Chin. Ann. Math.
34B(4), 2013, 491–514
DOI: 10.1007/s11401-013-0784-x

Chinese Annals of Mathematics, Series B

Relaxation of Certain Integral Functionals Depending on Strain and Chemical Composition*

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Abstract The authors provide a relaxation result in $BV \times L^q$, $1 \le q < +\infty$ as the first step towards the analysis of thermochemical equilibria.

Keywords Relaxation, Functions of bounded variation, Quasiconvexity
2000 MR Subject Classification 49J45, 74F99

1 Introduction

In this paper, we consider energies depending on two vector fields with different behaviors: $u \in W^{1,1}(\Omega;\mathbb{R}^d)\cap L^p(\Omega;\mathbb{R}^d)$, $v \in L^q(\Omega;\mathbb{R}^m)$, Ω being a bounded open set of $\mathbb{R}^N$. The functional $I : BV(\Omega;\mathbb{R}^d) \times L^q(\Omega;\mathbb{R}^m) \to \mathbb{R} \cup \{+\infty\}$ that we consider is defined by

$$I(u,v) = \begin{cases} \displaystyle\int_\Omega W(x,u(x),\nabla u(x))\mathrm{d}x + \int_\Omega \varphi(x,u(x),v(x))\mathrm{d}x, \\ \quad \text{if } (u,v) \in (W^{1,1}(\Omega;\mathbb{R}^d)\cap L^p(\Omega;\mathbb{R}^d)) \times L^q(\Omega;\mathbb{R}^m), \\ +\infty, \quad \text{otherwise,} \end{cases} \tag{1.1}$$

where $W : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to \mathbb{R}$ is a continuous function with linear growth from above and below in the gradient variable, $\varphi : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory function (that is, $\varphi(\cdot,u,v)$ is measurable for all $(u,v) \in \mathbb{R}^d \times \mathbb{R}^m$ and $\varphi(x,\cdot,\cdot)$ is continuous for a.e. $x \in \Omega$), with growth p and q respectively in the variables u and v.

Our results can be considered as a first step towards the analysis of functionals of the type $\int_\Omega V(x,u,\nabla u,v)\mathrm{d}x$, which generalizes those considered by [10, 14–15], to deal with equilibria for systems depending on elastic strain and chemical composition. In this context, a multiphase alloy is represented by the set Ω, the deformation gradient is given by ∇u, and v denotes the chemical composition of the system.

In [14], $V \equiv V(\nabla u, v)$ is a cross-quasiconvex function, while in our decoupled model we also take into account heterogeneities and the deformation without imposing any convexity

Manuscript received September 4, 2012.
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*Project supported by the Portuguese Foundation for Science and Technology Fundação para a Ciência e Tecnologia (No. SFRH/BPD/34477/2006), Financiamento Base 2010-ISFL/1/297 from FCT/MCTES/PT. Portuguese Foundation for Science and Technology Fundação para a Ciência e Tecnologia (Nos. UTA-CMU/MAT/0005/2009, PTDC/MAT/109973/2009).

restriction neither on W nor on φ. Moreover when $\varphi \equiv 0$, the functional in (1.1) recovers the one in [17] without quasiconvexity assumptions.

Additive models like the one we are addressing can also be found in imaging models, like those considered in [4–6], i.e.,

$$\inf_{u,v} \Big\{ |Du|(\Omega) + \frac{1}{2\lambda} \|\phi - u - v\|^2_{L^2} \Big\}, \tag{1.2}$$

where ϕ is a given image and λ a scaling factor for the L^2 norm of the fidelity term $\phi - (u+v)$.

In order to deal with the minimization of (1.1), since there may be a lack of lower semicontinuity, it is necessary to pass to the relaxed functional defined in $BV(\Omega;\mathbb{R}^d) \times L^q(\Omega;\mathbb{R}^m)$

$$\begin{aligned} \overline{I}(u,v) := \inf \Big\{ \liminf_{n\to+\infty} I(u_n, v_n) : (u_n, v_n) \in BV(\Omega;\mathbb{R}^d) \\ \times L^q(\Omega;\mathbb{R}^m) : u_n \to u \text{ in } L^1, v_n \rightharpoonup v \text{ in } L^q \Big\}, \end{aligned} \tag{1.3}$$

and prove a representation result for $\overline{I}$.

It is worthwhile to remark that for $q = 1$, the functional $\overline{I}$ may fail to be sequentially lower semicontinuous. However, as we will observe below, this can be achieved provided that φ is uniform continuous (cf. (1.10)).

We prove the following theorem.

Theorem 1.1 *Let $p \geq 1$ and $q \geq 1$ and let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Assume that $W : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to \mathbb{R}$ is a continuous function, satisfying that*

(i) $\exists\, C > 0 : \frac{1}{C}|\xi| - C \leq W(x,u,\xi) \leq C(1+|\xi|), \quad \forall\, (x,u,\xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$;

(ii) *for every compact subset K of $\Omega \times \mathbb{R}^d$, there exists a continuous function $\omega_K : [0,+\infty) \to \mathbb{R}$ with $\omega_K(0) = 0$, such that*

$$|W(x,u,\xi) - W(x',u',\xi)| \leq \omega_K(|x-x'| + |u-u'|)(1+|\xi|), \quad \forall\, (x,u,\xi), (x',u',\xi) \in K \times \mathbb{R}^{d\times N};$$

(iii) *for every $x_0 \in \Omega$ and for every $\varepsilon > 0$, there exists a $\delta > 0$, such that*

$$|x - x_0| < \delta \;\Rightarrow\; W(x,u,\xi) - W(x_0,u,\xi) \geq -\varepsilon(1+|\xi|), \quad \forall\, (u,\xi) \in \mathbb{R}^d \times \mathbb{R}^{d\times N};$$

(iv) *there exist $\alpha \in (0,1)$ and $C, L > 0$, such that*

$$t|\xi| > L \;\Rightarrow\; \left| W^\infty(x,u,\xi) - \frac{W(x,u,t\xi)}{t} \right| \leq C\frac{|\xi|^{1-\alpha}}{t^\alpha}, \quad \forall\, (x,u,\xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N},\; t \in \mathbb{R}.$$

Moreover, let $\varphi : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function, satisfying

(v) $\exists\, C > 0 : \frac{1}{C}(|u|^p + |v|^q) - C \leq \varphi(x,u,v) \leq C(1+|u|^p+|v|^q)$, $\forall\, (x,u,v) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^m$.

If I is defined by (1.1) and $\overline{I}$ is defined by (1.3) then, for every $u \in BV(\Omega;\mathbb{R}^d) \cap L^p(\Omega;\mathbb{R}^d)$ and $v \in L^q(\Omega;\mathbb{R}^m)$, the following identity holds:

$$\begin{aligned} \overline{I}(u,v) = \int_\Omega QW(x,u(x),\nabla u(x))\mathrm{d}x + \int_{J_u} \gamma(x,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1} \\ + \int_\Omega (QW)^\infty\Big(x,u(x),\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\mathrm{d}|D^c u| + \int_\Omega C\varphi(x,u(x),v(x))\mathrm{d}x. \end{aligned} \tag{1.4}$$

Remark 1.1 (i) An example of an integrand W satisfying the assumptions of Theorem 1.1 is given, by $W(x,u,F) := f(x)h(u)g(F)$, where $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ and $h : \mathbb{R}^2 \to \mathbb{R}$ are continuous bounded functions, bounded from below by a strictly positive constant, $g : \mathbb{R}^{2\times 2} \to \mathbb{R}$, where $g(F) := |F_{11} - F_{22}| + |F_{12} + F_{21}| + \min\{|F_{11} + F_{22}|, |F_{12} - F_{21}|\}$, is the function in [17, Example 2.18], which is not quasiconvex. For what concerns φ we can take $\varphi(x, u, v) \equiv f(x)(|u|^p + g_1(v))$, with f as above and $g_1 : \mathbb{R}^m \to \mathbb{R}$ any double well function with the required growth, as for example $g_1(v) = (|v| - 1)^p$.

(ii) In order to describe the right-hand side of (1.4) we recall that for every $x \in \Omega$, $QW(x, u, \cdot)$ stands for the quasiconvexification of W (cf. (2.1)), while $(QW)^\infty$ denotes the recession function of QW with respect to the last variable as introduced in Definition 2.2, and γ stands for the surface integral density, defined in (2.8). Finally, for every $(x, u) \in \Omega \times \mathbb{R}^d$, $C\varphi$ stands for the convex envelope (or convexification) of $\varphi(x, u, \cdot)$, namely

$$C\varphi(x, u, \cdot) := \sup\{g : \mathbb{R}^m \to \mathbb{R} : g \text{ convex}, \ g(v) \le \varphi(x, u, v) \ \forall\, v\}. \tag{1.5}$$

Classical results in Calculus of Variations ensure that, if φ takes only finite values then $C\varphi$ coincides with the bidual of φ, φ^{**}, whose characterization is given below

$$\begin{aligned}\varphi^{**}(x, u, \cdot) := \sup\{g : \mathbb{R}^m \to \mathbb{R} : g &\text{ convex and lower semicontinuous},\\ g(v) \le \varphi(x, u, v) \ \forall\, v\}.&\end{aligned} \tag{1.6}$$

(iii) We observe that if $\varphi \equiv 0$ our results extends [17, Theorem 2.16] (cf. also [2, Theorem 5.54]) to nonquasiconvex functions. We stress the fact that our hypotheses are made on the non-quasiconvex function W and thus we can not immediately apply the results in [17] to QW.

Remark 1.2 (1) We observe that in the Sobolev setting, Theorem 1.1 can be proven without coercivity assumptions on φ, indeed let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function satisfying

$$0 \le f(x, u, \xi, v) \le C(1 + |u|^p + |\xi|^p + |v|^q)$$

for a.e. $x \in \Omega$, for every $(u, \xi, v) \in \mathbb{R}^d \times \mathbb{R}^{d\times N} \times \mathbb{R}^m$ and for some $C > 0$. Consider for every $1 \le p, q < +\infty$ the following relaxed localized energy:

$$\begin{aligned}\mathcal{F}(u, v; A) := \inf\Big\{\liminf_{n\to\infty} \int_A f(x, u_n(x), \nabla u_n(x), v_n(x))\mathrm{d}x :&\\ u_n \rightharpoonup u \text{ in } W^{1,p}(A; \mathbb{R}^d),\ v_n \rightharpoonup v \text{ in } L^q(A; \mathbb{R}^m)\Big\}.&\end{aligned} \tag{1.7}$$

Then, in [10, Theorem 1.1] (cf. also [9]), it has been proven that, for every $u \in W^{1,p}(\Omega; \mathbb{R}^d)$, $v \in L^q(\Omega; \mathbb{R}^m)$ and $A \in \mathcal{A}(\Omega)$,

$$\mathcal{F}(u, v; A) = \int_A QCf(x, u(x), \nabla u(x), v(x))\mathrm{d}x,$$

where QCf stands for the quasiconvex-convex envelope of f with respect to the last two variables, namely,

$$\begin{aligned}QCf(x, u, \xi, v) = \inf\Big\{\frac{1}{|D|}\int_D f(x, u, \xi + \nabla\varphi(y), v + \eta(y))\mathrm{d}y :&\\ \varphi \in W_0^{1,\infty}(D; \mathbb{R}^d), \eta \in L^\infty(D; \mathbb{R}^m), \int_D \eta(y)\mathrm{d}y = 0\Big\},&\end{aligned} \tag{1.8}$$

where D is any bounded open set. Clearly this equality recovers our setting, since it suffices to define $f(x,u,\xi,v) := W(x,u,\xi) + \varphi(x,u,v)$ for every $(x,u,\xi,v) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \times \mathbb{R}^m$. In fact, it is easily seen that if f satisfies the above growth assumptions, then

$$QCf(x,u,\xi,v) = QW(x,u,\xi) + C\varphi(x,u,v).$$

(2) We notice that contrary to what one would expect from [15, 17], our density φ does not need to satisfy a property analogous to Theorem 1.1(ii) with respect to (x,u,v), indeed it is just a Carathéodory function.

(3) We emphasize that the arguments adopted to prove the previous theorem strongly rely on the fact that the energy densities are decoupled. In particular, in the case $q = 1$, we will approximate the functional I by adding an extra term with superlinear growth at ∞ in the v variable. This will ensure the sequentially weak lower semicontinuity of the relaxed approximating functional

$$\overline{I}_\varepsilon(u,v) := \inf\Big\{ \liminf_n \int_\Omega W(x,u_n,\nabla u_n)\mathrm{d}x + \int_\Omega (\varphi(x,u,v) + \varepsilon\theta(|v|))\mathrm{d}x : \\ u_n \to u \text{ in } L^1, v_n \rightharpoonup v \text{ in } L^1 \Big\},$$

allowing us to adopt arguments similar to those exploited in the proof for the case $q > 1$. These techniques are well suited for the convex setting but we are not aware if a similar procedure is possible in the quasiconvex-convex framework.

Having in mind the continuous embedding of $BV(\Omega;\mathbb{R}^d)$ in $L^{\frac{N}{N-1}}(\Omega;\mathbb{R}^d)$ (assuming $\Omega \subset \mathbb{R}^N$), we can obtain, in an easier way, the relaxation result as above. Indeed we can prove the following result.

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, and let $1 \le p \le \frac{N}{N-1}$ and $q \ge 1$. Let $W : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to \mathbb{R}$ be a continuous function satisfying Theorem* 1.1(i)/(iv). *Moreover let $\varphi : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function satisfying Theorem* 1.1(v) *in the weaker form*

$$\exists\, C > 0 : \quad \frac{1}{C}|v|^q - C \le \varphi(x,u,v) \le C(1 + |u|^p + |v|^q), \quad \forall\ (x,u,v) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^m. \tag{1.9}$$

Then, for every $(u,v) \in BV(\Omega;\mathbb{R}^d) \times L^q(\Omega;\mathbb{R}^m)$, (1.4) *holds.*

Remark 1.3 Theorem 1.2 allows us to consider very general growth conditions also in [10, Theorem 1.1] when decoupled energies are considered.

The continuous embedding of $BV(\Omega;\mathbb{R}^d)$ into $L^{\frac{N}{N-1}}(\Omega;\mathbb{R}^d)$ (with $\Omega \subset \mathbb{R}^N$) allows us to obtain Theorem 1.2, also replacing (1.9) by the following condition:

$$\exists\, C > 0 : \quad \frac{1}{C}|v|^q - C \le \varphi(x,u,v) \le C(1 + |u|^r + |v|^q), \quad \forall\ (x,u,v) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^m$$

and for some $r \in \left[1, \frac{N}{N-1}\right]$.

We observe that under assumptions Theorem 1.1(i)/(iv), [17, Theorem 2.16] ensures that the functional

$$\int_\Omega QW(x,u,\nabla u)\mathrm{d}x + \int_{J_u} \gamma(x,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1} + \int_\Omega (QW)^\infty\Big(x,u,\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\mathrm{d}|D^c u|$$

is lower semicontinuous with respect to the strong-L^1 topology. Moreover, [16, Theorem 7.5] guarantees that

$$\int_\Omega C\varphi(x,u,v)\mathrm{d}x$$

is sequentially weakly lower semicontinuous with respect to $L^1_{\text{strong}} \times L^1_{\text{weak}}$-topology provided that the function $C\varphi$ is convex in the last variable, satisfies suitable growth conditions, as those in (2.5)–(2.6), and that the function $C\varphi(x,\cdot,\cdot)$ is lower semicontinuous. We will observe in Remark 2.1 below that this latter condition may not be verified just under the assumptions of Theorems 1.1–1.2. On the other hand, an argument entirely similar to [11, Theorem 9.5] guarantees that $C\varphi(x,\cdot,\cdot)$ is lower semicontinuous (even continuous) by assuming additionally that

$$|\varphi(x,u,\xi)-\varphi(x,u',\xi)| \le \omega'(|u-u'|)(|\xi|+1) \tag{1.10}$$

for a suitable modulus of continuity ω', i.e., $\omega' : \mathbb{R}^+\cup\{0\} \to \mathbb{R}^+\cup\{0\}$ continuous and such that $\omega'(0)=0$.

Consequently, the superadditivity of lim inf implies the sequentially strong-weak lower semicontinuity of the right-hand side of (1.4) even for $q=1$.

2 Notations and General Facts

2.1 Properties of the integral density functions

In this subsection, we recall several notions applied to functions like quasiconvexity, envelopes and recession function, etc. We also recall or prove properties of those functions that will be useful through the paper. Such notions and related properties will apply to the density functions that will appear in the relaxed functionals that we characterize. We start recalling the notion of quasiconvex function due to Morrey.

Definition 2.1 *A Borel measurable function $h:\mathbb{R}^{d\times N}\to\mathbb{R}$ is said to be quasiconvex if there exists a bounded open set D of $\mathbb{R}^N$, such that*

$$h(\xi) \le \frac{1}{|D|}\int_D h(\xi+\nabla\varphi(x))\mathrm{d}x$$

for every $\xi\in\mathbb{R}^{d\times N}$ and for every $\varphi\in W_0^{1,\infty}(D;\mathbb{R}^d)$.

If $h:\mathbb{R}^{d\times N}\to\mathbb{R}$ is any given Borel measurable function bounded from below, it can be defined the quasiconvex envelope of h, that is, the largest quasiconvex function below h:

$$Qh(\xi) := \sup\{g(\xi):\ g\le h,\ g \text{ quasiconvex}\}.$$

Moreover, as well-known (cf. [11]),

$$Qh(\xi) := \inf\Big\{\frac{1}{|D|}\int_D h(\xi+\nabla\varphi(x))\mathrm{d}x:\ \varphi\in W_0^{1,\infty}(D;\mathbb{R}^d)\Big\} \tag{2.1}$$

for any bounded open set $D\subset\mathbb{R}^N$.

Proposition 2.1 *Let $\Omega\subset\mathbb{R}^N$ be a bounded open set and*

$$W:\Omega\times\mathbb{R}^d\times\mathbb{R}^{d\times N}\to[0,+\infty)$$

be a continuous function. Let QW be the quasiconvexification of W (cf. (2.1)). Then the validity of Theorem 1.1(i) guarantees that there exists a constant $C > 0$, such that

$$\frac{1}{C}|\xi| - C \le QW(x,u,\xi) \le C(1+|\xi|), \quad \forall\ (x,u,\xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}. \tag{2.2}$$

The validity of Theorem 1.1(i)–(ii) ensures that for every compact set $K \subset \Omega \times \mathbb{R}^d$, there exists a continuous function $\omega'_K : \mathbb{R} \to [0,+\infty)$, such that $\omega'_K(0) = 0$ and

$$|QW(x,u,\xi) - QW(x',u',\xi)| \le \omega'_K(|x-x'| + |u-u'|)(1+|\xi|),$$
$$\forall\ (x,u),(x',u') \in K,\ \forall\ \xi \in \mathbb{R}^{d\times N}. \tag{2.3}$$

Theorem 1.1(i) and (iii) imply that, for every $x_0 \in \Omega$ and $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$|x - x_0| \le \delta \ \Rightarrow\ QW(x,u,\xi) - QW(x_0,u,\xi) \ge -\varepsilon(1+|\xi|),\ \forall\ (u,\xi) \in \mathbb{R}^d \times \mathbb{R}^{d\times N}. \tag{2.4}$$

Moreover, if W satisfies conditions (i)–(ii) in Theorem 1.1, QW is a continuous function.

Remark 2.1 Analogous arguments imply that, under Theorem 1.1(v) and Theorem 1.2, respectively,

$$\exists\, C > 0: \quad \frac{1}{C}(|u|^p + |v|^q) - C \le C\varphi(x,u,v) \le C(1+|u|^p+|v|^q),$$
$$\forall\,(x,u,v) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^m \tag{2.5}$$

and

$$\exists\, C > 0: \quad \frac{1}{C}|v|^q - C \le C\varphi(x,u,v) \le C(1+|u|^p+|v|^q), \quad \forall\,(x,u,v) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^m. \tag{2.6}$$

On the other hand, we emphasize that φ is the same as in Theorems 1.1–1.2, namely a Carathéodory function, this is not enough to guarantee that $C\varphi$ is still a Carathéodory function (cf. [11, Example 9.6] and [16, Example 7.14]). In particular, $C\varphi$ turns out to be measurable in x, upper semicontinuous in u, convex and hence continuous in ξ. Furthermore if $q > 1$, [12, Lemma 4.3] guarantees that $C\varphi(x,\cdot,\cdot)$ is lower semicontinuous.

Proof of Proposition 2.1 By definition of the quasiconvex envelope of W, it is easily seen that Theorem 1.1(i) implies (2.2) with the same constant appearing in (i).

Next we prove (2.3). Let K be a compact set in $\Omega \times \mathbb{R}^d$ and take $(x,u),(x',u') \in K$. Let $\varepsilon > 0$. Then using condition (2.1), we find $\varphi_\varepsilon \in W_0^{1,\infty}(Q;\mathbb{R}^d)$, Q being the unitary cube, such that

$$QW(x,u,\xi) \ge -\varepsilon + \int_Q W(x,u,\xi + \nabla\varphi_\varepsilon(y))\mathrm{d}y.$$

Now, we observe that, by virtue of the coercivity condition expressed by Theorem 1.1(i) and (2.2), it follows that

$$\|\xi + \nabla\varphi_\varepsilon\|_{L^1} \le c(1+|\xi|).$$

By Theorem 1.1(ii), for every $(x,u),(x',u') \in K$ and for every $\xi \in \mathbb{R}^{d\times N}$, it implies

$$|W(x,u,\xi) - W(x',u',\xi)| \le \omega_K(|x-x'| + |u-u'|)(1+|\xi|).$$

Then we can write the following chain of inequalities:

$$QW(x,u,\xi) \geq -\varepsilon + \int_Q W(x,u,\xi+\nabla\varphi_\varepsilon(y))\mathrm{d}y$$
$$\geq -\varepsilon - \int_Q \lambda(y)\mathrm{d}y + \int_Q W(x',u',\xi+\nabla\varphi_\varepsilon(y))\mathrm{d}y,$$

where $\lambda(y) := |W(x,u,\xi+\nabla\varphi_\varepsilon(y)) - W(x',u',\xi+\nabla\varphi_\varepsilon(y))|$. Therefore, we get, from the definition of $QW(x',u',\xi)$, that

$$QW(x',u',\xi) - QW(x,u,\xi) \leq \varepsilon + \omega_K(|x-x'|+|u-u'|)(1+\|\xi+\nabla\varphi_\varepsilon\|_{L^1})$$
$$\leq \varepsilon + \omega_K(|x-x'|+|u-u'|)(1+c(1+|\xi|)).$$

Since ε is arbitrarily chosen, and since we can obtain in a similar way the same inequality with x in the place of x', and u in the place of u', we get (2.3).

In order to prove condition (2.4), we fix $x_0 \in \Omega$ and $\varepsilon > 0$. As before, for every $x \in \Omega$ and $\sigma > 0$, by (2.1), the coercivity condition expressed by Theorem 1.1(i), and by (2.2), there exist a constant $c > 0$ and a function $\varphi_\sigma \in W_0^{1,\infty}(Q;\mathbb{R}^d)$, such that

$$QW(x,u,\xi) \geq -\sigma + \int_Q W(x,u,\xi+\nabla\varphi_\sigma(y))\mathrm{d}y$$

with $\|\xi+\nabla\varphi_\sigma\|_{L^1} \leq c(1+|\xi|)$.

Thus arguing as above, and exploiting Theorem 1.1(iii), we have the following chain of inequalities, for $|x-x_0| < \delta$ with δ as in Theorem 1.1(iii),

$$QW(x_0,u,\xi) \leq \int_Q W(x_0,u,\xi+\nabla\varphi_\sigma(y))\mathrm{d}y$$
$$\leq \int_Q W(x,u,\xi+\nabla\varphi_\sigma(y))\mathrm{d}y + \varepsilon\int_Q (1+|\xi+\nabla\varphi_\sigma(y)|)\mathrm{d}y$$
$$\leq QW(x,u,\xi) + \sigma + \varepsilon(1+c(1+|\xi|)).$$

Thus it suffices to let σ go to 0 in order to achieve the statement.

Finally, we prove the continuity of QW. We need to show that, for every $\varepsilon > 0$ and $(x_0,u_0,\xi_0) \in \Omega\times\mathbb{R}^d\times\mathbb{R}^{d\times N}$, there exists a $\delta \equiv \delta(\varepsilon,x_0,u_0,\xi_0) > 0$, such that

$$|x-x_0|+|u-u_0|+|\xi-\xi_0| \leq \delta \Rightarrow |QW(x,u,\xi) - QW(x_0,u_0,\xi_0)| \leq \varepsilon. \tag{2.7}$$

Let $\varepsilon > 0$ be fixed. Since QW is quasiconvex on ξ, $QW(x_0,u_0,\cdot)$ is continuous. Thus we can find $\delta_1 = \delta_1(\varepsilon,x_0,u_0,\xi_0) > 0$, such that

$$|\xi-\xi_0| \leq \delta_1 \Rightarrow |QW(x_0,u_0,\xi) - QW(x_0,u_0,\xi_0)| \leq \frac{\varepsilon}{2}.$$

Moreover, by virtue of (2.3), defining $K := \overline{B}_\sigma(x_0,u_0)$ for some $\sigma > 0$ such that $K \subset \Omega\times\mathbb{R}^d$, one has

$$|\xi-\xi_0| \leq \delta_1 \Rightarrow |QW(x,u,\xi) - QW(x_0,u_0,\xi)| \leq \omega'_K(|x-x_0|+|u-u_0|)(1+|\xi_0|+\delta_1).$$

Since ω'_K is continuous and $\omega'_K(0) = 0$, there exists a $\delta_2 = \delta_2(\varepsilon,K,\xi_0) > 0$, such that

$$|x-x_0|+|u-u_0| \leq \delta_2 \Rightarrow \omega'_K(|x-x_0|+|u-u_0|) \leq \frac{\varepsilon}{2(1+|\xi_0|+1)}.$$

Consequently, by choosing δ as $\min\{\delta_1, \delta_2\}$, the above inequalities, and the triangular inequality give indeed (2.7).

We also recall the definition of the recession function.

Definition 2.2 *Let $h : \mathbb{R}^{d\times N} \to [0, +\infty)$. The recession function of h is denoted by $h^\infty : \mathbb{R}^{d\times N} \to [0, +\infty)$, and defined as*

$$h^\infty(\xi) := \limsup_{t\to+\infty} \frac{h(t\xi)}{t}.$$

Remark 2.2 (i) Recall that the recession function is a positively one homogeneous function, that is, $g(t\xi) = tg(\xi)$ for every $t \geq 0$ and $\xi \in \mathbb{R}^{d\times N}$.

(ii) Through this paper, we will work with functions $W : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to [0, +\infty)$ and W^∞ is the recession function with respect to the last variable:

$$W^\infty(x, u, \xi) := \limsup_{t\to+\infty} \frac{W(x, u, t\xi)}{t}.$$

We trivially observe that, if W satisfies the growth condition of Theorem 1.1(i), then W^∞ satisfies $\frac{1}{C}|\xi| \leq W^\infty(x, u, \xi) \leq C|\xi|$.

(iii) As showed in [17, Remark 2.2(ii)], if a function $h : \mathbb{R}^{d\times N} \to [0, +\infty)$ is quasiconvex and satisfies the growth condition $h(\xi) \leq c(1 + |\xi|)$ for some $c > 0$, then its recession function is also quasiconvex.

We now describe the surface energy density γ appearing in the characterization of $\overline{I}$. Let $W : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to \mathbb{R}$. By the notation above, $(QW)^\infty$ is the recession function of the quasiconvex envelope of W. Then $\gamma : \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times S^{N-1} \to \mathbb{R}$ is defined by

$$\gamma(x, a, b, \nu) = \inf\Big\{ \int_{Q_\nu} (QW)^\infty(x, \phi(y), \nabla\phi(y))\mathrm{d}y : \phi \in \mathcal{A}(a, b, \nu)\Big\}, \tag{2.8}$$

where Q_ν is the unit cube centered at the origin with faces parallel to $\nu, \nu_1, \cdots, \nu_{N-1}$, for some orthonormal basis of $\mathbb{R}^N$, $\{\nu_1, \cdots, \nu_{N-1}, \nu\}$, and where

$$\mathcal{A}(a, b, \nu) := \Big\{\phi \in W^{1,1}(Q_\nu, \mathbb{R}^d) : \phi(y) = a \text{ if } \langle y, \nu\rangle = \frac{1}{2},\ \phi(y) = b \text{ if } \langle y, \nu\rangle = -\frac{1}{2},$$
$$\phi \text{ is 1-periodic in the } \nu_1, \cdots, \nu_{N-1} \text{ directions}\Big\}.$$

We observe that the function γ is the same, whether we consider in the set $\mathcal{A}(a, b, \nu)$, $W^{1,1}(Q_\nu, \mathbb{R}^d)$ functions (as in [17–18]) or $W^{1,\infty}(Q_\nu, \mathbb{R}^d)$ functions (as in [2, p. 312]). Moreover, if W does not depend on u, $W : \Omega \times \mathbb{R}^{d\times N} \to \mathbb{R}$, then $\gamma(x, a, b, \nu) = (QW)^\infty(x, (a - b) \otimes \nu)$ (cf. [2, p. 313]).

Properties of the function $(QW)^\infty$ will be important to get the integral representation of the relaxed functionals under consideration. In particular, a proof entirely similar to [7, Proposition 3.4] ensures that for every $(x, u, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$, $Q(W^\infty)(x, u, \xi) = (QW)^\infty(x, u, \xi)$.

Proposition 2.2 *Let $W : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to [0, +\infty)$ be a continuous function satisfying Theorem* 1.1(i) *and* (iv). *Then*

$$Q(W^\infty)(x, u, \xi) = (QW)^\infty(x, u, \xi) \quad \textit{for every } (x, u, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}. \tag{2.9}$$

Proof The proof will be achieved by double inequality.

By definitions of the quasiconvex envelope and the recession function, one gets $(QW)^\infty \leq W^\infty$ and thus $Q(QW)^\infty \leq Q(W^\infty)$. Since the recession function of a quasiconvex envelope is still quasiconvex, under the hypothesis of Theorem 1.1(i) (cf. Remark 2.2(iii)), it follows that $(QW)^\infty \leq Q(W^\infty)$.

In order to prove the opposite inequality, we notice that, since by (i), the function W is bounded from below, we can assume without loss of generality that $W \geq 0$. Then fix $(x, u, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$ and, for every $t > 1$, take $\varphi_t \in W_0^{1,\infty}(Q; \mathbb{R}^d)$, such that

$$\int_Q W(x, u, t\xi + \nabla\varphi_t(y))\mathrm{d}y \leq QW(x, u, t\xi) + 1. \tag{2.10}$$

By (i) and (2.2) we have that $\left\|\nabla\left(\frac{1}{t}\varphi_t\right)\right\|_{L^1(Q)} \leq C$ for a constant independent of t but just on ξ.

Defining $\psi_t = \frac{1}{t}\varphi_t$, one has $\psi_t \in W_0^{1,\infty}(Q; \mathbb{R}^d)$ and thus

$$Q(W^\infty)(x, u, \xi) \leq \int_Q W^\infty(x, u, \xi + \nabla\psi_t(y))\mathrm{d}y.$$

Let L be the constant appearing in the condition of Theorem 1.1(iv), we split the cube Q in the set $\{y \in Q : \ t|\xi + \nabla\psi_t(y)| \leq L\}$ and its complement in Q. Then we apply the condition (iv) and the growth of W^∞ observed in Remark 2.2(ii) to get

$$Q(W^\infty)(x, u, \xi) \leq \int_Q \Big(C\frac{|\xi + \nabla\psi_t|^{1-\alpha}}{t^\alpha} + \frac{W(x, u, t\xi + \nabla\varphi_t(y))}{t} + C\frac{L}{t}\Big)\mathrm{d}y.$$

Applying Hölder inequality and (2.10), we get

$$Q(W^\infty)(x, u, \xi) \leq \frac{C}{t^\alpha}\Big(\int_Q |\xi + \nabla\psi_t|\mathrm{d}y\Big)^{1-\alpha} + \frac{QW(x, u, t\xi) + 1}{t} + C\frac{L}{t},$$

and the desired inequality follows by the definition of $(QW)^\infty$ and using the fact that $\nabla\psi_t$ has bounded L^1 norm, letting t go to $+\infty$.

The property of $(QW)^\infty$ stated next ensures that QW together with $(QW)^\infty$ satisfy the condition analogous to Theorem 1.1(iv). To this end, we first observe, as emphasized in [17], that Theorem 1.1(iv) is equivalent to saying that there exist $C > 0$ and $\alpha \in (0, 1)$, such that

$$|W^\infty(x, u, \xi) - W(x, u, \xi)| \leq C(1 + |\xi|^{1-\alpha}) \tag{2.11}$$

for every $(x, u, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$. Precisely, we have the following result.

Proposition 2.3 *Let $W : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to [0, +\infty)$ be a continuous function satisfying Theorem* 1.1(i) *and* (iv). *Then, there exist $\alpha \in (0, 1)$ and $C' > 0$, such that*

$$|(QW)^\infty(x, u, \xi) - QW(x, u, \xi)| \leq C(1 + |\xi|^{1-\alpha}), \quad \forall (x, u, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}.$$

Proof This paper will be achieved by double inequality. Let $\alpha \in (0, 1)$ be as in Theorem 1.1(iv) (cf. also (2.11)). Let $\xi \in \mathbb{R}^{d\times N}$, and Q be the unit cube in $\mathbb{R}^N$ and let c be a positive constant varying from line to line. For every $\varepsilon > 0$, by (2.1), find $\varphi \in W_0^{1,\infty}(Q; \mathbb{R}^d)$, such that

$$QW(x, u, \xi) > \int_Q W(x, u, \xi + \nabla\varphi(y))\mathrm{d}y - \varepsilon.$$

By Theorem 1.1(i) and (2.2), there exists a $c > 0$, such that

$$\|\xi + \nabla\varphi\|_{L^1} \leq c(1 + |\xi|). \tag{2.12}$$

Since, by Proposition 2.2, it implies

$$(QW)^\infty(x, u, \xi) \leq \int_Q W^\infty(x, u, \xi + \nabla\varphi(y))\mathrm{d}y,$$

we have

$$(QW)^\infty(x, u, \xi) - QW(x, u, \xi) \leq \int_Q (W^\infty(x, u, \xi + \nabla\varphi(y)) - W(x, u, \xi + \nabla\varphi(y)))\mathrm{d}y + \varepsilon.$$

Applying (2.11), we obtain

$$\begin{aligned}(QW)^\infty(x, u, \xi) - QW(x, u, \xi) &\leq \int_Q c(1 + |\xi + \nabla\varphi(y)|^{1-\alpha})\mathrm{d}y + \varepsilon \\ &\leq c\Big(1 + \int_Q |\xi + \nabla\varphi(y)|^{1-\alpha}\mathrm{d}y\Big) + \varepsilon \\ &\leq c + c\Big(\int_Q |\xi + \nabla\varphi(y)|\mathrm{d}y\Big)^{1-\alpha} + \varepsilon \\ &\leq c + c^2(1 + |\xi|^{1-\alpha}) + \varepsilon \\ &\leq C'(1 + |\xi|^{1-\alpha}) + \varepsilon,\end{aligned}$$

where we have applied Hölder inequality and (2.12) in the last lines, and we have estimated the term $(1 + |\xi|)^{1-\alpha}$ by separating the cases $|\xi| \leq 1$ and $|\xi| > 1$ and summing them up. To conclude this part it suffices to send ε to 0.

In order to prove the opposite inequality, we can argue in the same way. Let $\xi \in \mathbb{R}^{d\times N}$. For every $\varepsilon > 0$, by (2.1) and Proposition 2.2, there exists a $\psi \in W_0^{1,\infty}(Q; \mathbb{R}^d)$, such that

$$(QW)^\infty(x, u, \xi) > \int_Q W^\infty(x, u, \xi + \nabla\psi(y))\mathrm{d}y - \varepsilon.$$

Clearly, by (2.2), Theorem 1.1(i) and Remark 2.2(ii), there exists a $C > 0$, such that

$$\|\xi + \nabla\psi\|_{L^1} \leq C|\xi| + \varepsilon. \tag{2.13}$$

By (2.1), it implies

$$QW(x, u, \xi) \leq \int_Q W(x, u, \xi + \nabla\psi(y))\mathrm{d}y,$$

and hence

$$QW(x, u, \xi) - (QW)^\infty(x, u, \xi) \leq \int_Q (W(x, u, \xi + \nabla\psi(y)) - W^\infty(x, u, \xi + \nabla\psi(y)))\mathrm{d}y + \varepsilon.$$

Now, Theorem 1.1(iv) in the form (2.11) provides

$$\begin{aligned}QW(x, u, \xi) - (QW)^\infty(x, u, \xi) &\leq C\int_Q (1 + |\xi + \nabla\psi(y)|^{1-\alpha})\mathrm{d}y + \varepsilon \\ &\leq C'(1 + |\xi|^{1-\alpha}) + \varepsilon,\end{aligned}$$

where in the last line it has been used Hölder inequality, (2.13) and an argument entirely similar to the first part of the proof. By sending ε to 0, we conclude the proof.

2.2 Some results on measure theory and *BV* functions

Letting Ω be a generic open subset of $\mathbb{R}^N$, we denote by $\mathcal{M}(\Omega)$ the space of all signed Radon measures in Ω with bounded total variation. By the Riesz representation theorem, $\mathcal{M}(\Omega)$ can be identified to the dual of the separable space $\mathcal{C}_0(\Omega)$ of continuous functions on Ω vanishing on the boundary $\partial\Omega$. The N-dimensional Lebesgue measure in $\mathbb{R}^N$ is designated as $\mathcal{L}^N$ while $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure. If $\mu \in \mathcal{M}(\Omega)$ and $\lambda \in \mathcal{M}(\Omega)$ is a nonnegative Radon measure, we denote by $\frac{\mathrm{d}\mu}{d\lambda}$ the Radon-Nikodým derivative of μ with respect to λ. By a generalization of the Besicovich differentiation theorem (cf. [1, Proposition 2.2]), it can be proved that there exists a Borel set $E \subset \Omega$ such that $\lambda(E) = 0$ and

$$\frac{\mathrm{d}\mu}{d\lambda}(x) = \lim_{\rho \to 0^+} \frac{\mu(x + \rho\, C)}{\lambda(x + \rho\, C)} \quad \text{for all } x \in \text{Supp } \ \mu \setminus E \tag{2.14}$$

and any open convex set C containing the origin. (Recall that the set E is independent of C.)

We say that $u \in L^1(\Omega; \mathbb{R}^d)$ is a function of bounded variation, and we write $u \in BV(\Omega; \mathbb{R}^d)$, if all its first distributional derivatives $D_j u_i$ belong to $\mathcal{M}(\Omega)$ for $1 \le i \le d$ and $1 \le j \le N$. We refer to [2] for a detailed analysis of BV functions. The matrix-valued measure whose entries are $D_j u_i$ is denoted by Du and $|Du|$ stands for its total variation. By the Lebesgue decomposition theorem, we can split Du into the sum of two mutually singular measures $D^a u$ and $D^s u$ where $D^a u$ is the absolutely continuous part of Du with respect to the Lebesgue measure $\mathcal{L}^N$, while $D^s u$ is the singular part of Du with respect to $\mathcal{L}^N$. By ∇u, we denote the Radon-Nikodým derivative of $D^a u$ with respect to the Lebesgue measure, so that we can write

$$Du = \nabla u \mathcal{L}^N + D^s u.$$

The set S_u of points, where u does not have an approximate limit, is called the approximated discontinuity set, while $J_u \subseteq S_u$ is the so-called jump set of u defined as the set of points $x \in \Omega$, such that there exist $u^{\pm}(x) \in \mathbb{R}^d$ (with $u^+(x) \neq u^-(x)$) and $\nu_u(x) \in \mathbb{S}^{N-1}$ satisfying

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^N} \int_{\{y \in B_\varepsilon(x) : (y-x)\cdot \nu_u(x) > 0\}} |u(y) - u^+(x)| \mathrm{d}y = 0$$

and

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^N} \int_{\{y \in B_\varepsilon(x) : (y-x)\cdot \nu_u(x) < 0\}} |u(y) - u^-(x)| \mathrm{d}y = 0.$$

It is known that J_u is a countably $\mathcal{H}^{N-1}$-rectifiable Borel set. By Federer-Vol'pert theorem (cf. [2, Theorem 3.78]), $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$ for any $u \in BV(\Omega; \mathbb{R}^d)$. The measure $D^s u$ can in turn be decomposed into the sum of a jump part and a Cantor part defined by $D^j u := D^s u \llcorner J_u$ and $D^c u := D^s u \llcorner (\Omega \setminus S_u)$. We now recall the decomposition of Du:

$$Du = \nabla u \mathcal{L}^N + (u^+ - u^-) \otimes \nu_u \mathcal{H}^{N-1} \llcorner J_u + D^c u.$$

The three measures above are mutually singular. If $\mathcal{H}^{N-1}(B) < +\infty$, then $|D^c u|(B) = 0$ and there exists a Borel set E such that

$$\mathcal{L}^N(E) = 0, \quad |D^c u|(X) = |D^c u|(X \cap E)$$

for all Borel sets $X \subseteq \Omega$.

3 Relaxation

This section is devoted to the proof of the integral representation results dealing with the decoupled models described in the introduction.

To prove Theorems 1.1–1.2, we will use the characterization for the relaxed functional of $I_W : L^1(\Omega;\mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$I_W(u) := \begin{cases} \displaystyle\int_\Omega W(x,u(x),\nabla u(x))\mathrm{d}x, & \text{if } u \in W^{1,1}(\Omega;\mathbb{R}^d), \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.1}$$

The relaxed functional of I_W is defined by

$$\overline{I}_W(u) := \inf\Big\{ \liminf_n I_W(u_n) : \ u_n \in BV(\Omega;\mathbb{R}^d), \ u_n \to u \text{ in } L^1 \Big\},$$

and it was characterized by Fonseca-Müller [17], provided (among other hypotheses) that W is quasiconvex. In the next lemma, we establish conditions to obtain the representation of I_W in the general case, that is, with W not necessarily quasiconvex.

We will also use the following notation. The functional $I_{QW} : L^1(\Omega;\mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$I_{QW}(u) := \begin{cases} \displaystyle\int_\Omega QW(x,u(x),\nabla u(x))\mathrm{d}x, & \text{if } u \in W^{1,1}(\Omega;\mathbb{R}^d), \\ +\infty, & \text{otherwise,} \end{cases} \tag{3.2}$$

and its relaxed functional is

$$\overline{I}_{QW}(u) := \inf\Big\{ \liminf_n I_{QW}(u_n) : \ u_n \in BV(\Omega;\mathbb{R}^d), \ u_n \to u \text{ in } L^1 \Big\}.$$

We are now in position to establish the mentioned lemma, and we notice that we make no assumptions on the quasiconvexified function QW.

Lemma 3.1 *Let $W : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to [0,+\infty)$ be a continuous function and consider the functionals I_W and I_{QW} and their corresponding relaxed functionals defined as above. Then, if W satisfies conditions of Theorem* 1.1(i)/(iv)*, the two relaxed functionals coincide in $BV(\Omega,\mathbb{R}^d)$ and moreover*

$$\overline{I}_W(u) = \overline{I}_{QW}(u) = \int_\Omega QW(x,u(x),\nabla u(x))\mathrm{d}x + \int_{J_u} \gamma(x,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1} + \int_\Omega (QW)^\infty\Big(x,u(x),\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\mathrm{d}|D^c u|.$$

Proof First we observe that $\overline{I}_W(u) = \overline{I}_{QW}(u)$ for every $u \in BV(\Omega;\mathbb{R}^d)$. Indeed, since $QW \le W$, we have $\overline{I}_{QW} \le \overline{I}_W$. Next we prove the opposite inequality in the nontrivial case that $\overline{I}_{QW}(u) < +\infty$. For fixed $\delta > 0$, we can consider $u_n \in W^{1,1}(\Omega;\mathbb{R}^d)$ with $u_n \to u$ strongly in $L^1(\Omega;\mathbb{R}^d)$, such that

$$\overline{I}_{QW}(u) \ge \lim_n \int_\Omega QW(x,u_n(x),\nabla u_n(x))\mathrm{d}x - \delta.$$

Applying [11, Theorem 9.8], for each n there exists a sequence $\{u_{n,k}\}$ converging to u_n weakly in $W^{1,1}(\Omega;\mathbb{R}^d)$, such that

$$\int_\Omega QW(x,u_n(x),\nabla u_n(x))\mathrm{d}x = \lim_k \int_\Omega W(x,u_{n,k}(x),\nabla u_{n,k}(x))\mathrm{d}x.$$

Consequently,

$$\overline{I}_{QW}(u) \geq \lim_n \lim_k \int_\Omega W(x,u_{n,k}(x),\nabla u_{n,k}(x))\mathrm{d}x - \delta \tag{3.3}$$

and

$$\lim_n \lim_k \|u_{n,k}-u\|_{L^1} = 0.$$

Via a diagonal argument, there exists a sequence $\{u_{n,k_n}\}$ satisfying $u_{n,k_n} \to u$ in $L^1(\Omega;\mathbb{R}^d)$ and realizing the double limit in the right-hand side of (3.3). Thus, it implies

$$\overline{I}_{QW}(u) \geq \lim_n \int_\Omega W(x,u_{n,k_n}(x),\nabla u_{n,k_n}(x))\mathrm{d}x - \delta \geq \overline{I}_W(u) - \delta.$$

Letting δ go to 0 the conclusion follows.

Finally, we prove the integral representation for $\overline{I}_{QW}$ and consequently for $\overline{I}_W$. To this end, we invoke [17, Theorem 2.16] (cf. also [2, Theorem 5.54]).

By the hypotheses, and by Proposition 2.1 above, QW satisfies conditions (H1)–(H4) in [17], and the condition (H5) follows from Proposition 2.3. Applying [17, Theorem 2.16], we conclude the proof.

Let $I_{QW+\varphi} : BV(\Omega;\mathbb{R}^d) \times L^q(\Omega;\mathbb{R}^m) \to \mathbb{R} \cup \{+\infty\}$ be the functional defined by

$$I_{QW+\varphi}(u,v) := \begin{cases} \displaystyle\int_\Omega QW(x,u(x),\nabla u(x))\mathrm{d}x + \int_\Omega \varphi(x,u(x),v(x))\mathrm{d}x, \\ \qquad \text{if } (u,v) \in (W^{1,1}(\Omega;\mathbb{R}^d) \cap L^p(\Omega;\mathbb{R}^d)) \times L^q(\Omega;\mathbb{R}^m), \\ +\infty, \quad \text{otherwise,} \end{cases} \tag{3.4}$$

and its relaxed functional as

$$\begin{aligned} \overline{I}_{QW+\varphi}(u,v) := \inf\Big\{ &\liminf_n I_{QW+\varphi}(u_n,v_n) : (u_n,v_n) \in BV(\Omega;\mathbb{R}^d) \times L^q(\Omega;\mathbb{R}^m), \\ &u_n \to u \text{ in } L^1, v_n \rightharpoonup v \text{ in } L^q \Big\}. \end{aligned} \tag{3.5}$$

We can obtain, as in the first part of the proof of Lemma 3.1, the following result.

Corollary 3.1 *Let $p \geq 1$, $q \geq 1$ and $\Omega \subset \mathbb{R}^N$. Assume $W : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to [0,+\infty)$ and $\varphi : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \to [0,+\infty)$ satisfying Theorem* 1.1(i)/(iv) *and* (v)*, respectively. Let I and $\overline{I}$ be defined by* (1.1) *and* (1.3)*, respectively. Let $I_{QW+\varphi}$ and $\overline{I}_{Qw+\varphi}$ be as in* (3.4) *and* (3.5)*, respectively, then*

$$\overline{I}(u,v) = \overline{I}_{QW+\varphi}(u,v)$$

for every $(u,v) \in BV(\Omega;\mathbb{R}^d) \times L^q(\Omega;\mathbb{R}^m)$.

Remark 3.1 We observe that, in the case $1 \leq p < +\infty$, $1 < q < \infty$, given $W : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to \mathbb{R}$ and $\varphi : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$, Carathéodory functions satisfying Theorem 1.1(i) and (v), respectively, then, if one can provide that $C\varphi$ is still Carathéodory, an argument entirely similar to the first part of Lemma 3.1, implies that

$$\overline{I}(u,v) = \inf\Big\{ \liminf_{n\to+\infty} \int_\Omega (QW(x,u_n,\nabla u_n) + C\varphi(x,u_n,v_n))\mathrm{d}x : \\ (u_n,v_n) \in BV(\Omega;\mathbb{R}^d)\times L^q(\Omega;\mathbb{R}^m),\ u_n \to u \text{ in } L^1,\ v_n \rightharpoonup v \text{ in } L^q \Big\},$$

where $\overline{I}$ is the functional defined by (1.3), QW and $C\varphi$ are defined in (2.1) and (1.5), respectively. But we emphasize that since, assuming only (v) of Theorem 1.1, there may be a lack of continuity of $C\varphi(x,\cdot,\cdot)$ as observed in Remark 2.1, we focus just on the relaxation of the term $\int_\Omega W(x,u,\nabla u)\mathrm{d}x$ and we prove Lemma 3.1 (cf. also Corollary 3.1) in order to be allowed to assume W quasiconvex without loosing generality.

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1 The proof is divided in two parts. First we consider the case $q > 1$, and then we consider $q = 1$. In both cases, we first prove a lower bound for the relaxed energy $\overline{I}$, and then we prove that the lower bound obtained is also an upper bound for $\overline{I}$.

Preliminarly, we observe that by virtue of Corollary 3.1 and Propositions 2.1–2.3, we can assume without loss of generality, that W is quasiconvex in the last variable.

Part 1 $q > 1$.

(1) Lower bound. Let $u \in BV(\Omega;\mathbb{R}^d)\cap L^p(\Omega;\mathbb{R}^d)$ and $v \in L^q(\Omega;\mathbb{R}^m)$. We will prove that, for any sequences $u_n \in BV(\Omega;\mathbb{R}^d)$ and $v_n \in L^q(\Omega;\mathbb{R}^m)$, such that $u_n \to u$ in L^1 and $v_n \rightharpoonup v$ in L^q,

$$\liminf_{n\to+\infty} I(u_n,v_n) \geq \int_\Omega W(x,u,\nabla u)\mathrm{d}x + \int_{J_u} \gamma(x,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1} \\ + \int_\Omega W^\infty\Big(x,u,\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}(x)\Big)\mathrm{d}|D^c u| + \int_\Omega C\varphi(x,u,v)\mathrm{d}x.$$

Let u_n and v_n be two sequences in the conditions described above. Then, by [17, Theorem 2.16],

$$\int_\Omega W(x,u,\nabla u)\mathrm{d}x + \int_{J_u} \gamma(x,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1} + \int_\Omega W^\infty\Big(x,u,\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\mathrm{d}|D^c u| \\ \leq \liminf_{n\to+\infty} \int_\Omega W(x,u_n,\nabla u_n)\mathrm{d}x. \tag{3.6}$$

Moreover, since we can assume $\liminf_n \int_\Omega \varphi(x,u_n,v_n)\mathrm{d}x < +\infty$, the bound on $\|u_n\|_{L^p}$ provided by (v), the fact that $u_n \to u$ in $L^1(\Omega)$ and consequently pointwise, guarantee that $u_n \to u$ strongly in L^p. Furthermore, $v_n \rightharpoonup v$ weakly in L^q and because of the lower semi-continuity of $C\varphi(x,\cdot,\cdot)$ (cf. [12, Lemma 4.3]), it implies (cf. [16, Theorem 7.5] or [13])

$$\int_\Omega C\varphi(x,u,v)\mathrm{d}x \leq \liminf_{n\to+\infty} \int_\Omega C\varphi(x,u_n,v_n)\mathrm{d}x \leq \liminf_{n\to+\infty} \int_\Omega \varphi(x,u_n,v_n)\mathrm{d}x. \tag{3.7}$$

Consequently, the superadditivity of the $\liminf$, gives the desired lower bound.

(2) Upper bound. Let $u \in BV(\Omega;\mathbb{R}^d) \cap L^p(\Omega;\mathbb{R}^d)$ and $v \in L^q(\Omega;\mathbb{R}^m)$. We will prove that

$$\overline{I}(u,v) \le \int_\Omega W(x,u,\nabla u)\mathrm{d}x + \int_{J_u} \gamma(x,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1} + \int_\Omega W^\infty\Big(x,u,\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\mathrm{d}|D^c u| + \int_\Omega C\varphi(x,u,v)\mathrm{d}x, \tag{3.8}$$

constructing convenient sequences $u_n \in BV(\Omega;\mathbb{R}^d)$ such that $u_n \to u$ in L^1, and $v_n \in L^q(\Omega,\mathbb{R}^m)$ such that $v_n \rightharpoonup v$ in L^q.

We can assume, without loss of generality, that

$$\int_\Omega W(x,u,\nabla u)\mathrm{d}x + \int_{J_u} \gamma(x,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1} + \int_\Omega W^\infty\Big(x,u,\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\mathrm{d}|D^c u| + \int_\Omega C\varphi(x,u,v)\mathrm{d}x < +\infty. \tag{3.9}$$

In particular, from (v), it follows that $u \in L^p(\Omega;\mathbb{R}^d)$.

Moreover, we suppose, without loss of generality, that $W \ge 0$ and $\varphi \ge 0$. We will consider two cases.

Case 1 $u \in L^\infty(\Omega;\mathbb{R}^d)$.

Fix $M \in \mathbb{N}$. We will prove that, for some constant c (independent of M),

$$\overline{I}(u,v) \le \int_\Omega W(x,u,\nabla u)\mathrm{d}x + \int_{J_u} \gamma(x,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1} + \int_\Omega W^\infty\Big(x,u,\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\mathrm{d}|D^c u| + \int_\Omega C\varphi(x,u,v)\mathrm{d}x + \frac{c}{M}.$$

Then we get the desired inequality by letting M go to $+\infty$.

We proceed in three steps.

Step 1 Construction of a convenient sequence converging to u in $L^1(\Omega;\mathbb{R}^d)$.

Let $\{u_n\}$ be a sequence in $W^{1,1}(\Omega;\mathbb{R}^d)$, such that $u_n \to u$ in L^1 and

$$\lim \int_\Omega W(x,u_n,\nabla u_n)\mathrm{d}x = \int_\Omega W(x,u,\nabla u)\mathrm{d}x + \int_{J_u} \gamma(x,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1} + \int_\Omega W^\infty\Big(x,u,\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\mathrm{d}|D^c u|.$$

This holds by [17, Theorem 2.16]. Next we will truncate the sequence u_n.

Fix k such that $\mathrm{e}^k - 1 > 2\|u\|_{L^\infty}$. Then, (3.9) together with the coercivity condition of W on ξ (cf. (i)), and the fact that $\varphi \ge 0$, imply that $\sup \|\nabla u_n\|_{L^1}$ is bounded by a constant independent of the sequence u_n. Thus

$$\sum_{i=0}^{M-1} \int_{\{x\in\Omega:\ k+i\le \ln(1+|u_n|)<k+i+1\}} (1+|\nabla u_n|)\mathrm{d}x = \int_{\{x\in\Omega:\ k\le \ln(1+|u_n|)<k+M\}} (1+|\nabla u_n|)\mathrm{d}x \le |\Omega| + \sup_n \|\nabla u_n\|_{L^1},$$

and so, for each $n \in \mathbb{N}$, we can find $i = i(n) \in \{0,\cdots,M-1\}$, such that

$$\int_{\{x\in\Omega:\ k+i\le \ln(1+|u_n|)<k+i+1\}} (1+|\nabla u_n(x)|)\mathrm{d}x \le \frac{|\Omega| + \sup_n \|\nabla u_n\|_{L^1}}{M}. \tag{3.10}$$

For each n, and accordingly to the previous choice of $i(n)$, consider $\tau_n : \mathbb{R}_0^+ \to [0,1]$, such that $\tau_n \in C^1(\mathbb{R}_0^+)$, $|\tau_n'| \leq 1$,

$$\tau_n(t) = 1, \quad \text{if } 0 \leq t < k + i(n) \quad \text{and} \quad \tau_n(t) = 0, \quad \text{if } t \geq k + i(n) + 1.$$

We can now define the truncated sequence. Let $g_n(z) := \tau_n(\ln(1+|z|))\, z$, and $\overline{u}_n(x) = g_n(u_n)$. Since in a neighborhood of 0 the function $\tau_n(\ln(1+|\cdot|))$ is identically 1, g_n is a Lipschitz, C^1 function with

$$\nabla g_n(z) = \begin{cases} \tau_n(\ln(1+|z|))\,\mathbb{I} + \tau_n'(\ln(1+|z|))\dfrac{1}{1+|z|}\dfrac{z \otimes z}{|z|}, & \text{if } z \neq 0, \\ \mathbb{I}, & \text{if } z = 0 \end{cases}$$

and $|\nabla g_n(z)| \leq c$. So, by [2, Theorem 3.96], $\overline{u}_n \in W^{1,1}(\Omega;\mathbb{R}^d)$, $\nabla \overline{u}_n = \nabla g_n(u_n) \nabla u_n \mathcal{L}^N$ and $|\nabla \overline{u}_n| \leq c|\nabla u_n|$ which is bounded in L^1 as observed above. Moreover $\|\overline{u}_n\|_{L^\infty} \leq \mathrm{e}^{k+i(n)+1} - 1 \leq \mathrm{e}^{k+M} - 1$ and $\overline{u}_n \to u$ in L^1. Indeed, if $u \equiv 0$, then $\|\overline{u}_n\|_{L^1} \leq \|u_n\|_{L^1} \to 0$, otherwise, we have

$$\begin{aligned} \|\overline{u}_n - u\|_{L^1(\Omega)} &= \int_{\{x\in\Omega:\ 0 \leq \ln(1+|u_n|) < k+i(n)\}} |u_n(x) - u(x)| \mathrm{d}x \\ &\quad + \int_{\{x\in\Omega:\ k+i(n) \leq \ln(1+|u_n|) < k+i(n)+1\}} |\overline{u}_n(x) - u(x)| \mathrm{d}x \\ &\quad + \int_{\{x\in\Omega:\ \ln(1+|u_n|) \geq k+i(n)+1\}} |u(x)| \mathrm{d}x \\ &\leq \|u_n - u\|_{L^1(\Omega)} + \int_{\{x\in\Omega:\ k+i(n) \leq \ln(1+|u_n|) < k+i(n)+1\}} |\overline{u}_n(x) - u_n(x)| \mathrm{d}x \\ &\quad + \|u_n - u\|_{L^1(\Omega)} + \|u\|_{L^\infty(\Omega)} |\{x \in \Omega : \ln(1+|u_n|) \geq k+i(n)+1\}| \\ &\leq 2\|u_n - u\|_{L^1(\Omega)} + \int_{\{x\in\Omega:\ k+i(n) \leq \ln(1+|u_n|) < k+i(n)+1\}} |u_n(x)| \mathrm{d}x \\ &\quad + \|u\|_{L^\infty(\Omega)} \, |\{x \in \Omega :\ \ln(1+|u_n|) \geq k+i(n)+1\}|, \end{aligned}$$

where the last terms converging to zero because $u_n \to u$ in L^1 and because of the following estimates:

$$\begin{aligned} \int_{\{x\in\Omega:\ k+i(n) \leq \ln(1+|u_n|) < k+i(n)+1\}} |u_n(x)| \mathrm{d}x &\leq \int_{\{x\in\Omega:\ k+i(n) \leq \ln(1+|u_n|) < k+i(n)+1\}} (\mathrm{e}^{k+M} - 1) \mathrm{d}x \\ &\leq (\mathrm{e}^{k+M} - 1) |\{x \in \Omega : |u_n| \geq \mathrm{e}^{k+i(n)} - 1\}| \\ &\leq (\mathrm{e}^{k+M} - 1) |\{x \in \Omega :\ |u_n - u| \geq \|u\|_{L^\infty(\Omega)}\}| \\ &\leq (\mathrm{e}^{k+M} - 1) \frac{\|u_n - u\|_{L^1(\Omega)}}{\|u\|_{L^\infty(\Omega)}}, \\ |\{x \in \Omega : \ln(1+|u_n|) \geq k+i(n)+1\}| &= |\{x \in \Omega : |u_n| \geq \mathrm{e}^{k+i(n)+1} - 1\}| \\ &\leq |\{x \in \Omega : |u_n - u| \geq \|u\|_{L^\infty(\Omega)}\}| \\ &\leq \frac{\|u_n - u\|_{L^1(\Omega)}}{\|u\|_{L^\infty(\Omega)}}. \end{aligned}$$

So, we have, in particular, that $\overline{u}_n$ converges to u in L^1 and $\overline{u}_n$ clearly belongs to $L^p(\Omega;\mathbb{R}^d)$.

Step 2 Construction of a convenient sequence $\{v_n\}$ weakly converging to v in L^q.

We have, by (v), [16, Theorem 6.68, Remark 6.69(ii)], for any $w \in L^1(\Omega;\mathbb{R}^d)$,

$$\int_\Omega C\varphi(x,w,v)\mathrm{d}x = \inf\Big\{\liminf_{n\to+\infty}\int_\Omega \varphi(x,w,v_n)\mathrm{d}x:\ \{v_n\}\subset L^q(\Omega;\mathbb{R}^m),\ v_n \rightharpoonup v \text{ in } L^q\Big\},$$

whenever the second term is finite.

Since $q>1$ and thus $L^{q'}(\Omega;\mathbb{R}^m)$ is separable, we can consider a sequence $\{\psi_l\}$ of functions, dense in $L^{q'}(\Omega;\mathbb{R}^m)$.

Then, for each $n\in\mathbb{N}$ let $v_j^n \in L^q(\Omega;\mathbb{R}^m)$ be such that

$$\int_\Omega C\varphi(x,\overline{u}_n,v)\mathrm{d}x = \lim_{j\to+\infty}\int_\Omega \varphi(x,\overline{u}_n,v_j^n)\mathrm{d}x$$

and

$$\lim_{j\to+\infty}\int_\Omega (v_j^n - v)\psi_l \mathrm{d}x = 0,\quad \forall l\in\mathbb{N}.$$

We then extract a diagonalizing sequence v_n in the following way: for each $n\in\mathbb{N}$, consider $j(n)$ increasing and verifying

$$\Big|\int_\Omega (\varphi(x,\overline{u}_n,v_{j(n)}^n) - C\varphi(x,\overline{u}_n,v))\mathrm{d}x\Big| \le \frac{1}{n},$$
$$\Big|\int_\Omega (v_{j(n)}^n - v)\psi_l \mathrm{d}x\Big| \le \frac{1}{n},\quad l=1,\cdots,n.$$

Define then $v_n = v_{j(n)}^n$. We have that v_n is bounded in the L^q norm:

$$\int_\Omega |v_n|^q \mathrm{d}x \le C\int_\Omega \varphi(x,\overline{u}_n,v_n)\mathrm{d}x \le \frac{C}{n} + C\int_\Omega C\varphi(x,\overline{u}_n,v)\mathrm{d}x \le C + C\int_\Omega \varphi(x,\overline{u}_n,v)\mathrm{d}x,$$

where the last term is bounded because $\overline{u}_n$ is a bounded sequence in L^∞ and because of the growth condition (v) on φ.

Moreover, the density of ψ_l in $L^{q'}$ ensures that $v_n \rightharpoonup v$ in L^q. Indeed, let $\psi\in L^q(\Omega;\mathbb{R}^m)$ and let $\delta>0$. Consider $l\in\mathbb{N}$ such that $\|\psi_l-\psi\|_{L^{q'}}\le\delta$. Then, for sufficiently large n,

$$\Big|\int_\Omega (v_n-v)\psi\mathrm{d}x\Big| \le \Big|\int_\Omega (v_n-v)(\psi-\psi_l)\mathrm{d}x\Big| + \Big|\int_\Omega (v_n-v)\psi_l\mathrm{d}x\Big| \le \|v_n-v\|_{L^q}\|\psi_l-\psi\|_{L^{q'}}+\delta \le c\delta+\delta.$$

Step 3 Upper bound for $\overline{I}$.

Start remarking that

$$\limsup_{n\to+\infty}\int_\Omega \varphi(x,\overline{u}_n,v_n)\mathrm{d}x \le \int_\Omega C\varphi(x,u,v)\mathrm{d}x.$$

Indeed,

$$\begin{aligned}\limsup_{n\to+\infty}\int_\Omega \varphi(x,\overline{u}_n,v_n)\mathrm{d}x &= \limsup_{n\to+\infty}\int_\Omega (\varphi(x,\overline{u}_n,v_n) - C\varphi(x,\overline{u}_n,v) + C\varphi(x,\overline{u}_n,v))\mathrm{d}x \\ &\le \limsup_{n\to+\infty}\Big(\frac{1}{n} + \int_\Omega C\varphi(x,\overline{u}_n,v)\mathrm{d}x\Big).\end{aligned}$$

As observed in Remark 2.1, $C\varphi(x,\cdot,v)$ is upper semi-continuous. By the pointwise convergence of $\overline{u}_n$ towards u (up to a subsequence), we have

$$\limsup_{n\to+\infty} C\varphi(x,\overline{u}_n,v) \le C\varphi(x,u,v).$$

Moreover, the fact that $\overline{u}_n$ is bounded in L^∞ and the hypothesis (v) allows to apply the "inverted" Fatou's lemma and get the desired inequality.

Now we have

$$\begin{aligned}
&\int_\Omega W(x,\overline{u}_n,\nabla\overline{u}_n)\mathrm{d}x\\
&= \int_{\{x\in\Omega:\ 0\le \ln(1+|u_n|)<k+i(n)\}} W(x,u_n,\nabla u_n)\mathrm{d}x\\
&\quad+\int_{\{x\in\Omega:\ k+i(n)\le \ln(1+|u_n|)<k+i(n)+1\}} W(x,\overline{u}_n,\nabla\overline{u}_n)\mathrm{d}x\\
&\quad+\int_{\{x\in\Omega:\ \ln(1+|u_n|)\ge k+i(n)+1\}} W(x,0,0)\mathrm{d}x\\
&\le \int_\Omega W(x,u_n,\nabla u_n)\mathrm{d}x + \int_{\{x\in\Omega:\ k+i(n)\le \ln(1+|u_n|)<k+i(n)+1\}} C(1+|\nabla\overline{u}_n|)\mathrm{d}x\\
&\quad+C|\{x\in\Omega:\ \ln(1+|u_n|)\ge k+i(n)+1\}|,
\end{aligned}$$

where it has been used the growth condition (i). Using the expression of $\overline{u}_n$, by [2, Theorem 3.96], we have $|\nabla\overline{u}_n|\le c|\nabla u_n|$, and so, using (3.10), we get

$$\limsup_{n\to+\infty}\int_{\{x\in\Omega:\ k+i(n)\le \ln(1+|u_n|)<k+i(n)+1\}} C(1+|\nabla\overline{u}_n|)\mathrm{d}x \le c\,\frac{|\Omega|+\sup\|\nabla u_n\|_{L^1}}{M}=\frac{c}{M}$$

(note that c is independent of n and of the sequence u_n, and it does not represent always the same constant).

Moreover, since $|\{x\in\Omega:\ \ln(1+|u_n|)\ge k+i(n)+1\}|\to 0$ as $n\to+\infty$ (as already seen in the case where $\|u\|_{L^\infty}\neq 0$), we get

$$\limsup_{n\to+\infty}\int_\Omega W(x,\overline{u}_n,\nabla\overline{u}_n)\mathrm{d}x\le \lim_{n\to+\infty}\int_\Omega W(x,u_n,\nabla u_n)\mathrm{d}x+\frac{c}{M}.$$

Note that if $u=0$ we can still get $|\{x\in\Omega:\ \ln(1+|u_n|)\ge k+i(n)+1\}|\to 0$:

$$|\{x\in\Omega:\ \ln(1+|u_n|)\ge k+i(n)+1\}|\le|\{x\in\Omega:\ |u_n|\ge \mathrm{e}^{k+1}-1\}|\le\frac{\|u_n\|_{L^1}}{\mathrm{e}^{k+1}-1}\to 0,$$

since $u_n\to 0$ in L^1.

Finally, we get, as desired,

$$\begin{aligned}
\overline{I}(u,v)&\le\liminf_{n\to+\infty}\int_\Omega W(x,\overline{u}_n,\nabla\overline{u}_n)\mathrm{d}x+\int_\Omega\varphi(x,\overline{u}_n,v_n)\mathrm{d}x\\
&\le\limsup_{n\to+\infty}\int_\Omega W(x,\overline{u}_n,\nabla\overline{u}_n)\mathrm{d}x+\limsup_{n\to+\infty}\int_\Omega\varphi(x,\overline{u}_n,v_n)\mathrm{d}x\\
&\le\int_\Omega W(x,u,\nabla u)\mathrm{d}x+\int_{J_u}\gamma(x,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1}+\int_\Omega W^\infty\Big(x,u,\frac{\mathrm{d}D^cu}{\mathrm{d}|D^cu|}\Big)\mathrm{d}|D^cu|\\
&\quad+\frac{c}{M}+\int_\Omega C\varphi(x,u,v)\mathrm{d}x.
\end{aligned}$$

Case 2 Arbitrary $u\in BV(\Omega;\mathbb{R}^d)\cap L^p(\Omega;\mathbb{R}^d)$.

To achieve the upper bound on this case, we will reduce it to case 1 by means of a truncature argument developed in [17, Theorem 2.16, Step 4], in turn inspired by [3, Theorem 4.9]. We reproduce the same argument as in [17] for the readers' convenience.

Let $\phi_n \in C_0^1(\mathbb{R}^d;\mathbb{R}^d)$ be such that

$$\phi_n(y) = y, \quad \text{if } y \in B_n(0), \quad \|\nabla\phi_n\|_{L^\infty} \le 1,$$

and fix $u \in BV(\Omega;\mathbb{R}^d) \cap L^p(\Omega;\mathbb{R}^d)$.

As proven in [3, Theorem 4.9], directly from the definitions and properties for the approximate discontinuity set and the triplets (u^+, u^-, ν_u) (cf. Subection 2.2), it implies that

$$\begin{aligned} &J_{\phi_n(u)} \subset J_u, \\ &(\phi_n(u)^+, \phi_n(u)^-, \nu_{\phi_n(u)}) = (\phi_n(u^+), \phi_n(u^-), \nu_u), \quad \text{in } J_{\phi_n(u)}. \end{aligned}$$

Moreover, one has

$$|D\phi_n(u)|(B) \le |D(u)|(B) \quad \text{for every Borel set } B \subset \Omega. \tag{3.11}$$

Consequently,

$$\phi_n(u) \in BV(\Omega;\mathbb{R}^d) \cap L^\infty(\Omega;\mathbb{R}^d).$$

Since $\phi_n(u) \to u$ in L^1, by the lower semicontinuity of $\overline{I}$ (since $q > 1$) and by case 1, we get

$$\begin{aligned} \overline{I}(u,v) \le \liminf_{n\to+\infty} \Big[\int_\Omega W(x,\phi_n(u),\nabla\phi_n(u))\mathrm{d}x + \int_{J_{\phi_n(u)}} \gamma(x,\phi_n(u)^+,\phi_n(u)^-,\nu_{\phi_n(u)})\mathrm{d}\mathcal{H}^{N-1} \\ + \int_\Omega W^\infty\Big(x,\phi_n(u),\frac{\mathrm{d}D^c(\phi_n(u))}{\mathrm{d}|D^c(\phi_n(u))|}\Big)\mathrm{d}|D^c\phi_n(u)| + \int_\Omega C\varphi(x,\phi_n(u),v)\mathrm{d}x\Big]. \end{aligned}$$

By the upper semicontinuity of γ in all of its arguments as stated in [17, Lemma 2.15(c)] and by the fact that $\gamma(x,a,b,\nu) \le C|a-b|$ for every $(x,a,b,\nu) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1}$ (cf. [17, Lemma 2.15(d)]) and the properties of ϕ_n, we have

$$\gamma(x,\phi_n(u^+),\phi_n(u^-),\nu_u) \le C|u^+ - u^-|,$$

and so, by Fatou's lemma, we obtain

$$\limsup_{n\to+\infty} \int_{J_{\phi_n(u)}} \gamma(x,\phi_n(u)^+,\phi_n(u)^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1} \le \int_{J_u} \gamma(x,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1}.$$

Moreover, we have

$$\limsup_{n\to+\infty} \int_\Omega C\varphi(x,\phi_n(u),v)\mathrm{d}x = \int_\Omega C\varphi(x,u,v)\mathrm{d}x. \tag{3.12}$$

Indeed, as already observed in step 2, $C\varphi(x,\cdot,v)$ is upper semicontinuous and $\phi_n(u)$ is pointwise converging to u and thus we can apply the inverted Fatou's lemma.

For what concerns the other terms, setting $\Omega_n := \{x \in \Omega \setminus J_u : |u(x)| \le n\}$, we have

$$\begin{aligned} &\limsup_{n\to+\infty} \int_\Omega W(x,\phi_n(u),\nabla\phi_n(u))\mathrm{d}x \\ &= \limsup_{n\to+\infty} \Big[\int_{\Omega_n} W(x,\phi_n(u),\nabla\phi_n(u))\mathrm{d}x + \int_{(\Omega\setminus\Omega_n)\setminus J_u} W(x,\phi_n(u),\nabla\phi_n(u))\mathrm{d}x \Big] \\ &\le \int_\Omega W(x,u,\nabla u)\mathrm{d}x + \limsup_{n\to+\infty} C[|\Omega\setminus\Omega_n| + |D\phi_n(u)|((\Omega\setminus\Omega_n)\setminus J_u)]. \end{aligned}$$

On the other hand, by (3.11), we deduce that

$$\limsup_{n\to+\infty}|D\phi_n(u)|((\Omega\setminus\Omega_n)\setminus J_u)\leq\limsup_{n\to+\infty}|Du|(\Omega\setminus(\Omega_n\cup J_u))=0,$$

and so

$$\limsup_{n\to+\infty}\int_\Omega W(x,\phi_n(u),\nabla\phi_n(u))\mathrm{d}x\leq\int_\Omega W(x,u,\nabla u)\mathrm{d}x.$$

Similarly,

$$\begin{aligned}&\limsup_{n\to+\infty}\int_\Omega W^\infty\Big(x,\phi_n(u),\frac{\mathrm{d}D^c\phi_n(u)}{\mathrm{d}|D^c\phi_n(u)|}\Big)\mathrm{d}|D^c\phi_n(u)|\\&=\limsup_{n\to+\infty}\int_{\Omega_n} W^\infty\Big(x,\phi_n(u),\frac{\mathrm{d}D^c\phi_n(u)}{\mathrm{d}|D^c\phi_n(u)|}\Big)\mathrm{d}|D^c\phi_n(u)|\\&\quad+\limsup_{n\to+\infty}\int_{(\Omega\setminus\Omega_n)\setminus J_u} W^\infty\Big(x,\phi_n(u),\frac{\mathrm{d}D^c\phi_n(u)}{\mathrm{d}|D^c\phi_n(u)|}\Big)\mathrm{d}|D^c\phi_n(u)|\\&\leq\int_\Omega W^\infty\Big(x,u,\frac{\mathrm{d}D^cu}{\mathrm{d}|D^cu|}\Big)\mathrm{d}|D^cu|+C\limsup_{n\to+\infty}[|D\phi_n(u)|((\Omega\setminus\Omega_n)\setminus J_u)]\\&=\int_\Omega W^\infty\Big(x,u,\frac{\mathrm{d}D^cu}{\mathrm{d}|D^cu|}\Big)\mathrm{d}|D^cu|.\end{aligned}$$

This finishes the proof.

Part 2 $q=1$.

(1) Lower bound. Let $u\in BV(\Omega;\mathbb{R}^d)\cap L^p(\Omega;\mathbb{R}^d)$, $v\in L^1(\Omega;\mathbb{R}^m)$, $u_n\in BV(\Omega;\mathbb{R}^d)$ and $v_n\in L^1(\Omega;\mathbb{R}^m)$ such that $u_n\to u$ strongly in L^1 and $v_n\rightharpoonup v$ in L^1. Then by Lemma 3.1 exactly as in the case $q>1$, (3.6) continues to hold. Moreover, [8, Theorem 1.1] ensures that

$$\int_\Omega C\varphi(x,u,v)\mathrm{d}x\leq\liminf_{n\to+\infty}\int_\Omega\varphi(x,u_n,v_n)\mathrm{d}x.$$

Again the lower bound follows from the superadditivity of the lim inf.

(2) Upper bound. Let $u\in BV(\Omega;\mathbb{R}^d)\cap L^p(\Omega;\mathbb{R}^d)$ and $v\in L^1(\Omega;\mathbb{R}^m)$. We aim to prove (3.8), constructing convenient sequences $u_n\in BV(\Omega;\mathbb{R}^d)$ and $v_n\in L^1(\Omega;\mathbb{R}^m)$ with $u_n\to u$ in L^1 and $v_n\rightharpoonup v$ in L^1.

Case 1 As in the case $q>1$, we first assume that $u\in L^\infty(\Omega;\mathbb{R}^d)$ and develop our proof in three steps.

Step 1 This step 1 is identical to the step 1 of case 1 in part 1 proven for $q>1$.

Step 2 For what concerns this step, we preliminarly consider a continuous increasing function $\theta:[0,+\infty)\to[0,+\infty)$ such that

$$\lim_{t\to+\infty}\frac{\theta(t)}{t}=+\infty. \tag{3.13}$$

Then consider a decreasing sequence $\varepsilon\to0$ and take the functional $I_\varepsilon:BV(\Omega;\mathbb{R}^d)\times L^1(\Omega;\mathbb{R}^m)\to\mathbb{R}\cup\{+\infty\}$, defined as

$$I_\varepsilon(u,v):=I(u,v)+\varepsilon\int_\Omega\theta(|v|)\mathrm{d}x. \tag{3.14}$$

Let $C(\varphi(x,u,\cdot)+\varepsilon\theta(|\cdot|))$ be the convexification of $\varphi(x,u,\cdot)+\varepsilon\theta(|\cdot|)$ as in (1.5).

By [16, Theorem 6.68, Remark 6.69], we have that for every $w \in L^1(\Omega;\mathbb{R}^m)$,

$$\int_\Omega C(\varphi(x,w,v)+\varepsilon\theta(|v|))\mathrm{d}x = \inf\Big\{\liminf_{n\to+\infty}\int_\Omega(\varphi(x,w,v_n)+\varepsilon\theta(|v_n|))\mathrm{d}x : v_n \rightharpoonup v \text{ in } L^1\Big\},$$

whenever the second term is finite. Moreover, the left-hand side coincides with the sequentially weakly-L^1 lower semicontinuous envelope. Consequently, for every $n\in\mathbb{N}$, let $\overline{u}_n$ be the sequence constructed in the step 1 of case 1 in part 1, and let $v_j^n \in L^1(\Omega;\mathbb{R}^m)$ be such that $v_j^n \rightharpoonup v$ in L^1 as $j\to+\infty$ and

$$\int_\Omega C(\varphi(x,\overline{u}_n,v)+\varepsilon\theta(|v|))\mathrm{d}x = \lim_{j\to+\infty}\int_\Omega(\varphi(x,\overline{u}_n,v_j^n)+\varepsilon\theta(|v_j^n|))\mathrm{d}x.$$

The proof now develops as in [16, Proposition 3.18]. The growth condition (v) and the fact that $\overline{u}_n$ is bounded in L^∞ and thus in L^1, imply that there exists a constant M such that

$$\sup_{n,j\in\mathbb{N}}\int_\Omega\theta(|v_j^n|)\mathrm{d}x \le M. \tag{3.15}$$

We observe that the growth conditions on θ guarantee that $\sup\limits_{n,j\in\mathbb{N}}\|v_j^n\|_{L^1(\Omega)} \le C(M)$. Moreover, the separability of $C_0(\Omega)$ allows us to consider a dense sequence of functions $\{\psi_l\}$.

Next, in the way similar to the argument used in the analogous step for $q>1$, for every $\varepsilon>0$, we construct a diagonalizing sequence v_n as follows. For each $n\in\mathbb{N}$, consider $j(n)$ increasing, such that

$$\Big|\int_\Omega(\varphi(x,\overline{u}_n,v_{j(n)}^n)+\varepsilon\theta(|v_{j(n)}^n|)-C(\varphi(x,\overline{u}_n,v)+\varepsilon\theta(|v|)))\mathrm{d}x\Big| \le \frac{1}{n},$$
$$\Big|\int_\Omega(v_{j(n)}^n - v)\psi_l\mathrm{d}x\Big| \le \frac{1}{n},\quad l=1,\cdots,n.$$

Define $v_n := v_{j(n)}^n$. The bounds on θ, the fact that $\overline{u}_n$ is bounded in L^1 and the separability of $C_0(\Omega)$ guarantee that $v_n \overset{\star}{\rightharpoonup} v$ in $\mathcal{M}(\Omega)$, and moreover, (3.15), Dunford-Pettis' theorem imply that the convergence of v_n towards v is weak-L^1.

Step 3 Arguing as in the first part of the step 3 of case 1 in part 1 for $q>1$, we can prove that

$$\limsup_{n\to+\infty}\int_\Omega(\varphi(x,\overline{u}_n,v_n)+\varepsilon\theta(|v_n|))\mathrm{d}x \le \int_\Omega C(\varphi(x,u,v)+\varepsilon\theta(|v|))\mathrm{d}x.$$

Next we define

$$\begin{aligned}\overline{I}_\varepsilon(u,v) := \inf\Big\{&\liminf_{n\to+\infty} I_\varepsilon(u_n,v_n) : (u_n,v_n)\in BV(\Omega;\mathbb{R}^d)\times L^1(\Omega;\mathbb{R}^m),\\ &u_n\to u \text{ in } L^1,\ v_n\rightharpoonup v \text{ in } L^1\Big\}.\end{aligned}\tag{3.16}$$

The same argument in the last part of the step 3 of case 1 in part 1 for $q>1$, allows to prove that

$$\begin{aligned}\overline{I}_\varepsilon(u,v) \le &\int_\Omega W(x,u,\nabla u)\mathrm{d}x + \int_{J_u}\gamma(x,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1}\\ &+\int_\Omega W^\infty\Big(x,u,\frac{dD^cu}{\mathrm{d}|D^cu|}\Big)\mathrm{d}|D^cu| + \int_\Omega C(\varphi(x,u,v)+\varepsilon\theta(|v|))\mathrm{d}x\end{aligned}\tag{3.17}$$

for every $u \in BV(\Omega;\mathbb{R}^d) \cap L^\infty(\Omega;\mathbb{R}^d)$ and $v \in L^1(\Omega;\mathbb{R}^m)$. On the other hand, we observe that the sequence $\overline{I}_\varepsilon(u,v)$ is increasing in ε and $\overline{I} \leq \overline{I}_\varepsilon$ for every ε. Moreover, by virtue of the increasing behavior in ε of $\varphi + \varepsilon\theta$, invoking [16, Proposition 4.100], it results that for every $(x,u) \in \Omega \times \mathbb{R}^d$, we have

$$\inf_\varepsilon C(\varphi(x,u,v) + \varepsilon\theta(|v|)) = \lim_{\varepsilon\to 0} C(\varphi(x,u,v) + \varepsilon\theta(|v|)) = C\varphi(x,u,v).$$

Thus applying Lebesgue monotone convergence theorem, we have

$$\begin{aligned}
\overline{I}(u,v) &\leq \lim_{\varepsilon\to 0} \overline{I}_\varepsilon(u,v) \\
&= \lim_{\varepsilon\to 0} \Big(\int_\Omega W(x,u,\nabla u)\mathrm{d}x + \int_{J_u} \gamma(x,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1} \\
&\quad + \int_\Omega W^\infty\Big(x,u,\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\mathrm{d}|D^c u| + \int_\Omega C(\varphi(x,u,v) + \varepsilon\theta(|v|))\mathrm{d}x\Big) \\
&= \int_\Omega W(x,u,\nabla u)\mathrm{d}x + \int_{J_u} \gamma(x,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1} \\
&\quad + \int_\Omega W^\infty\Big(x,u,\frac{\mathrm{d}D^c u}{d|D^c u|}\Big)\mathrm{d}|D^c u| + \int_\Omega C\varphi(x,u,v)\mathrm{d}x
\end{aligned} \tag{3.18}$$

for every $(u,v) \in (BV(\Omega;\mathbb{R}^d) \cap L^\infty(\Omega;\mathbb{R}^d)) \times L^1(\Omega;\mathbb{R}^m)$.

Case 2 Now we consider $u \in BV(\Omega;\mathbb{R}^d) \cap L^p(\Omega;\mathbb{R}^d)$ and $v \in L^1(\Omega;\mathbb{R}^m)$.

To achieve the upper bound, we can preliminarily observe that, a proof entirely similar to [16, Proposition 3.18], guarantees that for every $\varepsilon > 0$, the functional $\overline{I}_\varepsilon(u,v)$, defined in (3.16) is sequentially weakly lower semicontinuous with respect to the topology $L^1(\Omega;\mathbb{R}^d)_{\text{strong}} \times L^1(\Omega;\mathbb{R}^m)_{\text{weak}}$. Thus, arguing exactly as in the case 2 in part 1 for $q > 1$, we have that

$$\begin{aligned}
\overline{I}_\varepsilon(u,v) \leq &\int_\Omega W(x,u,\nabla u)\mathrm{d}x + \int_{J_u} \gamma(x,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1} + \int_\Omega W^\infty\Big(x,u,\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\mathrm{d}|D^c u| \\
&+ \int_\Omega C(\varphi(x,u,v) + \varepsilon\theta(|v|))\mathrm{d}x.
\end{aligned} \tag{3.19}$$

Finally, the monotonicity argument for ε invoked in the step 3 of case 1 in part 2 for $q = 1$ can be recalled also in this context leading to the same inequality in (3.18) for every $u \in BV(\Omega;\mathbb{R}^d) \cap L^p(\Omega;\mathbb{R}^d)$ and for every $v \in L^1(\Omega;\mathbb{R}^m)$, and this concludes the proof of (3.8).

Now we present the proof of Theorem 1.2, which is much easier than the latter one, since, by virtue of the continuous embedding of $BV(\Omega;\mathbb{R}^d)$ in $L^{\frac{N}{N-1}}(\Omega;\mathbb{R}^d)$, it does not involve any truncature argument.

Proof of Theorem 1.2 We omit the details of the proof since it develops in the same way as that of Theorem 1.1. First we invoke Corollary 3.1 and assume without loss of generality that W is quasiconvex in the last variable. Then we prove a lower bound for the relaxed energy, and finally we show that the lower bound is also an upper bound. As in Theorem 1.1, we may consider two separate cases: $q > 1$ and $q = 1$.

(1) Lower bound for the cases $q = 1$ and $q > 1$. The proof of the lower bound is identical to that of Theorem 1.1.

(2) Upper bound.

(i) The case of $q > 1$. Let $u \in BV(\Omega;\mathbb{R}^d)$ and $v \in L^q(\Omega;\mathbb{R}^m)$. We can assume

$$\int_\Omega W(x,u,\nabla u)\mathrm{d}x + \int_{J_u} \gamma(x,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1} + \int_\Omega W^\infty\Big(x,u,\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\mathrm{d}|D^c u| + \int_\Omega C\varphi(x,u,v)\mathrm{d}x < +\infty. \tag{3.20}$$

Without loss of generality, we assume also that W and $\varphi \geq 0$. Applying [17, Theorem 2.16], we can get a sequence $\{u_n\}$ in $W^{1,1}(\Omega;\mathbb{R}^d)$ such that $u_n \to u$ in L^1 and

$$\lim \int_\Omega W(x,u_n,\nabla u_n)\mathrm{d}x = \int_\Omega W(x,u,\nabla u)\mathrm{d}x + \int_{J_u} \gamma(x,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1} + \int_\Omega W^\infty\Big(x,u,\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\mathrm{d}|D^c u|.$$

We observe that, by the coercivity condition on W and by (3.20), ∇u_n is bounded in L^1. Moreover, the continuous embedding of $BV(\Omega;\mathbb{R}^d)$ in $L^{\frac{N}{N-1}}(\Omega;\mathbb{R}^d)$, implies that u_n is bounded in $L^{\frac{N}{N-1}}(\Omega;\mathbb{R}^d)$ and thus in $L^p(\Omega;\mathbb{R}^d)$ since we are assuming $1 \leq p \leq \frac{N}{N-1}$.

Then, as in the proof of Theorem 1.1 (see the step 2 of case 1 in part 1 for $q > 1$), we can construct a recovery sequence v_n using the relaxation theorem in [16, Theorem 6.68] and the same diagonalizing argument. We emphasize that there is no need to make a preliminary truncature of the recovery sequence u_n. Indeed, to ensure that v_n is bounded in $L^q(\Omega;\mathbb{R}^m)$ (required to obtain the weak convergence of v_n towards v in L^q), it suffices to use the growth condition of φ and the fact that u_n is bounded in L^p.

Therefore, it is possible to get $v_n \rightharpoonup v$ in L^q and such that

$$\limsup_{n\to+\infty} \int_\Omega \varphi(x,u_n,v_n)\mathrm{d}x \leq \int_\Omega C\varphi(x,u,v)\mathrm{d}x.$$

The upper bound then follows by the sub-additivity of the $\limsup$.

(ii) The case of $q = 1$. In analogy with the case of $q > 1$, there is no need of truncature because of the continuous embedding of BV in $L^{\frac{N}{N-1}}$. As for Theorem 1.1, it suffices to approximate the functional I by I_ε in (3.14) and consequently it is enough to use, for the correspective relaxed functional, the diagonalization argument adopted in Theorem 1.1 (see the step 2 of case 1 in part 2 for $q = 1$) via an application of Dunford-Pettis' theorem. Finally the monotonicity behavior in ε of $\overline{I}_\varepsilon$, the approximation of the energy densities allowed by [16, Proposition 4.100] and the Lebegue monotone convergence theorem conclude the proof.

Acknowledgements The authors would like to thank Irene Fonseca and Giovanni Leoni for their invaluable comments on the subject of this paper and acknowledge the Center for Nonlinear Analysis, Carnegie Mellon University, Pittsburgh (PA) for the kind hospitality and support. A. M. Ribeiro thanks Università di Salerno and E. Zappale thanks Universidade Nova de Lisboa for their support and hospitality.

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