriction on $D^n_{a,b}$ is just f. In general, we have the following definition (see [10]).

Definition 5.1 *A complex manifold M^n is said to obey the Hartogs phenomenon, if for any $1 > a \geq 0$, any holomorphic map from $D^2_{a,1}$ into M can be extended to a holomorphic map from the unit ball B^2 into M.*

According to [10], if M obeys the Hartogs theorem, then for $m \geq 2$ and any $0 \leq a < b$, any holomorphic map from $D^m_{a,b}$ into M can be extended to a holomorphic map from B^m_b into M. One may restrict such a map to the intersection of a 2-dimensional complex plane with the spherical shell, and then apply the definition. There are complex manifolds that do not obey the Hartogs phenomenon. For example, when $n \geq 2$, $\mathbb{C}^n \setminus \{0\}$ does not obey it, and any Hopf manifold does not obey it, either.

Griffiths and Shiffman proved the following theorem (see [10]).

Theorem 5.1 (Griffiths-Shiffman) *Any complete Hermitian manifold with non-positive holomorphic sectional curvature obeys the Hartogs phenomenon.*

Based on the proof of the above theorem by Griffiths and Shiffman, we can prove the following theorem.

Theorem 5.2 *Any complete complex Finsler manifold with non-positive holomorphic curvature obeys the Hartogs phenomenon.*

Proof Let (M^n, F) be such a manifold. Let f be the holomorphic map from $D := D^2_{a,1}$ to M, where $D^2_{a,1} = \{z \in \mathbb{C}^2 \mid a < |z|^2 < 1\}$.

By the non-positive holomorphic curvature assumption, with the same argument, it follows that

$$0 \geq K_{f^*F}(v) = \sup_c \Big\{ -\frac{\Delta_z \ln \mu + \Delta_z \ln \lambda}{\mu \lambda} \Big\}.$$

Noticing $\mu > 0$, for any direction fixed,

$$\frac{\Delta_z \ln \mu}{\lambda} \geq \sup_c \Big(-\frac{\Delta_z \ln \lambda}{\lambda} \Big) = 0.$$

It means that for any fixed direction v, we have

$$\Delta \mu \geq 0.$$

When we fix v, $\mu(z,v)$ is a subharmonic function on D. Next we want to show that for any $\varepsilon > 0$, there is a constant $C > 0$, such that $f * F^2 \leq Cg_0$, where g_0 is the Hermitian metric on D.

Fix a_1, such that $a < a_1 < 1 - \varepsilon$, and denote $D' = D^2_{a,1-\varepsilon}$, $D'' = D^2_{a_1,1-\varepsilon}$. Let C be $\sup_{S\overline{D''}} \mu < \infty$ since $\overline{D''}$ is compact, and so does $S\overline{D''}$.

For any p in D' with $|p| \geq a_1$, it follows that $\mu \leq C$. When $|p| \leq a_1$, let P be the complex line path through point p and orthogonal to the line connecting 0 and p. We see that $P \cap D'$ is a disc of p, while $P \cap D''$ is an annulus. Take a loop γ in D'' around p. We have that for any fixed v, $\mu(p,v) \leq \frac{1}{|\gamma|}\int_\gamma \mu \mathrm{d}v \leq C$. With these two cases, we have that for a fixed direction v, $f^*F^2 \leq Cg_0$.

Any sequence approaching $p \in S_a$ in the radial direction is a Cauchy sequence. For any p_k, p_l in D can be jointed by a line with the tangent vector v. Then it follows that $d(f(p_k), f(p_l)) \leq Cd_0(p_k, p_l)$. $f(p_k)$ converges to a point in M by the complicity of M. Then, any holomorphic map f from D to M can be extended on the inner boundary S_a of D. Fixing $p \in S_a$, there is a neighborhood $U \subset D$, such that $\overline{U}$ is an open neighborhood of p in $\overline{D}$. f is given by n holomorphic functions on U. The classical Hartogs theorem shows that f can be extended onto $D^2_{a',1}$ with $a' < a$.

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