Chin. Ann. Math.
34B(4), 2013, 541–556
DOI: 10.1007/s11401-013-0781-0

Chinese Annals of Mathematics, Series B

The Presentation Problem of the Conjugate Cone of the Hardy Space H^p $(0 < p \leq 1)$*

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Abstract The Hardy space H^p is not locally convex if $0 < p < 1$, even though its conjugate space $(H^p)^*$ separates the points of H^p. But then it is locally p-convex, and its conjugate cone $(H^p)^*_p$ is large enough to separate the points of H^p. In this case, the conjugate cone can be used to replace its conjugate space to set up the duality theory in the p-convex analysis. This paper deals with the representation problem of the conjugate cone $(H^p)^*_p$ of H^p for $0 < p \leq 1$, and obtains the subrepresentation theorem $(H^p)^*_p \simeq L^\infty(T, \mathbf{C}^*_p)$.

Keywords Locally p-convex space, Hardy space, Normed conjugate cone, Shadow cone, Subrepresentation theorem
2000 MR Subject Classification 46A16, 46A20, 32A35, 42B30

1 Introduction

If X is a locally convex space, then its conjugate space X^* is large enough to separate the points of X. But a non-locally convex space may only have the trivial conjugate space $\{0\}$. $L^p(\mu)$ is just such an example if $0 < p < 1$ and μ is a non-atomic measure (see [1–2]). For a Hardy space H^p $(0 < p < 1)$, though its conjugate space $(H^p)^* \neq \{0\}$ (see [3, p. 115]), it is still not locally convex (see [1, p. 37]). Motivated by the oddness of $L^p(\mu)$ and H^p, Simmons [4] and Jarchow [5, p. 108] introduced the concept of the locally p-convex space in the sixties of the last century. To remedy the shortcoming that the conjugate space X^* of a non-locally convex space X may be trivial, we first introduced in [6] the concept of the conjugate cone X^*_p, and proved that X^*_p is large enough to separate the points of X if X is locally p-convex. For a locally p-convex X, we can use the conjugate cone X^*_p to replace its conjugate space X^* (which may be trivial or very small) to set up the duality theory in the p-convex analysis. It is one of the most important problems to represent the conjugate cone X^*_p of a locally p-convex space. The main purpose of this paper is to represent the conjugate cone $(H^p)^*_p$ of the Hardy space H^p for $0 < p \leq 1$. The necessary basic theories of the p-convex analysis is presented in Section 2, the subrepresentation theorem $(H^p)^*_p \simeq L^\infty(T, \mathbf{C}^*_p)$ is obtained in Section 3.

2 Some Basic Theories of p-Convex Analysis

The locally p-convex spaces and their conjugate cones are the main concepts in this paper, while the separating theorem and the Hahn-Banach extension theorem are the basic theorems

Manuscript received February 12, 2012. Revised July 26, 2012.
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*Project supported by the National Natural Science Foundation of China (No. 10871141).

of the p-convex analysis.

Let X be a vector space over the number field $\mathbf{K}$, Φ be the empty set, 0 be the zero vector, the zero functional or the number zero, and $0 < p \le 1$ be a constant. A set $A \subset X$ is called p-convex if $[x,y]_p \subset A$ for every $x, y \in A$, where

$$[x,y]_p = \{\lambda x + (1-\lambda^p)^{\frac{1}{p}} y : \lambda \in [0,1]\}$$

is the p-segment arc with the endpoints x and y. For a general set A, the smallest p-convex set $\mathrm{co}_p A$ containing A is called the p-convex hull of A. It is easy to show that

$$\mathrm{co}_p A = \Big\{ \sum_{k=1}^{n} \lambda_k x_k : \ x_k \in A, \ \lambda_k \ge 0, \ \sum_{k=1}^{n} \lambda_k^p = 1, \ n \in \mathbf{N} \Big\}.$$

A topological vector space X is called locally p-convex if there exists a 0-neighborhood basis consisting of p-convex sets (see [5, p. 108]). It is easy to see that any locally p-convex space has a 0-neighborhood basis consisting of circled open p-convex sets.

A real-valued functional f on X is called a p-subseminorm if

(a) $f(x) \ge 0, \ x \in X$;

(b) $f(tx) = t^p f(x), \ t \ge 0, \ x \in X$ (positive p-homogeneity);

(c) $f(x+y) \le f(x) + f(y), \ x, y \in X$.

We use X'_p to denote the convex cone consisting of all p-subseminorms on X. If X is a topological vector space, then X^*_p is used to denote the subcone of X'_p consisting of continuous p-subseminorms, called the p-conjugate cone of X. A typical example of p-subseminorm is the p-Minkowski functional

$$P_{B_p}(x) = \inf\{t > 0 : x \in t^{\frac{1}{p}} B\}, \quad x \in X \tag{2.1}$$

generated by a p-convex algebraic 0-neighborhood B. It is easy to check that $P_{B_p} \in X^*_p$ if and only if $0 \in \mathrm{int}\, B$ (see [5, p. 106]). The study of p-convexity is called the p-convex analysis.

If there is a nonnegative real-valued functional $\|\cdot\|_p$ on X satisfying

(a′) $\|x\|_p = 0 \Leftrightarrow x = 0$;

(b′) $\|tx\|_p = |t|^p \|x\|_p, \ t \in \mathbf{K}, \ x \in X$ (absolute p-homogeneity);

(c′) $\|x+y\|_p \le \|x\|_p + \|y\|_p, \ x, y \in X$,

then $\|\cdot\|_p$ is called a p-norm and $(X, \|\cdot\|_p)$ a p-normed space. With the 0-neighborhood basis consisting of the p-convex sets

$$U_\varepsilon = \{x \in X : \ \|x\|_p < \varepsilon\}, \quad \varepsilon > 0,$$

a p-normed space $(X, \|\cdot\|_p)$ is locally p-convex. A complete p-normed space is called a p-Banach space (see [1, p. 7]). The spaces $L^p(\mu)$, l^p and H^p $(0 < p < 1)$ are three typical classes of p-Banach spaces.

The second separation theorem and the separating theorem given by [6] are the theoretical basis of the p-convex analysis. For the sake of completeness, let us give their proofs here.

Theorem 2.1 (The Second Separation Theorem in [6]) *Let $0 < p \le 1$, and (X, τ) be a locally p-convex space. Suppose that A is a nonempty closed p-convex subset of X (for $p = 1$,* *still assume $0 \in A$), and that B is another nonempty closed subset of X with $A \cap B = \Phi$. If A or B is compact, then they can be strongly separated by some continuous p-subseminorm, i.e. there is $f \in X_p^*$ such that*

(i) *if A is compact, then*

$$\max\{f(x) : x \in A\} < 1 \le \inf\{f(x) : x \in B\};$$

(ii) *if B is compact, then*

$$\sup\{f(x) : x \in A\} \le 1 < \min\{f(x) : x \in B\}.$$

Proof Let $\mathcal{U}_\theta = \{U_\lambda : \lambda \in \Lambda\}$ be a 0-neighborhood basis in X consisting of circled open p-convex sets. Because A or B is compact, we assert that there is a $U_{\lambda_0} \in \mathcal{U}_\theta$ such that

$$(A + U_{\lambda_0}) \cap B = \Phi.$$

If the assertion is not true, then there exist $x_\lambda \in A$, $y_\lambda \in B$ and $z_\lambda \in U_{\lambda_0}$ such that $y_\lambda = x_\lambda + z_\lambda$ for any $\lambda \in \Lambda$. Under the set-theoretic relation $U_1 \prec U_2 \Leftrightarrow U_1 \supset U_2$, $\mathcal{U}_\theta$ is a semi-order set. As $\prec$ is directed, $\{x_\lambda\}$, $\{y_\lambda\}$ and $\{z_\lambda\}$ turn into three nets and $z_\lambda \to 0$. If A is compact, there exists some convergent subnet of $\{x_\lambda\}$. Without loss of generality, suppose $x_\lambda \to x_0$, and then $y_\lambda = x_\lambda + z_\lambda \to x_0$. By the closeness of A and B, we have $x_0 \in A \cap B$. This is contrary to the fact that $B \cap A = \Phi$. If B is compact, with the same argument we can also find a $U_{\lambda_0} \in \mathcal{U}_\theta$ such that $A \cap (B + U_{\lambda_0}) = \Phi$. This is equivalent to $(A + U_{\lambda_0}) \cap B = \Phi$ as U_{λ_0} is circled.

Now take a $U_{\lambda_0} \in \mathcal{U}_\theta$ such that $(A + U_{\lambda_0}) \cap B = \Phi$, and take a $U_{\lambda_1} \in \mathcal{U}_\theta$ such that $U_{\lambda_1} + U_{\lambda_1} \subset U_{\lambda_0}$. Then by $(A + U_{\lambda_1} + U_{\lambda_1}) \cap B = \Phi$ and the circled property of U_{λ_1} we have

$$(A + U_{\lambda_1}) \cap (B + U_{\lambda_1}) = \Phi \tag{2.2}$$

and

$$\overline{(A + U_{\lambda_1})} \cap B = \Phi. \tag{2.3}$$

As $A + U_{\lambda_1}$ is an open p-convex 0-neighborhood, the p-Minkowski functional generated by $A + U_{\lambda_1}$ is continuous, i.e. $f := P_{(A+U_{\lambda_1})_p} \in X_p^*$. The equality (2.3) implies

$$f(x) < f(y), \quad x \in A,\ y \in B. \tag{2.4}$$

If A is compact, assume that f takes its maximum at $x_0 \in A$, then

$$\max\{f(x) : x \in A\} = f(x_0) < 1 \le \inf\{f(x) : x \in B\}.$$

If B is compact, assume that f takes its minimum at $y_0 \in B$, then

$$\sup\{f(x) : x \in A\} \le 1 < f(y_0) = \min\{f(x) : x \in B\}.$$

This completes the proof.

Theorem 2.2 (Separating Theorem in [6]) *Let $0 < p \le 1$ and X be a locally p-convex Hausdorff space. Then the conjugate cone X_p^* separates the points of X, i.e. for each pair of $x, y \in X$, $x \neq y$, there is $f \in X_p^*$ such that $f(x) \neq f(y)$.*

Proof As $x \neq y$, we have $y \notin [\theta, x]$ or $x \notin [\theta, y]$. Without loss of generality assume $y \notin [\theta, x]$. As X is of Hausdorff, the sets $A = [\theta, x]$ and $B = \{y\}$ satisfy the conditions of Theorem 2.1, so there is $f \in X_p^*$ such that $f(x) \neq f(y)$.

The conjugate space X^* of a locally p-convex space X may be trivial, but by Theorem 2.2 the p-conjugate cone X_p^* of X is large enough to separate the points of X. In this case X_p^* is simply called the conjugate cone of X. We hope to use conjugate cone X_p^* to replace the conjugate space X^* to set up the duality theory in the p-convex analysis.

Let us recall the concept of normed cones and the further details can be found in [7].

Definition 2.1 *Let Y be a proper cone with abstract addition and nonnegative scalar multiplication, i.e. $x + y \in Y$, $tx \in Y$ for every $x, y \in Y$ and $t \ge 0$, and $x + y = 0 \Leftrightarrow x = y = 0$. If there is a nonnegative real-valued functional $\|\cdot\|$ on Y satisfying*

(n$_1$) $\|x\| = 0 \Leftrightarrow x = 0$;

(n$_2$) $\|tx\| = t\|x\|,\ x \in Y,\ t \ge 0$;

(n$_3$) $\|x\|, \|y\| \le \|x + y\| \le \|x\| + \|y\|,\ x, y \in Y$,

then $\|\cdot\|$ is called a (conical) norm and $(Y, \|\cdot\|)$ a normed cone.

If $(Y, \|\cdot\|)$ is a normed cone, then it is not difficult to see that the equation

$$\rho(x, y) = \inf\{t > 0 : \exists h, l \in Y,\ \|h\|, \|l\| \le t \text{ such that } x + h = y + l\}, \quad x, y \in Y \tag{2.5}$$

defines a translation invariant metric on Y. Thus $(Y, \|\cdot\|) = (Y, \rho)$ is called a normed topological cone. Two normed cones Y_1 and Y_2 are said to be norm-preserving isomorphic if there exists an algebra isomorphism $T : Y_1 \to Y_2$ such that $\|T(x)\| = \|x\|$ for all $x \in Y_1$.

The following is a basic proposition on normed topological cones.

Proposition 2.1 *Let $(Y, \|\cdot\|) = (Y, \rho)$ be a normed topological cone and let $\{g_n\} \subset Y$ and $g \in Y$. Then*

(i) $g_n \to 0 \Leftrightarrow \|g_n\| \to 0$;

(ii) *$g_n \to g$ if and only if there are $\{h_n\}, \{l_n\} \subset Y$ with $h_n, l_n \to 0$ such that $g_n + h_n = g + l_n$ for every $n \in \mathbf{N}$*;

(iii) *the norm $\|\cdot\|$ is continuous on Y.*

Proof Suppose $g_n \to 0$ in the conical topology. Then by $\rho(g_n, 0) \to 0$ and (2.5), there are $\{h_n\}, \{l_n\} \subset Y$ with $\|h_n\|, \|l_n\| \to 0$ such that $g_n + h_n = 0 + l_n$ for every n. By (n$_3$) we have

$$\|g_n\| \le \|g_n + h_n\| = \|l_n\| \to 0.$$

Conversely if $\|g_n\| \to 0$, then by $g_n + 0 = 0 + g_n$ and (2.5) we have

$$\rho(g_n, 0) \le \|g_n\| \to 0, \quad \text{i.e. } g_n \to 0.$$

Thus (i) holds. The result (ii) follows (i) and definition (2.5). Suppose $g_n \to g$. Then by (ii) there are $\{h_n\}, \{l_n\} \subset Y$ with $h_n \to 0$, $l_n \to 0$ such that $g_n + h_n = g + l_n$ for every n. By (n_3)

$$\|g_n\| - \|h_n\| \leq \|g_n\| \leq \|g_n + h_n\| = \|g + l_n\| \leq \|g\| + \|l_n\|,$$
$$\|g\| - \|l_n\| \leq \|g\| \leq \|g + l_n\| = \|g_n + h_n\| \leq \|g_n\| + \|h_n\|,$$

and hence

$$|\|g_n\| - \|g\|| \leq \|h_n\| + \|l_n\| \to 0.$$

This completes the proof.

If $(X, \|\cdot\|_p)$ is a p-normed space, then by Theorem 2.2 its conjugate cone X_p^* is large enough to separate the points of X. With the conical norm

$$\|f\| = \sup_{\|x\|_p = 1} f(x), \quad f \in X_p^*, \tag{2.6}$$

$(X_p^*, \|\cdot\|)$ forms a normed topological cone, called the normed conjugate cone of X (see [7]).

By definition, it is not difficult to verify the following proposition.

Proposition 2.2 (see [5–6]) *Let $0 < p \leq 1$, X be a topological vector space and $f \in X_p'$. Then the following conditions are equivalent:*

(i) $f \in X_p^*$ (i.e. *f is continuous*);

(ii) *f is continuous at* 0;

(iii) $0 \in \mathrm{int} U_f(\varepsilon)$ *for all $\varepsilon > 0$, where*

$$U_f(\varepsilon) = \{x \in X : \ f(x) < \varepsilon\}$$

is the f-open ball of radius ε;

(iv) $\|f\| < \infty$ *if X is a p-normed space, and then $f(x) \leq \|f\| \|x\|_p$ for all $x \in X$.*

The Hahn-Banach type extension theorems of p-subseminorms given by [8] are the most important materials to set up the theoretical basis of the p-convex analysis. Considering that some people may not be able to read Chinese, let us prove them here.

Lemma 2.1 (Control Extension Theorem in [8]) *Let $0 < p \leq 1$, and Y be a subspace of a linear space X. Suppose $f \in Y_p'$, $F \in X_p'$ with $f(x) \leq F(x)$, $x \in Y$. Then there exists a $g \in X_p'$ such that*

$$g(x) = f(x), \quad x \in Y; \quad g(x) \leq F(x), \quad x \in X. \tag{2.7}$$

Proof As f, F are nonnegative, the functional

$$g(x) = \inf_{y \in Y} \{F(x - y) + f(y)\}, \quad x \in X \tag{2.8}$$

is well-defined. For every $x \in X$, if $t > 0$, then

$$g(tx) = \inf_{y \in Y} \{F(tx - y) + f(y)\} = t^p \inf_{y \in Y} \left\{F\left(x - \frac{y}{t}\right) + f\left(\frac{y}{t}\right)\right\} = t^p g(x);$$

if $t = 0$, $g(tx) = t^p g(x)(= 0)$ is obvious, so g has positive p-homogeneity. If $x_1, x_2 \in X$, then for every $\varepsilon > 0$, there exist $y_1, y_2 \in Y$ such that

$$g(x_i) + \frac{\varepsilon}{2} \geq F(x_i - y_i) + f(y_i), \quad i = 1, 2.$$

Thus

$$g(x_1) + g(x_2) + \varepsilon \geq F((x_1 + x_2) - (y_1 + y_2)) + f(y_1 + y_2) \geq g(x_1 + x_2).$$

Letting $\varepsilon \to 0$, we have

$$g(x_1 + x_2) \leq g(x_1) + g(x_2),$$

and hence $g \in X'_p$. If $x \in X$,

$$g(x) = \inf_{y \in Y} \{F(x - y) + f(y)\} \leq F(x - 0) + f(0) = F(x).$$

If $x \in Y$,

$$g(x) = \inf_{y \in Y} \{F(x - y) + f(y)\} \leq F(x - x) + f(x) = f(x).$$

On the other hand, by $f \leq F$ on Y and the subadditivity of f, we have

$$g(x) = \inf_{y \in Y} \{F(x - y) + f(y)\} \geq \inf_{y \in Y} \{f(x - y) + f(y)\} \geq f(x),$$

so (2.7) holds.

Theorem 2.3 (Norm-Preserving Extension Theorem in [8]) *Let $0 < p \leq 1$, and Y be a subspace of a p-normed space $(X, \|\cdot\|_p)$. Then for every $f \in Y_p^*$, there exists a $g \in X_p^*$ such that*

$$g(x) = f(x), \quad x \in Y; \quad \|g\| = \|f\|. \tag{2.9}$$

Proof Let $f \in Y_p^*$. By Proposition 2.2, $\|f\| < \infty$. Define

$$F(x) = \|f\|\|x\|_p, \quad x \in X,$$

then $F \in X_p^*$, $\|F\| = \|f\|$ and $f(x) \leq F(x)$, $x \in Y$. By Lemma 2.1, there exists a control extension $g \in X'_p$ of f such that

$$g(x) = f(x), \quad x \in Y; \quad g(x) \leq F(x), \quad x \in X.$$

The facts $F \in X_p^*$ and $g(x) \leq F(x)$ imply that g is continuous at 0. For every $x, y \in X$, by the subadditivity of g

$$|g(x) - g(y)| \leq \max\{g(x - y), g(y - x)\}.$$

Hence g is continuous on X or $g \in X_p^*$ and $\|g\| \leq \|F\| = \|f\|$. On the other hand

$$\|g\| = \sup_{\substack{x \in X \\ \|x\|_p = 1}} g(x) \geq \sup_{\substack{x \in Y \\ \|x\|_p = 1}} g(x) = \sup_{\substack{x \in Y \\ \|x\|_p = 1}} f(x) = \|f\|,$$

so $\|g\| = \|f\|$.

3 The Subrepresentation Theorem of Conjugate Cones of Hardy Spaces

To represent the conjugate cone X_p^* of a p-normed space $(X, \|\cdot\|_p)$ is one of the most important problems in the p-convex analysis. Based on the theory of the above section, we are going to find out the specific representation of the conjugate cone $(H^p)_p^*$ of the Hardy space H^p for $0 < p \leq 1$ in this section.

Let D be the open unit disk in the complex plane $\mathbf{C}$, T the unit circle and i the imaginary unit. Let (Σ, μ) be the Lebesgue measure ring on $[0, 2\pi]$. For a positive number $0 < p < \infty$, the Hardy space H^p is the vector space of analytic functions $\varphi : D \to \mathbf{C}$ with

$$\|\varphi\|_p = \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} |\varphi(r\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta < \infty.$$

When $p \geq 1$, the Hardy space H^p is a Banach space with the norm $\|\cdot\|_p^{\frac{1}{p}}$. In this paper we are interested in the Hardy space H^p for $0 < p \leq 1$, which is a p-normed space with $\|\cdot\|_p$. By Hardy's convexity theorem in [3, p. 9] we know that $\frac{1}{2\pi}\int_0^{2\pi} |\varphi(r\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta$ is an increasing function of r, so

$$\|\varphi\|_p = \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} |\varphi(r\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta, \quad \varphi \in H^p. \tag{3.1}$$

Suppose $0 < p \leq 1$. Let $L^p(T)$ be the L_p-space of complex functions on T with the normalized Haar measure $\frac{\mathrm{d}\theta}{2\pi}$, $\theta \in [0, 2\pi]$. Then $L^p(T)$ is a p-normed space with

$$\|\varphi\|_p = \frac{1}{2\pi} \int_0^{2\pi} |\varphi(\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta, \quad \varphi \in L^p(T). \tag{3.2}$$

For $0 < p < 1$, abundant literature uses F-norm $\|\cdot\| = \|\cdot\|_p^{\frac{1}{p}}$ to replace p-norm $\|\cdot\|_p$ in deducing the same topology (see [1–3]). We prefer $\|\cdot\|_p$ to $\|\cdot\|_p^{\frac{1}{p}}$ because the former satisfies the triangle inequality, but the latter does not.

For a function φ analytic in D and a point $\mathrm{e}^{\mathrm{i}\theta} \in T$, if $\varphi(z)$ tends to a unique limit, say $\varphi(\mathrm{e}^{\mathrm{i}\theta})$, as z tends to $\mathrm{e}^{\mathrm{i}\theta}$ inside D along any path not tangent to the circle T, then φ is said to have the nontangential limit at $\mathrm{e}^{\mathrm{i}\theta}$ (see [3, p. 6]). Summing up a few results located at different places of [3] we can obtain the following lemma.

Lemma 3.1 *Suppose $0 < p \leq 1$ and $\varphi \in H^p$. Then*

(i) *the nontangential limit $\varphi(\mathrm{e}^{\mathrm{i}\theta})$ exists almost everywhere on T and $\varphi(\mathrm{e}^{\mathrm{i}\theta}) \in L^p(T)$* (see [3, p. 17]);

(ii) *the boundary function $\varphi(\mathrm{e}^{\mathrm{i}\theta})$ satisfies* (see [3, p. 21])

$$\lim_{r \to 1} \int_0^{2\pi} |\varphi(r\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta = \int_0^{2\pi} |\varphi(\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta \tag{3.3}$$

and

$$\lim_{r \to 1} \int_0^{2\pi} |\varphi(r\mathrm{e}^{\mathrm{i}\theta}) - \varphi(\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta = 0. \tag{3.4}$$

If we use $\mathcal{H}^p$ to denote the set of boundary functions $\varphi(\mathrm{e}^{\mathrm{i}\theta})$ of $\varphi \in H^p$, then $\mathcal{H}^p$ is a closed subspace of $L^p(T)$ (see [3, p. 36]) and H^p is isometric to $\mathcal{H}^p$ by (3.3)–(3.4). So H^p is a closed proper subspace of $L^p(T)$ in the sense of isometry.

If $0 < p \leq 1$, the Hardy space H^p is a p-Banach space with the p-norm of (3.1). Theorem 2.2 means that its conjugate cone $(H^p)_p^*$ is very large to constitute a normed topological cone with

$$\|f\| = \sup_{\|\varphi\|_p = 1} f(\varphi), \quad f \in (H^p)_p^*. \tag{3.5}$$

The main purpose of this paper is to represent the normed conjugate cone $((H^p)_p^*, \|\cdot\|)$. Every $f \in (H^p)_p^*$ has no linearity except 0, it is almost impossible to give $(H^p)_p^*$ a complete representation, so we need to introduce the concept of subrepresentation.

Definition 3.1 *Let $0 < p \leq 1$, and $(X, \|\cdot\|_p)$ be a p-normed space. Let M be a subcone of X_p^*.*

(s$_1$) *If M separates the points of X, i.e. for each pair of $x, y \in X$, $x \neq y$, there is $F \in M$ such that $F(x) \neq F(y)$, then M is called a separating subcone of X_p^*.*

(s$_2$) *If M is a separating subcone of X_p^* and for each $f \in X_p^*$, there is $F \in M$ such that $f \leq F$ and $\|f\| = \|F\|$, then M is called a shadow cone or the subrepresentation of X_p^*, denoted by $X_p^* \simeq M$.*

If M is a shadow cone of X_p^*, it follows from (s$_1$), (s$_2$) and Proposition 2.2 that the topological structure of X is determined completely by M. The condition (s$_2$) means that the metric structure of X_p^* is also determined by M to a great extent. So a shadow cone of X_p^* is very similar to its shadow carrying almost all its properties, and once we find a shadow cone and obtain its representation we shall grasp X_p^* itself to a great extent.

Let us recall some known results. Suppose that X is a Banach space. If $p \geq 1$ and (Ω, μ) is a finite measure space, then the X-valued function space $L^p(\mu, X)$ is a Banach space with the norm

$$\|\varphi\| = \Big(\int_\Omega \|\varphi(t)\|^p \mathrm{d}\mu(t)\Big)^{\frac{1}{p}}, \quad \varphi \in L^p(\mu, X).$$

Its conjugate space $[L^p(\mu, X)]^*$ can be represented as

$$[L^p(\mu, X)]^* = L^q(\mu, X^*)$$

if X^* has the Radon-Nikodym property with respect to μ, where $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ (see [9, p. 98]). Thus for $p = 1$ we have

$$[L^1(\mu, X)]^* = L^\infty(\mu, X^*). \tag{3.6}$$

If $0 < p < 1$, then the p-normed X-valued sequence space $l^p(X)$ is a p-Banach space with

$$\|x\|_p = \sum_{n=1}^\infty \|x_n\|^p, \quad x = \{x_n\} \in l^p(X).$$

Its conjugate cone has the subrepresentation (see [10])

$$[l^p(X)]_p^* \simeq l^\infty(X_p^*). \tag{3.7}$$

If $0 < p < 1$, by separating Theorem 2.2 the conjugate cone $(H^p)_p^*$ of H^p is large enough to separate the points of H^p. Thus it is quite natural for us to ask:

Can the conjugate cone $(H^p)_p^*$ of H^p $(0 < p \le 1)$ be represented with the formula similar to (3.6) or (3.7)?

Let us find the shadow cone of $(H^p)_p^*$ for $0 < p \le 1$ now. Under the usual topology, the complex number field $\mathbf{C}$ is a p-normed space with $|\cdot|^p$, and its conjugate cone $\mathbf{C}_p^*$ forms a normed topological cone with the conical norm

$$\|f\| = \sup_{z\in T} f(z), \quad f \in \mathbf{C}_p^*.$$

Let $L^\infty(T, \mathbf{C}_p^*)$ denote the positive cone formed by all $\mathbf{C}_p^*$-valued essentially norm-bounded measurable functions (equivalent classes) on the unit circle T. Then with the essential norm

$$\|f\|_\infty = \inf_{\mu(E)=0} \sup_{\theta\in[0,2\pi]\setminus E} \|f(\mathrm{e}^{\mathrm{i}\theta})\|,$$

$L^\infty(T, \mathbf{C}_p^*)$ forms a normed cone. If we use $\langle f(\mathrm{e}^{\mathrm{i}\theta}), z\rangle$ to denote $f(\mathrm{e}^{\mathrm{i}\theta})(z)$ formally, then

$$\|f\|_\infty = \inf_{\mu(E)=0} \sup_{\theta\in[0,2\pi]\setminus E} \sup_{z\in T} \langle f(\mathrm{e}^{\mathrm{i}\theta}), z\rangle, \quad f \in L^\infty(T, \mathbf{C}_p^*). \tag{3.8}$$

We should note that the function $\langle f(\mathrm{e}^{\mathrm{i}\theta}), \cdot\rangle = f(\mathrm{e}^{\mathrm{i}\theta})(\cdot)$ is nonlinear and it is only a p-subseminorm on $\mathbf{C}$ for each $\theta \in [0, 2\pi]$. From the following theorems we shall see that $L^\infty(T, \mathbf{C}_p^*)$ is the shadow cone of $(H^p)_p^*$.

Theorem 3.1 *Suppose $0 < p < 1$. For each $f \in L^\infty(T, \mathbf{C}_p^*)$, the equation*

$$F_f(\varphi) = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} \langle f(\mathrm{e}^{\mathrm{i}\theta}), \varphi(r\mathrm{e}^{\mathrm{i}\theta})\rangle \mathrm{d}\theta, \quad \varphi \in H^p \tag{3.9}$$

defines a continuous p-subseminorm $F_f \in (H^p)_p^*$ with $\|F_f\| \le \|f\|_\infty$.

Proof Suppose $f \in L^\infty(T, \mathbf{C}_p^*)$. By (3.8) there is a sequence of null sets E_n such that

$$\sup_{\theta\in[0,2\pi]\setminus E_n} \sup_{z\in T} \langle f(\mathrm{e}^{\mathrm{i}\theta}), z\rangle \le \|f\|_\infty + \frac{1}{n}$$

for every $n \in \mathbf{N}$. For the null set $E_0 = \bigcup_{n=1}^{\infty} E_n$,

$$\|f\|_\infty \le \sup_{\theta\in[0,2\pi]\setminus E_0} \sup_{z\in T} \langle f(\mathrm{e}^{\mathrm{i}\theta}), z\rangle \le \|f\|_\infty + \frac{1}{n}, \quad n \in \mathbf{N},$$

so

$$\|f\|_\infty = \sup_{\theta\in[0,2\pi]\setminus E_0} \sup_{z\in T} \langle f(\mathrm{e}^{\mathrm{i}\theta}), z\rangle.$$

If $\varphi \in H^p$,

$$\langle f(\mathrm{e}^{\mathrm{i}\theta}), \varphi(r\mathrm{e}^{\mathrm{i}\theta})\rangle \le \|f\|_\infty |\varphi(r\mathrm{e}^{\mathrm{i}\theta})|^p, \quad \theta \in [0, 2\pi]\backslash E_0, \quad r \in [0, 1),$$

so $\int_0^{2\pi} \langle f(\mathrm{e}^{\mathrm{i}\theta}), \varphi(r\mathrm{e}^{\mathrm{i}\theta})\rangle \mathrm{d}\theta$ defines a function of r. By Lemma 3.1, the boundary function $\varphi(\mathrm{e}^{\mathrm{i}\theta}) \in L^p(T)$. From the nonnegativity of $\langle f(\mathrm{e}^{\mathrm{i}\theta}), \varphi(\mathrm{e}^{\mathrm{i}\theta})\rangle$ and

$$\int_0^{2\pi} \langle f(\mathrm{e}^{\mathrm{i}\theta}), \varphi(\mathrm{e}^{\mathrm{i}\theta})\rangle \mathrm{d}\theta \le \int_{[0,2\pi]\backslash E_0} \|f\|_\infty |\varphi(\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta < \infty,$$

the integral $\int_0^{2\pi} \langle f(\mathrm{e}^{\mathrm{i}\theta}), \varphi(\mathrm{e}^{\mathrm{i}\theta})\rangle \mathrm{d}\theta$ is convergent. By (3.4)

$$\begin{aligned} &\Big| \int_0^{2\pi} \langle f(\mathrm{e}^{\mathrm{i}\theta}), \varphi(r\mathrm{e}^{\mathrm{i}\theta})\rangle \mathrm{d}\theta - \int_0^{2\pi} \langle f(\mathrm{e}^{\mathrm{i}\theta}), \varphi(\mathrm{e}^{\mathrm{i}\theta})\rangle \mathrm{d}\theta \Big| \\ &\le \|f\|_\infty \int_{[0,2\pi]\backslash E_0} |\varphi(r\mathrm{e}^{\mathrm{i}\theta}) - \varphi(\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta \to 0, \quad r \to 1, \end{aligned}$$

so $F_f(\varphi)$ is well-defined and

$$F_f(\varphi) = \frac{1}{2\pi} \int_0^{2\pi} \langle f(\mathrm{e}^{\mathrm{i}\theta}), \varphi(\mathrm{e}^{\mathrm{i}\theta})\rangle \mathrm{d}\theta, \quad \varphi \in H^p. \tag{3.10}$$

As the boundary function $\varphi(\mathrm{e}^{\mathrm{i}\theta})$ is more abstract than the original one $\varphi(r\mathrm{e}^{\mathrm{i}\theta})$, we would rather use (3.9) than (3.10) to define $F_f(\varphi)$. Because

$$\langle f(\mathrm{e}^{\mathrm{i}\theta}), \cdot\rangle = f(\mathrm{e}^{\mathrm{i}\theta})(\cdot) \in \mathbf{C}_p^* \text{ a.e.}, \quad \theta \in [0, 2\pi],$$

the positive p-homogeneity and the subadditivity of F_f are clear, i.e. $F_f \in (H^p)'_p$. For every $0 \ne \varphi \in H^p$, by (3.9)

$$F_f(\varphi) \le \lim_{r\to 1} \frac{1}{2\pi} \int_{[0,2\pi]\backslash E_0} |\varphi(r\mathrm{e}^{\mathrm{i}\theta})|^p \|f\|_\infty \mathrm{d}\theta = \|f\|_\infty \|\varphi\|_p,$$

so $\|F_f\| \le \|f\|_\infty$ and $F_f \in (H^p)_p^*$ by Proposition 2.2. This completes the proof.

By (3.9) we have $F_{f_1+f_2} = F_{f_1} + F_{f_2}$ and $F_{af_1} = aF_{f_1}$ for every $f_1, f_2 \in L^\infty(T, C_p^*)$, $a \in \mathbf{R}^+$ and $F_f = 0 \Leftrightarrow f = 0$. So the mapping $f \to F_f$ is an algebraic isomorphism between $L^\infty(T, C_p^*)$ and the subcone $\{F_f : f \in L^\infty(T, C_p^*)\}$ of $(H^p)_p^*$. If we endow $L^\infty(T, C_p^*)$ with another norm

$$\|f\| = \|F_f\|, \quad f \in L^\infty(T, C_p^*), \tag{3.11}$$

then $(L^\infty(T, C_p^*), \|\cdot\|)$ is a new normed topological cone. From now on, we always treat $L^\infty(T, C_p^*)$ as the normed cone with this norm, and make no distinction between each $f \in L^\infty(T, \mathbf{C}_p^*)$ and the corresponding functional $F_f \in (H^p)_p^*$. By Theorem 3.1 we have the following corollary.

Corollary 3.1 *The normed topological cone $(L^\infty(T, C_p^*), \|\cdot\|)$ is a subcone of $(H^p)_p^*$ in the sense of norm-preserving isomorphism.*

Theorem 3.2 *Suppose $0 < p \le 1$. Then $L^\infty(T, \mathbf{C}_p^*)$ is a separating subcone of $(H^p)_p^*$.*

Proof Theorem 3.1 implies that $L^\infty(T, \mathbf{C}_p^*)$ is a subcone of $(H^p)_p^*$. Let $\varphi, \psi \in H^p$, $\varphi \ne \psi$. Then their boundary functions, still denoted by φ and ψ, respectively, belong to $L^p(T)$ and are not equal, i.e.

$$\mu(\{\theta \in [0, 2\pi] :\ \varphi(\mathrm{e}^{\mathrm{i}\theta}) \ne \psi(\mathrm{e}^{\mathrm{i}\theta})\}) > 0.$$

We are going to construct $f \in L^\infty(T, \mathbf{C}_p^*)$ such that $F_f(\varphi) \ne F_f(\psi)$ in the following three cases. If $\|\varphi\|_p < \|\psi\|_p$, define $f \in L^\infty(T, \mathbf{C}_p^*)$ by

$$\langle f(\mathrm{e}^{\mathrm{i}\theta}), z\rangle = |z|^p, \quad \theta \in [0, 2\pi], \quad z \in \mathbf{C},$$

then by (3.10)

$$F_f(\varphi) = \|\varphi\|_p < \|\psi\|_p = F_f(\psi).$$

If $\|\varphi\|_p = \|\psi\|_p$ and there exists an $E_0 \in \Sigma$ with $\mu(E_0) > 0$ such that

$$|\varphi(\mathrm{e}^{\mathrm{i}\theta})| < |\psi(\mathrm{e}^{\mathrm{i}\theta})| \text{ a.e.}, \quad \theta \in E_0,$$

then by [11, p. 104]

$$\int_{E_0} |\varphi(\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta < \int_{E_0} |\psi(\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta.$$

Take a sufficiently large number $M > 0$ such that

$$\int_{[0,2\pi]\setminus E_0} |\varphi(\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta + M\int_{E_0} |\varphi(\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta < \int_{[0,2\pi]\setminus E_0} |\psi(\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta + M\int_{E_0} |\psi(\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta.$$

Define $f \in L^\infty(T, \mathbf{C}_p^*)$ by

$$\langle f(\mathrm{e}^{\mathrm{i}\theta}), z\rangle = \begin{cases} |z|^p, & \theta \in [0, 2\pi]\setminus E_0,\ z \in \mathbf{C}, \\ M|z|^p, & \theta \in E_0,\ z \in \mathbf{C}, \end{cases}$$

then $F_f(\varphi) < F_f(\psi)$ by (3.10) and the above inequality.

Now suppose $\|\varphi\|_p = \|\psi\|_p$ and $|\psi(\mathrm{e}^{\mathrm{i}\theta})| = |\varphi(\mathrm{e}^{\mathrm{i}\theta})|$ a.e., $\theta \in [0, 2\pi]$. By $\varphi \ne \psi$, the set

$$E_0 = \{\theta \in [0, 2\pi] :\ \varphi(\mathrm{e}^{\mathrm{i}\theta}) \ne \psi(\mathrm{e}^{\mathrm{i}\theta}),\ |\psi(\mathrm{e}^{\mathrm{i}\theta})| = |\varphi(\mathrm{e}^{\mathrm{i}\theta})|(> 0)\}$$

has positive measure. For each $\theta \in E_0$, let

$$L(\theta) = \mathrm{span}\{\varphi(\mathrm{e}^{\mathrm{i}\theta}), \psi(\mathrm{e}^{\mathrm{i}\theta})\}$$

be the real linear hull spanned by $\varphi(\mathrm{e}^{\mathrm{i}\theta})$ and $\psi(\mathrm{e}^{\mathrm{i}\theta})$. Then $L(\theta)$ is either the whole complex plane $\mathbf{C}$ or a line across 0 according to the real linear relationship between $\varphi(\mathrm{e}^{\mathrm{i}\theta})$ and $\psi(\mathrm{e}^{\mathrm{i}\theta})$. Let

$$B(\theta) = \mathrm{co}_p\Big((L(\theta) \cap \overline{D}) \cup \Big\{\frac{2^{\frac{1}{p}}\varphi(\mathrm{e}^{\mathrm{i}\theta})}{|\varphi(\mathrm{e}^{\mathrm{i}\theta})|}\Big\}\Big),$$

where $\overline{D}$ is the closed unit disk in $\mathbf{C}$. Then $B(\theta)$ is a p-convex 0-neighborhood in $L(\theta)$. By Proposition 2.2, the p-Minkowski functional $P_{B(\theta)_p}$ generated by $B(\theta)$ belongs to $[L(\theta)]_p^*$. The relation $L(\theta)\cap\overline{D}\subset B(\theta)$ means $P_{B(\theta)_p}(z)\le 1$ for all $z\in L(\theta)\cap\overline{D}$. By the construction, $P_{B(\theta)_p}$ takes its minimum $\frac{1}{2}$ at the only point $\frac{\varphi(\mathrm{e}^{\mathrm{i}\theta})}{|\varphi(\mathrm{e}^{\mathrm{i}\theta})|}$ on the compact set $L(\theta)\cap T$, so

$$\frac{1}{2}=P_{B(\theta)_p}\Big(\frac{\varphi(\mathrm{e}^{\mathrm{i}\theta})}{|\varphi(\mathrm{e}^{\mathrm{i}\theta})|}\Big)<P_{B(\theta)_p}\Big(\frac{\psi(\mathrm{e}^{\mathrm{i}\theta})}{|\psi(\mathrm{e}^{\mathrm{i}\theta})|}\Big),\quad \theta\in E_0.$$

By Theorem 2.3, $P_{B(\theta)_p}$ can be extended to a continuous p-subseminorm $f(\mathrm{e}^{\mathrm{i}\theta})\in\mathbf{C}_p^*$ with $\|f(\mathrm{e}^{\mathrm{i}\theta})\|=\|P_{B(\theta)_p}\|\le 1$ for every $\theta\in E_0$.

Let ω be the mapping from $[0,2\pi]$ to T defined by $w(\theta)=\mathrm{e}^{\mathrm{i}\theta}$. To verify the measurability of f on $\omega[E_0]$, let

$$\varphi_n=\sum_{j=1}^{m_n}z_j^{(n)}X_{\omega[E_j^{(n)}]},\quad \psi_n=\sum_{j=1}^{m_n}w_j^{(n)}X_{\omega[E_j^{(n)}]}\in L^p(T)$$

be two sequences of simple functions such that

$$\varphi_n(\mathrm{e}^{\mathrm{i}\theta})\to\varphi(\mathrm{e}^{\mathrm{i}\theta}),\quad \psi_n(\mathrm{e}^{\mathrm{i}\theta})\to\psi(\mathrm{e}^{\mathrm{i}\theta})\ (n\to\infty)\text{ a.e.},\quad \theta\in E_0,$$

where $\{E_j^{(n)}\}$ is a measurable partition of E_0 for each n, $X_{\omega[E_j^{(n)}]}$ is the the characteristic function of $\omega[E_j^{(n)}]$ and $z_j^{(n)},w_j^{(n)}\in\mathbf{C}$ with $z_j^{(n)}\ne w_j^{(n)}$, $|z_j^{(n)}|=|w_j^{(n)}|(>0)$. For each $n\in\mathbf{N}$ and $\theta\in E_j^{(n)}$, let

$$L_n(\theta)=\operatorname{span}\{z_j^{(n)},w_j^{(n)}\}$$

and

$$B_n(\theta)=\mathrm{co}_p\Big(\Big(L_n(\theta)\cap\overline{D}\Big)\cup\Big\{\frac{2^{\frac{1}{p}}z_j^{(n)}}{|z_j^{(n)}|}\Big\}\Big).$$

Then with the same argument we have $P_{B_n(\theta)_p}\in[L_n(\theta)]_p^*$ and its norm-preserving extension $f_n(\mathrm{e}^{\mathrm{i}\theta})\in\mathbf{C}_p^*$. By the construction, f_n can be chosen to be the $\mathbf{C}_p^*$-valued simple functions on $\omega[E_0]$ such that $f_n(\mathrm{e}^{\mathrm{i}\theta})\to f(\mathrm{e}^{\mathrm{i}\theta})(n\to\infty)$ a.e., $\theta\in E_0$, so f is measurable on $\omega[E_0]$.

If $\theta\in[0,2\pi]\backslash E_0$, define $f(\mathrm{e}^{\mathrm{i}\theta})=0\in\mathbf{C}_p^*$. Then $f\in L^\infty(T,\mathbf{C}_p^*)$ with $\|f\|_\infty\le 1$. Now from $|\varphi(\mathrm{e}^{\mathrm{i}\theta})|=|\psi(\mathrm{e}^{\mathrm{i}\theta})|$ a.e., $\theta\in E_0$ and

$$\frac{1}{2}=\Big\langle f(\mathrm{e}^{\mathrm{i}\theta}),\frac{\varphi(\mathrm{e}^{\mathrm{i}\theta})}{|\varphi(\mathrm{e}^{\mathrm{i}\theta})|}\Big\rangle<\Big\langle f(\mathrm{e}^{\mathrm{i}\theta}),\frac{\psi(\mathrm{e}^{\mathrm{i}\theta})}{|\psi(\mathrm{e}^{\mathrm{i}\theta})|}\Big\rangle,\quad \theta\in E_0,$$

we have

$$\begin{aligned}F_f(\varphi)&=\frac{1}{2\pi}\int_{E_0}|\varphi(\mathrm{e}^{\mathrm{i}\theta})|^p\Big\langle f(\mathrm{e}^{\mathrm{i}\theta}),\frac{\varphi(\mathrm{e}^{\mathrm{i}\theta})}{|\varphi(\mathrm{e}^{\mathrm{i}\theta})|}\Big\rangle\mathrm{d}\theta\\&<\frac{1}{2\pi}\int_{E_0}|\psi(\mathrm{e}^{\mathrm{i}\theta})|^p\Big\langle f(\mathrm{e}^{\mathrm{i}\theta}),\frac{\psi(\mathrm{e}^{\mathrm{i}\theta})}{|\psi(\mathrm{e}^{\mathrm{i}\theta})|}\Big\rangle\mathrm{d}\theta=F_f(\psi).\end{aligned}$$

This completes the proof of $L^\infty(T,\mathbf{C}_p^*)$ separating the points of H^p. This completes the proof.

Let us make a preparation for the next theorem.

Lemma 3.2 *Let $0 < p \leq 1$ and $f \in [L^p(T)]_p^*$. Define*

$$q(z) = \begin{cases} 0, & z = 0, \\ \sup_{\|l\|_p \leq \frac{1}{|z|^p}} f(lz), & z \neq 0 \end{cases} \tag{3.12}$$

and

$$Q(z) = q(z)|z|^p, \quad z \in \mathbf{C}, \tag{3.13}$$

where $l \in L^p(T)$ is a nonnegative real-valued function on T. Then

(i) $q(tz) = q(z)$ *for all* $t > 0$ *and* $z \in \mathbf{C}$;
(ii) $q(z) \leq \|f\|$ *for all* $z \in \mathbf{C}$;
(iii) $Q \in \mathbf{C}_p^*$ *and* $\|Q\| \leq \|f\|$.

Proof The conclusions (i)–(ii) can be obtained directly from (3.12) and (2.6). The positive p-homogeneity of Q follows (i). Suppose that Q satisfies the subadditivity. Then the result (ii) means $\|Q\| \leq \|f\|$ and $Q \in \mathbf{C}_p^*$ by Proposition 2.2.

Now nothing remains but to check the subadditivity of Q, i.e.

$$q(z+w)|z+w|^p \leq q(z)|z|^p + q(w)|w|^p, \quad z, w \in \mathbf{C}. \tag{3.14}$$

If $z = 0$, $w = 0$ or $z+w = 0$, the inequality (3.14) is clear. Suppose $z \neq 0$, $w \neq 0$ and $z+w \neq 0$. Let α be the included angle between z and $z+w$, and β be that between w and $z+w$. If $\alpha = 0$, then there is a $t > 0$ such that $z + w = tz$ and $q(z+w) = q(z)$. By $w \neq 0$ we know $t \neq 1$. If $0 < t < 1$, then $|z+w| < |z|$ and (3.14) is clear; if $t > 1$, then

$$w = (t-1)z, \quad q(z) = q(w) = q(z+w),$$

and thus we have (3.14) by $|z+w|^p \leq |z|^p + |w|^p$. If $\alpha = \pi$, without loss of generality, assume that $|w| > |z|$. Thus we have (3.14) by $|z+w| < |w|$ and $q(z+w) = q(w)$. If $\beta = 0$ or π. Then (3.14) also holds with the same argument. If $\alpha, \beta \in (0, \pi)$, then by the Sine theorem

$$|z| = \frac{\sin\beta}{\sin(\alpha+\beta)}|z+w|$$

and

$$|w| = \frac{\sin\alpha}{\sin(\alpha+\beta)}|z+w|.$$

Then (3.14) is equivalent to

$$q(z+w) \leq \Big(\frac{\sin\beta}{\sin(\alpha+\beta)}\Big)^p q(z) + \Big(\frac{\sin\alpha}{\sin(\alpha+\beta)}\Big)^p q(w). \tag{3.15}$$

By

$$\frac{z+w}{|z+w|} = \frac{\sin\beta}{\sin(\alpha+\beta)}\frac{z}{|z|} + \frac{\sin\alpha}{\sin(\alpha+\beta)}\frac{w}{|w|},$$

we have

$$\begin{aligned} q(z+w) &= q\Big(\frac{z+w}{|z+w|}\Big) = \sup_{\|l\|_p\le 1} f\Big(l\frac{z+w}{|z+w|}\Big) \\ &\le \sup_{\|l\|_p\le 1} f\Big(l\frac{\sin\beta}{\sin(\alpha+\beta)}\frac{z}{|z|}\Big) + \sup_{\|l\|_p\le 1} f\Big(l\frac{\sin\alpha}{\sin(\alpha+\beta)}\frac{w}{|w|}\Big) \\ &= \Big(\frac{\sin\beta}{\sin(\alpha+\beta)}\Big)^p q\Big(\frac{z}{|z|}\Big) + \Big(\frac{\sin\alpha}{\sin(\alpha+\beta)}\Big)^p q\Big(\frac{w}{|w|}\Big) \\ &= \Big(\frac{\sin\beta}{\sin(\alpha+\beta)}\Big)^p q(z) + \Big(\frac{\sin\alpha}{\sin(\alpha+\beta)}\Big)^p q(w). \end{aligned}$$

Thus we have shown the inequalities (3.14)–(3.15) and the conclusion (iii).

Theorem 3.3 *Suppose* $0 < p \le 1$. *Then for each* $f \in (H^p)^*_p$, *there is* $F \in L^\infty(T, \mathbf{C}^*_p)$ *such that* $f \le F$ *and* $\|F\| = \|f\|$.

Proof Suppose $f \in (H^p)^*_p$. As H^p is a closed subspace of $L^p(T)$, by Theorem 2.3 f can be extended norm-preservingly to a continuous p-subseminorm on $L^p(T)$, still denoted by f. Use the function $Q \in \mathbf{C}^*_p$ in Lemma 3.2 to define a $\mathbf{C}^*_p$-valued constant function g on T by

$$\langle g(\mathrm{e}^{\mathrm{i}\theta}), z\rangle = g(\mathrm{e}^{\mathrm{i}\theta})(z) = Q(z), \quad \theta \in [0, 2\pi],\ z \in \mathbf{C}.$$

Then $g \in L^\infty(T, \mathbf{C}^*_p)$, $\|g\|_\infty = \|Q\| \le \|f\|$ and

$$\frac{1}{2\pi} Q(z)\mu(E) = \frac{1}{2\pi}\int_E \langle g(\mathrm{e}^{\mathrm{i}\theta}), z\rangle \mathrm{d}\theta, \quad E \in \Sigma,\ z \in \mathbf{C}. \tag{3.16}$$

By Theorem 3.1, the equation

$$F_g(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \langle g(\mathrm{e}^{\mathrm{i}\theta}), \varphi(\mathrm{e}^{\mathrm{i}\theta})\rangle \mathrm{d}\theta, \quad \varphi \in H^p \tag{3.17}$$

defines a functional $F_g \in L^\infty(T, \mathbf{C}^*_p) \subset (H^p)^*_p$ with $\|F_g\| \le \|g\|_\infty \le \|f\|$. Now if we can verify the inequality of $f \le F_g$ on H^p, then we have the equality $\|F_g\| = \|f\|$, and this completes the proof.

As $H^p \subset L^p(T)$, and f and F_g (being extended norm-preservingly to $L^p(T)$) are continuous on $L^p(T)$, let us verify the inequality $f(\varphi) \le F_g(\varphi)$ for all $\varphi \in L^p(T)$ in the following three cases. If $\varphi = \sum\limits_{j=1}^n z_j X_{\omega[E_j]} \in L^p(T)$ is a simple function, then by the equalities (3.16)–(3.17),

$$\begin{aligned} f(\varphi) &\le \sum_{j=1}^n f(z_j X_{\omega[E_j]}) \le \sum_{j=1}^n \sup_{\|l\|_p \le \frac{1}{|z_j|^p}} f(lz_j)\|z_j X_{\omega[E_j]}\|_p \\ &= \frac{1}{2\pi}\sum_{j=1}^n Q(z_j)\mu(E_j) = \frac{1}{2\pi}\int_0^{2\pi} \langle g(\mathrm{e}^{\mathrm{i}\theta}), \varphi(\mathrm{e}^{\mathrm{i}\theta})\rangle \mathrm{d}\theta = F_g(\varphi). \end{aligned}$$

If $\varphi \in L^p(T)$ is bounded, then there exists a sequence of uniformly bounded simple functions $\{\varphi_n\}$ on T such that

$$\varphi_n(\mathrm{e}^{\mathrm{i}\theta}) \to \varphi(\mathrm{e}^{\mathrm{i}\theta})(n \to \infty), \quad \text{a.e. } \theta \in [0, 2\pi].$$

By Egoroff's theorem in [9, p. 41], $\{\varphi_n\}$ also converges in mean to φ, i.e.

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} |\varphi_n(\mathrm{e}^{\mathrm{i}\theta}) - \varphi(\mathrm{e}^{\mathrm{i}\theta})| \mathrm{d}\theta = 0.$$

By the Hölder's inequality

$$\begin{aligned} \|\varphi_n - \varphi\|_p &= \frac{1}{2\pi} \int_0^{2\pi} |\varphi_n(\mathrm{e}^{\mathrm{i}\theta}) - \varphi(\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta \\ &\le \frac{1}{(2\pi)^p} \Big(\int_0^{2\pi} |\varphi_n(\mathrm{e}^{\mathrm{i}\theta}) - \varphi(\mathrm{e}^{\mathrm{i}\theta})| \mathrm{d}\theta \Big)^p \to 0, \end{aligned}$$

i.e. $\{\varphi_n\}$ converges in the p-norm to φ. We have already shown $f(\varphi_n) \le F_g(\varphi_n)$ for all n, so by the continuity of f and F_g we have

$$f(\varphi) = \lim_{n\to\infty} f(\varphi_n) \le \lim_{n\to\infty} F_g(\varphi_n) = F_g(\varphi).$$

If $\varphi \in L^p(T)$ is a general complex function on T, then the measure ν defined by the indefinite integral

$$\nu(E) = \frac{1}{2\pi} \int_E |\varphi(\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta, \quad E \in \Sigma$$

is absolutely continuous with respect to the Lebesgue measure μ, and the series

$$\sum_{n=1}^{\infty} \mu(\{\theta \in [0, 2\pi] : \ |\varphi(\mathrm{e}^{\mathrm{i}\theta})|^p > n\})$$

is convergent (see [11, p. 115]). So $\mu(\{\theta \in [0, 2\pi] : |\varphi(\mathrm{e}^{\mathrm{i}\theta})|^p > n\}) \to 0 (n \to \infty)$. Let

$$\varphi_n(\mathrm{e}^{\mathrm{i}\theta}) = \begin{cases} \varphi(\mathrm{e}^{\mathrm{i}\theta}), & |\varphi(\mathrm{e}^{\mathrm{i}\theta})|^p \le n, \\ 0, & |\varphi(\mathrm{e}^{\mathrm{i}\theta})|^p > n, \end{cases} \quad \theta \in [0, 2\pi].$$

Then φ_n is bounded and

$$\begin{aligned} \|\varphi_n - \varphi\|_p &= \frac{1}{2\pi} \int_0^{2\pi} |\varphi_n(\mathrm{e}^{\mathrm{i}\theta}) - \varphi(\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta \\ &= \frac{1}{2\pi} \int_{\{\theta\in[0,2\pi]:|\varphi(\mathrm{e}^{\mathrm{i}\theta})|^p > n\}} |\varphi(\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta \to 0. \end{aligned}$$

Thus by the continuity of f and F_g, we have

$$f(\varphi) = \lim_{n\to\infty} f(\varphi_n) \le \lim_{n\to\infty} F_g(\varphi_n) = F_g(\varphi).$$

This completes the proof of Theorem 3.3.

Now we are in the position to show the main result of this paper.

Theorem 3.4 (The Subrepresentation Theorem) *Suppose* $0 < p \le 1$. *Then* $L^\infty(T, C_p^*)$ *is a shadow cone of* $(H^p)_p^*$ *in the sense of norm-preserving isomorphism, or* $(H^p)_p^*$ *has the subrepresentation*

$$(H^p)_p^* \simeq L^\infty(T, \mathbf{C}_p^*). \tag{3.18}$$

Proof By Theorems 3.1–3.2, $L^\infty(T, \mathbf{C}_p^*)$ is a separating subcone of $(H^p)_p^*$. By Theorem 3.3 for each $f \in (H^p)_p^*$, there exists an $F \in L^\infty(T, \mathbf{C}_p^*)$ such that $f(\varphi) \leq F(\varphi)$ for all $\varphi \in H^p$ and $\|f\| = \|F\|$. Thus by Definition 3.1, $L^\infty(T, \mathbf{C}_p^*)$ is a shadow cone of $(H^p)_p^*$. So we have the subrepresentation (3.18).

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