Chin. Ann. Math.
34B(4), 2013, 479–490
DOI: 10.1007/s11401-013-0785-9

Chinese Annals of Mathematics, Series B

Exact Boundary Controllability and Exact Boundary Observability for a Coupled System of Quasilinear Wave Equations

Long HU[1] Fanqiong JI[1] Ke WANG[1]

Abstract Based on the theory of semi-global classical solutions to quasilinear hyperbolic systems, the authors apply a unified constructive method to establish the local exact boundary (null) controllability and the local boundary (weak) observability for a coupled system of 1-D quasilinear wave equations with various types of boundary conditions.

Keywords Coupled system of quasilinear wave equations, Exact boundary controllability, Exact boundary observability

1 Introduction

There are many publications concerning the exact boundary controllability for linear hyperbolic systems (see [17–19] and the references therein). As a special case of second-order hyperbolic equations, the exact boundary controllability for linear wave equations was obtained in a complete manner (see [17–18]). Zuazua [25–26], Emanuilov [1] and Lasiecka and Triggiani [5] established the exact boundary controllability for some semilinear wave equations. In the quasilinear case, based on the result about the semi-global C^2 solution, by a direct constructive method, Li and Yu established the local exact boundary controllability for a single 1-D quasilinear wave equation with various types of boundary conditions (see [11–13]). Later, this result was applied to get the exact boundary controllability of the nodal profile and the exact boundary controllability on a tree-like network for quasilinear wave equations, respectively (see [3, 20]). For the following second-order quasilinear hyperbolic system

$$u_{tt} + A(u, u_x, u_t)u_{tx} + B(u, u_x, u_t)u_{xx} = C(u, u_x, u_t), \tag{1.1}$$

under different hypotheses on matrices A and B, the corresponding local exact boundary controllability was obtained by Yu [22] and Wang [21], respectively.

On the other hand, as a dual problem of controllability, the exact boundary observability for wave equations has been widely studied (see [2, 6, 17–18]). In fact, The essence of J.-L. Lions' HUM method is to use the duality to get the controllability by a corresponding observability inequality. Based on the result about semi-global classical solutions to quasilinear hyperbolic systems, by a constructive method, the exact boundary observability for a single quasilinear wave equation was established by Li [12, 14] and Guo [4], respectively, and some

Manuscript received January 16, 2013.
[1]School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
E-mail: scofieldhu88@gmail.com jifanqiongfdu@gmail.com kwang0815@gmail.com

implicit dualities have been shown between the exact boundary controllability and the exact boundary observability in the quasilinear case. As to the second-order quasilinear hyperbolic system (1.1), the corresponding local exact boundary observability was obtained by Yu [23] under some hypotheses on matrices A and B.

In this paper, we consider a kind of coupled system of 1-D quasilinear wave equations, which can be rewritten in the form of second-order quasilinear hyperbolic systems discussed in [21–23], but the systems given in [21–23] are too general, which is not convenient to get the desired results. Therefore, for the coupled system of quasilinear wave equations under consideration, we restudy its controllability and observability to show the corresponding results in a clear manner. Moreover, the corresponding discussions on the exact boundary null controllability and the weak observability are added. Based on the existence and uniqueness of semi-global C^2 solutions, by a constructive method developed by Li [8–14], we obtain the local exact boundary (null) controllability and the local exact boundary (weak) observability for a coupled system of quasilinear wave equations with different types of boundary conditions. The conclusions will provide a foundation for studying the exact boundary synchronization for a coupled system of wave equations (see [15]).

Consider the following coupled system of 1-D quasilinear wave equations:

$$\frac{\partial^2 w_i}{\partial t^2} - a_i^2(w)\frac{\partial^2 w_i}{\partial x^2} + \sum_{j=1}^{n} a_{ij}(w)w_j = 0, \tag{1.2}$$

where $w = (w_1, \dots, w_n)^{\mathrm{T}}$ is the unknown vector function of (t, x), $a_i(w)$ and $a_{ij}(w)$ $(i, j = 1, \cdots, n)$ are all C^1 functions of w, satisfying

$$a_i(0) > 0, \quad i = 1, \cdots, n. \tag{1.3}$$

On one end $x = 0$, we prescribe any one of the following boundary conditions of Dirichlet type, Neumann type, coupled third type and coupled dissipative type, respectively:

$$x = 0: \quad w_i = h_i(t), \quad i = 1, \cdots, n, \tag{1.4a}$$

$$x = 0: \quad \frac{\partial w_i}{\partial x} = h_i(t), \quad i = 1, \cdots, n, \tag{1.4b}$$

$$x = 0: \quad \frac{\partial w_i}{\partial x} - \sum_{j=1}^{n} b_{ij}(w)w_j = h_i(t), \quad i = 1, \cdots, n, \tag{1.4c}$$

$$x = 0: \quad \frac{\partial w_i}{\partial x} - \sum_{j=1}^{n} c_{ij}(w)\frac{\partial w_j}{\partial t} = h_i(t), \quad i = 1, \cdots, n, \tag{1.4d}$$

where $b_{ij} = b_{ij}(w)$ and $c_{ij} = c_{ij}(w)$ are C^1 functions of w, $h_i(t)$ are C^2 (in case (1.4a)) or C^1 (in cases (1.4b)–(1.4d)) functions.

Similarly, on another end $x = L$, the boundary conditions are given as

$$x = L: \quad w_i = \overline{h}_i(t), \quad i = 1, \cdots, n, \tag{1.5a}$$

$$x = L: \quad \frac{\partial w_i}{\partial x} = \overline{h}_i(t), \quad i = 1, \cdots, n, \tag{1.5b}$$

$$x = L: \quad \frac{\partial w_i}{\partial x} + \sum_{j=1}^{n} \overline{b}_{ij}(w)w_j = \overline{h}_i(t), \quad i = 1, \cdots, n, \tag{1.5c}$$

$$x = L: \quad \frac{\partial w_i}{\partial x} + \sum_{j=1}^{n} \overline{c}_{ij}(w) \frac{\partial w_j}{\partial t} = \overline{h}_i(t), \quad i = 1, \cdots, n, \tag{1.5d}$$

where $\overline{b}_{ij} = \overline{b}_{ij}(w)$ and $\overline{c}_{ij} = \overline{c}_{ij}(w)$ are C^1 functions of w, $\overline{h}_i(t)$ are C^2 (in case (1.5a)) or C^1 (in cases (1.5b)–(1.5d)) functions.

The initial conditions are given by

$$t = 0: \quad (w, w_t) = (\varphi(x), \psi(x)), \quad 0 \le x \le L, \tag{1.6}$$

where $\varphi = (\varphi_1, \cdots, \varphi_n)^{\mathrm{T}}$ is a C^2 vector function of x with small C^2 norm on $[0, L]$, $\psi = (\psi_1, \cdots, \psi_n)^{\mathrm{T}}$ is a C^1 vector function of x with small C^1 norm on $[0, L]$, such that the conditions of C^2 compatibility at the points $(t, x) = (0, 0)$ and $(0, L)$ are satisfied, respectively.

Obviously, $w = 0$ is an equilibrium of system (1.2). Based on the theory of semi-global C^2 solutions, by a constructive method (see [8–14]), we will establish the local exact boundary controllability and the local exact boundary observability around $w = 0$.

This paper is organized as follows. The existence and uniqueness of semi-global C^2 solution to the coupled system (1.2) of quasilinear wave equations with boundary conditions (1.4) and (1.5) will be given in Section 2. Based on this, in Section 3, we obtain the corresponding local exact boundary (null) controllability with boundary controls on one end or on two ends, and in Section 4, we obtain the corresponding local exact boundary (weak) observability with observed values on one end or on two ends.

2 Existence and Uniqueness of Semi-global C^2 Solution

For the purpose of getting the local exact boundary controllability and observability for system (1.2) with boundary conditions (1.4)–(1.5), we should first prove the existence and uniqueness of semi-global C^2 solution to the mixed initial-boundary value problem (1.2) and (1.4)–(1.6). In order to get it in a unified manner, the best way is to reduce the system to a first-order quasilinear hyperbolic system and use the corresponding results of semi-global C^1 solutions.

Setting

$$u_i = \frac{\partial w_i}{\partial x}, \quad v_i = \frac{\partial w_i}{\partial t}, \quad i = 1, \cdots, n, \tag{2.1}$$

$$u = (u_1, \cdots, u_n)^{\mathrm{T}}, \quad v = (v_1, \cdots, v_n)^{\mathrm{T}}, \tag{2.2}$$

system (1.2) can be reduced to the following first-order quasilinear system:

$$\begin{cases} \dfrac{\partial w}{\partial t} = v, \\ \dfrac{\partial u}{\partial t} - \dfrac{\partial v}{\partial x} = 0, \\ \dfrac{\partial v}{\partial t} - \Lambda^2(w) \dfrac{\partial u}{\partial x} = -A(w)w, \end{cases} \tag{2.3}$$

where $\Lambda(w) = \mathrm{diag}\{a_1(w), \cdots, a_n(w)\}$ and $A(w) = (a_{ij}(w))_{n \times n}$. Its equivalent matrix form is

$$\frac{\partial}{\partial t} \begin{pmatrix} w \\ u \\ v \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -I_n \\ 0 & -\Lambda^2(w) & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} w \\ u \\ v \end{pmatrix} = \begin{pmatrix} v \\ 0 \\ -A(w)w \end{pmatrix}. \tag{2.4}$$

The characteristic equation of (2.3) or (2.4) is

$$\det\begin{pmatrix}\lambda I_n & 0 & 0\\ 0 & \lambda I_n & I_n\\ 0 & \Lambda^2(w) & \lambda I_n\end{pmatrix} = \lambda^n|\lambda^2 I_n - \Lambda^2(w)| = 0, \tag{2.5}$$

whose solutions, the real eigenvalues of system (2.3) or (2.4), are

$$\lambda_i^- = -a_i(w), \quad \lambda_i^0 = 0, \quad \lambda_i^+ = a_i(w), \quad i = 1, \cdots, n, \tag{2.6}$$

the corresponding left eigenvectors, which constitute a complete set, can be chosen as

$$l_i^- = (\mathbf{0}, a_i(w)\boldsymbol{e}_i, \quad \boldsymbol{e}_i), \quad l_i^0 = (\boldsymbol{e}_i, \mathbf{0}, \mathbf{0}), \quad l_i^+ = (\mathbf{0}, -a_i(w)\boldsymbol{e}_i, \boldsymbol{e}_i), \quad i = 1, \cdots, n, \tag{2.7}$$

in which $\mathbf{0} = (0, \cdots, 0)$ is the zero vector of order n, $\boldsymbol{e}_i = (0, \cdots, \overset{(i)}{1}, \cdots, 0)$ is a row vector of order n. Thus, (2.3) or (2.4) reduced from system (1.2) is a first-order quasilinear hyperbolic system.

Let

$$U = (w, u, v)^{\mathrm{T}}. \tag{2.8}$$

Taking

$$\begin{cases} V_i^- = l_i^- U = a_i(w)u_i + v_i,\\ V_i^0 = l_i^0 U = w_i, \quad i = 1, \cdots, n,\\ V_i^+ = l_i^+ U = -a_i(w)u_i + v_i,\end{cases} \tag{2.9}$$

namely,

$$\begin{cases} V^- = \Lambda(w)u + v,\\ V^0 = w,\\ V^+ = -\Lambda(w)u + v,\end{cases} \tag{2.10}$$

we have

$$\begin{cases} w = V^0,\\ u = \dfrac{1}{2}\Lambda^{-1}(V^0)(V^- - V^+),\\ v = \dfrac{1}{2}(V^- + V^+).\end{cases} \tag{2.11}$$

When system (1.2) is reduced to system (2.3) or (2.4), the boundary condition (1.4) will be correspondingly replaced by

$$x = 0: \quad v = \dot{H}(t), \tag{2.12a}$$

$$x = 0: \quad u = H(t), \tag{2.12b}$$

$$x = 0: \quad u - B(w)w = H(t), \tag{2.12c}$$

$$x = 0: \quad u - C(w)v = H(t), \tag{2.12d}$$

in which $H(t) = (h_1(t), \cdots, h_n(t))^{\mathrm{T}}$, $B(w) = (b_{ij}(w))_{n\times n}$ and $C(w) = (c_{ij}(w))_{n\times n}$.

Noting (2.11), if

$$\det(\Lambda^{-1}(0) + C(0)) \neq 0, \tag{2.13}$$

then in a neighborhood of $U = 0$, the boundary condition (2.12) (namely, (1.4)) on $x = 0$ can be rewritten as

$$x = 0: \quad V^+ = -V^- + 2\dot{H}(t), \tag{2.14a}$$

$$x = 0: \quad V^+ = V^- - 2\Lambda(V^0)H(t), \tag{2.14b}$$

$$x = 0: \quad V^+ = V^- - 2\Lambda(V^0)B(V^0)V^0 - 2\Lambda(V^0)H(t), \tag{2.14c}$$

$$\begin{aligned} x = 0: \quad V^+ = {} & (\Lambda^{-1}(V^0) + C(V^0))^{-1}(\Lambda^{-1}(V^0) - C(V^0))V^- \\ & - 2(\Lambda^{-1}(V^0) + C(V^0))^{-1}H(t), \end{aligned} \tag{2.14d}$$

which can be uniformly expressed as

$$x = 0: \quad V^+ = G(t, V^-, V^0) + \widetilde{H}(t), \tag{2.15}$$

where G and $\widetilde{H}$ are C^1 functions with respect to their arguments, and without loss of generality, we may assume that

$$G(t, 0, 0) \equiv 0. \tag{2.16}$$

Similarly, if

$$\det(\Lambda^{-1}(0) + \overline{C}(0)) \neq 0, \tag{2.17}$$

then in a neighborhood of $U = 0$, the boundary condition (1.5) on $x = L$ can be written in the form

$$x = L: \quad V^- = \overline{G}(t, V^0, V^+) + \widetilde{\overline{H}}(t), \tag{2.18}$$

in which $\overline{G}$ and $\widetilde{\overline{H}}$ are C^1 functions with respect to their arguments, and without loss of generality, we may assume that

$$\overline{G}(t, 0, 0) \equiv 0. \tag{2.19}$$

Meanwhile, the corresponding initial condition (1.6) can be written as

$$t = 0: \quad U = (\varphi(x), \varphi'(x), \psi(x))^{\mathrm{T}}, \quad 0 \leq x \leq L. \tag{2.20}$$

Together with the conditions of C^2 compatibility at the points $(t, x) = (0, 0)$ and $(0, L)$ for the coupled system of wave equations (1.2) with the boundary conditions (1.4)–(1.5) on $x = 0$ and $x = L$, respectively, and the initial condition (1.6), it is easy to see that the conditions of C^1 compatibility at these two points are also satisfied for the mixed initial-boundary value problem (2.3), (2.15), (2.18) and (2.20).

For the convenience of statement, in the whole paper, we denote that

$$d = \begin{cases} 2 & \text{for (1.4a)}, \\ 1 & \text{for (1.4b)–(1.4d)} \end{cases} \tag{2.21}$$

and

$$\overline{d} = \begin{cases} 2 & \text{for (1.5a)}, \\ 1 & \text{for (1.5b)}-\text{(1.5d)}. \end{cases} \tag{2.22}$$

Based on the theory of semi-global C^1 solutions to the first-order quasilinear hyperbolic system with zero eigenvalues (see [7, 11–14, 24]), it is easy to get the following

Lemma 2.1 *Under the hypotheses given in Section* 1*, suppose that the conditions of C^2 compatibility are satisfied at the points $(t, x) = (0, 0)$ and $(0, L)$, respectively. Suppose furthermore that* (2.13) *and* (2.17) *hold. For any given and possibly quite large $T > 0$, if*

$$\|(\varphi_i, \psi_i)\|_{C^2[0,L]\times C^1[0,L]}, \quad \|h_i\|_{C^d[0,T]}, \quad \|\overline{h}_i\|_{C^{\overline{d}}[0,T]}, \quad i = 1, \cdots, n$$

are suitably small, then the forward mixed initial-boundary value problem (1.2) *and* (1.4)–(1.6) *admits a unique semi-global C^2 solution $w = w(t, x)$ on the domain $R(T) = \{(t, x) \mid 0 \le t \le T, 0 \le x \le L\}$, and*

$$\|w\|_{C^2[R(T)]} \le C\Big(\sum_{i=1}^{n} \|(\varphi_i, \psi_i)\|_{C^2[0,L]\times C^1[0,L]} + \sum_{i=1}^{n} \|(h_i, \overline{h}_i)\|_{C^d[0,L]\times C^{\overline{d}}[0,L]}\Big), \tag{2.23}$$

where C is a positive constant possibly depending on T.

Corollary 2.1 *Under the hypotheses given in Section* 1*, if $\|(\varphi_i, \psi_i)\|_{C^2[0,L]\times C^1[0,L]}$ $(i = 1, \cdots, n)$ are suitably small, then Cauchy problem* (1.2) *and* (1.6) *admits a unique global C^2 solution $w = w(t, x)$ on its whole maximum determinate domain and*

$$\|w\|_{C^2} \le C \sum_{i=1}^{n} \|(\varphi_i, \psi_i)\|_{C^2[0,L]\times C^1[0,L]}, \tag{2.24}$$

where C is a positive constant.

As for the backward mixed initial-boundary value problem (1.2), (1.4)–(1.5) with the final condition

$$t = T: \quad (w, w_t) = (\Phi(x), \Psi(x)), \quad 0 \le x \le L, \tag{2.25}$$

in which $\Phi = (\Phi_1, \cdots, \Phi_n)^{\mathrm{T}}$ is a C^2 vector function of x with small C^2 norm on $[0, L]$, $\Psi = (\Psi_1, \cdots, \Psi_n)^{\mathrm{T}}$ is a C^1 vector function of x with small C^1 norm on $[0, L]$, such that the conditions of C^2 compatibility at the points $(t, x) = (T, 0)$ and (T, L) are satisfied, respectively, similarly we have

Lemma 2.2 *Under the hypotheses given in Section* 1*, suppose that the conditions of C^2 compatibility are satisfied at the points $(t, x) = (T, 0)$ and (T, L), respectively. Suppose furthermore that*

$$\det(\Lambda^{-1}(0) - C(0)) \neq 0 \tag{2.26}$$

and

$$\det(\Lambda^{-1}(0) - \overline{C}(0)) \neq 0. \tag{2.27}$$

For any given and possibly quite large $T > 0$, if

$$\|(\Phi_i, \Psi_i)\|_{C^2[0,L]\times C^1[0,L]}, \quad \|h_i\|_{C^d[0,T]}, \quad \|\overline{h}_i\|_{C^{\overline{d}}[0,T]}, \quad i = 1, \cdots, n$$

are suitably small, then the backward mixed initial-boundary value problem (1.2), (2.25) *and* (1.4)–(1.5) *admits a unique semi-global C^2 solution $w = w(t, x)$ on the domain $R(T) = \{(t, x) \mid 0 \le t \le T, 0 \le x \le L\}$ and*

$$\|w\|_{C^2[R(T)]} \le C\Big(\sum_{i=1}^{n} \|(\Phi_i, \Psi_i)\|_{C^2[0,L]\times C^1[0,L]} + \sum_{i=1}^{n} \|(h_i, \overline{h}_i)\|_{C^d[0,L]\times C^{\overline{d}}[0,L]}\Big), \tag{2.28}$$

where C is a positive constant possibly depending on T.

Remark 2.1 If $a_i, a_{ij}, b_{ij}, \overline{b}_{ij}, c_{ij}, \overline{c}_{ij}$ are all independent of w, then the problem is linear and it is not necessary to assume the smallness of initial (resp. finial) data and boundary functions in Lemma 2.1 (resp. Lemma 2.2). Moreover, the C^2 solution exists globally in time.

Remark 2.2 Suppose that $a_i(w) \equiv a > 0$ $(i = 1, \cdots, n)$ are the same constant, the conditions (2.13) and (2.17) in Lemma 2.1 mean that $-a$ is not an eigenvalue of both matrices $C(0)$ and $\overline{C}(0)$, while the conditions (2.26)–(2.27) in Lemma 2.2 mean that a is not an eigenvalue of both matrices $C(0)$ and $\overline{C}(0)$.

Remark 2.3 By Lemma 2.1, conditions (2.13) and (2.17) for matrices C and $\overline{C}$ in boundary conditions (1.4d) and (1.5d) of coupled dissipative type are imposed to guarantee the well-posedness of the corresponding forward mixed initial-boundary value problem, while, by Lemma 2.2, conditions (2.26) and (2.27) for matrices C and $\overline{C}$ in boundary conditions (1.4d) and (1.5d) of coupled dissipative type are imposed to guarantee the well-posedness of the corresponding backward mixed initial-boundary value problem. Other coupled matrices A, B and $\overline{B}$, however, can be completely arbitrary.

Remark 2.4 Lemmas 2.1–2.2 and Corollary 2.1 are still valid for the following systems:

$$\frac{\partial^2 w_i}{\partial t^2} - a_i^2(w)\frac{\partial^2 w_i}{\partial x^2} + \sum_{j=1}^{n} \overline{a}_{ij}(w)\frac{\partial w_j}{\partial t} = 0 \tag{2.29}$$

or

$$\frac{\partial^2 w_i}{\partial t^2} - a_i^2(w)\frac{\partial^2 w_i}{\partial x^2} + \sum_{j=1}^{n} a_{ij}(w) w_j + \sum_{j=1}^{n} \overline{a}_{ij}(w)\frac{\partial w_j}{\partial t} = 0, \tag{2.30}$$

in which $a_i(w)$ and $a_{ij}(w)$, $\overline{a}_{ij}(w)$ are all C^1 functions with respect to their arguments, and (1.3) holds.

3 Local Exact Boundary Controllability

Theorem 3.1 (Two-Sided Control) *Under the hypotheses given in Section* 1*, suppose furthermore that* (2.13) *and* (2.17) *hold. Let*

$$T > L \max_{i=1,\cdots,n}\Big(\frac{1}{a_i(0)}\Big). \tag{3.1}$$

For any given initial data (φ, ψ) *and final data* (Φ, Ψ) *with small norms* $\|(\varphi_i, \psi_i)\|_{C^2[0,L]\times C^1[0,L]}$ *and* $\|(\Phi_i, \Psi_i)\|_{C^2[0,L]\times C^1[0,L]}$ $(i = 1, \cdots, n)$*, there exist boundary controls* $H = (h_1, \cdots, h_n)$ *and* $\overline{H} = (\overline{h}_1, \cdots, \overline{h}_n)$ *with small norms* $\|h_i\|_{C^d[0,T]}$ *and* $\|\overline{h}_i\|_{C^{\overline{d}}[0,T]}$ $(i = 1, \cdots, n)$*, such that the mixed initial-boundary value problem* (1.2) *and* (1.4)–(1.6) *admits a unique* C^2 *solution* $w = w(t, x)$ *with small* C^2 *norm on the domain* $R(T) = \{(t, x) \mid 0 \le t \le T, 0 \le x \le L\}$*, which exactly satisfies the final condition* (2.25)*.*

In order to prove Theorem 3.1, it suffices to use the constructive method suggested in [8–13] to prove the following lemma, we omit the details here.

Lemma 3.1 *Under the assumptions of Theorem* 3.1*, for any given initial data* (φ, ψ) *and final data* (Φ, Ψ) *with small norms* $\|(\varphi_i, \psi_i)\|_{C^2[0,L]\times C^1[0,L]}$ *and* $\|(\Phi_i, \Psi_i)\|_{C^2[0,L]\times C^1[0,L]}$ $(i = 1, \cdots, n)$*, the coupled system of quasilinear wave equations* (1.2) *admits a* C^2 *solution* $w = w(t, x)$ *with small* C^2 *norm on the domain* $R(T) = \{(t, x) \mid 0 \le t \le T, 0 \le x \le L\}$*, which satisfies simultaneously the initial condition* (1.6) *and the final condition* (2.25)*.*

Remark 3.1 If $a_i, a_{ij}, b_{ij}, \overline{b}_{ij}, c_{ij}, \overline{c}_{ij}$ are all independent of w, then the problem is linear, Theorem 3.1 implies the corresponding global exact boundary controllability.

Remark 3.2 The exact controllability time (3.1) given in Theorem 3.1 is sharp.

Theorem 3.2 (One-Sided Control) *Under the hypotheses given in Section* 1*, suppose furthermore that* (2.13)*,* (2.17) *and* (2.26) *hold. Let*

$$T > 2L \max_{i=1,\cdots,n} \Big(\frac{1}{a_i(0)}\Big). \tag{3.2}$$

For any given initial data (φ, ψ) *and final data* (Φ, Ψ) *with small norms* $\|(\varphi_i, \psi_i)\|_{C^2[0,L]\times C^1[0,L]}$ *and* $\|(\Phi_i, \Psi_i)\|_{C^2[0,L]\times C^1[0,L]}$ $(i = 1, \cdots, n)$*, for any given boundary functions* $H = (h_1, \cdots, h_n)$ *on* $x = 0$ *with small norms* $\|h_i\|_{C^d[0,T]}$ $(i = 1, \cdots, n)$*, such that the conditions of* C^2 *compatibility are satisfied at the points* $(t, x) = (0, 0)$ *and* $(T, 0)$ *respectively, there exist boundary controls* $\overline{H} = (\overline{h}_1, \cdots, \overline{h}_n)$ *on* $x = L$ *with small norms* $\|\overline{h}_i\|_{C^{\overline{d}}[0,T]}$ $(i = 1, \cdots, n)$*, such that the mixed initial-boundary value problem* (1.2) *and* (1.4)–(1.6) *admits a unique* C^2 *solution* $w = w(t, x)$ *with small* C^2 *norm on the domain* $R(T) = \{(t, x) \mid 0 \le t \le T, 0 \le x \le L\}$*, which exactly satisfies the final condition* (2.25)*.*

In order to get Theorem 3.2, similarly it suffices to prove the following lemma.

Lemma 3.2 *Under the assumptions of Theorem* 3.2*, for any given initial data* (φ, ψ) *and final data* (Φ, Ψ) *with small norms* $\|(\varphi_i, \psi_i)\|_{C^2[0,L]\times C^1[0,L]}$ *and* $\|(\Phi_i, \Psi_i)\|_{C^2[0,L]\times C^1[0,L]}$ $(i = 1, \cdots, n)$*, for any given boundary functions* $H = (h_1, \cdots, h_n)$ *on* $x = 0$ *with small norm* $\|h_i\|_{C^d[0,T]}$ $(i = 1, \cdots, n)$*, such that the conditions of* C^2 *compatibility are satisfied at the points* $(t, x) = (0, 0)$ *and* $(T, 0)$*, respectively, the coupled system of quasilinear wave equations* (1.2) *with the boundary condition* (1.4) *on* $x = 0$ *admits a* C^2 *solution* $w = w(t, x)$ *with small* C^2 *norm on the domain* $R(T) = \{(t, x) \mid 0 \le t \le T, 0 \le x \le L\}$*, which satisfies simultaneously the initial condition* (1.6) *and the final condition* (2.25)*.*

Remark 3.3 Similar results hold if we take the boundary controls $H(t)$ $(0 \le t \le T)$ at $x = 0$ instead of $\overline{H}(t)$ at $x = L$ and hypothesis (2.26) is replaced by (2.27).

Remark 3.4 If $a_i, a_{ij}, b_{ij}, \overline{b}_{ij}, c_{ij}, \overline{c}_{ij}$ are all independent of w, then the problem is linear, Theorem 3.2 implies the corresponding global exact boundary controllability.

Remark 3.5 The exact controllability time (3.2) given in Theorem 3.2 is sharp.

If we only consider the corresponding null controllability (see [10]), for which the final data (2.25) are specially taken as

$$t = T: \quad w = 0, \quad w_t = 0, \quad 0 \le x \le L, \tag{3.3}$$

then for boundary conditions (1.4d) (resp. (1.5d)) of coupled dissipative type, the condition (2.26) (resp. (2.27)) is not necessary. In fact, in this situation, similar to [10], we have the following theorem.

Theorem 3.3 (One-Sided Null Control) *Let $T > 0$ satisfy* (3.2)*. Suppose that* (2.13) *and* (2.17) *hold. Suppose furthermore that*

$$H(t) \equiv 0. \tag{3.4}$$

For any given initial data (φ, ψ) with small $\|(\varphi_i, \psi_i)\|_{C^2[0,L]\times C^1[0,L]}$ $(i = 1, \cdots, n)$, such that the conditions of C^2 compatibility are satisfied at the point $(t, x) = (0, 0)$, there exist boundary controls $\overline{H} = (\overline{h}_1, \cdots, \overline{h}_n)$ on $x = L$ with small norms $\|\overline{h}_i\|_{C^{\overline{d}}[0,T]}$ $(i = 1, \cdots, n)$, such that the mixed initial-boundary value problem (1.2) *and* (1.4)–(1.6) *admits a unique C^2 solution $w = w(t, x)$ with small C^2 norm on the domain $R(T) = \{(t, x) \mid 0 \le t \le T, 0 \le x \le L\}$, which exactly satisfies the null final condition* (3.3)*.*

Remark 3.6 Similar results hold if we take the boundary controls $H(t)$ $(0 \le t \le T)$ on $x = 0$ instead of $\overline{H}(t)$ at $x = L$, and hypothesis (3.4) is replaced by

$$\overline{H}(t) \equiv 0. \tag{3.5}$$

Remark 3.7 Theorems 3.1–3.3 are still valid for the coupled system (2.29) or (2.30).

4 Local Exact Boundary Observability

We now consider the exact boundary observability for the mixed initial-boundary value problem (1.2) and (1.4)–(1.6), in which the boundary functions $h_i(t)$ and $\overline{h}_i(t)$ $(i = 1, \cdots, n)$ are given.

The principle of choosing the observed values on the boundary is that the observed values together with the boundary conditions can uniquely determine the values (w, w_x) on the boundary.

Hence, the observed values at $x = 0$ can be taken as

(1) $w_{ix} = k_i(t)$ $(i = 1, \cdots, n)$, for boundary conditions (1.4a) of Dirichlet type, then

$$x = 0: \quad (w_i, w_{ix}) = (h_i(t), k_i(t)); \tag{4.1a}$$

(2) $w_i = k_i(t)$ $(i = 1, \cdots, n)$, for boundary conditions (1.4b) of Neumann type, then

$$x = 0: \quad (w_i, w_{ix}) = (k_i(t), h_i(t)); \tag{4.1b}$$

(3) $w_i = k_i(t)$ $(i = 1, \cdots, n)$, for boundary conditions (1.4c) of coupled third type, then

$$x = 0: \quad (w_i, w_{ix}) = \Big(k_i(t), \sum_{j=1}^{n} b_{ij} k_j(t) + h_i(t)\Big); \tag{4.1c}$$

(4) $w_i = k_i(t)$ $(i = 1, \cdots, n)$, for boundary conditions (1.4d) of coupled dissipative type, then

$$x = 0: \quad (w_i, w_{ix}) = \Big(k_i(t), \sum_{j=1}^{n} c_{ij} k_j'(t) + h_i(t)\Big), \tag{4.1d}$$

where $k_i(t)$ $(i = 1, \cdots, n) \in C^1$ for (1.4a) or $\in C^2$ for (1.4b)–(1.4d). Thus, by means of the observed values at $x = 0$, we have

$$x = 0: \quad (w, w_x) = (a(t), b(t)), \tag{4.2}$$

and for any given $T > 0$,

$$\|(a, b)\|_{C^2[0,T]\times C^1[0,T]} \le C\Big(\sum_{i=1}^{n} \|k_i\|_{C^l[0,T]} + \sum_{i=1}^{n} \|h_i\|_{C^d[0,T]}\Big), \tag{4.3}$$

where C is a positive constant, d is given by (2.21) and

$$l = \begin{cases} 1 & \text{for (1.4a)}, \\ 2 & \text{for (1.4b)–(1.4d)}. \end{cases} \tag{4.4}$$

The corresponding observed values $\overline{k}_i(t)$ $(i = 1, \cdots, n)$ at $x = L$ can be similarly taken, then we have

$$x = 0: \quad (w, w_x) = (\overline{a}(t), \overline{b}(t)), \tag{4.5}$$

and for any given $T > 0$,

$$\|(\overline{a}, \overline{b})\|_{C^2[0,T]\times C^1[0,T]} \le C\Big(\sum_{i=1}^{n} \|\overline{k}_i\|_{C^{\overline{l}}[0,T]} + \sum_{i=1}^{n} \|\overline{h}_i\|_{C^{\overline{d}}[0,T]}\Big), \tag{4.6}$$

where C is a positive constant, $\overline{d}$ is given by (2.22) and

$$\overline{l} = \begin{cases} 1 & \text{for (1.5a)}, \\ 2 & \text{for (1.5b)–(1.5d)}. \end{cases} \tag{4.7}$$

By the constructive method suggested in [4, 10, 12, 14], we can prove the following theorems.

Theorem 4.1 (Two-Sided Observation) *Under the hypotheses given in Section* 1*, suppose furthermore that* (2.13) *and* (2.17) *hold. Let* $T > 0$ *satisfy* (3.1)*. For any given initial condition* (φ, ψ)*, such that* $\|(\varphi_i, \psi_i)\|_{C^2[0,L]\times C^1[0,L]}$ *are suitably small and the conditions of* C^2 *compatibility are satisfied at the points* $(t, x) = (0, 0)$ *and* $(0, L)$*, respectively. If we have the observed values* $k_i(t)$ $(i = 1, \cdots, n)$ *at* $x = 0$ *and* $\overline{k}_i(t)$ $(i = 1, \cdots, n)$ *at* $x = L$ *on the interval* $[0, T]$*, then the initial data* (φ, ψ) *can be uniquely determined and we have the following observability inequality:*

$$\|(\varphi, \psi)\|_{C^2[0,L]\times C^1[0,L]} \le C\Big(\sum_{i=1}^{n} \|(k_i, \overline{k}_i)\|_{C^l[0,T]\times C^{\overline{l}}[0,T]} + \sum_{i=1}^{n} \|(h_i, \overline{h}_i)\|_{C^d[0,T]\times C^{\overline{d}}[0,T]}\Big), \tag{4.8}$$

where C *is a positive constant.*

Remark 4.1 If $a_i, a_{ij}, b_{ij}, \overline{b}_{ij}, c_{ij}, \overline{c}_{ij}$ are all independent of w, then the problem is linear, Theorem 4.1 implies the corresponding global exact boundary observability.

Remark 4.2 The exact observability time (3.1) given in Theorem 4.1 is sharp.

Theorem 4.2 (One-Sided Observability) *Under the hypotheses given in Section* 1*, suppose furthermore that* (2.13)*,* (2.17) *and* (2.27) *hold. Let $T > 0$ satisfy* (3.2)*. For any given initial condition (φ, ψ), such that $\|(\varphi_i, \psi_i)\|_{C^2[0,L]\times C^1[0,L]}$ are suitably small and the conditions of C^2 compatibility are satisfied at the points $(t, x) = (0, 0)$ and $(0, L)$, respectively. If we have the observed values $k_i(t)$ $(i = 1, \cdots, n)$ at $x = 0$ on the interval $[0, T]$, then the initial data (φ, ψ) can be uniquely determined and we have the following observability inequality:*

$$\|(\varphi, \psi)\|_{C^2[0,L]\times C^1[0,L]} \leq C\Big(\sum_{i=1}^{n} \|k_i\|_{C^l[0,T]} + \sum_{i=1}^{n} \|(h_i, \overline{h}_i)\|_{C^d[0,T]\times C^{\overline{d}}[0,T]}\Big), \tag{4.9}$$

where C is a positive constant.

Remark 4.3 Similar results hold if we take observed values $\overline{k}_i(t)$ $(i = 1, \cdots, n)$ at $x = L$ instead of $k_i(t)$ $(i = 1, \cdots, n)$ at $x = 0$ and hypothesis (2.27) is replaced by (2.26).

Remark 4.4 If $a_i, a_{ij}, b_{ij}, \overline{b}_{ij}, c_{ij}, \overline{c}_{ij}$ are all independent of w, then the problem is linear, Theorem 3.2 implies the corresponding global exact boundary observability.

Remark 4.5 The exact observability time (3.2) given in Theorem 4.2 is sharp.

If we only consider the corresponding weak observability (see [10]), for which the final condition (2.25) can be uniquely determined by the observed values $k_i(t)$ $(i = 1, \cdots, n)$ at $x = 0$, then for boundary conditions (1.4d) (resp. (1.5d)) of coupled dissipative type, the condition (2.26) (resp. (2.27)) is not necessary. In fact, in this situation, we have the following theorem.

Theorem 4.3 (One-Sided Weak Observability) *Under the hypotheses given in Section* 1*, suppose furthermore that* (2.13)*,* (2.17) *hold. Let $T > 0$ satisfy* (3.2)*. For any given initial condition (φ, ψ), such that $\|(\varphi_i, \psi_i)\|_{C^2[0,L]\times C^1[0,L]}$ are suitably small and the conditions of C^2 compatibility are satisfied at the points $(t, x) = (0, 0)$ and $(0, L)$, respectively. Then the final data (Φ, Ψ) can be uniquely determined by the observed values $k_i(t)$ $(i = 1, \cdots, n)$ at $x = 0$ and the boundary conditions $h_i(t)$ and $\overline{h}_i(t)$ on the interval $[0, T]$, and we have the following weak observability inequality:*

$$\|(\Phi, \Psi)\|_{C^2[0,L]\times C^1[0,L]} \leq C\Big(\sum_{i=1}^{n} \|k_i\|_{C^l[0,T]} + \sum_{i=1}^{n} \|(h_i, \overline{h}_i)\|_{C^d[0,T]\times C^{\overline{d}}[0,T]}\Big), \tag{4.10}$$

where C is a positive constant.

Remark 4.6 Similar results hold if we take observed values $k_i(t)$ $(i = 1, \cdots, n)$ at $x = 0$ instead of $\overline{k}_i(t)$ $(i = 1, \cdots, n)$ at $x = L$.

Remark 4.7 Theorems 4.1–4.3 are still valid for the coupled system (2.29) or (2.30).

Acknowledgement The authors would like to thank Professor Tatsien Li for his valuable suggestions and support.

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