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Analytic Functions Related with the Hyperbola

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Abstract The author considers a new class $SH^m_{\lambda\mu}(\alpha)$ of normalized analytic functions defined by a differential operator. Several basic properties and characteristics of the functions belonging to the class $SH^m_{\lambda\mu}(\alpha)$ are investigated. These include integral representations, coefficient bounds, the Fekete-Szegö problem, class-preserving operators and T_δ-neighborhoods.

Keywords Analytic function, Differential operator, Hyperbolic domain, Fekete-Szegö problem, Neighborhood
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1 Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Let S^* and K denote the usual classes of starlike and convex functions in $\mathbb{U}$, respectively.

Suppose that $f, g \in \mathcal{A}$. Then f is said to be subordinate to g, written as $f \prec g$, if $f(z) = g(\omega(z)),\ z \in \mathbb{U}$ for some analytic function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1,\ z \in \mathbb{U}$.

The Hadamard product or convolution of the functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z), \quad z \in \mathbb{U}.$$

In [19] Stankiewicz and Wiśniowska studied the class of functions $SH(\alpha)$ defined as folllows.

A function $f \in \mathcal{A}$ is said to be in the class $SH(\alpha)$ if it satisfies the condition

$$\Big|\frac{zf'(z)}{f(z)} - 2\alpha(\sqrt{2}-1)\Big| < \sqrt{2}\Re\frac{zf'(z)}{f(z)} + 2\alpha(\sqrt{2}-1) \tag{1.2}$$

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for some α ($\alpha > 0$) and for all $z \in \mathbb{U}$.

Note that $f \in SH(\alpha)$ if and only if $\frac{zf'(z)}{f(z)}$ lies in the hyperbolic domain

$$\Omega(\alpha) = \{w = u + \mathrm{i}v : v^2 < 4\alpha u + u^2,\ u > 0\} \tag{1.3}$$

which is included in the right half-plane, is symmetric about the real axis with a vertex at the origin.

It is easy to see that $SH(\alpha) \subset S^*$ for all $\alpha > 0$.

Denote by $\mathcal{P}(P_\alpha)$ ($\alpha > 0$) the family of functions p such that $p \in \mathcal{P}$ and $p \prec P_\alpha$ in $\mathbb{U}$, where $\mathcal{P}$ is the well-known class of Carathéodory functions and P_α maps the unit disk conformally onto the domain $\Omega(\alpha)$ such that $P_\alpha(0) = 1$ and $P'_\alpha(0) > 0$.

The function which plays the role of the extremal function for the class $\mathcal{P}(P_\alpha)$ was obtained in [19] and was given by

$$P_\alpha(z) = (1 + 2\alpha)\sqrt{\frac{1 + bz}{1 - z}} - 2\alpha, \tag{1.4}$$

where

$$b = b(\alpha) = \frac{1 + 4\alpha - 4\alpha^2}{(1 + 2\alpha)^2}, \tag{1.5}$$

the branch of the square root $\sqrt{w}$ being chosen such that $\Im\sqrt{w} \geq 0$.

If $P_\alpha(z) = 1 + B_1 z + B_2 z^2 + \cdots$, then (see [19])

$$B_1 = \frac{1 + 4\alpha}{1 + 2\alpha} \quad \text{and} \quad B_2 = \frac{(1 + 4\alpha)(1 + 4\alpha + 8\alpha^2)}{2(1 + 2\alpha)^3}. \tag{1.6}$$

Denote by F_α (see [19]) the function satisfying

$$\frac{zF'_\alpha(z)}{F_\alpha(z)} = P_\alpha(z) \quad \text{and} \quad F'_\alpha(0) = 1, \tag{1.7}$$

where P_α is defined by (1.4). Elementary calculation shows that

$$F_\alpha(z) = z\Big[\frac{(\sqrt{1 + bz} + \mathrm{i}\sqrt{b - bz})^{\mathrm{i}\sqrt{b}}}{\sqrt{1 + bz} + \sqrt{1 - z}}\Big]^{2(1+2\alpha)}\Big[\frac{2}{(1 + \mathrm{i}\sqrt{b})^{\mathrm{i}\sqrt{b}}}\Big]^{2(1+2\alpha)}, \tag{1.8}$$

where b is given by (1.5) and the branch of $\sqrt{w}$ is chosen such that $\Im\sqrt{w} \geq 0$.

It is easy to see that the function F_α plays the role of the extremal function for the class $SH(\alpha)$. Note that since b is real ($-1 < b < 1$), both functions P_α and F_α have real coefficients.

Let $f \in \mathcal{A}$. We consider the following differential operator introduced by Răducanu and Orhan in [11]:

$$\begin{aligned} &D^0_{\lambda\mu} f(z) = f(z), \\ &D^1_{\lambda\mu} f(z) = D_{\lambda\mu} f(z) = \lambda\mu z^2 f''(z) + (\lambda - \mu) z f'(z) + (1 - \lambda + \mu) f(z), \\ &D^m_{\lambda\mu} f(z) = D_{\lambda\mu}(D^{m-1}_{\lambda\mu} f(z)), \end{aligned} \tag{1.9}$$

where $0 \leq \mu \leq \lambda$ and $m \in \mathbb{N} := \{1, 2, \cdots\}$.

If the function f is given by (1.1), then from (1.9) we see that

$$D_{\lambda\mu}^m f(z) = z + \sum_{n=2}^{\infty} \Phi_n(\lambda, \mu, m) a_n z^n, \tag{1.10}$$

where

$$\Phi_n(\lambda, \mu, m) = [1 + (\lambda\mu n + \lambda - \mu)(n-1)]^m, \quad n \geq 2. \tag{1.11}$$

From (1.10) it follows that $D_{\lambda\mu}^m f(z)$ can be written in terms of convolution as

$$D_{\lambda\mu}^m f(z) = (f * g_{\lambda\mu})(z), \tag{1.12}$$

where

$$g_{\lambda\mu}(z) = z + \sum_{n=2}^{\infty} \Phi_n(\lambda, \mu, m) z^n, \quad z \in \mathbb{U}. \tag{1.13}$$

When $\lambda = 1$ and $\mu = 0$, we obtain the Sălăgean differential operator (see [15]); when $\mu = 0$, we get the differential operator defined by Al-Oboudi [2].

Making use of the operator $D_{\lambda\mu}^m$, we define the following class of functions.

Definition 1.1 *A function $f \in \mathcal{A}$ is said to be in the class $SH_{\lambda\mu}^m(\alpha)$, if $D_{\lambda\mu}^m f$ belongs to $SH(\alpha)$, that is*

$$\left|\frac{z(D_{\lambda\mu}^m f(z))'}{D_{\lambda\mu}^m f(z)} - 2\alpha(\sqrt{2}-1)\right| < \sqrt{2}\Re\left\{\frac{z(D_{\lambda\mu}^m f(z))'}{D_{\lambda\mu}^m f(z)}\right\} + 2\alpha(\sqrt{2}-1) \tag{1.14}$$

for some $\alpha > 0,\ 0 \leq \mu \leq \lambda,\ m \in \mathbb{N}_0 := \{0, 1, 2, \cdots\}$ and for all $z \in \mathbb{U}$.

When $m = 0$, we have $D_{\lambda\mu}^0 f = f$ and thus the class $SH_{\lambda\mu}^0(\alpha)$ reduces to the class $SH(\alpha)$. Since $SH(\alpha) \subset S^*$, it follows that if $f \in SH_{\lambda\mu}^m(\alpha)$, then $D_{\lambda\mu}^m f \in S^*$.

The main objective of this paper is to present a systematic investigation of the class $SH_{\lambda\mu}^m(\alpha)$. In particular, for this class of functions we obtain integral representations, coefficient bounds, class preserving operators, sharp estimates of the functional $|a_3 - \eta a_2^2|$ and T_δ-neighborhoods.

2 Integral Representations

In this section we provide integral representations for $D_{\lambda\mu}^m f$ and f, respectively.

Theorem 2.1 *Let $f \in SH_{\lambda\mu}^m(\alpha)$. Then*

$$D_{\lambda\mu}^m f(z) = z \exp\left\{\int_0^z \frac{P_\alpha(\omega(\zeta)) - 1}{\zeta} \mathrm{d}\zeta\right\}, \tag{2.1}$$

where ω is analytic with $\omega(0) = 0,\ |\omega(z)| < 1,\ z \in \mathbb{U}$, and P_α is given by (1.4).

Proof Suppose $f \in SH_{\lambda\mu}^m(\alpha)$. From Definition 1.1, we have

$$\frac{z(D_{\lambda\mu}^m f(z))'}{D_{\lambda\mu}^m f(z)} \prec P_\alpha(z), \quad z \in \mathbb{U}.$$

It follows that there exists an analytic function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1,\ z \in \mathbb{U}$ such that

$$\frac{z(D^m_{\lambda\mu}f(z))'}{D^m_{\lambda\mu}f(z)} = P_\alpha(\omega(z)), \quad z \in \mathbb{U}$$

or equivalently

$$\log\frac{D^m_{\lambda\mu}f(z)}{z} = \int_0^z \frac{P_\alpha(\omega(\zeta)) - 1}{\zeta}\mathrm{d}\zeta.$$

From the last equality, we get

$$D^m_{\lambda\mu}f(z) = z\exp\Big\{\int_0^z \frac{P_\alpha(\omega(\zeta)) - 1}{\zeta}\mathrm{d}\zeta\Big\},$$

and thus the proof is completed.

Making use of Theorem 2.1, (1.12) and (1.13), we obtain the next integral representation for a function in $SH^m_{\lambda\mu}(\alpha)$.

Corollary 2.1 *Let $f \in SH^m_{\lambda\mu}(\alpha)$. Then*

$$f(z) = h_{\lambda\mu}(z) * \Big[z\exp\Big\{\int_0^z \frac{P_\alpha(\omega(\zeta)) - 1}{\zeta}\mathrm{d}\zeta\Big\}\Big], \tag{2.2}$$

where ω is analytic with $\omega(0) = 0,\ |\omega(z)| < 1,\ z \in \mathbb{U}$, P_α is given by (1.4), *and $h_{\lambda\mu}$ is defined by*

$$h_{\lambda\mu}(z) = z + \sum_{n=2}^{\infty}\frac{z^n}{\Phi_n(\lambda,\mu,m)}. \tag{2.3}$$

Theorem 2.2 *Let $f \in SH^m_{\lambda\mu}(\alpha)$. Then*

$$D^m_{\lambda\mu}f(z) = z^{2\alpha(\sqrt{2}-1)}\exp\Big\{\int_X \log(1-\sqrt{2}xz)^{-\sqrt{2}\alpha}\mathrm{d}\mu(x)\Big\}, \tag{2.4}$$

where $\mu(x)$ is a probability measure on $X = \{x : |x| = 1\}$.

Proof Let $f \in SH^m_{\lambda\mu}(\alpha)$ and denote $w = \frac{z(D^m_{\lambda\mu}f(z))'}{D^m_{\lambda\mu}f(z)}$. Then we have

$$|w - 2\alpha(\sqrt{2}-1)| < \sqrt{2}\Re w + 2\alpha(\sqrt{2}-1).$$

Therefore

$$\Big|\frac{w - 2\alpha(\sqrt{2}-1)}{\sqrt{2}w + 2\alpha(\sqrt{2}-1)}\Big| < 1$$

or

$$\frac{w - 2\alpha(\sqrt{2}-1)}{\sqrt{2}w + 2\alpha(\sqrt{2}-1)} = xz$$

for $z \in \mathbb{U}$ and $x \in X = \{x : |x| = 1\}$. This yields

$$\frac{(D_{\lambda\mu}^m f(z))'}{D_{\lambda\mu}^m f(z)} = 2\alpha(\sqrt{2}-1)\frac{1+xz}{z(1-\sqrt{2}xz)},$$

and therefore

$$\log \frac{D_{\lambda\mu}^m f(z)}{z^{2\alpha(\sqrt{2}-1)}} = -\sqrt{2}\alpha \log(1-\sqrt{2}xz).$$

If $\mu(x)$ is a probability measure on X, then

$$D_{\lambda\mu}^m f(z) = z^{2\alpha(\sqrt{2}-1)} \exp\Big\{ \int_X \log(1-\sqrt{2}xz)^{-\sqrt{2}\alpha} \mathrm{d}\mu(x) \Big\},$$

and thus the proof is completed.

The next result follows from Theorem 2.2, (1.12) and (1.13).

Corollary 2.2 *Let $f \in SH_{\lambda\mu}^m(\alpha)$. Then*

$$f(z) = h_{\lambda\mu}(z) * \Big[z^{2\alpha(\sqrt{2}-1)} \exp\Big\{ \int_X \log(1-\sqrt{2}xz)^{-\sqrt{2}\alpha} \mathrm{d}\mu(x) \Big\} \Big], \tag{2.5}$$

where $\mu(x)$ is a probability measure on $X = \{x : |x| = 1\}$ and $h_{\lambda\mu}$ is given by (2.3).

3 Class-Preserving Operators

In order to prove the main result of this section, we need the following lemma due to Ruscheweyh and Sheil-Small.

Lemma 3.1 (see [13]) *Suppose $g \in K$ and $h \in S^*$. Then for any analytic function G in $\mathbb{U}$, we have*

$$\frac{(g * hG)(z)}{(g * h)(z)} \in \overline{\mathrm{co}}\, G(\mathbb{U}), \quad z \in \mathbb{U},$$

where $\overline{\mathrm{co}}\, G(\mathbb{U})$ is the closed convex hull of $G(\mathbb{U})$.

The next theorem shows that the class $SH_{\lambda\mu}^m(\alpha)$ is invariant under convolution with convex functions.

Theorem 3.1 *Let $f \in SH_{\lambda\mu}^m(\alpha)$ and $g \in K$. Then $g * f \in SH_{\lambda\mu}^m(\alpha)$.*

Proof Suppose $f \in SH_{\lambda\mu}^m(\alpha)$. Then

$$\frac{z(D_{\lambda\mu}^m f(z))'}{D_{\lambda\mu}^m f(z)} \prec P_\alpha(z), \quad z \in \mathbb{U}$$

and $D_{\lambda\mu}^m f(z) \in S^*$. Let $g \in K$. We have

$$\frac{z(D_{\lambda\mu}^m (f*g)(z))'}{D_{\lambda\mu}^m (f*g)(z)} = \frac{g(z) * z(D_{\lambda\mu}^m f(z))'}{g(z) * D_{\lambda\mu}^m f(z)} = \frac{g(z) * \Big(\frac{z(D_{\lambda\mu}^m f(z))'}{D_{\lambda\mu}^m f(z)} \Big) D_{\lambda\mu}^m f(z)}{g(z) * D_{\lambda\mu}^m f(z)}.$$

Since $g \in K$, $D^m_{\lambda\mu} f(z) \in S^*$ and $\Omega(\alpha)$ is convex, it follows from Lemma 3.1 that

$$\frac{z(D^m_{\lambda\mu}(f*g)(z))'}{D^m_{\lambda\mu}(f*g)(z)} \prec P_\alpha(z), \quad z \in \mathbb{U}.$$

Thus, $g * f \in SH^m_{\lambda\mu}(\alpha)$ and the proof of our theorem is completed.

Consider

$$g_1(z) = -\log(1-z), \quad \log 1 = 0,$$
$$g_2(z) = -2\Big[\frac{z + \log(1+z)}{z}\Big]$$

and

$$g_3(z) = \sum_{n=1}^{\infty} \frac{\gamma+1}{\gamma+n} z^n, \quad \Re\gamma > 0.$$

Note that the convolutions

$$(f*g_1)(z) = \int_0^z \frac{f(t)}{t}\mathrm{d}t,$$
$$(f*g_2)(z) = \frac{2}{z}\int_0^z f(t)\mathrm{d}t$$

and

$$(f*g_3)(z) = \frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1} f(t)\mathrm{d}t$$

are the familiar Alexander, Libera and Bernardi operators, respectively.

Corollary 3.1 *If $f \in SH^m_{\lambda\mu}(\alpha)$, then $f * g_i \in SH^m_{\lambda\mu}(\alpha)$ for each $i = 1, 2, 3$.*

Proof It is well-known that the functions g_1, g_2, g_3 are convex (see for example [4]). Thus, the proof of the corollary follows as an application of Theorem 3.1.

4 Coefficient Bounds

Let $f_{\lambda\mu\alpha}(z)$ be defined by

$$f_{\lambda\mu\alpha}(z) = (h_{\lambda\mu} * F_\alpha)(z), \quad z \in \mathbb{U}, \tag{4.1}$$

where the functions F_α and $h_{\lambda\mu}$ are given by (1.7) and (2.3), respectively. It is easy to check that

$$\frac{z(D^m_{\lambda\mu} f_{\lambda\mu\alpha}(z))'}{D^m_{\lambda\mu} f_{\lambda\mu\alpha}(z)} = P_\alpha(z), \quad z \in \mathbb{U}.$$

Thus, the function $f_{\lambda\mu\alpha}(z)$ is the extremal function in the class $SH^m_{\lambda\mu}(\alpha)$.

Taking into account the relation between the extremal functions in the classes $\mathcal{P}(P_\alpha)$ and $SH^m_{\lambda\mu}(\alpha)$ and in view of (1.10), for $f_{\lambda\mu\alpha}(z) = z + A_2 z^2 + A_3 z^3 + \cdots$ and $P_\alpha(z) = 1 + B_1 z + B_2 z^2 + \cdots$ we have the following coefficient relation

$$(n-1)\Phi_n(\lambda,\mu,m)A_n = \sum_{k=1}^{n-1} \Phi_k(\lambda,\mu,m) A_k B_{n-k}, \quad A_1 = 1,\ n \ge 2. \tag{4.2}$$

In particular, by straightforward computation, we obtain

$$A_2 = \frac{B_1}{\Phi_2(\lambda,\mu,m)} \tag{4.3}$$

and

$$A_3 = \frac{B_2 + B_1^2}{2\Phi_3(\lambda,\mu,m)}, \tag{4.4}$$

where coefficients B_1 and B_2 are given by (1.6).

Note that the coefficients A_n and B_n are nonnegative.

Theorem 4.1 *Let f given by* (1.1) *be in $SH_{\lambda\mu}^m(\alpha)$. Then*

$$|a_2| \le A_2, \quad |a_3| \le A_3. \tag{4.5}$$

Proof Assume $f \in SH_{\lambda\mu}^m(\alpha)$. Let $p(z) = \frac{z(D_{\lambda\mu}^m f(z))'}{D_{\lambda\mu}^m f(z)} = 1 + p_1 z + p_2 z^2 + \cdots$. From the relation between f and p, we have

$$(n-1)\Phi_n(\lambda,\mu,m)a_n = \sum_{k=1}^{n-1} \Phi_k(\lambda,\mu,m)a_k p_{n-k}, \quad a_1 = 1,\ n \ge 2. \tag{4.6}$$

Since P_α is univalent, the function

$$q(z) = \frac{1 + P_\alpha^{-1}(p(z))}{1 - P_\alpha^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \cdots$$

is analytic in $\mathbb{U}$ and $\Re q(z) > 0,\ z \in \mathbb{U}$. Equivalently, we can write

$$p(z) = P_\alpha\Big(\frac{q(z)-1}{q(z)+1}\Big) = 1 + \frac{1}{2}c_1 B_1 z + \Big[\frac{1}{2}c_2 B_1 + \frac{1}{4}c_1^2(B_2 - B_1)\Big]z^2 + \cdots.$$

In particular,

$$p_1 = \frac{1}{2}c_1 B_1, \quad p_2 = \frac{1}{2}c_2 B_1 + \frac{1}{4}c_1^2(B_2 - B_1). \tag{4.7}$$

From (4.6) we have

$$a_2 = \frac{p_1}{\Phi_2(\lambda,\mu,m)} \tag{4.8}$$

and

$$a_3 = \frac{p_2 + p_1^2}{2\Phi_3(\lambda,\mu,m)}. \tag{4.9}$$

Making use of (4.3) and (4.7)–(4.8), we obtain

$$|a_2| = \frac{|c_1|}{2}\frac{B_1}{\Phi_2(\lambda,\mu,m)} = \frac{|c_1|}{2}A_2 \le A_2,$$

where we have used the inequality $|c_n| \le 2,\ n \ge 1$. By virtue of the relation $|p_1|^2 + |p_2| \le B_1^2 + B_2$ (see [19]), (4.4) and (4.9), we have

$$|a_3| \le \frac{|p_2| + |p_1|^2}{2\Phi_3(\lambda,\mu,m)} \le \frac{B_2 + B_1^2}{2\Phi_3(\lambda,\mu,m)} = A_3.$$

Thus, the proof is completed.

Theorem 4.2 *Let f of the form* (1.1) *be in the class $SH^m_{\lambda\mu}(\alpha)$. Then*

$$|a_n| \le \frac{(B_1)_{n-1}}{(n-1)!\Phi_n(\lambda,\mu,m)}, \quad n \ge 2, \tag{4.10}$$

where $(\tau)_n$ is the Pochhammer symbol, and $\Phi_n(\lambda,\mu,m)$ is given by (1.11).

Proof In view of Theorem 4.1, the result is true for $n=2$. Assume that the inequality (4.10) is true for all integers $k \le n-1,\ n \ge 2$. Making use of (4.6), we have

$$\begin{aligned}|a_n| &= \Big|\frac{1}{(n-1)\Phi_n(\lambda,\mu,m)}\sum_{k=1}^{n-1}\Phi_k(\lambda,\mu,m)a_k p_{n-k}\Big| \\ &\le \frac{1}{(n-1)\Phi_n(\lambda,\mu,m)}\sum_{k=1}^{n-1}\Phi_k(\lambda,\mu,m)\frac{(B_1)_{k-1}}{(k-1)!\Phi_k(\lambda,\mu,m)}B_1 \\ &= \frac{B_1}{(n-1)\Phi_n(\lambda,\mu,m)}\Big[1+\sum_{k=2}^{n-1}\frac{(B_1)_{k-1}}{(k-1)!}\Big],\end{aligned}$$

where we have applied the induction hypothesis to $|a_k|$ and the Rogosinski result $|p_j| \le B_1$ (see [12]). To complete the proof of the theorem, it sufficies to show that

$$\frac{B_1}{(n-1)}\Big[1+\sum_{k=2}^{n-1}\frac{(B_1)_{k-1}}{(k-1)!}\Big] = \frac{(B_1)_{n-1}}{(n-1)!}$$

or

$$1+\sum_{k=2}^{n-1}\frac{(B_1)_{k-1}}{(k-1)!} = \frac{(B_1+1)_{n-2}}{(n-2)!}. \tag{4.11}$$

Equality (4.11) follows from the sequence of calculations listed below

$$\begin{aligned}&1+\sum_{k=2}^{n-1}\frac{(B_1)_{k-1}}{(k-1)!} \\ &= \frac{1}{(n-2)!}\Big[(n-2)!+(n-2)!B_1+\frac{(n-2)!}{2}B_1(B_1+1)+\cdots+B_1(B_1+1)\cdots(B_1+n-3)\Big] \\ &= \frac{B_1+1}{(n-2)!}\Big[(n-2)!+\frac{(n-2)!}{2}B_1+\frac{(n-2)!}{3!}B_1(B_1+2)+\cdots+B_1(B_1+2)\cdots(B_1+n-3)\Big] \\ &= \frac{(B_1+1)(B_1+2)}{(n-2)!}\Big[\frac{(n-2)!}{2}+\frac{(n-2)!}{3!}B_1+\cdots+B_1(B_1+3)\cdots(B_1+n-3)\Big] \\ &= \cdots = \frac{(B_1+1)_{n-2}}{(n-2)!}\end{aligned}$$

as asserted in (4.11).

5 The Fekete-Szegö Problem

During the time, many authors have considered the problem of finding sharp upper bounds for the functional $|a_3-\eta a_2^2|$ for different subclasses of $\mathcal{A}$ (see, for instance [7–10]).

In this section we consider the Fekete-Szegö problem for functions in the class $SH^m_{\lambda\mu}(\alpha)$.

For the class $\mathcal{P}$ of Carathéodory functions, the next result is well-known.

Lemma 5.1 (see [6]) *Let p be a function in the class $\mathcal{P}$. If $p(z) = 1+p_1z+p_2z^2+\cdots$, $z \in \mathbb{U}$, then for $-\infty < u < \infty$*

$$|p_2 - up_1^2| \leq \begin{cases} 2+(u-1)|p_1|^2, & u > \dfrac{1}{2}, \\ 2-\dfrac{1}{2}|p_1|^2, & u = \dfrac{1}{2}, \\ 2-u|p_1|^2, & u < \dfrac{1}{2}. \end{cases} \tag{5.1}$$

Similar estimates with (5.1) for a subclass of Carathóodory functions defined by conical domains were obtained by Kanas [7] and more recently by Mishra and Gochhayat [9].

A coefficient inequality for the subclass $\mathcal{P}(P_\alpha)$ is given in the following theorem.

Theorem 5.1 *Let $\alpha > 0$ be fixed and let $P_\alpha(z) = 1 + B_1z + B_2z^2 + \cdots$ be defined by (1.4). If the function $p(z) = 1 + p_1z + p_2z^2 + \cdots$ is a member of $\mathcal{P}(P_\alpha)$, then for $-\infty < u < \infty$*

$$|p_2 - up_1^2| \leq \begin{cases} uB_1^2 - B_2, & u > \delta_1, \\ B_1, & \delta_2 \leq u \leq \delta_1, \\ B_2 - uB_1^2, & u < \delta_2, \end{cases} \tag{5.2}$$

where

$$\delta_1 = \frac{B_1 + B_2}{B_1^2}, \quad \delta_2 = \frac{B_2 - B_1}{B_1^2}, \tag{5.3}$$

and B_1, B_2 are given by (1.6). All estimates in (5.2) are sharp.

Since the proof is similar to the proof of Theorem 3.1 in [9], we omit it.

Theorem 5.1 enables us to obtain a short and direct proof of the Fekete-Szegö inequalities for the class $SH^m_{\lambda\mu}(\alpha)$.

Theorem 5.2 *Let f given by (1.1) be in the class $SH^m_{\lambda\mu}(\alpha)$. Then*

$$|a_3 - \eta a_2^2| \leq \begin{cases} \dfrac{(1+4\alpha)^2}{(1+2\alpha)^2\Phi_3(\lambda,\mu,m)}\Big[\dfrac{\Phi_3(\lambda,\mu,m)}{\Phi_2^2(\lambda,\mu,m)}\eta - \dfrac{1+4\alpha}{4(1+2\alpha)} - \dfrac{1+2\alpha}{2(1+4\alpha)}\Big], & \eta > \delta_1(\alpha,\lambda,\mu,m), \\ \dfrac{1+4\alpha}{2(1+2\alpha)\Phi_3(\lambda,\mu,m)}, \quad \delta_2(\alpha,\lambda,\mu,m) \leq \eta \leq \delta_1(\alpha,\lambda,\mu,m), \\ \dfrac{(1+4\alpha)^2}{(1+2\alpha)^2\Phi_3(\lambda,\mu,m)}\Big[\dfrac{1+4\alpha}{4(1+2\alpha)} + \dfrac{1+2\alpha}{2(1+4\alpha)} - \dfrac{\Phi_3(\lambda,\mu,m)}{\Phi_2^2(\lambda,\mu,m)}\eta\Big], & \eta < \delta_2(\alpha,\lambda,\mu,m), \end{cases}$$

where

$$\delta_1(\alpha,\lambda,\mu,m) = \frac{\Phi_2^2(\lambda,\mu,m)}{\Phi_3(\lambda,\mu,m)}\Big[\frac{1+4\alpha}{4(1+2\alpha)} + \frac{1+2\alpha}{1+4\alpha}\Big],$$
$$\delta_2(\alpha,\lambda,\mu,m) = \frac{\Phi_2^2(\lambda,\mu,m)}{\Phi_3(\lambda,\mu,m)}\frac{1+4\alpha}{4(1+2\alpha)},$$

and $\Phi_2(\lambda,\mu,m), \Phi_3(\lambda,\mu,m)$ are given by (1.11) with $n = 2$ and $n = 3$, respectively. All estimates are sharp.

Proof Assume $f \in SH_{\lambda\mu}^m(\alpha)$ and let

$$p(z) = \frac{z(D_{\lambda\mu}^m f(z))'}{D_{\lambda\mu}^m f(z)} = 1 + p_1 z + p_2 z^2 + \cdots . \tag{5.4}$$

Then $p(z) \prec P_\alpha(z)$ and thus $p \in \mathcal{P}(P_\alpha)$. Equating the coefficients of z and z^2 in (5.4), we obtain

$$a_2 = \frac{p_1}{\Phi_2(\lambda,\mu,m)}, \quad a_3 = \frac{p_2 + p_1^2}{2\Phi_3(\lambda,\mu,m)}.$$

We have

$$|a_3 - \eta a_2^2| = \frac{1}{2\Phi_3(\lambda,\mu,m)}\Big|p_2 - \Big(\frac{2\Phi_3(\lambda,\mu,m)}{\Phi_2^2(\lambda,\mu,m)}\eta - 1\Big)p_1^2\Big|.$$

Making use of Theorem 5.1 with

$$u = \frac{2\Phi_3(\lambda,\mu,m)}{\Phi_2^2(\lambda,\mu,m)}\eta - 1,$$

we obtain the desired result.

Remark 5.1 For $m = 0$, our Theorem 5.2 yields the Fekete-Szegö inequalities for the class $SH(\alpha)$ found in [19].

6 T_δ-Neighborhoods

In this section we investigate the T_δ-neighborhoods of functions in $SH_{\lambda\mu}^m(\alpha)$.

Given a sequence $T = \{T_n\}_{n=2}^\infty$ consisting of positive real numbers, the T_δ-neighborhoods $(\delta > 0)$ of a function f given by (1.1) is defined by

$$T_\delta(f) = \Big\{g \in \mathcal{A};\ g(z) = z + \sum_{n=2}^\infty b_n z^n : \sum_{n=2}^\infty T_n |a_n - b_n| \le \delta\Big\}. \tag{6.1}$$

The notion of the T_δ-neighborhoods was introduced in [16].

Note that if $T = \{n\}_{n=2}^\infty$, then the T_δ-neighborhood becomes the δ-neighborhoods $N_\delta(f)$ introduced by Ruscheweyh in [14].

The problem of δ-neighborhoods or T_δ-neighborhoods for typical subclasses of $\mathcal{A}$ was studied by many authors (see [1, 3, 5, 18]).

In the following theorem, a necessary and sufficient condition for a function $f \in \mathcal{A}$ to be in the class $SH_{\lambda\mu}^m(\alpha)$ is given.

Theorem 6.1 *Let $0 \le \mu \le \lambda$, $m \in \mathbb{N}$ and $\alpha > 0$. Then a function $f \in \mathcal{A}$ belongs to the class $SH_{\lambda\mu}^m(\alpha)$ if and only if $\frac{(f * H_{\lambda\mu})(z)}{z} \ne 0$ in $\mathbb{U}$, where*

$$H_{\lambda\mu}(z) = (g_{\lambda\mu} * h)(z) \tag{6.2}$$

with $g_{\lambda\mu}(z)$ given by (1.13),

$$h(z) = \frac{z}{(1-z)^2}\Big[1 + \frac{w(t)z}{1 - w(t)}\Big] \tag{6.3}$$

and

$$w(t) = t \pm \mathrm{i}\sqrt{t^2 + \alpha t}, \quad t > 0. \tag{6.4}$$

Proof Suppose $f \in SH^m_{\lambda\mu}(\alpha)$. Then from (1.14) we have that the values of $\frac{z(D^m_{\lambda\mu}f(z))'}{D^m_{\lambda\mu}f(z)}$ lie in the domain $\Omega(\alpha)$ defined by (1.3). Therefore

$$\frac{z(D^m_{\lambda\mu}f(z))'}{D^m_{\lambda\mu}f(z)} \neq t \pm \mathrm{i}\sqrt{t^2 + \alpha t} = w(t) \tag{6.5}$$

with $z \in \mathbb{U}$ and $t > 0$. Applying the properties of the Hadamard product and (1.12), the condition (6.5) will hold if

$$f(z) * \frac{zg'_{\lambda\mu}(z) - w(t)g_{\lambda\mu}(z)}{z(1 - w(t))} \neq 0. \tag{6.6}$$

Making use of (1.13) it follows from (6.6) that $\frac{(f*H_{\lambda\mu})(z)}{z} \neq 0$, where $H_{\lambda\mu}$ is given by (6.2).

Conversely, if $\frac{(f*H_{\lambda\mu})(z)}{z} \neq 0$ in $\mathbb{U}$, then the values of $\frac{z(D^m_{\lambda\mu}f(z))'}{D^m_{\lambda\mu}f(z)}$ lie completely inside $\Omega(\alpha)$ or its complement. Since

$$\frac{z(D^m_{\lambda\mu}f(z))'}{D^m_{\lambda\mu}f(z)}\Big|_{z=0} = 1 \in \Omega(\alpha),$$

we obtain $\frac{z(D^m_{\lambda\mu}f(z))'}{D^m_{\lambda\mu}f(z)} \in \Omega(\alpha)$ which shows that $f \in SH^m_{\lambda\mu}(\alpha)$.

Theorem 6.2 *The coefficients h_n of the function $H_{\lambda\mu}$ given by* (6.2) *satisfy the inequality*

$$|h_n| \leq \begin{cases} \dfrac{n\Phi_n(\lambda,\mu,m)}{\sqrt{2\alpha(1-\alpha)}}, & 0 < \alpha < \dfrac{1}{2}, \\ n\Phi_n(\lambda,\mu,m), & \alpha \geq \dfrac{1}{2}, \end{cases}$$

where $\Phi_n(\lambda,\mu,m)$, $n \geq 2$ is given by (1.11).

Proof From the Taylor expansion of the function $H_{\lambda\mu}$, we have

$$h_n = \Phi_n(\lambda,\mu,m)\frac{n - w(t)}{1 - w(t)}$$

and therefore

$$|h_n|^2 = [\Phi_n(\lambda,\mu,m)]^2\left|\frac{n - w(t)}{1 - w(t)}\right|^2 = [\Phi_n(\lambda,\mu,m)]^2 V(t),$$

where

$$V(t) = \frac{2t^2 + 2t(2\alpha - n) + n^2}{2t^2 + 2t(2\alpha - 1) + 1}.$$

We have

$$V(t) = 1 + \frac{(n-1)(n+1-2t)}{2t^2 + 2t(2\alpha-1) + 1} \leq 1 + \frac{n^2 - 1}{2t^2 + 2t(2\alpha - 1) + 1}.$$

It is easy to see that

$$2t^2 + 2t(2\alpha - 1) + 1 \geq \begin{cases} 2\alpha(1-\alpha), & 0 < \alpha < \dfrac{1}{2}, \\ 1, & \alpha \geq \dfrac{1}{2}. \end{cases}$$

Therefore, for $0 < \alpha < \frac{1}{2}$, we have

$$|h_n|^2 \le [\Phi_n(\lambda,\mu,m)]^2\Big[1 + \frac{n^2-1}{2\alpha(1-\alpha)}\Big] \le [\Phi_n(\lambda,\mu,m)]^2 \frac{n^2}{2\alpha(1-\alpha)},$$

and for $\alpha \ge \frac{1}{2}$,

$$|h_n|^2 \le [\Phi_n(\lambda,\mu,m)]^2(1+n^2-1) = [\Phi_n(\lambda,\mu,m)]^2 n^2.$$

Thus, the proof of our theorem is completed.

Corollary 6.1 *Let $f(z) = z + az^n$. If*

$$|a| \le \begin{cases} \dfrac{\sqrt{2\alpha(1-\alpha)}}{n\Phi_n(\lambda,\mu,m)}, & 0 < \alpha < \dfrac{1}{2}, \\ \dfrac{1}{n\Phi_n(\lambda,\mu,m)}, & \alpha \ge \dfrac{1}{2}, \end{cases}$$

then $f \in SH^m_{\lambda\mu}(\alpha)$.

Proof Since

$$\Big|\frac{(f * H_{\lambda\mu})(z)}{z}\Big| = |1 + ah_n z^{n-1}| \ge 1 - |h_n||a||z| \ge 1 - |z| > 0, \quad z \in \mathbb{U},$$

it folllows that $f \in SH^m_{\lambda\mu}(\alpha)$.

In order to establish the T_δ-neighborhoods of functions belonging to the class $SH^m_{\lambda\mu}(\alpha)$, we need the following lemma.

Lemma 6.1 *Let $f \in \mathcal{A}$ and $\epsilon \in \mathbb{C}$ with $|\epsilon| < \gamma$ for some $\gamma > 0$. If the function*

$$F_\epsilon(z) = \frac{f(z) + \epsilon z}{1+\epsilon}, \quad z \in \mathbb{U} \tag{6.7}$$

belongs to the class $SH^m_{\lambda\mu}(\alpha)$, then

$$\Big|\frac{1}{z}(f * H_{\lambda\mu})(z)\Big| \ge \gamma, \quad z \in \mathbb{U},$$

where $H_{\lambda\mu}$ is given by (6.2).

Proof Assume $F_\epsilon(z) \in SH^m_{\lambda\mu}(\alpha)$. Then by Theorem 6.1 it follows

$$\frac{1}{z}(F_\epsilon * H_{\lambda\mu})(z) \ne 0, \quad z \in \mathbb{U}.$$

Equivalently

$$\frac{(f * H_{\lambda\mu})(z) + \epsilon z}{(1+\epsilon)z} \ne 0 \quad \text{or} \quad \frac{(f * H_{\lambda\mu})(z)}{z} \ne -\epsilon.$$

Since $|\epsilon| < \gamma$, it easily follows that

$$\Big|\frac{1}{z}(f * H_{\lambda\mu})(z)\Big| \ge \gamma$$

and thus, the proof is completed.

In the sequence, we give the definition of the T_δ-neighborhoods of a function in $f \in SH^m_{\lambda\mu}(\alpha)$ of the form (1.1).

Let $\delta > 0$ and let the sequence $\{T_n\}_{n=2}^{\infty}$ of positive real numbers be defined by

$$T_n := n\Phi_n(\lambda, \mu, m), \quad n \geq 2,\ 0 \leq \mu \leq \lambda,\ m \in \mathbb{N}.$$

Then the T_δ-neighborhoods defined in (6.1) becomes

$$T_\delta(f) = \Big\{ g \in \mathcal{A}; g(z) = z + \sum_{n=2}^{\infty} b_n z^n : \sum_{n=2}^{\infty} n\Phi_n(\lambda, \mu, m)|b_n - a_n| \leq \delta \Big\}.$$

We have the following result on the T_δ-neighborhoods for the class $SH^m_{\lambda\mu}(\alpha)$.

Theorem 6.3 *Let $f \in \mathcal{A}$ and $\gamma > 0$. For $\epsilon \in \mathbb{C}$ with $|\epsilon| < \gamma$, let the function F_ϵ defined by* (6.7) *be in $SH^m_{\lambda\mu}(\alpha)$. Then $T_\delta(f) \subset SH^m_{\lambda\mu}(\alpha)$ for*

$$\delta := \begin{cases} \gamma\sqrt{2\alpha(1-\alpha)}, & 0 < \alpha < \dfrac{1}{2}\ , \\ \gamma, & \alpha \geq \dfrac{1}{2}. \end{cases}$$

Proof Let $g \in \mathcal{A}$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be in $T_\delta(f)$. Then

$$\begin{aligned} \Big|\frac{1}{z}(g * H_{\lambda\mu})(z)\Big| &= \Big|\frac{1}{z}(f * H_{\lambda\mu})(z) + \frac{1}{z}((g-f) * H_{\lambda\mu})(z)\Big| \\ &\geq \Big|\frac{1}{z}(f * H_{\lambda\mu})(z)\Big| - \Big|\frac{1}{z}((g-f) * H_{\lambda\mu})(z)\Big|. \end{aligned}$$

Making use of Lemma 6.1, we obtain

$$\begin{aligned} \Big|\frac{1}{z}(g * H_{\lambda\mu})(z)\Big| &\geq \gamma - \Big|\sum_{n=2}^{\infty} \frac{(b_n - a_n)h_n z^n}{z}\Big| \\ &> \gamma - \sum_{n=2}^{\infty} |b_n - a_n||h_n|. \end{aligned}$$

We have

$$\sum_{n=2}^{\infty} |b_n - a_n||h_n| \leq \begin{cases} \dfrac{1}{\sqrt{2\alpha(1-\alpha)}} \displaystyle\sum_{n=2}^{\infty} n\Phi_n(\lambda, \mu, m)|b_n - a_n|, & 0 < \alpha < \dfrac{1}{2}, \\ \displaystyle\sum_{n=2}^{\infty} n\Phi_n(\lambda, \mu, m)|b_n - a_n|, & \alpha \geq \dfrac{1}{2}. \end{cases}$$

Since $g \in T_\delta(f)$, it follows that

$$\sum_{n=2}^{\infty} n\Phi_n(\lambda, \mu, m)|b_n - a_n| \leq \delta$$

and thus

$$\Big|\frac{1}{z}(g * H_{\lambda\mu})(z)\Big| > \gamma - \gamma = 0.$$

Therefore, $\big|\frac{1}{z}(g * H_{\lambda\mu})(z)\big| \neq 0$. In virtue of Theorem 6.1, we obtain $g \in SH^m_{\lambda\mu}(\alpha)$ which proves that $T_\delta(f) \subset SH^m_{\lambda\mu}(\alpha)$.

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