*Chin. Ann. Math.*
34B(4), 2013, 529–540
DOI: 10.1007/s11401-013-0782-z

**Chinese Annals of Mathematics, Series B**


# The Asymptotic Behavior and the Quasineutral Limit for the Bipolar Euler-Poisson System with Boundary Effects and a Vacuum*

Yeping LI[1]

**Abstract** In this paper, a one-dimensional bipolar Euler-Poisson system (a hydrodynamic model) from semiconductors or plasmas with boundary effects is considered. This system takes the form of Euler-Poisson with an electric field and frictional damping added to the momentum equations. The large-time behavior of uniformly bounded weak solutions to the initial-boundary value problem for the one-dimensional bipolar Euler-Poisson system is firstly presented. Next, two particle densities and the corresponding current momenta are verified to satisfy the porous medium equation and the classical Darcy's law time asymptotically. Finally, as a by-product, the quasineutral limit of the weak solutions to the initial-boundary value problem is investigated in the sense that the bounded $L^\infty$ entropy solution to the one-dimensional bipolar Euler-Poisson system converges to that of the corresponding one-dimensional compressible Euler equations with damping exponentially fast as $t \to +\infty$. As far as we know, this is the first result about the asymptotic behavior and the quasineutral limit for the one-dimensional bipolar Euler-Poisson system with boundary effects and a vacuum.

**Keywords** Bipolar hydrodynamic model, Asymptotic behavior, Quasineutral limit, Entropy, Energy estimate
**2000 MR Subject Classification** 35M20, 35Q35, 76W05

## 1 Introduction

In this paper, we consider a bipolar hydrodynamic model in a one-dimensional space. Denoting by $n_i, j_i, P(n_i), i = 1, 2$, and $E$ the charge densities, current densities, pressures and the electric field, the scaled equations of a one-dimensional bipolar hydrodynamic model (cf. [10, 12, 16]) are given by

$$\begin{cases} n_{1t} + j_{1x} = 0, \\ j_{1t} + \Big(\dfrac{j_1^2}{n_1} + P(n_1)\Big)_x = n_1 E - \dfrac{j_1}{\tau}, \\ n_{2t} + j_{2x} = 0, \\ j_{2t} + \Big(\dfrac{j_2^2}{n_2} + P(n_2)\Big)_x = -n_2 E - \dfrac{j_2}{\tau}, \\ \lambda^2 E_x = n_1 - n_2. \end{cases} \tag{1.1}$$

Manuscript received March 22, 2012. Revised August 8, 2012.
[1]Department of Mathematics, Shanghai Normal University, Shanghai 200234, China.
E-mail: ypleemei@yahoo.com.cn
*Project supported by the National Natural Science Foundation of China (No. 11171223) and the Innovation Program of Shanghai Municipal Education Commission (No. 13ZZ109).

The positive constants $\tau$ and $\lambda$ denote the relaxation time and the Debye length, respectively. The relaxation terms describe the damping effect in a very rough manner. The Debye length is related to the Coulomb screening of the charged particles.

Recently, many efforts have been made for the one-dimensional bipolar hydrodynamic equations from semiconductors or plasmas. More precisely, Zhou-Li [19] and Tsuge [17] discussed the unique existence of the stationary solutions to the one-dimensional bipolar hydrodynamic model with proper boundary value conditions, respectively. Natalini [13] and Hsiao-Zhang [5] established the global entropy weak solutions in the framework of compensated compactness on the whole real line and the spatial bounded domain, respectively. Hattori and Zhu [20] proved the stability of steady-state solutions to a recombined one-dimensional bipolar hydrodynamical model. Gasser-Hsiao-Li [3] investigated the large-time behavior of smooth "small" solutions to the initial value problem for the one-dimensional bipolar hydrodynamic model, and they found that the frictional damping is the key to the nonlinear diffusive phenomena of hyperbolic waves. Li [11] studied the similar results for the initial-boundary problem of the one-dimensional bipolar hydrodynamic model in the quarter plane. Huang-Mei-Wang [7] discussed the large-time behavior of the solution to the bipolar hydrodynamic model for semiconductors with the switch-on case. Lastly, Huang and Li studied the large-time behavior and the quasi-neutral limit of $L^\infty$ solution to the Cauchy problem with a vacuum and large data in [6]. As far as we know, no result on the large-time behavior of weak solutions to the initial-boundary value problem of the bipolar Euler-Poisson system can be found.

In [3–4], we have known that the re-scaled Debye limit and the relaxation limit of the bipolar hydrodynamic model lead to a diffusive approximate model. Let us start with the (pure) quasineutral limit $\lambda \to 0$ for fixed $\tau$. It is obvious that the Poisson equation requires the limits of $n_1$ and $n_2$ to be equal, and we denote them by $m$. Taking the difference of the two continuity equations and assuming the current densities to be equal at $x = -\infty$ (or $x = +\infty$), one concludes the same limit $j$ of the current densities $j_1$ and $j_2$. Moreover, the sum of the momentum equations implies the elimination of the electric field. Therefore, we expect that the system (1.1) converges to

$$\begin{cases} m_t + j_x = 0, \\ j_t + \Big(\dfrac{j^2}{m} + (P(m)\Big)_x = -\dfrac{j}{\tau}. \end{cases} \tag{1.2}$$

The justification of the formal limit is an open challenging problem. The mathematically rigorous results regarding this problem only concern the local smooth solution (cf. [2, 15, 18]) (i.e., up to the breakdown of classical solutions in time).

In (1.2), one can perform a relaxation limit, i.e., rescaling

$$s = \tau t, \quad m^\tau = m\Big(\frac{s}{\tau}, x\Big), \quad j^\tau = \frac{1}{\tau} j\Big(\frac{s}{\tau}, x\Big),$$

we have

$$\begin{cases} m^\tau_s + j^\tau_x = 0, \\ \tau^2 j^\tau_s + \Big(\tau^2 \dfrac{j^{\tau 2}}{m^\tau} + (P(m^\tau))\Big)_x = -j^\tau, \end{cases} \tag{1.3}$$

and letting $\tau \to 0$, it is easy to see that the limit problem for the limit $n$ of $m^\tau$ is

$$n_s = (P(n))_{xx}. \tag{1.4}$$

On the other hand, if we let $\tau \to 0$ and let $\lambda \to 0$, then we can also obtain (1.4) from (1.1) at least at a formal level. From the above procedures, we can directly apply the re-scaling Debye limit and the relaxation limit as follows. Assuming in (1.1) $\tau_1 = \tau_2 = \tau,\ \lambda^2 = \tau^{1+\alpha}$ with $\alpha > -1$ and rescaling

$$\begin{aligned} &t \to \tau t, \\ &n_i^\tau = n_i\Big(\frac{t}{\tau}, x\Big), \quad j_i^\tau = \frac{1}{\tau} j_i\Big(\frac{t}{\tau}, x\Big), \quad i = 1, 2, \\ &E^\tau = E\Big(\frac{t}{\tau}, x\Big), \end{aligned}$$

we have

$$\begin{cases} n_{1s}^\tau + j_{1x}^\tau = 0, \\ \tau^2 j_{1s}^\tau + \Big(\tau^2 \frac{j_1^{\tau 2}}{n_1^\tau} + P(n_1^\tau)\Big)_x = n_1^\tau E^\tau - j_1^\tau, \\ n_{2s}^\tau + j_{2x}^\tau = 0, \\ \tau^2 j_{2s}^\tau + \Big(\tau^2 \frac{j_2^{\tau 2}}{n_2^\tau} + P(n_2^\tau)\Big)_x = -n_2^\tau E^\tau - j_2^\tau, \\ \tau^{1+\alpha} E_x^\tau = n_1^\tau - n_2^\tau. \end{cases} \tag{1.5}$$

Note that this is a long-time scaling. Formally when $\tau \to 0$ for $\alpha > -1$, the Poisson equation implies that the limit $n$ for $n_1^\tau$ is equal to that for $n_2^\tau$. Then, we get the following porous media equation (1.4) for $n$.

In this paper, we first study the large-time behavior of uniformly bounded weak solutions to the initial-boundary value problem for the one-dimensional bipolar Euler-Poisson system (1.1). Next, we show that two particle densities and the corresponding current momenta satisfy the porous medium equation and the classical Darcy's law time asymptotically. Finally, as a by-product, we present the quasineutral limit of the weak solutions to the initial-boundary value problem in the sense that the bounded $L^\infty$ entropy solution of the one-dimensional bipolar Euler-Poisson system converges to that of the corresponding one-dimensional compressible Euler equations with damping exponentially fast as $t \to +\infty$. To begin with, we assume in the present paper that the pressure-density functions satisfy

$$P(n_i) = n_i^\gamma \quad i = 1, 2, \quad 1 < \gamma < 3,$$

and we set $\tau$ and $\lambda$ as one for simplicity. Hence, the system (1.1) is simplified as

$$\begin{cases} n_{1t} + j_{1x} = 0, \\ j_{1t} + \Big(\dfrac{j_1^2}{n_1} + P(n_1)\Big)_x = n_1 E - j_1, \\ n_{2t} + j_{2x} = 0, \\ j_{2t} + \Big(\dfrac{j_2^2}{n_2} + P(n_2)\Big)_x = -n_2 E - j_2, \\ E_x = n_1 - n_2 \end{cases} \tag{1.6}$$

for $x \in (0, 1),\ t \in (0, \infty)$. The initial conditions are prescribed as

$$n_i(0, x) = n_{i0}(x) \geq 0, \quad j_i(0, x) = j_{i0}(x), \quad 0 < x < 1, \tag{1.7}$$

satisfying

$$\int_0^1 n_{10}(x)\mathrm{d}x = n_* = \int_0^1 n_{20}(x)\mathrm{d}x, \tag{1.8}$$

where $n_*$ is a given positive constant, and (1.8) can avoid the trivial case $n_1 = n_2 = 0$. The boundary conditions for $j_i$ $(i = 1, 2)$ and $E$ are

$$j_i(0, t) = 0 = j_i(1, t) \tag{1.9}$$

and

$$E(0, t) = 0 \quad \text{a.e. } t > 0, \tag{1.10}$$

respectively.

Due to dissipation of the momentum equation and the boundary condition, the kinetic energy is expected to vanish as time tends to infinity while the potential energy will converge to a constant. Furthermore, it is easy to see

$$\int_0^1 n_1(x, t)\mathrm{d}x = \int_0^1 n_{10}(x) = \int_0^1 n_2(x, t)\mathrm{d}x = \int_0^1 n_{20}(x)\mathrm{d}x = n_*,$$

due to the conservation law of the total mass. This suggests that the asymptotic state of $(n_1, j_1, n_2, j_2, E)(x, t)$ should be $(n_*, 0, n_*, 0, 0)$. In this paper, we first investigate the large-time behavior of weak solutions to (1.6)–(1.10), based on the existence results in [5].

Before stating the main results, we first introduce the definition of the entropy solutions to (1.6).

**Definition 1.1** *For every $T > 0$, we define a weak solution to* (1.6)–(1.10) *to be a pair of bounded measurable functions $(n_1, j_1, n_2, j_2, E)(x, t)$ which satisfies the following pair integral identities:*

$$\begin{aligned}
&\int_0^T\int_0^1 (n_1\phi_t + j_1\phi_x)\mathrm{d}x\mathrm{d}t + \int_0^1 n_{10}(x)\phi(x, 0)\mathrm{d}x = 0,\\
&\int_0^T\int_0^1 \Big(j_1\phi_t + \Big(\frac{j_1^2}{n_1} + P(n_1)\Big)\phi_x - j_1\phi + n_1E\phi\Big)\mathrm{d}x\mathrm{d}t + \int_0^1 j_{10}(x)\phi(x, 0)\mathrm{d}x = 0,\\
&\int_0^T\int_0^1 (n_2\phi_t + j_2\phi_x)\mathrm{d}x\mathrm{d}t + \int_0^1 n_{20}(x)\phi(x, 0)\mathrm{d}x = 0,\\
&\int_0^T\int_0^1 \Big(j_2\phi_t + \Big(\frac{j_2^2}{n_1} + P(n_2)\Big)\phi_x - j_2\phi - n_2E\phi\Big)\mathrm{d}x\mathrm{d}t + \int_0^1 j_{20}(x)\phi(x, 0)\mathrm{d}x = 0,\\
&E(x, t) = \int_{-\infty}^x (n_1 - n_2)(y, t)\mathrm{d}y + E(0, t)
\end{aligned}$$

*for any test function $\phi \in C_0^\infty(I_T)$ satisfying $\phi(x, T) = 0$ for $0 \le x \le 1$, and $\phi(0, T) = \phi(1, T) = 0$ for $t \ge 0$, where $I_T = (0, 1) \times (0, T)$, and $\frac{j_i}{n_i}$ $(i = 1, 2)$ vanish when $n_i = 0$ $(i = 1, 2)$. Moreover, $j_1$ and $j_2$ satisfy the boundary condition* (1.9) *in the sense of trace, and the following entropy inequality*

$$\eta_{et} + q_{ex} + \Big(\frac{j_1^2}{n_1} + \frac{j_2^2}{n_2} - j_1E + j_2E\Big) \le 0 \tag{1.11}$$

*holds in the sense of distribution. Here, the entropy-flux pair* $(\eta_e, q_e)$ *for* (1.11) *is associated with mechanical energy*

$$\begin{aligned} \eta_e &=: \sum_{i=1}^{2} \eta(n_i, j_i), \quad \eta(n, j) = \frac{j^2}{2n} + \frac{1}{\gamma - 1} n^\gamma, \\ q_e &=: \sum_{i=1}^{2} q(n_i, j_i), \quad q(n, j) = \frac{j^3}{2n^2} + \frac{\gamma}{\gamma - 1} n^{\gamma - 1} j. \end{aligned} \tag{1.12}$$

We are ready to state our main results.

**Theorem 1.1** *Let* $(n_1, j_1, n_2, j_2, E)(x, t)$ *be the* $L^\infty$ *entropy solution to the initial-boundary problem* (1.6)–(1.10)*, defined in Definition* 1.1*, satisfying*

$$\begin{aligned} &0 \le n_1, n_2 \le a_1 < \infty, \\ &|j_1| \le a_2 n_1, \quad |j_2| \le a_2 n_2, \end{aligned} \tag{1.13}$$

*where* $a_1, a_2$ *are positive constants. Then, there exist constants* $b_1, c_1 > 0$ *depending on* $\gamma, n_*, a_1$ *and the initial data such that*

$$\|(n_1 - n_*, j_1, n_2 - n_*, j_2, E)(\cdot, t)\|^2_{L^2([0,1])} \le b_1 \mathrm{e}^{-c_1 t}. \tag{1.14}$$

Next, we have known that the solutions to (1.6) with the initial data in large-time can be captured by the decoupled system

$$\begin{cases} \widetilde{n}_t = P(\widetilde{n})_{xx}, \\ \widetilde{j} = -P(\widetilde{n})_x, \end{cases} \tag{1.15}$$

where the first equation is the well-known porous medium equation while the second equation states Darcy's law. More precisely, when the initial data is small, smooth and away from a vacuum, the global existence and diffusion wave phenomena of the solutions to (1.6) with (1.7) were established by [3]. However, when the initial data is large or rough, shock will develop in finite time, and one has to consider weak entropy solutions. Using and modifying the arguments in [8–9], we recently studied the large-time behavior and the quasineutral limit of $L^\infty$ solution to the Cauchy problem in [6]. Here we also believe that the weak solutions to (1.6)–(1.10) in large-time converge to (1.15) with the initial-boundary conditions

$$\widetilde{n}(x, 0) = \widetilde{n}_0(x), \quad 0 < x < 1, \quad \widetilde{n}_x|_{x=0} = 0 = \widetilde{n}_x|_{x=1}. \tag{1.16}$$

As the second aim of this paper, we show that the weak entropy solutions to (1.6)–(1.10) converge to those of (1.15)–(1.16).

**Theorem 1.2** *Assume that the assumptions in Theorem* 1.1 *hold, and let* $(\widetilde{n}, \widetilde{j})$ *be the weak solution to* (1.15)–(1.16) *with* $\int_0^1 \widetilde{n}_0(x)\mathrm{d}x = n_*$, $0 \le \widetilde{n}_0(x) \le a_1$, *and* $\widetilde{j} = -P(\widetilde{n})_x$*. Then, there exist constants* $b_2, c_2 > 0$ *depending on* $\gamma, n_*, a_1$ *and the initial data such that*

$$\|(n_1 - \widetilde{n}, j_1 - \widetilde{j}, n_2 - \widetilde{n}, j_2 - \widetilde{j})(\cdot, t)\|^2_{L^2([0,1])} \le b_2 \mathrm{e}^{-c_2 t}. \tag{1.17}$$

Finally, as a by-product, we have the following quasi-neutral limit of the hydrodynamic models in the sense that the bounded $L^\infty$ entropy solution to the one-dimensional bipolar Euler-Poisson system converges to that of the corresponding one-dimensional compressible Euler equations with damping exponentially fast as $t \to +\infty$.

**Theorem 1.3** *Let the conditions in Theorem* 1.1 *hold. In addition, suppose that* $n_{10}(x) = n_{20}(x) = n_0(x)$ *and* $j_{10}(x) = j_{20}(x) = j_0(x)$*. Let* $(m, j)$ *be any bounded entropy solution to* (1.2) *with the initial data* $m(0, x) = n_0(x)$*,* $j(0, x) = j_0(x)$ *and the boundary condition* $j(0, t) = j(1, t) = 0$*. Then there exist constants* $b_3, c_3 > 0$ *such that*

$$\|(n_1 - m, j_1 - j, n_2 - m, j_2 - j)\|^2_{L^2([0,1])} \leq b_3 \mathrm{e}^{-c_3 t}. \tag{1.18}$$

**Remark 1.1** The assumption $0 \leq n_1, n_2 \leq a_1$ is essential in Theorems 1.1–1.3. Although this assumption seems natural, the uniform $L^\infty$ bounds of $n_i$ with respect to $t$ is still an open problem. However, away from a vacuum, the "small" smooth solution has been shown to exist in [11], and such a solution satisfies the assumption (1.13). The boundary conditions (1.9) and (1.10) are natural from the existence theorem. Furthermore, our initial data can contain a vacuum and can be arbitrarily large.

Using and modifying the arguments in [8–9, 14], we can show Theorem 1.1. However, in contrast with [14], we should overcome the difficulty from the coupling and cancellation interactions between $n_1$ and $n_2$, and face the additional electric field. Finally, directly applying the result in [14] and the triangle inequality, we also obtain the nonlinear diffusive wave phenomena and the quasineutral limit of general entropy solutions to the bipolar hydrodynamic model (1.6)–(1.10).

The rest of this paper is outlined as follows. In Section 2, the large-time behavior is established by the energy method and the entropy analysis. The asymptotic behavior and the quasineutral limit of solutions are presented in Section 3.

## 2 Large-Time Behavior

This section is devoted to the proof of Theorem 1.1. First we set

$$w_i = n_i - n_*, \quad z_i = j_i, \quad i = 1, 2,$$

which from (1.6) and (1.8) satisfy

$$\begin{cases} w_{1t} + z_{1x} = 0, \\ z_{1t} + \Big(\dfrac{j_1^2}{n_1}\Big)_x + (P(n_1) - P(n_*))_x + z_1 = (w_1 + n_*)E, \\ w_{2t} + z_{2x} = 0, \\ z_{2t} + \Big(\dfrac{j_2^2}{n_2}\Big)_x + (P(n_2) - P(n_*))_x + z_2 = -(w_2 + n_*)E, \\ E_x = w_1 - w_2 \end{cases} \tag{2.1}$$

and

$$\int_0^1 w_1(x, t)\mathrm{d}x = \int_0^1 w_2(x, t)\mathrm{d}x = 0.$$

Define the stream type functions

$$y_i(x, t) = -\int_0^x w_i(\xi, t)\mathrm{d}\xi, \quad i = 1, 2,$$

which imply that

$$y_{1x} = -w_1 = n_* - n_1, y_{1t} = z_1, \quad y_{2x} = -w_2 = n_* - n_2, \quad y_{2t} = z_2. \tag{2.2}$$

Since

$$\int_0^1 n_1(x,t)\mathrm{d}x = \int_0^1 n_2(x,t)\mathrm{d}x = \int_0^1 n_{10}(x)\mathrm{d}x = \int_0^1 n_{20}(x)\mathrm{d}x = n_*,$$

we have

$$y_1(0) = y_1(1) = y_2(0) = y_2(1) = 0, \tag{2.3}$$

which together with $E(0,t) = 0$ implies

$$E = y_2 - y_1. \tag{2.4}$$

Therefore, the second and fourth equations of (2.1) turn into

$$y_{1tt} + \Big(\frac{j_1^2}{n_1}\Big)_x + (P(n_1) - P(n_*))_x + y_{1t} = (w_1 + n_*)(y_2 - y_1), \tag{2.5}$$

$$y_{2tt} + \Big(\frac{j_2^2}{n_2}\Big)_x + (P(n_2) - P(n_*))_x + y_{2t} = -(w_2 + n_*)(y_2 - y_1). \tag{2.6}$$

Multiplying (2.5) and (2.6) by $y_1$ and $y_2$, and integrating over $(0,1)$, respectively, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \Big(y_{1t}y_1 + \frac{1}{2}y_1^2\Big)\mathrm{d}x - \int_0^1 y_{1t}^2\mathrm{d}x + \int_0^1 (P(n_1) - P(n_*))(n_1 - n_*)\mathrm{d}x\\ &= \int_0^1 \frac{j_1^2}{n_1}y_{1x}\mathrm{d}x + \int_0^1 (w_1 + n_*)(y_2 - y_1)y_1\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \Big(y_{2t}y_2 + \frac{1}{2}y_2^2\Big)\mathrm{d}x - \int_0^1 y_{2t}^2\mathrm{d}x + \int_0^1 (P(n_2) - P(n_*))(n_2 - n_*)\mathrm{d}x\\ &= \int_0^1 \frac{j_2^2}{n_2}y_{2x}\mathrm{d}x - \int_0^1 (w_2 + n_*)(y_2 - y_1)y_2\mathrm{d}x.\end{aligned}$$

Since $n_i, j_i = y_{it} \in L^\infty([0,1])$ $(i = 1,2)$, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \Big(y_{1t}y_1 + \frac{1}{2}y_1^2\Big)\mathrm{d}x - \int_0^1 y_{1t}^2\mathrm{d}x + \int_0^1 (P(n_1) - P(n_*))(n_1 - n_*)\mathrm{d}x\\ &= \int_0^1 \frac{n_*}{n_1}y_{1t}^2\mathrm{d}x - \int_0^1 y_{1t}^2\mathrm{d}x + \int_0^1 (w_1 + n_*)(y_2 - y_1)y_1\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \Big(y_{2t}y_2 + \frac{1}{2}y_2^2\Big)\mathrm{d}x - \int_0^1 y_{2t}^2\mathrm{d}x + \int_0^1 (P(n_2) - P(n_*))(n_2 - n_*)\mathrm{d}x\\ &= \int_0^1 \frac{n_*}{n_2}y_{2t}^2\mathrm{d}x - \int_0^1 y_{2t}^2\mathrm{d}x - \int_0^1 (w_2 + n_*)(y_2 - y_1)y_2\mathrm{d}x,\end{aligned}$$

which lead to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\Big(y_{1t}y_1+\frac{1}{2}y_1^2\Big)\mathrm{d}x+\int_0^1(P(n_1)-P(n_*))(n_1-n_*)\mathrm{d}x\\=&\int_0^1\frac{n_*}{n_1}y_{1t}^2\mathrm{d}x+\int_0^1(w_1+n_*)(y_2-y_1)y_1\mathrm{d}x\end{aligned}\tag{2.7}$$

and

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\Big(y_{2t}y_2+\frac{1}{2}y_2^2\Big)\mathrm{d}x+\int_0^1(P(n_2)-P(n_*))(n_2-n_*)\mathrm{d}x\\=&\int_0^1\frac{n_*}{n_2}y_{2t}^2\mathrm{d}x-\int_0^1(w_2+n_*)(y_2-y_1)y_2\mathrm{d}x.\end{aligned}\tag{2.8}$$

On the other hand, from $E=y_2-y_1$ and (2.2) we have

$$\begin{aligned}&\int_0^1(w_1+n_*)(y_2-y_1)y_1\mathrm{d}x-\int_0^1(w_2+n_*)(y_2-y_1)y_2\mathrm{d}x\\=&-\int_0^1 n_*(y_2-y_1)^2\mathrm{d}x+\int_0^1(y_2-y_1)(w_1y_1-w_2y_2)\mathrm{d}x\\=&-\int_0^1 n_*(y_2-y_1)^2\mathrm{d}x-\frac{1}{4}\int_0^1(y_1+y_2)_x(y_2-y_1)^2\mathrm{d}x\\=&-\int_0^1\frac{n_1+n_2+2n_*}{4}(y_1-y_2)^2\mathrm{d}x.\end{aligned}\tag{2.9}$$

Therefore, combination of (2.7)–(2.9) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\Big(y_{1t}y_1+\frac{1}{2}y_1^2+y_{2t}y_2+\frac{1}{2}y_2^2\Big)\mathrm{d}x+\int_0^1(P(n_1)-P(n_*))(n_1-n_*)\mathrm{d}x\\&+\int_0^1(P(n_2)-P(n_*))(n_2-n_*)\mathrm{d}x+\int_0^1\frac{n_1+n_2+2n_*}{4}(y_1-y_2)^2\mathrm{d}x\\=&\int_0^1\Big(\frac{n_*}{n_1}y_{1t}^2+\frac{n_*}{n_2}y_{2t}^2\Big)\mathrm{d}x.\end{aligned}\tag{2.10}$$

In order to deal with the nonlinearity, we now use the entropy inequality, rather than the usual energy method. We denote

$$\eta_*=\eta_e-\frac{1}{\gamma-1}P'(n_*)(n_1-n_*)-\frac{1}{\gamma-1}P'(n_*)(n_2-n_*)-\frac{2}{\gamma-1}P(n_*).$$

Thus, by the definition of the weak entropy solution in (1.12), the following entropy inequality holds in the sense of distribution

$$\eta_{*t}+\frac{P'(n_*)}{\gamma-1}(n_1-n_*)_t+\frac{P'(n_*)}{\gamma-1}(n_2-n_*)_t+q_{ex}+\frac{j_1^2}{n_1}+\frac{j_2^2}{n_2}-j_1E+j_2E\le 0.\tag{2.11}$$

By the conservation of mass and the theory of divergence-measure fields (cf. [1, 14]), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\eta_*(x,t)\mathrm{d}x+\int_0^1\Big(\frac{j_1^2}{n_1}+\frac{j_2^2}{n_2}\Big)\mathrm{d}x-\int_0^1E(j_1-j_2)\mathrm{d}x\le 0.\tag{2.12}$$

Moreover, we can compute

$$
\begin{aligned}
-\int_0^1 (j_1-j_2)E\mathrm{d}x &= \int_0^1 (z_1-z_2)(y_1-y_2)\mathrm{d}x = \int_0^1 (y_{1t}-y_{2t})(y_1-y_2)\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{(y_1-y_2)^2}{2}\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{E^2}{2}\mathrm{d}x. \qquad (2.13)
\end{aligned}
$$

Therefore, (2.12)–(2.13) yield

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \Big(\eta_*(x,t)+\frac{1}{2}E^2\Big)\mathrm{d}x + \int_0^1 \Big(\frac{y_{1t}^2}{n_1}+\frac{y_{2t}^2}{n_2}\Big)\mathrm{d}x \le 0. \qquad (2.14)
$$

Choosing $K=\max\{2, 2a_1+n_*\}$, and then adding (2.10) to (2.14)$\times K$, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \Big(K\eta_* + y_1y_{1t} + \frac{1}{2}y_1^2 + y_2y_{2t} + \frac{1}{2}y_2^2 + \frac{K}{2}E^2\Big)\mathrm{d}x \\
&+ \int_0^1 (P(n_1)-P(n_*))(n_1-n_*)\mathrm{d}x + \int_0^1 (P(n_2)-P(n_*))(n_2-n_*)\mathrm{d}x \\
&+ \int_0^1 \frac{n_1+n_2+2n_*}{4}E^2\mathrm{d}x + \int_0^1 \Big(\frac{K-n_*}{n_1}y_{1t}^2 + \frac{K-n_*}{n_2}y_{2t}^2\Big)\mathrm{d}x \le 0. \qquad (2.15)
\end{aligned}
$$

Further, using the expression of $\eta_*$, we get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \Big(\frac{K}{2n_1}y_{1t}^2 + y_1y_{1t} + \frac{1}{2}y_1^2 + \frac{K}{2n_2}y_{2t}^2 + y_2y_{2t} + \frac{1}{2}y_2^2 + \frac{K}{2}E^2 \\
&+ \frac{K}{\gamma-1}(P(n_1)-P(n_*)-P'(n_*)(n_1-n_*)+P(n_2)-P(n_*)-P'(n_*)(n_2-n_*))\Big)\mathrm{d}x \\
&+ \int_0^1 (P(n_1)-P(n_*))(n_1-n_*)\mathrm{d}x + \int_0^1 (P(n_2)-P(n_*))(n_2-n_*)\mathrm{d}x \\
&+ \int_0^1 \frac{n_1+n_2+2n_*}{4}(y_1-y_2)^2\mathrm{d}x + \int_0^1 \Big(\frac{K-n_*}{n_1}y_{1t}^2 + \frac{K-n_*}{n_2}y_{2t}^2\Big)\mathrm{d}x \le 0. \qquad (2.16)
\end{aligned}
$$

Clearly, Lemma 5.2 in [14] implies

$$
\begin{aligned}
&\int_0^1 \frac{K}{\gamma-1}(P(n_1)-P(n_*)-P'(n_*)(n_1-n_*)+P(n_2)-P(n_*)-P'(n_*)(n_2-n_*))\mathrm{d}x \\
\le{}& \frac{C_1K}{\gamma-1}\int_0^1 ((P(n_1)-P(n_*))(n_1-n_*)+(P(n_2)-P(n_*))(n_2-n_*))\mathrm{d}x.
\end{aligned}
$$

On the other hand, since $P$ is a convex function, Lemma 4.1 of [8–9] and Poincaré's inequality imply that there exist positive constants $C_2$ and $C_3$ such that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \Big(\frac{K}{2n_1}y_{1t}^2 + y_1y_{1t} + \frac{1}{2}y_1^2 + \frac{K}{2n_2}y_{2t}^2 + y_2y_{2t} + \frac{1}{2}y_2^2 + \frac{K}{2}E^2\Big)\mathrm{d}x \\
\le{}& C_2\int_0^1 \Big(\frac{K-n_*}{n_1}y_{1t}^2 + \frac{K-n_*}{n_2}y_{2t}^2\Big)\mathrm{d}x + \int_0^1 (y_1^2+y_2^2+\frac{1}{2}E^2)\mathrm{d}x \\
\le{}& C_2\int_0^1 \Big(\frac{K-n_*}{n_1}y_{1t}^2 + \frac{K-n_*}{n_2}y_{2t}^2 + \frac{1}{2}E^2\Big)\mathrm{d}x + \int_0^1 (y_{1x}^2+y_{2x}^2)\mathrm{d}x \\
\le{}& C_2\int_0^1 \Big(\frac{K-n_*}{n_1}y_{1t}^2 + \frac{K-n_*}{n_2}y_{2t}^2 + \frac{1}{2}E^2\Big)\mathrm{d}x
\end{aligned}
$$

$$+ C_3 \int_0^1 ((P(n_1) - P(n_*))(n_1 - n_*) + (P(n_2) - P(n_*))(n_2 - n_*))\mathrm{d}x. \tag{2.17}$$

Therefore, it follows that for $C_4 = \max\{C_2, C_3\}$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \Big(K\eta_* + y_1 y_{1t} + \frac{1}{2}y_1^2 + y_2 y_{2t} + \frac{1}{2}y_2^2 + \frac{1}{2}E^2\Big)\mathrm{d}x \\
&\le C_4\Big(\int_0^1 \Big(\frac{K - n_*}{n_1}y_{1t}^2 + \frac{K - n_*}{n_2}y_{2t}^2 + \frac{1}{2}E^2\Big)\mathrm{d}x \\
&\quad + \int_0^1 ((P(n_1) - P(n_*))(n_1 - n_*) + (P(n_2) - P(n_*))(n_2 - n_*))\mathrm{d}x\Big).
\end{aligned} \tag{2.18}$$

Therefore, from (2.16)–(2.18), we conclude that there exists a positive constant $C_5$ such that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \Big(K\eta_* + y_1 y_{1t} + \frac{1}{2}y_1^2 + y_2 y_{2t} + \frac{1}{2}y_2^2 + \frac{K}{2}E^2\Big)\mathrm{d}x \\
&+ C_5 \int_0^1 \Big(K\eta_* + y_1 y_{1t} + \frac{1}{2}y_1^2 + y_2 y_{2t} + \frac{1}{2}y_2^2 + \frac{K}{2}E^2\Big)\mathrm{d}x \le 0.
\end{aligned} \tag{2.19}$$

Furthermore, since $K > 2a_1 > 2n_i$ $(i = 1, 2)$, we know that

$$\begin{aligned}
&K\eta_* + y_1 y_{1t} + \frac{1}{2}y_1^2 + y_2 y_{2t} + \frac{1}{2}y_2^2 + \frac{1}{2}E^2 \\
&\ge 2y_{1t}^2 + y_1 y_{1t} + \frac{1}{2}y_1^2 + 2y_{2t}^2 + y_2 y_{2t} + \frac{1}{2}y_2^2 + \frac{1}{2}E^2 + \frac{K}{\gamma - 1}(P(n_1) \\
&\quad - P(n_*) - P'(n_*)(n_1 - n_*) + P(n_2) - P(n_*) - P'(n_*)(n_2 - n_*)) \\
&\ge y_{1t}^2 + y_{2t}^2 + E^2 + C_6((n_1 - n_*)^2 + (n_2 - n_*)^2),
\end{aligned} \tag{2.20}$$

where $C_6$ is a positive constant. Hence, (2.20) implies that

$$\int_0^1 \Big(K\eta_* + y_1 y_{1t} + \frac{1}{2}y_1^2 + y_2 y_{2t} + \frac{1}{2}y_2^2 + \frac{1}{2}E^2\Big)\mathrm{d}x \le C_7 \mathrm{e}^{-C_5 t} \tag{2.21}$$

and

$$\int_0^1 (y_{1t}^2 + y_{2t}^2 + E^2 + (n_1 - n_*)^2 + (n_2 - n_*)^2)\mathrm{d}x \le C_7 \mathrm{e}^{-C_5 t}. \tag{2.22}$$

This completes the proof of Theorem 1.1.

## 3 The Asymptotic Behavior and the Quasineutral Limit

In this section we are going to prove Theorems 1.2–1.3. We first state the large-time behavior of (1.15)–(1.16).

**Lemma 3.1** (cf. [14]) *Let $\widetilde{n}_0(x)$ satisfy $0 \le \widetilde{n}_0(x) \le a_3$ and $\int_0^1 \widetilde{n}_0(x)\mathrm{d}x = n_*$. Then for the global weak solution $\widetilde{n}(x,t)$ to (1.15)–(1.16) and $\widetilde{j} = -\widetilde{P}_x$, there exist positive constants $b_4$ and $c_4 > 0$ such that*

$$\int_0^1 ((\widetilde{n} - n_*)^2 + \widetilde{j}^2)\mathrm{d}x \le b_4 \mathrm{e}^{-c_4 t}. \tag{3.1}$$

Next, [14] also established the weak entropy solution and its asymptotic behavior for the compressible Euler equation (1.2) with proper initial-boundary value conditions. Let us introduce it as follows.

**Lemma 3.2** *Let $(m, j)(x, t)$ be the $L^\infty$ entropy solution to* (1.2) *with the initial data $m(0, x) = n_0(x),\ j(0, x) = j_0(x)$ and the boundary condition $j(0, t) = j(1, t) = 0$, satisfying*

$$\begin{aligned} &0 \le m \le a_4 < \infty, \\ &|j| \le a_5 m, \end{aligned}$$

*where $a_4, a_5$ are positive constants. Then, there exist constants $b_5, c_5 > 0$ depending on $\gamma, n_*, a_4$ and the initial data such that*

$$\|(m - n_*, j)(\cdot, t)\|^2_{L^2([0,1])} \le b_5 \mathrm{e}^{-c_5 t}. \tag{3.2}$$

**Proof of Theorem 1.2** Let $(n_1, j_1, n_2, j_2, E)$ be any $L^\infty$ entropy solution to (1.6)–(1.10). Theorem 1.1 implies,

$$\|(n_1 - n_*, n_2 - n_*)(x, t)\|^2_{L^2([0,1])} + \|(j_1, j_2)(x, t)\|^2_{L^2([0,1])} \le b_1 \mathrm{e}^{-c_1 t}. \tag{3.3}$$

On the other hand, if $\widetilde{n}(x, t)$ is the global weak solution to (1.15)–(1.16) and $\widetilde{j} = -\widetilde{P}_x$, Lemma 3.1 implies

$$\int_0^1 ((\widetilde{n} - n_*)^2 + \widetilde{j}^2)\mathrm{d}x \le b_4 \mathrm{e}^{-c_4 t}. \tag{3.4}$$

Hence, combining (3.3) and (3.4), and using the triangle inequality, we complete the proof of Theorem 1.2.

**Proof of Theorem 1.3** Let $(n_1, j_1, n_2, j_2, E)$ be any $L^\infty$ entropy solution to (1.6)–(1.10). Theorem 1.1 implies

$$\|(n_1 - n_*, n_2 - n_*)(x, t)\|^2_{L^2([0,1])} + \|(j_1, j_2)(x, t)\|^2_{L^2([0,1])} \le b_1 \mathrm{e}^{-c_1 t}. \tag{3.5}$$

On the other hand, if $(m, j)$ is any bound entropy solution to (1.2) with the initial data $m(0, x) = n_0(x),\ j(0, x) = j_0(x)$ and the boundary condition $j(0, t) = j(1, t) = 0$, Lemma 3.2 implies

$$\|m - n_*(x, t)\|^2_{L^2([0,1])} + \|j(x, t)\|^2_{L^2([0,1])} \le b_5 \mathrm{e}^{-c_5 t}. \tag{3.6}$$

Hence, combining (3.5) and (3.6), and using the triangle inequality, we complete the proof of Theorem 1.3.

**Acknowledgement** The author would like to express his sincere thanks to the referees for the suggestive comments on this paper.

## References

[1] Chen, G. Q. and Frid, H., Divergence-measure fields and hyperbolic conservation laws, *Arch. Rational Mech. Anal.*, **147**, 1999, 294–307.

[2] Cordier, S. and Grenier, E., Quasineutral limit of an Euler-Poisson system arising from plasma physics, *Comm. Partial Differential Equations*, **25**, 2000, 1099–1113.

[3] Gasser, I., Hsiao, L. and Li, H. L., Large-time behavior of solutions of the bipolar hydrodynamical model for semiconductors, *J. Differential Equations*, **192**, 2003, 326–359.

[4] Gasser, I. and Marcati, P., The combined relaxation and vanishing Debye length limit in the hydrodynamic model for semiconductors, *Math. Meth. Appl. Sci.*, **24**, 2001, 81–92.

[5] Hsiao, L. and Zhang, K. J., The global weak solution and relaxation limits of the initial-boundary value problem to the bipolar hydrodynamic model for semiconductors, *Math. Models Methods Appl. Sci.*, **10**, 2000, 1333–1361.

[6] Huang, F. M. and Li, Y. P., Large-time behavior and quasineutral limit of solutions to a bipolar hydrodynamic model with large data and vacuum, *Dis. Cont. Dyn. Sys. Sec. A*, **24**, 2009, 455–470.

[7] Huang, F. M., Mei, M. and Wang, Y., Large time behavior of solution to $n$-dimensional bipolar hydrodynamic model for semiconductors, *SIAM J. Math. Anal.*, **43**, 2011, 1595–1630.

[8] Huang, F. M. and Pan, R. H., Convergence rate for compressible Euler equations with damping and vacuum, *Arch. Rational Mech. Anal.*, **166**, 2003, 359–376.

[9] Huang, F. M. and Pan, R. H., Asymptotic behavior of the solutions to the damped compressible Euler equations with vacuum, *J. Differential Equations*, **220**, 2006, 207–233.

[10] Jüngel, A., Quasi-hydrodynamic semiconductor equations, Progress in Nonlinear Differential Equations, Birkhäuser Verlag, Basel, 2001.

[11] Li, Y. P., Global existence and asymptotic behavior of smooth solutions for a bipolar Euler-Poisson system in the quarter plane, *Boundary Value Problems*, **21**, 2012, 1–21.

[12] Markowich, P. A., Ringhofev, C. A. and Schmeiser, C., Semiconductor Equations, Springer-Verlag, Wien, New York, 1990.

[13] Natalini, R., The bipolar hydrodynamic model for semiconductors and the drift-diffusion equation, *J. Math. Anal. Appl.*, **198**, 1996, 262–281.

[14] Pan, R. H. and Zhao, K., Initial boundary value problem for compressible Euler equations with damping, *Indiana Univ. J. Math.*, **57**, 2008, 2257–2282.

[15] Peng, Y. J. and Wang, Y. G., Convergence of compressible Euler-Poisson equations to incompressible type Euler equations, *Asymptotic Anal.*, **41**, 2005, 141–160.

[16] Sitnko, A. and Malnev, V., Plasma Physics Theory, Chapman & Hall, London, 1995.

[17] Tsuge, N., Existence and uniqueness of stationary solutions to a one-dimensional bipolar hydrodynamic models of semiconductors, *Nonlinear Anal. TMA*, **73**, 2010, 779–787.

[18] Wang, S., Quasineutral limit of Euler-Poisson system with and without viscosity, *Comm. Partial Differential Equations*, **29**, 2004, 419–456.

[19] Zhou, F. and Li, Y. P., Existence and some limits of stationary solutions to a one-dimensional bipolar Euler-Poisson system, *J. Math. Anal. Appl.*, **351**, 2009, 480–490.

[20] Zhu, C. and Hattori, H., Stability of steady state solutions for an isentropic hydrodynamic model of semiconductors of two species, *J. Differential Equations*, **166**, 2000, 1–32.