Strong Unique Continuation of Sub-elliptic Operator on the Heisenberg Group*

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Abstract In this paper, the Almgren's frequency function of the following sub-elliptic equation with singular potential on the Heisenberg group:

$$-\mathcal{L}u + V(z,t)u = -X_i(a_{ij}(z,t)X_j u) + V(z,t)u = 0$$

is introduced. The monotonicity property of the frequency is established and a doubling condition is obtained. Consequently, a quantitative proof of the strong unique continuation property for such equation is given.

Keywords Heisenberg group, Frequency function, Doubling condition, Strong unique continuation principle
2000 MR Subject Classification 35A02, 35H10, 35R03

1 Introduction

The Heisenberg group H^n is a nilpotent Lie group of step two whose underlying manifold is $R^{2n} \times R$ with coordinates $(z,t) = (x,y,t) = (x_1,\cdots,x_n,y_1,\cdots,y_n,t)$ and its group action ∘ is given by

$$(x_0,y_0,t_0) \circ (x,y,t) = \Big(x + x_0, y + y_0, t + t_0 + 2\sum_{i=1}^{n}(x_i y_{0_i} - y_i x_{0_i})\Big). \tag{1.1}$$

A basis for the Lie algebra of left-invariant vector fields on H^n is given by

$$\begin{cases} X_i = \dfrac{\partial}{\partial x_i} + 2y_i \dfrac{\partial}{\partial t}, & i = 1,\cdots,n, \\[2mm] X_{n+i} = \dfrac{\partial}{\partial y_i} - 2x_i \dfrac{\partial}{\partial t}, & i = 1,\cdots,n, \\[2mm] T = \dfrac{\partial}{\partial t}. \end{cases} \tag{1.2}$$

From (1.2), it is easy to check that X_i and X_{n+j} satisfy

$$[X_i, X_{n+j}] = -4T\delta_{ij}, \quad [X_i, X_j] = [X_{n+i}, X_{n+j}] = 0, \quad i,j = 1,\cdots,n.$$

Therefore, the vector fields X_i, X_{n+i} $(i = 1, \cdots, n)$ and their first order commutators span the whole Lie algebra. The horizontal gradient of a function f is defined as

$$\nabla_{H^n} f = Xf = (X_1 f, \cdots, X_n f, X_{n+1} f, \cdots, X_{2n} f).$$

Let us denote by δ_λ the Heisenberg group dilation

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t), \quad \lambda > 0, \tag{1.3}$$

which leads to a homogeneous dimension $Q = 2n + 2$.

For $(z, t) \in H^n$, we define the gauge norm from the origin

$$d(z, t) = \left[\left(\sum_{i=1}^n (x_i^2 + y_i^2) \right)^2 + t^2 \right]^{\frac{1}{4}} \equiv (|z|^4 + t^2)^{\frac{1}{4}}, \tag{1.4}$$

which satisfies $d(\delta_\lambda(z, t)) = \lambda d(z, t)$, and means that d is homogeneous of degree one with respect to the dilation δ_λ (see [4, 17, 19]).

In the sequel, we let

$$B_r = \{(z, t) \in H^n \mid d(z, t) < r\}, \quad \partial B_r = \{(z, t) \in H^n \mid d(z, t) = r\},$$

and call these sets a Heisenberg-ball and a sphere centered at the origin with radius r respectively. Since $d \in C^\infty(H^n \backslash \{(0, 0)\})$, the outer unit normal on ∂B_r is given by $\overrightarrow{n} = |\nabla d|^{-1} \nabla d$, where ∇d means the ordinary Euclidean gradient of d.

Introducing the function

$$\psi(z, t) = |\nabla_{H^n} d|^2 = \frac{|z|^2}{d(z, t)^2}, \tag{1.5}$$

we define

$$|B_r| = \int_{B_r} \psi \mathrm{d}z \mathrm{d}t \quad \text{and} \quad |\partial B_r| = \frac{\mathrm{d}}{\mathrm{d}r} |B_r|.$$

Using the polar coordinates adapted to H^n introduced by Greiner [12], it is easy to obtain that there exists a constant $\omega_Q > 0$ depending only on Q such that

$$|B_r| = \omega_Q r^Q. \tag{1.6}$$

The Kohn-Laplacian on H^n is

$$\Delta_{H^n} = \sum_{i=1}^{2n} X_i^2,$$

which is the sum of squares of vector fields. Since Hörmander's work [13], the study of operators of the type sum of squares of vector fields has received a strong impetus. Among the large body of literature dedicated to sub-elliptic operators and Carnot-Caratheodory geometry, we briefly recall Bony [5], Folland and Stein [7], Rothschild and Stein [17], Nagel, Stein and Wainger [16], Sanchez-Calle [18] and Jerison [14]. The sub-elliptic operators have a wide range of applications, from several complex variables and CR geometry (see for instance [7]) to control theory and financial mathematics (see for instance [3, 11]).

In this paper, we study the strong unique continuation property of the following sub-elliptic equation on $B_{R_0} \subset H^n$:

$$-\mathcal{L}u + V(z, t)u = -\sum_{i,j=1}^{2n} X_i(a_{ij}(z, t)X_j u) + V(z, t)u = 0. \tag{1.7}$$

Our main concern is whether, under suitable assumptions on the coefficients (a_{ij}) and the potential V, the strong unique continuation property holds for the equation (1.7).

We assume that $A = (a_{ij}(z,t))$ is a $2n \times 2n$ matrix-valued function on B_{R_0}, and for simplicity, we assume that $A(0) = I$ (note that this assumption really involves with no loss of generality, because we can always achieve it with a suitable linear transformation, provided that the original equation is at least elliptic at 0). We shall denote by B the matrix

$$B = A - I_{2n \times 2n}.$$

Furthermore, we assume that A is symmetric and satisfies the following hypotheses:

(i) There exist $0 < \lambda \le \Lambda < \infty$ such that for any $\eta \in R^{2n}$,

$$\lambda |\eta|^2 \le \sum_{i,j=1}^{2n} a_{ij} \eta_i \eta_j \le \Lambda |\eta|^2. \tag{1.8}$$

(ii) There exist positive constants C_1 and C_2 such that

$$\begin{cases} |b_{ij}| \equiv |a_{ij} - \delta_{ij}| \le C_1 \psi d, & i,j = 1, \cdots, 2n, \\ |X_k a_{ij}| \le C_2 \psi^{\frac{1}{2}}, & i,j,k = 1, \cdots, 2n. \end{cases} \tag{1.9}$$

We note that the condition (1.9) means that a_{ij} is ψ-Lipschitz.

We require that the potential V in (1.7) satisfies the following assumption: There exist a constant $M > 0$ and an increasing function $f : (0, R_0) \to R^+$ such that

$$\int_0^{R_0} \frac{f(r)}{r} dr < \infty, \tag{1.10}$$

for which

$$|V(z,t)| \le M \frac{f(d(z,t))}{d(z,t)^2} \psi(z,t) \quad \text{for a.e. } (z,t) \in B_{R_0}. \tag{1.11}$$

According to (1.10)–(1.11), the potential V is allowed to be singular (see [8]).

We need to introduce the following definitions.

Definition 1.1 *A weak solution to (1.7) is a function $u \in C(B_{R_0}) \cap L^2(B_{R_0})$ such that the horizontal gradient $Xu \in L^2(B_{R_0})$, and (1.7) is satisfied in the distribution sense, i.e.,*

$$\int_{B_{R_0}} a_{ij} X_i u X_j \phi \, dz dt + \int_{B_{R_0}} V u \phi \, dz dt = 0$$

for every $\phi \in C_0^\infty(B_{R_0})$.

Definition 1.2 *We say that u is polyradial in H^n if for any $(z,t) = (z_1, \cdots, z_n, t) \in H^n$ where $z_j = x_j + i y_j$ and $|z_j| = (x_j^2 + y_j^2)^{\frac{1}{2}}$, we have $u(z,t) = u^*(|z_1|, \cdots, |z_n|, t)$ for some u^*.*

Definition 1.3 *We say that $u \in L^2(B_{R_0})$ vanishes up to infinite order at the origin, if for every $k > 0$ one has*

$$\lim_{r \to 0} \frac{1}{r^k} \int_{B_r} u^2 \psi \, dz dt = 0.$$

We are ready to state our main results.

Theorem 1.1 *Let V satisfy* (1.11) *for some M and f, A be a symmetric matrix satisfying* (1.8)–(1.9)*, and u be a polyradial solution to* (1.7) *in B_{R_0}. Then there exist positive constants $\Gamma = \Gamma(u, \lambda, \Lambda)$ and $r_0 = r_0(\lambda, \Lambda)$, such that for any $2r \leq r_0$, we have*

$$\int_{B_{2r}} u^2 \psi \mathrm{d}z \mathrm{d}t \leq \Gamma \int_{B_r} u^2 \psi \mathrm{d}z \mathrm{d}t. \tag{1.12}$$

Note that (1.12) is often referred to as the doubling condition and it yields quantitative information on the vanishing order at the origin of u. As well-known (see [8–9] and etc.), Theorem 1.1 implies the following strong unique continuation property.

Theorem 1.2 *With the assumptions of the Theorem* 1.1*, if u vanishes up to infinite order at the origin, then $u \equiv 0$ in B_{r_0}, where r_0 is as in the statement of Theorem* 1.1*.*

We mention that when the horizontal gradient is replaced by the classical gradient on R^n, i.e.,

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) = 0, \tag{1.13}$$

a result due to [2] states that if the matrix (a_{ij}) is Lipschitz continuous, then the equation (1.13) possesses the strong unique continuation property. Furthermore, it was shown in [15] that the Lipschitz continuous assumption on the coefficients is optimal. Our results can be seen as a generalization of those in [2]. The approach, however, is different from that in [2], which is based on Carleman inequalities that do not seem to be adaptable to our operator due to the lack of ellipticity. Instead, we have used the ideas of Almgren's frequency function that goes back to Almgren [1] and has been developed in [8–10].

We note that when $a_{ij} = \delta_{ij}$, the equation (1.7) becomes the Kohn-Laplace equation with potential V on H^n

$$-\sum_{i=1}^{2n} X_i^2 u + V u = 0, \tag{1.14}$$

which was studied by Garofalo and Lanconelli [8], under some assumptions of V and with the weak solution u, they proved the strong unique continuation property of equation (1.14). So our results can also be seen as a generalization of those in [8]. Because of the variable coefficients, we should overcome more difficulties to obtain the monotonicity of the frequency function by using the refined geometry properties of H^n.

The rest of the paper is organized as follows. In Section 2, we give some notations and various technical estimates. In Section 3, we introduce the frequency function and prove its monotonicity and the doubling condition, and finally we give the proofs of Theorems 1.1 and 1.2.

2 Preliminary Facts

We begin this section by giving some basic facts about the horizontal gradient ∇_{H^n} and the operator $\mathcal{L}$ on H^n.

We denote by S the $2n \times (2n + 1)$ matrix relating the horizontal gradient ∇_{H^n} and the standard gradient ∇ in R^{2n+1}, i.e., $\nabla_{H^n} = S \cdot \nabla$, where

$$S = \begin{pmatrix} I_{n\times n} & 0_{n\times n} & (2y)^{\mathrm{T}} \\ 0_{n\times n} & I_{n\times n} & (-2x)^{\mathrm{T}} \end{pmatrix}. \tag{2.1}$$

Hence we have

$$\mathcal{L}u = \sum_{i=1}^{2n} X_i(a_{ij}X_ju) = \mathrm{div}(S^{\mathrm{T}}A\nabla_{H^n}u) = \mathrm{div}(S^{\mathrm{T}}AS\nabla u). \tag{2.2}$$

Now we define a vector field

$$Z = \sum_{i=1}^{n}\left(x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i}\right) + 2t\frac{\partial}{\partial t}. \tag{2.3}$$

A direct calculation yields

$$Zu = \frac{d}{\psi}\sum_{i=1}^{2n}X_i dX_i u, \quad \text{if } \widetilde{T}u \equiv \sum_{i=1}^{n}\left(y_i\frac{\partial u}{\partial x_i} - x_i\frac{\partial u}{\partial y_i}\right) = 0. \tag{2.4}$$

Proposition 2.1 *Letting Z and $\widetilde{T}$ be the above vector fields on H^n, we have*

$$[X_i, Z]u = X_i u, \quad i = 1, \cdots, 2n \tag{2.5}$$

and

$$\left|[X_i, X_j]u\right| \le \frac{1}{|z|}|Xu|, \quad \text{if } \widetilde{T}u = 0, \ i, j = 1, \cdots, 2n. \tag{2.6}$$

Proof By the definitions of the vector fields X_i and Z, we have

$$\begin{aligned}
[X_i, Z]u &= X_i Z u - Z X_i u \\
&= \left(\frac{\partial}{\partial x_i} + 2y_i\frac{\partial}{\partial t}\right)\left(\sum_{k=1}^{n} x_k\frac{\partial u}{\partial x_k} + y_k\frac{\partial u}{\partial y_k} + 2t\frac{\partial u}{\partial t}\right) \\
&\quad - \left(\sum_{k=1}^{n} x_k\frac{\partial}{\partial x_k} + y_k\frac{\partial}{\partial y_k} + 2t\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial x_i} + 2y_i\frac{\partial u}{\partial t}\right) \\
&= \sum_{k=1}^{n}\delta_{ki}\frac{\partial u}{\partial x_k} + 2\delta_{ki}y_k\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x_i} + 2y_i\frac{\partial u}{\partial t} = X_i u
\end{aligned}$$

for any $u \in C^\infty$. Hence $[X_i, Z] = X_i$ for $i = 1, \cdots, n$.

Similarly, we prove $[X_{n+i}, Z] = X_{n+i}$ for $i = 1, \cdots, n$.

A direct calculation yields

$$\begin{aligned}
\sum_{i=1}^{n} y_i X_i u - x_i X_{n+i}u &= \sum_{i=1}^{n}\left(y_i\frac{\partial u}{\partial x_i} - x_i\frac{\partial u}{\partial y_i}\right) + 2\sum_{i=1}^{n}(x_i^2 + y_i^2)\frac{\partial u}{\partial t} \\
&= 2|z|^2\frac{\partial u}{\partial t},
\end{aligned}$$

where we use $\widetilde{T}u \equiv \sum\limits_{i=1}^{n}\left(y_i\frac{\partial u}{\partial x_i} - x_i\frac{\partial u}{\partial y_i}\right) = 0$ in the last equality. Thus

$$\left|[X_i, X_j]u\right| \le 4\left|\frac{\partial u}{\partial t}\right| \le \frac{1}{|z|}|Xu|.$$

Next we prove some basic estimates that will be used later.

Proposition 2.2 (1) *The horizontal derivatives of the distance function d and the angle function ψ satisfy*

$$|X_i d| \leq C\psi^{\frac{1}{2}}, \quad i = 1, \cdots, 2n,$$

$$|X_i \psi| \leq C\frac{1}{d}\psi^{\frac{1}{2}}, \quad i = 1, \cdots, 2n.$$

(2) *The second horizontal derivatives of d satisfy*

$$|X_i X_j d| \leq C\frac{1}{d}\psi, \quad i, j = 1, \cdots, n,$$

$$|X_i X_{n+j} d| \leq C\frac{1}{d}, \quad i, j = 1, \cdots, n,$$

$$|X_{n+i} X_j d| \leq C\frac{1}{d}, \quad i, j = 1, \cdots, n,$$

$$|X_{n+i} X_{n+j} d| \leq C\frac{1}{d}\psi, \quad i, j = 1, \cdots, n$$

and

$$\sum_{i,j=1}^{2n} X_i d(X_i X_j d) X_j d = 0.$$

Proof By the definitions of d in (1.4) and ψ in (1.5),

$$X_i d = \left(\frac{\partial}{\partial x_i} + 2y_i\frac{\partial}{\partial t}\right)d = \frac{1}{d^3}(|z|^2 x_i + y_i t), \quad i = 1, \cdots, n.$$

$$X_{n+i} d = \left(\frac{\partial}{\partial y_i} - 2x_i\frac{\partial}{\partial t}\right)d = \frac{1}{d^3}(|z|^2 y_i - x_i t), \quad i = 1, \cdots, n,$$

$$X_i \psi = X_i\left(\frac{|z|^2}{d^2}\right) = \frac{2x_i}{d^2} - \frac{2|z|^2}{d^3}X_i d, \quad i = 1, \cdots, n,$$

$$X_{n+i} \psi = X_{n+i}\left(\frac{|z|^2}{d^2}\right) = \frac{2y_i}{d^2} - \frac{2|z|^2}{d^3}X_{n+i} d, \quad i = 1, \cdots, n.$$

This shows that

$$|X_i d| = \left|\frac{|z|^2}{d^2} \cdot \frac{x_i}{d} + \frac{t}{d^2} \cdot \frac{y_i}{d}\right| \leq C\psi^{\frac{1}{2}}, \quad i = 1, \cdots, n,$$

$$|X_{n+i} d| = \left|\frac{|z|^2}{d^2} \cdot \frac{y_i}{d} - \frac{t}{d^2} \cdot \frac{x_i}{d}\right| \leq C\psi^{\frac{1}{2}}, \quad i = 1, \cdots, n$$

and

$$|X_i \psi| \leq C\frac{1}{d}\psi^{\frac{1}{2}}, \quad i = 1, \cdots, 2n.$$

We continue to compute the second derivative of d and this is done easily by using the product rule. We only write the expressions for the second derivatives

$$X_i(X_j d) = \frac{2x_i x_j + 2y_i y_j + |z|^2\delta_{ij}}{d^3} - 3\frac{(|z|^2 x_i + y_i t)(|z|^2 x_j + y_j t)}{d^7},$$

$$X_i(X_{n+j} d) = \frac{2x_i y_j - 2y_i x_j - \delta_{ij}t}{d^3} - 3\frac{(|z|^2 x_i + y_i t)(|z|^2 y_j - x_j t)}{d^7},$$

$$X_{n+i}(X_j d) = \frac{2y_i x_j - 2x_i y_j + \delta_{ij}t}{d^3} - 3\frac{(|z|^2 y_i - x_i t)(|z|^2 x_j + y_j t)}{d^7},$$

$$X_{n+i}(X_{n+j} d) = \frac{2y_i y_j + 2x_i x_j + |z|^2\delta_{ij}}{d^3} - 3\frac{(|z|^2 y_i - x_i t)(|z|^2 y_j - x_j t)}{d^7}.$$

Hence,

$$|X_i X_j d| \leq C\frac{1}{d}\psi, \quad i,j = 1, \cdots, n,$$

$$|X_i X_{n+j} d| \leq C\frac{1}{d}, \quad i,j = 1, \cdots, n,$$

$$|X_{n+i} X_j d| \leq C\frac{1}{d}, \quad i,j = 1, \cdots, n,$$

$$|X_{n+i} X_{n+j} d| \leq C\frac{1}{d}\psi, \quad i,j = 1, \cdots, n.$$

Finally, with the first and second horizontal derivatives of d in hand, we obtain

$$\sum_{i,j=1}^{2n} X_i d(X_i X_j d) X_j d$$
$$= \sum_{i,j=1}^{n} [X_i d(X_i X_j d) X_j d + X_i d(X_i X_{n+j} d) X_{n+j} d$$
$$+ X_{n+i} d(X_{n+i} X_j d) X_j d + X_{n+i} d(X_{n+i} X_{n+j} d) X_{n+j} d] = 0.$$

At the end of this section, we give an inequality (see [8, Theorem 2.2]).

Lemma 2.1 *For every $u \in C_0^\infty(H^n \setminus \{(0,0)\})$ and every $r > 0$, we have*

$$\int_{B_r} \frac{u(z,t)^2}{d(z,t)^2}\psi(z,t)\mathrm{d}z\mathrm{d}t \leq \left(\frac{2}{Q-2}\right)^2\Big\{\int_{B_r} |\nabla_{H^n} u(z,t)|^2 \mathrm{d}z\mathrm{d}t$$
$$+ \left(\frac{Q-2}{2}\right)\frac{1}{r}\int_{\partial B_r} u(z,t)^2 \frac{\psi(z,t)}{|\nabla d(z,t)|^2}\mathrm{d}H^{2n}\Big\}. \tag{2.7}$$

3 The Frequency Function and Unique Continuation

The purpose of this section is to prove Theorems 1.1 and 1.2. The main step is to show a monotonicity of the frequency function, which was first discovered by Almgren [1]. We begin by introducing the relevant quantities that will appear in the proofs. Hereafter, the summation convention over repeated indices will be adopted.

Definition 3.1 *For a weak solution u of (1.7) in B_{R_0} and $0 < r < R_0$, we define its height in B_r as follows:*

$$H(r) = \int_{\partial B_r} u^2 \frac{\langle A\nabla_{H^n} d, \nabla_{H^n} d\rangle}{|\nabla d|}\mathrm{d}H^{2n}.$$

We also let

$$D(r) = \int_{B_r} \langle A\nabla_{H^n} u, \nabla_{H^n} u\rangle \mathrm{d}z\mathrm{d}t,$$

$$I(r) = \int_{B_r} (\langle A\nabla_{H^n} u, \nabla_{H^n} u\rangle + V u^2)\mathrm{d}z\mathrm{d}t,$$

and call these quantities the Dirichlet integral and the total energy of u in B_r, respectively.

Remark 3.1 In view of the elliptic assumption (1.8) and the fact $|\nabla_{H^n} d|^2 = \psi$, there exist constants C_1 and C_2, such that

$$C_1 \int_{\partial B_r} u^2 \frac{\psi}{|\nabla d|} \mathrm{d}H^{2n} \leq H(r) \leq C_2 \int_{\partial B_r} u^2 \frac{\psi}{|\nabla d|} \mathrm{d}H^{2n}, \tag{3.1}$$

$$C_1 \int_{B_r} |\nabla_{H^n} u|^2 \mathrm{d}z\mathrm{d}t \leq D(r) \leq C_2 \int_{B_r} |\nabla_{H^n} u|^2 \mathrm{d}z\mathrm{d}t. \tag{3.2}$$

Lemma 3.1 *Let u be a weak solution of (1.7) in B_{R_0}. Then there exists an $r_0 > 0$ depending only on Q, M and f in (1.11), such that either $u \equiv 0$ in B_{r_0} or $H(r) \neq 0$ for every $r \in (0, r_0)$.*

Proof Suppose that for some $r_0 < R_0$, $H(r_0) = 0$. Then $u = 0$ a.e. on ∂B_{r_0}. Therefore, from (2.2), the divergence theorem and the outer unit normal on ∂B_r being $\overrightarrow{n} = |\nabla d|^{-1} \nabla d$, we have

$$\begin{aligned}
D(r_0) &= I(r_0) - \int_{B_{r_0}} V(z, t) u^2(z, t) \mathrm{d}z\mathrm{d}t \\
&\leq \frac{1}{2} \int_{B_{r_0}} \mathcal{L}(u^2) \mathrm{d}z\mathrm{d}t + \int_{B_{r_0}} |V(z, t)| u^2(z, t) \mathrm{d}z\mathrm{d}t \\
&= \frac{1}{2} \int_{B_{r_0}} \mathrm{div}(S^T A \nabla_{H^n} u^2) \mathrm{d}z\mathrm{d}t + \int_{B_{r_0}} |V(z, t)| u^2(z, t) \mathrm{d}z\mathrm{d}t \\
&= \int_{\partial B_{r_0}} u \frac{\langle A \nabla_{H^n} u, \nabla_{H^n} d \rangle}{|\nabla d|} \mathrm{d}H^{2n} + \int_{B_{r_0}} |V(z, t)| u^2(z, t) \mathrm{d}z\mathrm{d}t \\
&= \int_{B_{r_0}} |V(z, t)| u^2(z, t) \mathrm{d}z\mathrm{d}t. \tag{3.3}
\end{aligned}$$

Now we use the assumption of V (1.11) and (2.7) of Lemma 2.1 to get the bound

$$\begin{aligned}
&\int_{B_{r_0}} |V(z, t)| u^2(z, t) \mathrm{d}z\mathrm{d}t \\
&\leq M f(r_0) \int_{B_{r_0}} \frac{\psi}{d^2} u^2 \mathrm{d}z\mathrm{d}t \\
&\leq \left(\frac{2}{Q-2}\right)^2 \left\{ \left(\frac{Q-2}{2}\right) \frac{1}{r_0} \int_{\partial B_{r_0}} u^2 \frac{\psi}{|\nabla d|^2} \mathrm{d}H^{2n} + \int_{B_{r_0}} |\nabla_{H^n} u|^2 \mathrm{d}z\mathrm{d}t \right\} \\
&\leq C f(r_0) D(r_0). \tag{3.4}
\end{aligned}$$

Since by (1.10) $\lim_{r \to 0^+} f(r) = 0$, we obtain a contradiction from (3.3)–(3.4) unless $D(r_0) = 0$, which implies that $u \equiv 0$ in B_{r_0}. This completes the proof of the lemma.

Lemma 3.1 allows us to introduce the Almgren's generalized frequency of u on B_r as following:

$$N(r) = \begin{cases} \dfrac{rI(r)}{H(r)}, & \text{if } H \neq 0, \\ 0, & \text{if } H = 0. \end{cases}$$

Lemma 3.1 also implies that $r \to N(r)$ is absolutely continuous on $(0, r_0)$. Therefore, if we set

$$\Omega_{r_0} = \{ r \in (0, r_0) \mid N(r) > \max\{1, N(r_0)\} \}, \tag{3.5}$$

then Ω_{r_0} is an open subset of $\mathbb{R}$. Hence there holds a decomposition

$$\Omega_{r_0} = \bigcup_{j=1}^{\infty}(a_j, b_j) \quad \text{with } a_j, b_j \notin \Omega_{r_0}. \tag{3.6}$$

Obviously in Ω_{r_0}, we have $N(r) > 1$, i.e.,

$$\frac{H(r)}{r} < I(r) \quad \text{for every } r \in \Omega_{r_0}. \tag{3.7}$$

Lemma 3.2 *There exists a constant $C = C(Q, M, f) > 0$, such that for every $r \in \Omega_{r_0}$, we have*

$$D(r) \leq CI(r). \tag{3.8}$$

Proof As in the proof of (3.4), we have

$$D(r) \leq I(r) + \int_{B_r} |V(z,t)|u(z,t)^2 \mathrm{d}z\mathrm{d}t$$

$$\leq I(r) + \left(\frac{2}{Q-2}\right)^2 Mf(r)\left\{\left(\frac{Q-2}{2}\right)\frac{H(r)}{r} + D(r)\right\}$$

$$\leq \left[1 + \frac{2}{Q-2}Mf(r)\right]I(r) + \left(\frac{2}{Q-2}\right)^2 Mf(r)D(r),$$

where in the last inequality we have used (3.7). We can choose small $r_0 > 0$ such that $\left(\frac{2}{Q-2}\right)^2 Mf(r_0) < 1$, and thus we prove $D(r) \leq CI(r)$.

Proposition 3.1 *For a.e. $r \in (0, R_0)$, the total energy of u on B_r can be expressed by the surface integral*

$$I(r) = \int_{\partial B_r} u\frac{\langle A\nabla_{H^n}u, \nabla_{H^n}d\rangle}{|\nabla d|}\mathrm{d}H^{2n}. \tag{3.9}$$

Proof By using the divergence theorem, (2.2) and the fact $\mathcal{L}u = Vu$,

$$\int_{\partial B_r} u\frac{\langle A\nabla_{H^n}u, \nabla_{H^n}d\rangle}{|\nabla d|}\mathrm{d}H^{2n}$$

$$= \int_{\partial B_r} u\langle S^{\mathrm{T}}A\nabla_{H^n}u, \overrightarrow{n}\rangle\mathrm{d}H^{2n}$$

$$= \int_{B_r} \mathrm{div}(uS^{\mathrm{T}}A\nabla_{H^n}u)\mathrm{d}z\mathrm{d}t$$

$$= \int_{B_r} u\mathrm{div}(S^{\mathrm{T}}A\nabla_{H^n}u)\mathrm{d}z\mathrm{d}t + \int_{B_r} \nabla uS^{\mathrm{T}}A\nabla_{H^n}u\mathrm{d}z\mathrm{d}t$$

$$= \int_{B_r} Vu^2\mathrm{d}z\mathrm{d}t + \int_{B_r} \langle A\nabla_{H^n}u, \nabla_{H^n}u\rangle\mathrm{d}z\mathrm{d}t = I(r).$$

This completes the proof of the proposition.

By the definition of $N(r)$, we get

$$\frac{N'(r)}{N(r)} = \frac{1}{r} + \frac{I'(r)}{I(r)} - \frac{H'(r)}{H(r)}. \tag{3.10}$$

Next, we compute $H'(r)$ and $I'(r)$ respectively by using the conditions (1.8)–(1.9) on the coefficients (a_{ij}) and some estimates given in Section 2.

Theorem 3.1 *There exists a positive constant $C_1 = C_1(\lambda, \Lambda)$, such that for a.e. $r \in (0, R_0)$, one has*

$$\left| H'(r) - \frac{Q-1}{r}H(r) - 2I(r) \right| \leq C_1 H(r). \tag{3.11}$$

Proof The divergence theorem gives

$$\begin{aligned}
H(r) &= \int_{\partial B_r} u^2 \frac{\langle A\nabla_{H^n} d, \nabla_{H^n} d\rangle}{|\nabla d|} dH^{2n} \\
&= \int_{\partial B_r} u^2 \langle S^T A\nabla_{H^n} d, \overrightarrow{n}\rangle dH^{2n} \\
&= \int_{B_r} \mathrm{div}(u^2 S^T A\nabla_{H^n} d) dz dt \\
&= \int_{B_r} u^2 \mathrm{div}(S^T A\nabla_{H^n} d) dz dt + \int_{B_r} \langle S^T A\nabla_{H^n} d, \nabla u^2\rangle dz dt \\
&= \int_{B_r} u^2 \mathcal{L} d\, dz dt + \int_{B_r} 2u\langle A\nabla_{H^n} d, \nabla_{H^n} u\rangle dz dt.
\end{aligned}$$

We now recall Federer's co-area formula (see [6]): Let $f \in L^1(\mathbb{R}^N)$ and $g \in \mathrm{Lip}(\mathbb{R}^N)$. Then

$$\int_{\mathbb{R}^N} f(x) dx = \int_{-\infty}^{+\infty} ds \int_{\{g=s\}} \frac{f(x)}{|\nabla g(x)|} dH^{N-1}, \tag{3.12}$$

provided that ∇g does not vanish on the set $\{g = s\}$ for a.e. $s \in \mathbb{R}$.

Using (3.12) and Proposition 3.1, and differentiating $H(r)$ with respect to r, we obtain

$$\begin{aligned}
H'(r) &= \int_{\partial B_r} \frac{u^2 \mathcal{L} d}{|\nabla d|} dH^{2n} + \int_{\partial B_r} \frac{2u\langle A\nabla_{H^n} d, \nabla_{H^n} u\rangle}{|\nabla d|} dH^{2n} \\
&= 2I(r) + \int_{\partial B_r} \frac{u^2 \mathcal{L} d}{|\nabla d|} dH^{2n}.
\end{aligned}$$

This implies

$$\begin{aligned}
&H'(r) - \frac{Q-1}{r}H(r) - 2I(r) \\
&= \int_{\partial B_r} \frac{u^2 \mathcal{L} d}{|\nabla d|} dH^{2n} - \frac{Q-1}{r} \int_{\partial B_r} u^2 \frac{\langle A\nabla_{H^n} d, \nabla_{H^n} d\rangle}{|\nabla d|} dH^{2n} \\
&= \int_{\partial B_r} u^2 \frac{\mathrm{div}(S^T B\nabla_{H^n} d)}{|\nabla d|} dH^{2n} + \int_{\partial B_r} u^2 \frac{\Delta_{H^n} d}{|\nabla d|} dH^{2n} \\
&\quad - \frac{Q-1}{r} \int_{\partial B_r} u^2 \frac{\langle B\nabla_{H^n} d, \nabla_{H^n} d\rangle}{|\nabla d|} dH^{2n} - \frac{Q-1}{r} \int_{\partial B_r} u^2 \frac{|\nabla_{H^n} d|^2}{|\nabla d|} dH^{2n}.
\end{aligned}$$

Using the formula

$$\Delta_{H^n} d = \frac{Q-1}{d} |\nabla_{H^n} d|^2,$$

we get

$$\begin{aligned}
&H'(r) - \frac{Q-1}{r}H(r) - 2I(r) \\
&= \int_{\partial B_r} u^2 \frac{\mathrm{div}(S^T B\nabla_{H^n} d)}{|\nabla d|} dH^{2n} - \frac{Q-1}{r} \int_{\partial B_r} u^2 \frac{\langle B\nabla_{H^n} d, \nabla_{H^n} d\rangle}{|\nabla d|} dH^{2n}.
\end{aligned}$$

We estimate the two terms on the right-hand side, which thanks to (1.8)–(1.9), yields

$$\frac{\langle B\nabla_{H^n}d, \nabla_{H^n}d\rangle}{|\nabla d|} \leq Cd\frac{\langle \nabla_{H^n}d, \nabla_{H^n}d\rangle}{|\nabla d|} \leq Cd\frac{\langle A\nabla_{H^n}d, \nabla_{H^n}d\rangle}{|\nabla d|},$$

and thus

$$\left|\frac{Q-1}{r}\int_{\partial B_r}u^2\frac{\langle B\nabla_{H^n}d, \nabla_{H^n}d\rangle}{|\nabla d|}\mathrm{d}H^{2n}\right| \leq CH(r).$$

Finally, we estimate the first term on the right-hand side. Writing the divergence term as

$$\mathrm{div}(S^{\mathrm{T}}B\nabla_{H^n}d) = X_i(b_{ij}X_jd) = (X_ib_{ij})\cdot X_jd + b_{ij}\cdot(X_iX_jd)$$

and taking into account the assumption (1.9) and Proposition 2.1, we get the following inequalities:

$$|(X_ib_{ij})X_jd| \leq |X_ib_{ij}|\cdot|X_jd| \leq C\psi^{\frac{1}{2}}\cdot\psi^{\frac{1}{2}} \leq C\psi$$

and

$$|b_{ij}(X_iX_jd)| \leq |b_{ij}|\cdot|X_iX_jd| \leq C\psi d\cdot\frac{1}{d} \leq C\psi.$$

Hence,

$$\int_{\partial B_r}u^2\frac{\mathrm{div}(S^{\mathrm{T}}B\nabla_{H^n}d)}{|\nabla d|}\mathrm{d}H^{2n} \leq C\int_{\partial B_r}u^2\frac{\psi}{|\nabla d|}\mathrm{d}H^{2n} \leq CH(r).$$

Therefore,

$$\left|H'(r) - \frac{Q-1}{r}H(r) - 2I(r)\right| \leq C_1H(r).$$

Next, we need to estimate $I'(r)$, letting

$$\mu = \langle A\nabla_{H^n}d, \nabla_{H^n}d\rangle, \quad \nu = \langle B\nabla_{H^n}d, \nabla_{H^n}d\rangle, \tag{3.13}$$

and note that $\lambda\psi \leq \mu \leq \Lambda\psi$, and moreover $\psi = \mu - \nu$.

Consider the vector field F defined as follows:

$$F = \frac{d}{\mu}\sum_{i,j=1}^{2n}a_{ij}X_j\mathrm{d}X_i, \tag{3.14}$$

i.e.,

$$Fu = \frac{d}{\mu}\langle A\nabla_{H^n}d, \nabla_{H^n}u\rangle. \tag{3.15}$$

Hence

$$Fd = \langle F, \nabla d\rangle = d. \tag{3.16}$$

Lemma 3.3 (Rellich-Type Identity)   *Let F be the above considered vector field in H^n. Then we have the following identity:*

$$\int_{\partial B_r} \langle A\nabla_{H^n} u, \nabla_{H^n} u\rangle \langle F, \overrightarrow{n}\rangle \mathrm{d}H^{2n}$$

$$= 2\int_{\partial B_r} a_{jk} X_j u \langle X_k, \overrightarrow{n}\rangle F u \mathrm{d}H^{2n}$$

$$- 2\int_{B_r} a_{jk} X_j u[X_k, F] u \mathrm{d}z\mathrm{d}t + \int_{B_r} \mathrm{div}(F) \langle A\nabla_{H^n} u, \nabla_{H^n} u\rangle \mathrm{d}z\mathrm{d}t$$

$$+ \int_{B_r} \langle (FA)\nabla_{H^n} u, \nabla_{H^n} u\rangle \mathrm{d}z\mathrm{d}t - 2\int_{B_r} F u \mathcal{L} u \mathrm{d}z\mathrm{d}t.$$

Proof   The divergence theorem yields

$$\int_{\partial B_r} \langle A\nabla_{H^n} u, \nabla_{H^n} u\rangle \langle F, \overrightarrow{n}\rangle \mathrm{d}H^{2n}$$

$$= \int_{B_r} \mathrm{div}(F\langle A\nabla_{H^n} u, \nabla_{H^n} u\rangle) \mathrm{d}z\mathrm{d}t$$

$$= \int_{B_r} (\mathrm{div} F)\langle A\nabla_{H^n} u, \nabla_{H^n} u\rangle \mathrm{d}z\mathrm{d}t + \int_{B_r} \langle (FA)\nabla_{H^n} u, \nabla_{H^n} u\rangle \mathrm{d}z\mathrm{d}t$$

$$+ 2\int_{B_r} \langle A\nabla_{H^n} u, F\nabla_{H^n} u\rangle \mathrm{d}z\mathrm{d}t.$$

On the other hand,

$$2\int_{\partial B_r} a_{jk} X_j u \langle X_k, \overrightarrow{n}\rangle F u \mathrm{d}H^{2n}$$

$$= 2\int_{B_r} \mathrm{div}(a_{jk} X_j u F u X_k) \mathrm{d}z\mathrm{d}t$$

$$= 2\int_{B_r} \mathrm{div}(X_k) a_{jk} X_j u F u \mathrm{d}z\mathrm{d}t + 2\int_{B_r} X_k(a_{jk} X_j u) F u \mathrm{d}z\mathrm{d}t + 2\int_{B_r} a_{jk} X_j u(X_k F u) \mathrm{d}z\mathrm{d}t.$$

In view of the fact $\mathrm{div}(X_k) = 0$, and the identities

$$X_k F u - F X_k u = [X_k, F] u \quad \text{for any } k,$$

we obtain

$$\int_{\partial B_r} \langle A\nabla_{H^n} u, \nabla_{H^n} u\rangle \langle F, \overrightarrow{n}\rangle \mathrm{d}H^{2n}$$

$$= 2\int_{\partial B_r} a_{jk} X_j u \langle X_k, \overrightarrow{n}\rangle F u \mathrm{d}H^{2n} - 2\int_{B_r} a_{jk} X_j u[X_k, F] u \mathrm{d}z\mathrm{d}t$$

$$+ \int_{B_r} \mathrm{div}(F)\langle A\nabla_{H^n} u, \nabla_{H^n} u\rangle \mathrm{d}z\mathrm{d}t + \int_{B_r} \langle (FA)\nabla_{H^n} u, \nabla_{H^n} u\rangle \mathrm{d}z\mathrm{d}t$$

$$- 2\int_{B_r} F u \mathcal{L} u \mathrm{d}z\mathrm{d}t.$$

This completes the proof of Lemma 3.3.

Theorem 3.2 *Let u be a polyradial solution to (1.7), then there exist positive constants $C_2 = C_2(\lambda, \Lambda)$ and $C_3 = C_3(\lambda, \Lambda)$, such that for a.e. $r \in \Omega_{r_0}$, we have*

$$I'(r) \geq 2 \int_{\partial B_r} \frac{1}{\mu} \frac{\langle A\nabla_{H^n} u, \nabla_{H^n} d \rangle^2}{|\nabla d|} \mathrm{d}H_2 - C_2 \frac{f(r)}{r} I(r) - C_3 I(r), \tag{3.17}$$

where μ is defined in (3.13).

Remark 3.2 On H^1, a polyradial function u is equivalent to $\widetilde{T}u = \left(y_i \frac{\partial u}{\partial x_i} - x_i \frac{\partial u}{\partial y_i} \right) = 0$. On H^n ($n > 1$), a polyradial function u implies that $\widetilde{T}u = 0$, however, the converse is false. For example, for $(x_1, x_2, y_1, y_2, t) \in H^2$, take $u = x_1 x_2 + y_1 y_2 + t$. A direct calculation yields $\widetilde{T}u = 0$, but u is not a polyradial function. However, over the course of our proof of Theorem 3.2, we only need the condition $\widetilde{T}u = 0$. Thus we have the following more general result.

Corollary 3.1 *Let u be a solution to (1.7) satisfying $\widetilde{T}u = 0$. Then Theorem 3.2 also holds.*

Proof of Theorem 3.2 By using the co-area formula (3.12),

$$I(r) = \int_0^r \int_{\partial B_\rho} \frac{\langle A\nabla_{H^n} u, \nabla_{H^n} u \rangle}{|\nabla d|} \mathrm{d}H^{2n} \mathrm{d}\rho + \int_0^r \int_{\partial B_\rho} \frac{Vu^2}{|\nabla d|} \mathrm{d}H^{2n} \mathrm{d}\rho.$$

Differentiating $I(r)$ with respect to r, we get

$$I'(r) = \int_{\partial B_r} \frac{\langle A\nabla_{H^n} u, \nabla_{H^n} u \rangle}{|\nabla d|} \mathrm{d}H^{2n} + \int_{\partial B_r} \frac{Vu^2}{|\nabla d|} \mathrm{d}H^{2n}.$$

From (3.16), $\mathcal{L}u = Vu$ and Rellich-type identity Lemma 3.3, we obtain

$$\begin{aligned}
I'(r) &= \frac{1}{r} \int_{\partial B_r} \langle A\nabla_{H^n} u, \nabla_{H^n} u \rangle \left\langle F, \frac{\nabla d}{|\nabla d|} \right\rangle \mathrm{d}H^{2n} + \int_{\partial B_r} \frac{Vu^2}{|\nabla d|} \mathrm{d}H^{2n} \\
&= \frac{2}{r} \int_{\partial B_r} a_{jk} X_j u \langle X_k, \overrightarrow{n} \rangle Fu \mathrm{d}H^{2n} - \frac{2}{r} \int_{B_r} a_{jk} X_j u [X_k, F] u \mathrm{d}z\mathrm{d}t \\
&\quad + \frac{1}{r} \int_{B_r} \mathrm{div}(F) \langle A\nabla_{H^n} u, \nabla_{H^n} u \rangle \mathrm{d}z\mathrm{d}t + \frac{1}{r} \int_{B_r} \langle (FA)\nabla_{H^n} u, \nabla_{H^n} u \rangle \mathrm{d}z\mathrm{d}t \\
&\quad - \frac{2}{r} \int_{B_r} FuVu \mathrm{d}z\mathrm{d}t + \int_{\partial B_r} \frac{Vu^2}{|\nabla d|} \mathrm{d}H^{2n} \\
&= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4 + \mathrm{I}_5 + \mathrm{I}_6. \tag{3.18}
\end{aligned}$$

First, by using the definition of F, we deal with I_1 as follows:

$$\begin{aligned}
\mathrm{I}_1 &= \frac{2}{r} \int_{\partial B_r} a_{jk} X_j u \langle X_k, \overrightarrow{n} \rangle Fu \mathrm{d}H^{2n} \\
&= \frac{2}{r} \int_{\partial B_r} a_{jk} X_j u \frac{X_k d}{|\nabla d|} \cdot \frac{d}{\mu} \langle A\nabla_{H^n} d, \nabla_{H^n} u \rangle \mathrm{d}H^{2n} \\
&= 2 \int_{\partial B_r} \frac{1}{\mu} \cdot \frac{\langle A\nabla_{H^n} d, \nabla_{H^n} u \rangle^2}{|\nabla d|} \mathrm{d}H^{2n}. \tag{3.19}
\end{aligned}$$

Now we use (1.11), (3.1) and (3.7) to get the bound of I_6 as follows:

$$\begin{aligned}
|\mathrm{I}_6| &\leq \int_{\partial B_r} \frac{|V|u^2}{|\nabla d|} \mathrm{d}H^{2n} \\
&\leq C \frac{f(r)}{r^2} \int_{\partial B_r} u^2 \frac{\psi}{|\nabla d|} \mathrm{d}H^{2n} \leq C \frac{f(r)}{r} \frac{H(r)}{r} \leq \frac{f(r)}{r} I(r). \tag{3.20}
\end{aligned}$$

By using conditions (1.9) and (3.13), we estimate $|Fu|$ as follows:

$$|Fu| = \frac{d}{\mu}a_{ij}X_idX_ju = \frac{d}{\mu}b_{ij}X_idX_ju + \frac{d}{\mu}X_idX_iu$$

$$\leq \frac{d}{\mu}|b_{ij}||X_id||X_ju| + \frac{d}{\mu}|X_id||X_iu|$$

$$\leq Cd\psi^{-\frac{1}{2}}|Xu|.$$

With the help of (1.11) and (2.7), we can estimate I_5 as follows:

$$|I_5| \leq \frac{2}{r}\int_{B_r}|Fu||V(z,t)||u(z,t)|\mathrm{d}z\mathrm{d}t$$

$$\leq C\frac{f(r)}{r}\int_{B_r}\frac{\psi^{\frac{1}{2}}|u|}{d}|Xu|\mathrm{d}z\mathrm{d}t$$

$$\leq C\frac{f(r)}{r}\int_{B_r}\frac{u^2\psi}{d^2}\mathrm{d}z\mathrm{d}t + C\frac{f(r)}{r}\int_{B_r}|\nabla_{H^n}u|^2\mathrm{d}z\mathrm{d}t$$

$$\leq Cf(r)\frac{H(r)}{r} + C\frac{f(r)}{r}D(r) \leq C\frac{f(r)}{r}I(r). \tag{3.21}$$

Next we get a bound for I_4: For every $r,s = 1,\cdots,2n$, we have

$$|Fa_{rs}| = \left|\frac{d}{\mu}a_{ij}X_idX_j(a_{rs})\right|$$

$$\leq \left|\frac{d}{\mu}b_{ij}X_idX_j(a_{rs})\right| + \left|\frac{d}{\mu}X_idX_ib_{rs}\right|$$

$$\leq \frac{d}{\mu}|b_{ij}||X_id||X_j(a_{rs})| + \frac{d}{\mu}|X_id||X_ib_{rs}|$$

$$\leq \frac{d}{\mu}d\psi\psi^{\frac{1}{2}}\psi^{\frac{1}{2}} + \frac{d}{\mu}\psi^{\frac{1}{2}}\psi^{\frac{1}{2}} \leq Cd.$$

Hence

$$|I_4| \leq \frac{1}{r}\int_{B_r}|\langle(FA)\nabla_{H^n}u, \nabla_{H^n}u\rangle|\mathrm{d}z\mathrm{d}t$$

$$\leq C\frac{1}{r}\int_{B_r}d\langle\nabla_{H^n}u, \nabla_{H^n}u\rangle\mathrm{d}z\mathrm{d}t$$

$$\leq CD(r) \leq CI(r). \tag{3.22}$$

We now would like to get a bound for I_3: Because of $\mathrm{div}(X_i) = 0$ for every $i = 1,\cdots,2n$, we get

$$\mathrm{div}F = \mathrm{div}\left(\frac{d}{\mu}a_{ij}X_jdX_i\right) = \frac{d}{\mu}a_{ij}X_jd(\mathrm{div}X_i) + X_i\left(\frac{d}{\mu}a_{ij}X_jd\right) = X_i\left(\frac{d}{\mu}(a_{ij})X_jd\right)$$

$$= \frac{1}{\mu}X_id(a_{ij})X_jd - \frac{d}{\mu^2}X_i\mu a_{ij}X_jd + \frac{d}{\mu}(X_ia_{ij})X_jd + \frac{d}{\mu}a_{ij}(X_iX_jd)$$

$$= \frac{1}{\mu}X_id(b_{ij})X_jd + \frac{1}{\mu}X_idX_id - \frac{d}{\mu^2}X_i\mu(b_{ij})X_jd - \frac{d}{\mu^2}X_i\mu X_id$$

$$+ \frac{d}{\mu}(X_ia_{ij})X_jd + \frac{d}{\mu}b_{ij}(X_iX_jd) + \frac{d}{\mu}(X_iX_id).$$

Here

$$X_i \mu = X_i \langle A \nabla_{H^n} d, \nabla_{H^n} d \rangle$$
$$= X_i (a_{kl} X_k d X_l d)$$
$$= X_i (a_{kl}) X_k d X_l d + 2 a_{kl} (X_i X_k d) X_l d$$
$$= X_i (a_{kl}) X_k d X_l d + 2 b_{kl} (X_i X_k d) X_l d + 2 (X_i X_k d) X_k d,$$

and then

$$|X_i \mu| \le C \frac{1}{d} \psi^{\frac{1}{2}}. \tag{3.23}$$

According to $X_i d (X_i X_k d) X_k d = 0$, we have

$$\left| \frac{d}{\mu^2} X_i \mu (a_{ij}) X_j d + \frac{d}{\mu^2} X_i \mu X_i d \right|$$

$$= \frac{d}{\mu^2} \{ X_i (a_{kl}) X_k d X_l d (b_{ij}) X_j d + 2 b_{kl} (X_i X_k d) X_l d (b_{ij}) X_j d + 2 (X_i X_k d) X_k d (b_{ij}) X_j d$$

$$+ (X_i a_{kl}) X_k d X_l d X_i d + 2 b_{kl} (X_i X_k d) X_l d X_i d + 2 (X_i X_k d) X_k d X_i d \}$$

$$\le C \frac{d}{\mu^2} \psi^2 \le Cd.$$

Therefore,

$$|\mathrm{div}\, F| \le \frac{1}{\mu} (b_{ij} X_j d X_i d + (X_i d)^2) + \frac{d}{\mu} (X_i b_{ij} X_j d + b_{ij} X_i X_j d + X_i X_i d) + Cd$$
$$\le C,$$

and thus

$$|I_3| \le C \frac{1}{r} \int_{B_r} \langle A \nabla_{H^n} u, \nabla_{H^n} u \rangle \mathrm{d}z \mathrm{d}t$$

$$\le C \frac{1}{r} D(r) \le C \frac{1}{r} I(r). \tag{3.24}$$

Finally, under the assumption $\widetilde{T} u = 0$, we get (2.4). Let us estimate I_2 as follows:

$$Fu = \frac{d}{\mu} a_{ij} X_i d X_j u = \frac{d}{\mu} b_{ij} X_i d X_j u + \frac{d}{\mu} X_i d X_i u$$

$$= \frac{d}{\mu} b_{ij} X_i d X_j u + \frac{d}{\mu} \left(\frac{1}{d} \psi Z u \right) = \frac{d}{\mu} b_{ij} X_i d X_j u + \frac{\psi}{\mu} Z u$$

$$= \frac{d}{\mu} b_{ij} X_i d X_j u + Z u - \frac{\nu}{\mu} Z u.$$

Hence

$$[X_k, F] u = \left[X_k, \frac{d}{\mu} b_{ij} X_i d X_j \right] u + [X_k, Z] u - \left[X_k, \frac{\nu}{\mu} Z \right] u. \tag{3.25}$$

We estimate the first term of (3.25) as follows:

$$\left| \left[X_k, \frac{d}{\mu} b_{ij} X_i dX_j \right] u \right|$$

$$\leq \left| X_k \left(\frac{d}{\mu} b_{ij} X_i d \right) X_j u \right| + \left| \frac{d}{\mu} b_{ij} X_i d [X_k, X_j] u \right|$$

$$\leq \left| \frac{1}{\mu} X_k d b_{ij} X_i dX_j u \right| + \left| \frac{d}{\mu^2} X_k \mu b_{ij} X_i dX_j u \right| + \left| \frac{d}{\mu} X_k b_{ij} X_i dX_j u \right|$$

$$+ \left| \frac{d}{\mu} b_{ij} (X_k X_I d) X_j u \right| + \left| \frac{d}{\mu} b_{ij} X_i d [X_k, X_j] u \right|$$

$$\leq C d |Xu| + \left| \frac{d}{\mu} b_{ij} X_i d [X_k, X_j] u \right|$$

$$\leq C d |Xu| + \frac{d}{\mu} |b_{ij}| |X_i d| \frac{1}{|z|} |Xu|$$

$$\leq C d |Xu|.$$

Because of (2.4) and $|X_k \nu| = X_k(b_{rs} X_r dX_s d) \leq C \psi^{\frac{3}{2}}$, the last term of (3.25) is

$$\left| \left[X_k, \frac{\nu}{\mu} Z \right] u \right| = \left| \left[X_k, \frac{\nu}{\mu} \cdot \frac{d}{\psi} X_i dX_i \right] u \right|$$

$$\leq \left| X_k \left(\frac{\nu}{\mu} \cdot \frac{d}{\psi} X_i d \right) \right| |X_i u| + \left| \frac{\nu}{\mu} \cdot \frac{d}{\psi} X_i d [X_k, X_i] u \right|$$

$$\leq C |Xu|.$$

Therefore,

$$I_2 \leq C \frac{1}{r} \int_{B_r} |\nabla_{H^n} u|^2 dz dt \leq C \frac{1}{r} I(r). \tag{3.26}$$

Substituting (3.19)–(3.22), (3.24) and (3.26) in (3.18), we obtain

$$I'(r) \geq 2 \int_{\partial B_r} \frac{1}{\mu} \cdot \frac{\langle A \nabla_{H^n} d, \nabla_{H^n} u \rangle^2}{|\nabla d|} dH^{2n} - C_2 \frac{f(r)}{r} I(r) - C_3 \frac{1}{r} I(r).$$

With the help of (3.10), Theorems 3.1 and 3.2, we get the monotonicity of the frequency function.

Theorem 3.3 *Let u be a polyradial solution to (1.7). Then there exist positive constants $r_0 = r_0(\lambda, \Lambda)$ and $C = C(\lambda, \Lambda)$, such that for a.e. $r \in \Omega_{r_0}$ we have*

$$\frac{N'(r)}{N(r)} \geq -C_4 \frac{f(r)}{r} - C_5 \frac{1}{r}. \tag{3.27}$$

Proof Applying (3.11) and (3.17) in (3.10), we obtain

$$\frac{N'(r)}{N(r)} \geq \frac{1}{r} + \frac{2 \int_{\partial B_r} \frac{1}{\mu} \cdot \frac{\langle A \nabla_{H^n} d, \nabla_{H^n} u \rangle^2}{|\nabla d|} dH^{2n}}{I(r)} - C_2 \frac{f(r)}{r} - C_3 \frac{1}{r}$$

$$- \frac{Q-1}{r} - C_1 - 2 \frac{I(r)}{H(r)}$$

$$\geq -C_4 \frac{f(r)}{r} - C_5 \frac{1}{r},$$

where we have applied Proposition 3.1 and the Cauchy-Schwarz inequality.

By using the monotonicity of $N(r)$, it is easy to prove Theorems 1.1–1.2, and we give the proof in the current setting for completeness.

Proof of Theorem 1.1 We rewrite inequality (3.11) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\ln\Big(\frac{H(t)}{t^{Q-1}}\Big)\Big) \leq C_1 + 2\frac{N(t)}{t}, \quad t \in (0, r_0). \tag{3.28}$$

Integrating (3.28) between r and $2r$, with $2r < r_0$, we have

$$\ln\Big(\frac{H(2r)}{H(r)}2^{1-Q}\Big) \leq C_1 r + 2\int_r^{2r} N(t)\frac{1}{t}\mathrm{d}t$$

$$\leq C_1 r + 2\int_{(r,2r)\cap\Omega_{r_0}} N(t)\frac{1}{t}\mathrm{d}t + 2\int_{J_r} N(t)\frac{1}{t}\mathrm{d}t, \tag{3.29}$$

where $J_r = \{t \in (r, 2r) \mid t \notin \Omega_{r_0}, N(t) \geq 0\}$. Due to (3.5), on J_r we have $0 \leq N(t) \leq \max(1, N(r_0))$,

$$\int_{J_r} N(t)\frac{1}{t}\mathrm{d}t \leq \max(1, N(r_0))\int_r^{2r}\frac{1}{t}\mathrm{d}t = \max(1, N(r_0))\ln 2. \tag{3.30}$$

On the other hand, by using the monotonicity of $N(r)$, i.e., (3.27), we have

$$\ln\frac{N(b_j)}{N(r)} = \int_r^{b_j}\frac{N'(t)}{N(t)}\mathrm{d}t \geq -C_4\int_0^{r_0}\frac{f(t)}{t}\mathrm{d}t - C_5\int_r^{r_0}\frac{1}{t}\mathrm{d}t.$$

From the above inequality, recalling that $b_j \notin \Omega_{r_0}$, we have

$$N(r) \leq \exp\Big(C_4\int_0^{R_0}\frac{f(t)}{t}\mathrm{d}t + C_5\ln\frac{r_0}{r}\Big)\max(1, N(r_0)) \quad \text{for every } r \in \Omega_{r_0}.$$

Hence

$$\int_{(r,2r)\cap\Omega_{r_0}} N(t)\frac{1}{t}\mathrm{d}t \leq \exp\Big(C_4\int_0^{R_0}\frac{f(t)}{t}\mathrm{d}t + C_5\ln\frac{r_0}{r}\Big)\max(1, N(r_0))\ln 2. \tag{3.31}$$

Applying (3.30)–(3.31) in (3.29), we finally obtain

$$H(2r) \leq \Gamma H(r).$$

Integrating with respect to r and using the co-area formula we finally prove (1.12).

Proof of Theorem 1.2 Letting r_0 be as in Theorem 1.1, we obtain after k interactions of (1.12)

$$\int_{B_{r_0}} u^2\psi(z,t)\mathrm{d}z\mathrm{d}t \leq \cdots \leq \Gamma^k\int_{B_{2^{-k}r_0}} u^2\psi(z,t)\mathrm{d}z\mathrm{d}t \tag{3.32}$$

$$= \Gamma^k|B_{2^{-k}r_0}|^\beta\frac{1}{|B_{2^{-k}r_0}|^\beta}\int_{B_{2^{-k}r_0}} u^2\psi(z,t)\mathrm{d}z\mathrm{d}t, \tag{3.33}$$

where $\beta > 0$ is a number to be suitably chosen later. By (1.6), we have

$$\Gamma^k|B_{2^{-k}r_0}|^\beta = \omega_Q^\beta r_0^{\beta Q}\Big(\frac{\Gamma}{2^{\beta Q}}\Big)^k = |B_{r_0}|^\beta\Big(\frac{\Gamma}{2^{\beta Q}}\Big)^k.$$

We now choose β such that $\frac{\Gamma}{2^{\beta Q}} = 1$. Then, (3.32) becomes

$$\int_{B_{r_0}} u^2 \psi(z,t) \mathrm{d}z \mathrm{d}t \le |B_{r_0}|^\beta \frac{1}{|B_{2^{-k}r_0}|^\beta} \int_{B_{2^{-k}r_0}} u^2 \psi(z,t) \mathrm{d}z \mathrm{d}t. \tag{3.34}$$

Let $k \to \infty$. Then the right-hand side of (3.34) goes to zero, because of the assumption that u vanishes to infinite order at the origin. We conclude that it must be $u \equiv 0$ in B_{r_0}.

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